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# Inequalities for Fractional Integrals of a Generalized Class of Strongly Convex Functions

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**Abstract:** Fractional integral operators are useful tools for generalizing classical integral inequalities. Convex functions play very important role in the theory of mathematical inequalities. This paper aims to investigate the Hadamard type inequalities for a generalized class of functions namely strongly $(\alpha, h-m)$-$p$-convex functions by using Riemann–Liouville fractional integrals. The results established in this paper give refinements of various well-known inequalities which have been published in the recent past.

**Keywords:** Riemann–Liouville integrals; Hadamard inequality; strongly convex function; convex function

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## 1. Introduction

Convex functions are very important in the study of mathematical inequalities and in solving the problems of optimization theory. Many well-known inequalities are direct consequences of these functions. Convex functions are further analyzed to define new classes of functions which are helpful in studying the extensions and generalizations of classical results. Because of their fascinating properties, convex functions are used in almost all areas of mathematics including analysis, optimization theory, and graph theory, etc. A refinement of convex function is the notion of strongly convex function introduced by Polyak in [1]. For more details one can see [1,2]. By using the definition of strongly convex functions it is quite possible to obtain refinements of inequalities which have been established for convex functions in the literature.

The Hadamard inequality is the most popular inequality which is studied for new classes of functions defined after motivating by analytic representation of convex functions given in (1). Consequently, a lot of generalizations, refinements, and extensions of the Hadamard inequality can be found in the literature.

Inspired by a rich literature dedicated to convex functions and the Hadamard inequality, in this paper we aim to define a new class of functions namely strongly $(\alpha, h-m)$-$p$-convex functions. This will generate many well known classes of functions such as; $(\alpha, m)$-convex [3], $(s, m)$-convex [4], $(h-m)$-convex [5], $(p-h)$-convex [6], $h$-convex [7], $p$-convex [8], and harmonically convex functions [9] which are further linked with classical definitions of several types of convex functions. The newly defined class of functions also provide refinements of all aforementioned functions. Hence a strongly $(\alpha, h-m)$-$p$-convex function unifies several types of convexities and strongly convexities.

Moreover, it is used to establish new versions of the Hadamard type inequalities for Riemann–Liouville fractional integrals. These inequalities will work as generalizations as well as refinements of a lot of versions of this inequality which have been established in recent decades.

The Hadamard type inequalities which we have proved in this paper unify numerous published versions of such inequalities already exist in the literature of fractional integral inequalities. Following definitions and results will be useful to obtain connections and understandings with the findings of this paper.

**Definition 1.** *Let $I \subseteq \mathbb{R}$ be an interval in $\mathbb{R}$. Then a real valued function $f : I \to \mathbb{R}$ is said to be convex function, if*

$$f(at + (1-t)b) \leq tf(a) + (1-t)f(b), \tag{1}$$

*for all $a, b \in I$ and $t \in [0,1]$. If (1) holds in reverse direction, then $f$ will be called concave function.*

Every convex function $f$ on an interval $[a, b]$ can be modified at the endpoints to become convex and continuous. An immediate consequence of this fact is the Riemann integrability of $f$. The Riemann integral of convex function $f$ is estimated by the Hadamard inequality stated in the following theorem, see (Section 1.9 [10]).

**Theorem 1.** *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

*If $f$ is concave, then the inequality (2) holds in reverse direction.*

**Definition 2** ([11])**.** *A function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0,1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \tag{3}$$

*for all $x, y \in [0, b]$ and $t \in [0,1]$. We say that $f$ is m-concave if $-f$ is m-convex.*

The definitions of strongly convex functions of different kinds are given as follows:

**Definition 3** ([1])**.** *Let $(\mathbb{X}, ||.||)$ be a normed space and $E$ be a convex subset of $\mathbb{X}$. A function $f : E \to \mathbb{R}$ will be called strongly convex function with modulus $c \geq 0$, if*

$$f(xt + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)||y-x||^2 \tag{4}$$

*holds $\forall\ x, y \in E$, $t \in [0,1]$.*

**Definition 4** ([12])**.** *A function $\psi : I \to \mathbb{R}$ is called strongly m-convex function with modulus $C \geq 0$ if*

$$\psi(za + m(1-z)b) \leq z\psi(a) + m(1-z)\psi(b) - Cmz(1-z)|a-b|^2.$$

*for $a, b \in I$ and $z \in [0,1]$.*

**Definition 5** ([13])**.** *A function $f : [0, +\infty) \to \mathbb{R}$ is said to be strongly $(s, m)$-convex function, with modulus $\lambda \geq 0$, for $(s, m) \in [0,1]^2$, if*

$$f(ta + m(1-t)b) \leq t^s f(a) + m(1-t)^s f(b) - \lambda mt(1-t)|b-a|^2. \tag{5}$$

*holds for all $a, b \in [0, +\infty)$ and $t \in [0,1]$.*

**Definition 6** ([14])**.** *A function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be strongly $(\alpha, m)$-convex, where $(\alpha, m) \in [0, 1]^2$ if*

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y) - \lambda m t^\alpha (1-t^\alpha)|y-x|^2, \tag{6}$$

*holds for all $x, y \in [0, b]$ and $t \in [0, 1]$.*

**Definition 7** ([15])**.** *Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : [0, b] \to \mathbb{R}$ is called strongly $(h - m)$-convex function with modulus $\lambda \geq 0$, if $f$ is non-negative and for all $x, y \in [0, b], t \in (0, 1)$ and $m \in [0, 1]$, one has*

$$f(tx + m(1-t)y) \leq h(t)f(x) + mh(1-t)f(y) - m\lambda h(t)h(1-t)|y-x|^2. \tag{7}$$

Several variants of inequality (2) in the form of generalizations, extensions and refinements have been published, see [6,16–19] and references therein. In [20], the notion of $p$-convex function is introduced and in [6] it is extended to the notion of $(p, h)$-convex function. Likewise many such classes of functions are defined to establish the Hadamard type inequalities, see [17,21–23]. In recent decades, the Hadamard inequality has been studied for different kinds of fractional integral operators, see [16,24–28].

The classical Riemann–Liouville integrals of fractional order and the Hadamard inequalities for these integrals are given in the following definition and theorem, respectively:

**Definition 8** ([29])**.** *Let $f \in L_1[a, b]$. Then the left and right sided Riemann–Liouville integrals of the function $f$ of fractional order $\mu > 0$ are defined as follows:*

$$I^\mu_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)dt, \quad x > a, \tag{8}$$

*and*

$$I^\mu_{b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t)dt, \quad x < b. \tag{9}$$

The Hadamard inequality for Riemann–Liouville fractional integrals is composed in the following theorem:

**Theorem 2** ([30])**.** *Let $f \in L_1[a, b]$ be positive and convex function on $[a, b]$. Then, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\mu+1)}{2(b-a)^\mu}\left[I^\mu_{a^+} f(b) + I^\mu_{b^-} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{10}$$

*with $\mu > 0$.*

The definition of $k$-analogue of the Riemann–Liouville integrals is stated as follows:

**Definition 9** ([31])**.** *Let $f \in L_1[a, b]$. Then the left and right sided $k$-fractional Riemann–Liouville integrals of the function $f$ of fractional order $\mu > 0$, $k > 0$ are defined as follows:*

$$_kI^\mu_{a^+} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} f(t)dt, \quad x > a, \tag{11}$$

*and*

$$_kI^\mu_{b^-} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1} f(t)dt, \quad x < b, \tag{12}$$

*where $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt$.*

Using the fact $\Gamma_k(\mu) = k^{\frac{\mu}{k}-1}\Gamma\left(\frac{\mu}{k}\right)$ in (8) and (9) after replacing $\mu$ by $\frac{\mu}{k}$, we obtain

$$k^{\frac{-\mu}{k}} I^{\frac{\mu}{k}}_{\kappa_1^+}\varphi(x) = {}_kI^{\mu}_{\kappa_1^+}\varphi(x) \tag{13}$$

$$k^{\frac{-\mu}{k}} I^{\frac{\mu}{k}}_{\kappa_2^-}\varphi(x) = {}_kI^{\mu}_{\kappa_2^-}\varphi(x) \tag{14}$$

In the upcoming section, we define a generalize class of functions will be called strongly $(\alpha, h-m)$-$p$-convex functions, and discuss their consequences in the form of classical and new classes of functions. In Section 3, we prove Hadamard type inequalities for strongly $(\alpha, h-m)$-$p$-convex and related functions via integrals (8) and (9). Furthermore, we obtain refinements of some fractional versions of Hadamard inequalities proved in [6,9,20,23,26–28,30,32–38]. By using a parameter substitution, $k$-fractional versions of Hadamard inequalities which have proved in Section 3 can be obtained.

## 2. Some New Definitions

In this Section, we define a generalized class of functions, which reproduce several kinds of convex and strongly convex functions along with some new deduced definitions.

**Definition 10.** *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : (0,d] \to \mathbb{R}$, $d > 0$, is called strongly ($\alpha$, h-m)-p-convex with modulus $c \geq 0$, if $f$ is non-negative and*

$$f\Big((tx^p + m(1-t)y^p)^{\frac{1}{p}}\Big) \leq h(t^\alpha)f(x) + mh(1-t^\alpha)f(y) - cmh(t^\alpha)h(1-t^\alpha)|y^p - x^p|^2, \tag{15}$$

*holds for all $x, y, (tx^p + m(1-t)y^p)^{\frac{1}{p}} \in (0,d]$, $t \in (0,1)$, $p \in \mathbb{R}\setminus\{0\}$ and $(\alpha, m) \in (0,1] \times [0,1]$.*

It is interesting to note that a number of already known definitions are direct consequences of the above definition. For specific settings of involved symbols and function $h$ in the inequality (15), one can obtain easily strongly $p$-convex [36], strongly convex [1], strongly harmonic convex [37], $p$-convex [20], $(p,h)$-convex [6], $(\alpha, h-m)$-$p$-convex [38], $(\alpha, h-m)$-convex [26], $(s,m)$-$p$-Godunova–Levin [39], $(s,m)$-$p$-convex, harmonically convex [9], P-function, harmonic $s$-convex [40], harmonically $h$-convex [41], $(\alpha, m)$-HA-convex [42], and reciprocally $(s,m)$-convex [22] functions.

It can be noted that $f(my) \leq mf(y)$ when $p = 1$, $\alpha = 1$, $h(t) = t$, $c = 0$, $x = my$, also for $p = 1$, $m = 0$, $\alpha = 1$, $h(t) = t$, we obtain the definition of star-shaped function. Some new induced definitions are given as follows:

We will say that the function $f$ is strongly $(h-m)$-$p$-convex with modulus $c \geq 0$, if (15) is considered for $\alpha = 1$.

We will say that the function $f$ is strongly $(\alpha, m)$-$p$-convex with modulus $c \geq 0$, in the second sense if (15) is considered for $h(t) = t$.

We will say that the function $f$ is strongly $(s,m)$-$p$-convex with modulus $c \geq 0$, if (15) is considered for $\alpha = 1$ and $h(t) = t^s$.

We will say that the function $f$ is strongly $(p,h)$-convex with modulus $c \geq 0$, in the second sense if (15) is considered for $\alpha = m = 1$.

We will say that the function $f$ is strongly $(p,P)$-function with modulus $c \geq 0$, in the second sense if (15) is considered for $\alpha = m = 1$, $h(t) = 1$.

We will say that the function $f$ is Godunova–Levin type of strongly harmonic convex with modulus $c \geq 0$, in the second sense if (15) is considered for $\alpha = m = 1$, $p = -1$ and $h(t) = t^{-1}$.

We will say that the function $f$ is strongly $(p,P)$-function with modulus $c \geq 0$, in the second sense if (15) is considered for $\alpha = m = 1$, $h(t) = 1$.

We will say that the function $f$ is strongly harmonic $s$-convex with modulus $c \geq 0$, in the second sense if (15) is considered for $m = \alpha = 1$, $p = -1$ and $h(t) = t^s$.

We will say that the function $f$ is strongly harmonic $P$-function with modulus $c \geq 0$, in the second sense if (15) is considered for $\alpha = m = 1$ and $h(t) = 1$.

We will say that the function $f$ is strongly $(\alpha, m)$-HA-convex with modulus $c \geq 0$, if (15) is considered for $p = -1$ and $h(t) = t$.

We will say that the function $f$ is strongly $(s, m)$-HA-convex with modulus $c \geq 0$, if (15) is considered for $\alpha = 1$, $p = -1$ and $h(t) = t^s$.

We will say that the function $f$ is Godunova–Levin type of strongly $(s, m)$-HA-convex with modulus $c \geq 0$, if (15) is considered for $\alpha = 1$, $p = -1$ and $h(t) = t^{-s}$.

## 3. Fractional Versions of Hadamard-Type Inequalities for Strongly $(\alpha, h-m)$-$p$-Convex Functions

In this section, we prove the Hadamard-type inequality for strongly $(\alpha, h-m)$-$p$-convex functions and further study its consequences.

**Theorem 3.** *Let $f : [a,b] \subseteq (0,d] \to \mathbb{R}$ be a positive function such that $f \in L_1[a,b]$. If $f$ is a strongly $(\alpha, h-m)$-$p$-convex function on $(0,d]$ and $p \in \mathbb{R} \setminus \{0\}$. Then, for $(\alpha, m) \in (0,1]^2$ and $c \geq 0$, the following inequalities hold:*

*(i) If $p > 0$,*

$$\begin{aligned} &f\left(\left(\frac{a^p + mb^p}{2}\right)^{\frac{1}{p}}\right) + \frac{ch\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}{(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p - a^p)^2 + 2\left(\frac{a^p}{m} - mb^p\right)^2\right.\\ &\left.+ 2\mu(b^p - a^p)\left(\frac{a^p}{m} - mb^p\right)\right] \leq \frac{\Gamma(\mu+1)}{(mb^p - a^p)^\mu}\left[h\left(\frac{1}{2^\alpha}\right)I^\mu_{(a^p)^+}(f \circ \phi)(mb^p)\right.\\ &\left.+ m^{\mu+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\mu_{(b^p)^-}(f \circ \phi)\left(\frac{a^p}{m}\right)\right] \leq \mu\left[h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right]\\ &\times \int_0^1 h(t^\alpha)t^{\mu-1}dt + m\mu\left[h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right]\int_0^1 h(1-t^\alpha)t^{\mu-1}dt\\ &- cm\mu\left[h\left(\frac{1}{2^\alpha}\right)(b^p - a^p)^2 + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(b^p - \frac{a^p}{m^2}\right)^2\right]\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\mu-1}dt, \end{aligned} \tag{16}$$

*with $\mu > 0$, $\phi(z) = z^{\frac{1}{p}}$ for all $z \in [a^p, mb^p]$.*

*(ii) If $p < 0$,*

$$\begin{aligned} &f\left(\left(\frac{a^p + mb^p}{2}\right)^{\frac{1}{p}}\right) + \frac{cmh\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}{(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p - a^p)^2 + 2\left(\frac{a^p}{m} - mb^p\right)^2\right.\\ &\left.+ 2\mu(b^p - a^p)\left(\frac{a^p}{m} - mb^p\right)\right] \leq \frac{\Gamma(\mu+1)}{(a^p - mb^p)^\mu}\left[h\left(\frac{1}{2^\alpha}\right)I^\mu_{(a^p)^-}(f \circ \phi)(mb^p)\right.\\ &\left.+ m^{\mu+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\mu_{(b^p)^+}(f \circ \phi)\left(\frac{a^p}{m}\right)\right] \leq \mu\left[h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right]\\ &\times \int_0^1 h(t^\alpha)t^{\mu-1}dt + m\mu\left[h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right]\int_0^1 h(1-t^\alpha)t^{\mu-1}dt\\ &- cm\mu\left[h\left(\frac{1}{2^\alpha}\right)(b^p - a^p)^2 + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(b^p - \frac{a^p}{m^2}\right)^2\right]\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\mu-1}dt, \end{aligned} \tag{17}$$

*with $\mu > 0$, $\phi(z) = z^{\frac{1}{p}}$ for all $z \in [mb^p, a^p]$.*

**Proof.** (i) The following inequality holds for strongly $(\alpha, h-m)$-$p$-convex function

$$f\left(\left(\frac{x^p + my^p}{2}\right)^{\frac{1}{p}}\right) \leq h\left(\frac{1}{2^\alpha}\right)f(x) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(y) - cmh\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(y^p - x^p)^2. \tag{18}$$

By setting $x = (a^p t + m(1-t)b^p)^{\frac{1}{p}}, y = \left(\frac{a^p}{m}(1-t) + b^p t\right)^{\frac{1}{p}}$ in (18) and integrating the resulting inequality over the interval $[0,1]$ after multiplying with $t^{\mu-1}$, we obtain

$$\begin{aligned} &\frac{1}{\mu}f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \leq h\left(\frac{1}{2^\alpha}\right)\int_0^1 f((a^p t+m(1-t)b^p))^{\frac{1}{p}})t^{\mu-1}dt + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\times \int_0^1 f\left(\left(\frac{a^p}{m}(1-t)+b^p)t\right)^{\frac{1}{p}}\right)t^{\mu-1}dt - cmh\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\times \int_0^1 \left((1-t)\left(\frac{a^p}{m}-mb^p\right)+t(b^p-a^p)\right)^2 t^{\mu-1}dt. \end{aligned} \tag{19}$$

Now, let $u \in [a^p, mb^p]$ such that $u = a^p t + m(1-t)b^p$, that is, $t = \frac{mb^p-u}{mb^p-a^p}$ and let $v \in [\frac{a^p}{m}, b^p]$ such that $v = \frac{a^p}{m}(1-t) + b^p t$, that is, $t = \frac{v-\frac{a^p}{m}}{b^p-\frac{a^p}{m}}$ in (19), then multiplying by $\mu$ after applying Definition 8, we obtain the following inequality

$$\begin{aligned} &f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Gamma(\mu+1)}{(mb^p-a^p)^\mu}\left[h\left(\frac{1}{2^\alpha}\right)I^\mu_{(a^p)^+}(f\circ\phi)(mb^p) + m^{\mu+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\right. \\ &\left.\times I^\mu_{(b^p)^-}(f\circ\phi)\left(\frac{a^p}{m}\right)\right] - \frac{cmh\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}{(\mu+1)(\mu+2)}\left[2\left(\frac{a^p}{m}-mb^p\right)^2 + \mu(\mu+1)(b^p-a^p)^2\right. \\ &\left.+2\mu(b^p-a^p)\left(\frac{a^p}{m}-mb^p\right)^2\right]. \end{aligned} \tag{20}$$

From which one can obtain the first inequality of (16). Again using strongly $(\alpha, h-m)$-$p$-convexity of $f$ and integrating the resulting inequality over the interval $[0,1]$ after multiplying with $t^{\mu-1}$, we obtain

$$\begin{aligned} &h\left(\frac{1}{2^\alpha}\right)\int_0^1 f((a^p t+m(1-t)b^p)^{\frac{1}{p}})t^{\mu-1}dt + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 f((a^p t+m(1-t)b^p)^{\frac{1}{p}})t^{\mu-1}dt \\ &\leq \left[h\left(\frac{1}{2^\alpha}\right)f(a) + h\left(\frac{2^{\alpha-1}}{2^{2\alpha}}\right)f(b)\right]\int_0^1 h(t^\alpha)t^{\mu-1}dt + m\left[h\left(\frac{1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right] \\ &\times \int_0^1 h(1-t^\alpha)t^{\mu-1}dt - cm\left[h\left(\frac{1}{2^\alpha}\right)(b^p-a^p)^2 + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(b^p-\frac{a^p}{m^2}\right)^2\right] \\ &\times \int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\mu-1}dt. \end{aligned} \tag{21}$$

Again using substitution as considered in (19), the above inequality leads to the second inequality of (16).

(ii) Proof is similar to the proof of (i). □

The following remark establishes the connection with several already published inequalities.

**Remark 1.** *(i) If $c = 0$ in (17), then one can obtain (Theorem 2.2 [38]). (ii) If $p = -1, m = \alpha = 1$, $c = 0$ and $h(t) = t$ in (17), then one can obtain (Theorem 4 [34]).*

*(iii) If $p = -1$, $m = \mu = 1$, $c = 0$ and $h(t) = t$ in (17), then one can obtain (Theorem 2.4 [9]).*

*(iv) If $c = 0$ and $h(t) = t$ in (16), then one can obtain (Theorem 3.10 [38]).*

*(v) If $\alpha = p = 1$ and $c = 0$ in (16), then one can obtain (Corollary 2.2 [26]).*

*(vi) If $\alpha = m = p = 1$, $c = 0$ and $h(t) = t$ in (16), then Theorem 2 is obtained.*

*(vii) If $\alpha = \mu = m = 1$, $c = 0$ and $h(t) = t$ in (16), then one can obtain the Hadamard inequality.*

*(viii) If $\alpha = p = 1$, $c = 0$ and $h(t) = t$ in (16), then one can obtain (Theorem 2.1 [33]).*

*(ix) If $m = \alpha = \mu = 1$, $p = -1$, $c = 0$ and $h(t) = t^s$ in (17), then one can obtain (Theorem 2.1 [20]).*

*(x) If $m = \alpha = p = \mu = 1$, $c = 0$ and $h(t) = t^s$ in (16), then one can obtain (Theorem 2.1 [23]).*

*(xi) If $m = \alpha = \mu = 1$ and $h(t) = t$ in (16) and (17), then one can obtain (Theorem 2.3 [36]).*

*(xii) If $m = \alpha = \mu = p = 1$ and $h(t) = t$ in (16), then one can obtain (Theorem 6 [32]).*

*(xiii) If $\alpha = p = 1$ and $h(t) = t$ in (16), then one can obtain (Theorem 6 [35]).*

*(xiv) If $\alpha = m = \mu = 1$, $p = -1$ and $h(t) = t$ in (17), then one can obtain (Theorem 2.1 [37]).*

*(xv) If $\alpha = m = p = 1$ and $h(t) = t$ in (16), then one can obtain (Theorem 2.1 [28]).*

**Corollary 1.** *The following inequalities hold for strongly $(h-m)$-p-convex function:*
*(i) If $p > 0$,*

$$\begin{aligned}
&f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)+\frac{cmh^2\left(\frac{1}{2}\right)}{(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p-a^p)^2+2\left(\frac{a^p}{m}-mb^p\right)^2+2\mu(b^p-a^p)\right.\\
&\left.\times\left(\frac{a^p}{m}-mb^p\right)\right]\leq\frac{h\left(\frac{1}{2}\right)\Gamma(\mu+1)}{(mb^p-a^p)^\mu}\left[I^{\mu}_{(a^p)^+}(f\circ\phi)(mb^p)+m^{\mu+1}I^{\mu}_{(b^p)^-}(f\circ\phi)\left(\frac{a^p}{m}\right)\right]\\
&\leq\mu h\left(\frac{1}{2}\right)[f(a)+mf(b)]\int_0^1 h(t)t^{\mu-1}dt+m\mu h\left(\frac{1}{2}\right)\left[f(b)+mf\left(\frac{a}{m^2}\right)\right]\int_0^1 h(1-t)t^{\mu-1}dt\\
&-cm\mu h\left(\frac{1}{2}\right)\left[(b^p-a^p)^2+m\left(b^p-\frac{a^p}{m^2}\right)^2\right]\int_0^1 h(t)h(1-t)t^{\mu-1}dt.
\end{aligned}\tag{22}$$

*(ii) If $p < 0$,*

$$\begin{aligned}
&f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)+\frac{cmh^2\left(\frac{1}{2}\right)}{(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p-a^p)^2+2\left(\frac{a^p}{m}-mb^p\right)^2\right.\\
&\left.+2\mu(b^p-a^p)\left(\frac{a^p}{m}-mb^p\right)\right]\leq\frac{h\left(\frac{1}{2}\right)\Gamma(\mu+1)}{(a^p-mb^p)^\mu}\left[I^{\mu}_{(a^p)^-}(f\circ\phi)(mb^p)+m^{\mu+1}\right.\\
&\left.\times I^{\mu}_{(b^p)^+}(f\circ\phi)\left(\frac{a^p}{m}\right)\right]\leq\mu h\left(\frac{1}{2}\right)[f(a)+mf(b)]\int_0^1 h(t)t^{\mu-1}dt+m\mu h\left(\frac{1}{2}\right)\\
&\times\left[f(b)+mf\left(\frac{a}{m^2}\right)\right]\int_0^1 h(1-t)t^{\mu-1}dt-cm\mu h\left(\frac{1}{2}\right)\left[(b^p-a^p)^2+m\left(b^p-\frac{a^p}{m^2}\right)^2\right]\\
&\times\int_0^1 h(t)h(1-t)t^{\mu-1}dt.
\end{aligned}\tag{23}$$

**Proof.** If $\alpha = 1$ in (16), then after some computations the inequality (22) of (i) can be obtained. If $\alpha = 1$ in (17), then after some computations the inequality (23) of (ii) can be obtained. □

**Remark 2.** *If $m = 1$ in (22) and (23), then the results for strongly $(p,h)$-convex function in second sense can be obtained.*

**Corollary 2.** *The following inequalities hold for strongly Godunova–Levin type $(s,m)$-p-convex function:*
*(i) If $p > 0$,*

$$\begin{aligned}
&2^{-s}f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)+\frac{2^s cm}{(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p-a^p)^2+2\left(\frac{a^p}{m}-mb^p\right)^2+2\mu(b^p-a^p)\right.\\
&\left.\times\left(\frac{a^p}{m}-mb^p\right)\right]\leq\frac{\Gamma(\mu+1)}{(mb^p-a^p)^\mu}\left[I^{\mu}_{(a^p)^+}(f\circ\phi)(mb^p)+m^{\mu+1}I^{\mu}_{(b^p)^-}(f\circ\phi)\left(\frac{a^p}{m}\right)\right]\leq\frac{\mu[f(a)+mf(b)]}{(\mu-s)}\\
&+m\mu B(1-s,\mu)\left[f(b)+mf\left(\frac{a}{m^2}\right)\right]-cm\mu B(1-s,\mu-s)\left[(b^p-a^p)^2+m\left(b^p-\frac{a^p}{m^2}\right)^2\right].
\end{aligned}\tag{24}$$

*(ii) If $p < 0$,*

$$2^{-s} f\left(\left(\frac{a^p + mb^p}{2}\right)^{\frac{1}{p}}\right) + \frac{2^s cm}{(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p - a^p)^2 + 2\left(\frac{a^p}{m} - mb^p\right)^2 + 2\mu(b^p - a^p)\right.$$
$$\left.\times\left(\frac{a^p}{m} - mb^p\right)\right] \le \frac{\Gamma(\mu+1)}{(a^p - mb^p)^\mu}\left[I^\mu_{(a^p)^-}(f\circ\phi)(mb^p) + m^{\mu+1} I^\mu_{(b^p)^+}(f\circ\phi)\left(\frac{a^p}{m}\right)\right] \le \frac{\mu[f(a) + mf(b)]}{(\mu - s)} \tag{25}$$
$$+ m\mu B(1-s,\mu)\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] - cm\mu B(1-s,\mu-s)\left[(b^p - a^p)^2 + m\left(b^p - \frac{a^p}{m^2}\right)^2\right].$$

**Proof.** If $\alpha = 1$ and $h(t) = t^{-s}$ in (16), then after some computations the inequality (24) of (i) can be obtained. If $\alpha = 1$ and $h(t) = t^{-s}$ in (17), then after some computations the inequality (25) of (ii) can be obtained. □

**Remark 3.** *If $m = 1$ in (24) and (25), then the inequalities for strongly Godunova–Levin type of $(p,s)$-convex function can be obtained.*

**Corollary 3.** *The following inequalities hold for strongly $(s,m)$-p-convex function in third sense:*
*(i) If $p > 0$,*

$$f\left(\left(\frac{a^p + mb^p}{2}\right)^{\frac{1}{p}}\right) + \frac{cm}{2^{2s}(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p - a^p)^2 + 2\left(\frac{a^p}{m} - mb^p\right)^2 + 2\mu(b^p - a^p)\right.$$
$$\left.\times\left(\frac{a^p}{m} - mb^p\right)\right] \le \frac{\Gamma(\mu+1)}{(mb^p - a^p)^\mu}\left[I^\mu_{(a^p)^+}(f\circ\phi)(mb^p) + m^{\mu+1} I^\mu_{(b^p)^-}(f\circ\phi)\left(\frac{a^p}{m}\right)\right] \le \frac{\mu[f(a) + mf(b)]}{2^s(\mu+s)} \tag{26}$$
$$+ \frac{m\mu B(1+s,\mu)}{2^s}\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] - \frac{cm\mu B(1+s,\mu+s)}{2^s}\left[(b^p - a^p)^2 + m\left(b^p - \frac{a^p}{m^2}\right)^2\right].$$

*(ii) If $p < 0$,*

$$2^{-s} f\left(\left(\frac{a^p + mb^p}{2}\right)^{\frac{1}{p}}\right) + \frac{cm}{2^s(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p - a^p)^2 + 2\left(\frac{a^p}{m} - mb^p\right)^2 + 2\mu(b^p - a^p)\right.$$
$$\left.\times\left(\frac{a^p}{m} - mb^p\right)\right] \le \frac{\Gamma(\mu+1)}{(a^p - mb^p)^\mu}\left[I^\mu_{(a^p)^-}(f\circ\phi)(mb^p) + m^{\mu+1} I^\mu_{(b^p)^+}(f\circ\phi)\left(\frac{a^p}{m}\right)\right] \le \frac{\mu[f(a) + mf(b)]}{(\mu+s)} \tag{27}$$
$$+ m\mu B(1-s,\mu)\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] - cm\mu B(1-s,\mu-s)\left[(b^p - a^p)^2 + m\left(b^p - \frac{a^p}{m^2}\right)^2\right].$$

**Proof.** If $\alpha = 1$ and $h(t) = t^s$ in (16), then after some computations the inequality (26) of (i) can be obtained. If $\alpha = 1$ and $h(t) = t^s$ in (17), then after some computations the inequality (27) of (ii) can be obtained. □

**Corollary 4.** *The following inequalities hold for strongly $(\alpha,m)$-p-convex function:*
*(i) If $p > 0$,*

$$2^\alpha f\left(\left(\frac{a^p + mb^p}{2}\right)^{\frac{1}{p}}\right) + \frac{cm(2^\alpha - 1)}{2^\alpha(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p - a^p)^2 + 2\left(\frac{a^p}{m} - mb^p\right)^2 + 2\mu(b^p - a^p)\right.$$
$$\left.\times\left(\frac{a^p}{m} - mb^p\right)\right] \le \frac{\Gamma(\mu+1)}{(mb^p - a^p)^\mu}\left[I^\mu_{(a^p)^+}(f\circ\phi)(mb^p) + m^{\mu+1}(2^\alpha - 1) I^\mu_{(b^p)^-}(f\circ\phi)\left(\frac{a^p}{m}\right)\right]$$
$$\le \frac{\mu}{(\mu+\alpha)}[f(a) + m(2^\alpha - 1)f(b)] + \frac{m\alpha}{(\mu+\alpha)}\left[f(b) + m(2^\alpha - 1)f\left(\frac{a}{m^2}\right)\right] \tag{28}$$
$$- \frac{cm\mu\alpha}{(\mu+\alpha)(\mu+2\alpha)}\left[h\left(\frac{1}{2^\alpha}\right)(b^p - a^p)^2 + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\left(b^p - \frac{a^p}{m^2}\right)^2\right].$$

*(ii) If $p < 0$,*

$$
\begin{aligned}
&2^{\alpha} f\left(\left(\frac{a^p + mb^p}{2}\right)^{\frac{1}{p}}\right) + \frac{cm(2^{\alpha}-1)}{2^{\alpha}(\mu+1)(\mu+2)}\left[\mu(\mu+1)(b^p-a^p)^2 + 2\left(\frac{a^p}{m} - mb^p\right)^2 + 2\mu(b^p-a^p)\right.\\
&\left.\times\left(\frac{a^p}{m} - mb^p\right)\right] \leq \frac{\Gamma(\mu+1)}{(a^p - mb^p)^{\mu}}\left[I^{\mu}_{(a^p)^-}(f\circ\phi)(mb^p) + m^{\mu+1}(2^{\alpha}-1) I^{\mu}_{(b^p)^+}(f\circ\phi)\left(\frac{a^p}{m}\right)\right]\\
&\leq \frac{\mu}{(\mu+\alpha)}[m(2^{\alpha}-1)f(b) + f(a)] + \frac{m\alpha}{(\mu+\alpha)}\left[f(b) + m(2^{\alpha}-1)f\left(\frac{a}{m^2}\right)\right] - \frac{cm\mu\alpha}{(\mu+\alpha)(\mu+2\alpha)}\\
&\times\left[(b^p-a^p)^2 + m\left(b^p - \frac{a^p}{m^2}\right)^2\right].
\end{aligned} \tag{29}
$$

**Proof.** If $h(t) = t$ in (16), then after some computations the inequality (4) of (i) can be obtained. If $h(t) = t$ in (17), then after some computations the inequality (29) of (ii) can be obtained. □

**Corollary 5.** *The following inequality holds for strongly $(\alpha, m)$-HA-convex functions in second sense:*

$$
\begin{aligned}
&2^{\alpha} f\left(\frac{2ab}{b+am}\right) + \frac{cm(2^{\alpha}-1)}{2^{\alpha}(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{b-a}{ab}\right)^2 + 2\left(\frac{b-am^2}{abm}\right)^2 + \frac{2\mu(a-b)(b-am^2)}{m(ab)^2}\right]\\
&\leq \frac{\Gamma(\mu+1)(ab)^{\mu}}{(b-am)^{\mu}}\left[I^{\mu}_{\left(\frac{1}{a}\right)^-}(f\circ\phi)\left(\frac{m}{b}\right) + m^{\mu+1}(2^{\alpha}-1) I^{\mu}_{\left(\frac{1}{b}\right)^+}(f\circ\phi)\left(\frac{1}{ma}\right)\right] \leq \frac{\mu[f(a) + m(2^{\alpha}-1)f(b)]}{(\mu+\alpha)}\\
&+ \frac{m\alpha}{(\mu+\alpha)}\left[f(b) + m(2^{\alpha}-1)f\left(\frac{a}{m^2}\right)\right] - \frac{cm\mu\alpha}{(\mu+\alpha)(\mu+2\alpha)}\left[\left(\frac{1}{b} - \frac{1}{a}\right)^2 + m\left(\frac{1}{b} - \frac{1}{am^2}\right)^2\right].
\end{aligned} \tag{30}
$$

**Proof.** If $p = -1$ and $h(t) = t$ in (17), then after some computations the above inequality can be obtained. □

**Corollary 6.** *The following inequality holds for strongly $(s, m)$-HA-convex functions in second sense:*

$$
\begin{aligned}
&f\left(\frac{2ab}{b+am}\right) + \frac{cm}{2^{2s}(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{b-a}{ab}\right)^2 + 2\left(\frac{b-am^2}{abm}\right)^2 + \frac{2\mu(a-b)(b-am^2)}{m(ab)^2}\right]\\
&\leq \frac{\Gamma(\mu+1)(ab)^{\mu}}{(b-am)^{\mu}}\left[I^{\mu}_{\left(\frac{1}{a}\right)^-}(f\circ\phi)\left(\frac{m}{b}\right) + m^{\mu+1} I^{\mu}_{\left(\frac{1}{b}\right)^+}(f\circ\phi)\left(\frac{1}{ma}\right)\right] \leq \frac{\mu[f(a) + mf(b)]}{2^s(\mu+s)}\\
&+ \frac{m\mu B(1+s,\mu)}{2^s}\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] - \frac{cm\mu B(1+s,\mu+s)}{2^s}\left[\left(\frac{1}{b} - \frac{1}{a}\right)^2 + m\left(\frac{1}{b} - \frac{1}{am^2}\right)^2\right].
\end{aligned} \tag{31}
$$

**Proof.** If $p = -1$, $\alpha = 1$ and $h(t) = t^s$ in (17), then after some computations the above inequality can be obtained. □

**Corollary 7.** *The following inequality holds for Godunova–Levin type of strongly $(s, m)$-HA-convex functions:*

$$
\begin{aligned}
&2^{-s} f\left(\frac{2ab}{b+am}\right) + \frac{2^s cm}{(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{b-a}{ab}\right)^2 + 2\left(\frac{b-am^2}{abm}\right)^2 + \frac{2\mu(a-b)(b-am^2)}{m(ab)^2}\right]\\
&\leq \frac{\Gamma(\mu+1)(ab)^{\mu}}{(b-am)^{\mu}}\left[I^{\mu}_{\left(\frac{1}{a}\right)^-}(f\circ\phi)\left(\frac{m}{b}\right) + m^{\mu+1} I^{\mu}_{\left(\frac{1}{b}\right)^+}(f\circ\phi)\left(\frac{1}{ma}\right)\right] \leq \mu(\mu-s)[f(a) + mf(b)]\\
&+ m\mu B(1-s,\mu)\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] - cm\mu B(1-s,\mu-s)\left[\left(\frac{1}{b} - \frac{1}{a}\right)^2 + m\left(\frac{1}{b} - \frac{1}{am^2}\right)^2\right].
\end{aligned} \tag{32}
$$

**Proof.** If $p = -1$, $\alpha = 1$ and $h(t) = t^{-s}$ in (17), then after some computations the above inequality can be obtained. □

**Remark 4.** *Using (13) and (14) after replacing $\mu$ by $\frac{\mu}{k}$ in all of the above inequalities, one can obtain k-fractional versions of all above results.*

**Lemma 1.** *Let $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$. Furthermore, suppose that $f' \in L[a,b]$ and $m \in (0,1)$. Then, the following identities hold:*

*(i) If $p > 0$,*

$$\begin{aligned}&\frac{(f\circ\phi)(a^p)+(f\circ\phi)(mb^p)}{2}-\frac{\Gamma(\mu+1)}{2(mb^p-a^p)^{\mu}}\Big[I^{\mu}_{(a^p)^+}(f\circ\phi)(mb^p)+I^{\mu}_{(mb^p)^-}(f\circ\phi)(a^p)\Big]\\&=\frac{mb^p-a^p}{2p}\int_0^1\frac{((1-t)^{\mu}-t^{\mu})f'((ta^p+m(1-t)b^p)^{\frac{1}{p}})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt.\end{aligned} \tag{33}$$

*with $\mu > 0$, $\phi(z) = z^{\frac{1}{p}}$ for all $z \in [a^p, mb^p]$.*

*(ii) If $p < 0$,*

$$\begin{aligned}&\frac{(f\circ\phi)(a^p)+(f\circ\phi)(mb^p)}{2}-\frac{\Gamma(\mu+1)}{2(a^p-mb^p)^{\mu}}\Big[I^{\mu}_{(a^p)^-}(f\circ\phi)(mb^p)+I^{\mu}_{(mb^p)^+}(f\circ\phi)(a^p)\Big]\\&=\frac{mb^p-a^p}{2p}\int_0^1\frac{((1-t)^{\mu}-t^{\mu})f'((ta^p+m(1-t)b^p)^{\frac{1}{p}})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt.\end{aligned} \tag{34}$$

*with $\mu > 0$, $\phi(z) = z^{\frac{1}{p}}$ for all $z \in [mb^p, a^p]$.*

**Proof.** Consider

$$\frac{mb^p-a^p}{2p}\int_0^1\frac{((1-t)^{\mu}-t^{\mu})f'((ta^p+m(1-t)b^p)^{\frac{1}{p}})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt.$$

$$\begin{aligned}&=\frac{mb^p-a^p}{2p}\int_0^1\frac{(1-t)^{\mu}f'((ta^p+m(1-t)b^p)^{\frac{1}{p}})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\\&-\frac{mb^p-a^p}{2p}\int_0^1\frac{t^{\mu}f'((ta^p+m(1-t)b^p)^{\frac{1}{p}})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt=I_1+I_2.\end{aligned}$$

Integrating by parts, we have

$$I_1=\frac{1}{2}\Big[-(1-t)^{\mu-1}f((ta^p+m(1-t)b^p)^{\frac{1}{p}})\Big|_0^1+\mu\int_0^1(1-t)^{\mu-1}f((ta^p+m(1-t)b^p)^{\frac{1}{p}})dt\Big].$$

Setting $x = a^p t + m(1-t)b^p$ in above equation, we have

$$I_1=\frac{1}{2}\Big[f((mb^p)^{\frac{1}{p}})+\frac{\mu}{(mb^p-a^p)^{\mu}}\int_{a^p}^{mb^p}(x-a^p)^{\mu-1}f((x)^{\frac{1}{p}})dt\Big].$$

$$I_1=\frac{1}{2}\Big[(f\circ\phi)(mb^p)+\frac{\Gamma(\mu+1)}{(mb^p-a^p)^{\mu}}I^{\mu}_{(mb^p)^-}(f\circ\phi)(a^p)\Big]. \tag{35}$$

Now we will evaluate $I_2$

$$I_2=-\frac{1}{2}\Big[-t^{\mu-1}f((ta^p+m(1-t)b^p)^{\frac{1}{p}})\Big|_0^1+\mu\int_0^1(1-t)^{\mu-1}f((ta^p+m(1-t)b^p)^{\frac{1}{p}})dt\Big].$$

$$I_2 = \frac{1}{2}\left[f((a^p)^{\frac{1}{p}}) + \frac{\mu}{(mb^p - a^p)^\mu}\int_{a^p}^{mb^p}(mb^p - x)^{\mu-1}(f\circ\phi)(x)dt\right].$$

$$I_2 = \frac{1}{2}\left[(f\circ\phi)(a^p) + \frac{\Gamma(\mu+1)}{(mb^p - a^p)^\mu} I^{\mu}_{(a^p)^+}(f\circ\phi)(mb^p)\right]. \tag{36}$$

By adding (35) and (36), we obtain the required equality.

(ii) The proof is similar with (i). □

**Remark 5.** *(i) If $m = p = 1$, then one can obtain (Lemma 2 [30]).*

*(ii) If $p = 1$, then one can obtain (Lemma 2.4 [33]).*

**Theorem 4.** *Let $f : [a,b] \subseteq (0,d] \to \mathbb{R}$ be a positive function such that $f \in L_1[a,b]$. If $f$ is a strongly $(\alpha, h-m)$-$p$-convex function on $(0,d]$ and $p \in \mathbb{R}\setminus\{0\}$. Then, for $(\alpha,m) \in (0,1]^2$ and $c \geq 0$, the following inequalities hold:*

*(i) $p > 0$,*

$$\begin{aligned}
&\left|\frac{(f\circ\phi)(a^p) + (f\circ\phi)(mb^p)}{2} - \frac{\Gamma(\mu+1)}{2(mb^p - a^p)^\mu}\left[I^{\mu}_{(a^p)^+}(f\circ\phi)(mb^p) + I^{\mu}_{(mb^p)^-}(f\circ\phi)(a^p)\right]\right| \\
&\leq \frac{mb^p - a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{h(t^\alpha)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1\frac{h(t^\alpha)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right. \\
&+ m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{h(1-t^\alpha)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1\frac{h(1-t^\alpha)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right) \\
&\left.- cm(b^p - a^p)^2\left(\int_0^{\frac{1}{2}}\frac{h(t^\alpha)h(1-t^\alpha)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1\frac{h(t^\alpha)h(1-t^\alpha)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].
\end{aligned} \tag{37}$$

*with $\mu > 0$, $\phi(z) = z^{\frac{1}{p}}$ for all $z \in [a^p, mb^p]$.*

*(ii) If $p < 0$,*

$$\begin{aligned}
&\left|\frac{(f\circ\phi)(a^p) + (f\circ\phi)(mb^p)}{2} - \frac{\Gamma(\mu+1)}{2(a^p - mb^p)^\mu}\left[I^{\mu}_{(a^p)^-}(f\circ\phi)(mb^p) + I^{\mu}_{(mb^p)^+}(f\circ\phi)(a^p)\right]\right| \\
&\leq \frac{mb^p - a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{h(t^\alpha)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1\frac{h(t^\alpha)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right. \\
&+ m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{h(1-t^\alpha)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1\frac{h(1-t^\alpha)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right) \\
&- cm(b^p - a^p)^2\left(\int_0^{\frac{1}{2}}\frac{h(t^\alpha)h(1-t^\alpha)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right. \\
&\left.\left.+ \int_{\frac{1}{2}}^1\frac{h(t^\alpha)h(1-t^\alpha)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].
\end{aligned} \tag{38}$$

*with $\mu > 0$, $\phi(z) = z^{\frac{1}{p}}$ for all $z \in [mb^p, a^p]$.*

**Proof.** From Lemma 1 and using strongly $(\alpha, h-m)$-$p$-convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{(f\circ\phi)(a^p) + (f\circ\phi)(mb^p)}{2} - \frac{\Gamma(\mu+1)}{2(mb^p - a^p)^\mu}\left[I^{\mu}_{(a^p)^+}(f\circ\phi)(mb^p) + I^{\mu}_{(mb^p)^-}(f\circ\phi)(a^p)\right]\right| \\
&\leq \frac{mb^p - a^p}{2p}\int_0^1\frac{|(1-t)^\mu - t^\mu||f'((ta^p + m(1-t)b^p)^{\frac{1}{p}})|}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt.
\end{aligned}$$

$$\leq \frac{mb^p - a^p}{2p}\int_0^1 \frac{|(1-t)^\mu - t^\mu|[h(t^\alpha)|f'(a)| + mh(1-t^\alpha)|f'(b)| - cmh(t^\alpha)h(1-t^\alpha)(b^p-a^p)^2]}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt.$$

$$\leq \frac{mb^p - a^p}{2p}\int_0^{\frac{1}{2}} \frac{[(1-t)^\mu - t^\mu][h(t^\alpha)|f'(a)| + mh(1-t^\alpha)|f'(b)| - cmh(t^\alpha)h(1-t^\alpha)(b^p-a^p)^2}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt$$
$$+\int_{\frac{1}{2}}^1 \frac{[t^\mu - (1-t)^\mu][h(t^\alpha)|f'(a)| + mh(1-t^\alpha)|f'(b)| - cmh(t^\alpha)h(1-t^\alpha)(b^p-a^p)^2}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt.$$

□

**Remark 6.** *(i) If $p = \alpha = m = 1$, $c = 0$ and $h(t) = t$ in (37), then one can obtain (Theorem 3 [30]).*

*(ii) If $p = \alpha = \mu = m = 1$, $c = 0$ and $h(t) = t$ in (37), then one can obtain (Theorem 2.2 [43]).*

**Corollary 8.** *The following inequalities hold for strongly $(h-m)$-$p$-convex function:*

*(i) If $p > 0$,*

$$\left|\frac{(f\circ\phi)(a^p) + (f\circ\phi)(mb^p)}{2} - \frac{\Gamma(\mu+1)}{2(mb^p - a^p)^\mu}\left[I^\mu_{(a^p)^+}(f\circ\phi)(mb^p) + I^\mu_{(mb^p)^-}(f\circ\phi)(a^p)\right]\right|$$
$$\leq \frac{mb^p - a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}} \frac{h(t)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1 \frac{h(t)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right.$$
$$+ m|f'(b)|\left(\int_0^{\frac{1}{2}} \frac{h(1-t)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1 \frac{h(1-t)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)$$
$$\left.- cm(b^p - a^p)^2\left(\int_0^{\frac{1}{2}} \frac{h(t)h(1-t)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1 \frac{h(t)h(1-t)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].$$

*(ii) If $p < 0$,*

$$\left|\frac{(f\circ\phi)(a^p) + (f\circ\phi)(mb^p)}{2} - \frac{\Gamma(\mu+1)}{2(a^p - mb^p)^\mu}\left[I^\mu_{(a^p)^-}(f\circ\phi)(mb^p) + I^\mu_{(mb^p)^+}(f\circ\phi)(a^p)\right]\right|$$
$$\leq \frac{mb^p - a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}} \frac{h(t)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1 \frac{h(t)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right.$$
$$+ m|f'(b)|\left(\int_0^{\frac{1}{2}} \frac{h(1-t)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1 \frac{h(1-t)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)$$
$$\left.- cm(b^p - a^p)^2\left(\int_0^{\frac{1}{2}} \frac{h(t)h(1-t)((1-t)^\mu - t^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1 \frac{h(t)h(1-t)(t^\mu - (1-t)^\mu)}{(ta^p + m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].$$

**Proof.** If $\alpha = 1$ in (37), then one can obtain the above inequality in (i). If $\alpha = 1$ in (38), then one can obtain the above inequality in (ii). □

**Corollary 9.** *The following inequalities hold for strongly Godunova–Levin type $(s, m)$-$p$-convex function:*

*(i) If $p > 0$,*

$$\left|\frac{(f\circ\phi)(a^p)+(f\circ\phi)(mb^p)}{2}-\frac{\Gamma(\mu+1)}{2(mb^p-a^p)^\mu}\left[I^\mu_{(a^p)^+}(f\circ\phi)(mb^p)+I^\mu_{(mb^p)^-}(f\circ\phi)(a^p)\right]\right|$$
$$\leq\frac{mb^p-a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{t^{-s}((1-t)^\mu-t^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t^{-s}(t^\mu-(1-t)^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right.$$
$$+m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{(1-t)^{-s}((1-t)^\mu-t^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{(1-t)^{-s}(t^\mu-(1-t)^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)$$
$$\left.-cm(b^p-a^p)^2\left(\int_0^{\frac{1}{2}}\frac{t^{-s}(1-t)^{-s}((1-t)^\mu-t^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t^{-s}(1-t)^{-s}(t^\mu-(1-t)^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].$$

*(ii) If $p < 0$,*

$$\left|\frac{(f\circ\phi)(a^p)+(f\circ\phi)(mb^p)}{2}-\frac{\Gamma(\mu+1)}{2(a^p-mb^p)^\mu}\left[I^\mu_{(a^p)^-}(f\circ\phi)(mb^p)+I^\mu_{(mb^p)^+}(f\circ\phi)(a^p)\right]\right|$$
$$\leq\frac{mb^p-a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{t^{-s}((1-t)^\mu-t^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t^{-s}(t^\mu-(1-t)^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right.$$
$$+m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{(1-t)^{-s}((1-t)^\mu-t^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{(1-t)^{-s}(t^\mu-(1-t)^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)$$
$$\left.-cm(b^p-a^p)^2\left(\int_0^{\frac{1}{2}}\frac{t^{-s}(1-t)^{-s}((1-t)^\mu-t^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t^{-s}(1-t)^{-s}(t^\mu-(1-t)^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].$$

**Proof.** If $\alpha = 1$ and $h(t) = t^{-s}$ in (37), then one can obtain the above inequality in (i). If $\alpha = 1$ and $h(t) = t^{-s}$ in (38), then one can obtain the above inequality in (ii). □

**Corollary 10.** *The following inequalities hold for strongly $(s,m)$-p-convex function in third sense:*
*(i) If $p > 0$,*

$$\left|\frac{(f\circ\phi)(a^p)+(f\circ\phi)(mb^p)}{2}-\frac{\Gamma(\mu+1)}{2(mb^p-a^p)^\mu}\left[I^\mu_{(a^p)^+}(f\circ\phi)(mb^p)+I^\mu_{(mb^p)^-}(f\circ\phi)(a^p)\right]\right|$$
$$\leq\frac{mb^p-a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{t^{s}((1-t)^\mu-t^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t^{s}(t^\mu-(1-t)^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right.$$
$$+m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{(1-t)^{s}((1-t)^\mu-t^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{(1-t)^{s}(t^\mu-(1-t)^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)$$
$$\left.-cm(b^p-a^p)^2\left(\int_0^{\frac{1}{2}}\frac{t^{s}(1-t)^{s}((1-t)^\mu-t^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t^{s}(1-t)^{s}(t^\mu-(1-t)^\mu)}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].$$

*(ii) If $p < 0$,*

$$\left|\frac{(f\circ\phi)(a^p)+(f\circ\phi)(mb^p)}{2}-\frac{\Gamma(\mu+1)}{2(a^p-mb^p)^{\mu}}\left[I^{\mu}_{(a^p)^-}(f\circ\phi)(mb^p)+I^{\mu}_{(mb^p)^+}(f\circ\phi)(a^p)\right]\right|$$
$$\leq\frac{mb^p-a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{t^s((1-t)^{\mu}-t^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t^s(t^{\mu}-(1-t)^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right.$$
$$+m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{(1-t)^s((1-t)^{\mu}-t^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{(1-t)^s(t^{\mu}-(1-t)^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)$$
$$\left.-cm(b^p-a^p)^2\left(\int_0^{\frac{1}{2}}\frac{t^s(1-t)^s((1-t)^{\mu}-t^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t^s(1-t)^s(t^{\mu}-(1-t)^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].$$

**Proof.** If $\alpha = 1$ and $h(t) = t^s$ in (37), then one can obtain the above inequality of (i). If $\alpha = 1$ and $h(t) = t^s$ in (38), then one can obtain the above inequality of (ii). □

**Corollary 11.** *The following inequalities hold for strongly $(\alpha, m)$-p-convex function:*
*(i) If $p > 0$,*

$$\left|\frac{(f\circ\phi)(a^p)+(f\circ\phi)(mb^p)}{2}-\frac{\Gamma(\mu+1)}{2(mb^p-a^p)^{\mu}}\left[I^{\mu}_{(a^p)^+}(f\circ\phi)(mb^p)+I^{\mu}_{(mb^p)^-}(f\circ\phi)(a^p)\right]\right|$$
$$\leq\frac{mb^p-a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{t((1-t)^{\mu}-t^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t(t^{\mu}-(1-t)^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right.$$
$$+m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{(1-t)((1-t)^{\mu}-t^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{(1-t)(t^{\mu}-(1-t)^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)$$
$$\left.-cm(b^p-a^p)^2\left(\int_0^{\frac{1}{2}}\frac{t(1-t)((1-t)^{\mu}-t^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t(1-t)(t^{\mu}-(1-t)^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].$$

**Proof.** If $h(t) = t$ in (37), then one can obtain the above inequality. □

*(ii) $p < 0$,*

$$\left|\frac{(f\circ\phi)(a^p)+(f\circ\phi)(mb^p)}{2}-\frac{\Gamma(\mu+1)}{2(a^p-mb^p)^{\mu}}\left[I^{\mu}_{(a^p)^-}(f\circ\phi)(mb^p)+I^{\mu}_{(mb^p)^+}(f\circ\phi)(a^p)\right]\right|$$
$$\leq\frac{mb^p-a^p}{2}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{t((1-t)^{\mu}-t^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t(t^{\mu}-(1-t)^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right.$$
$$+m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{(1-t)((1-t)^{\mu}-t^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{(1-t)(t^{\mu}-(1-t)^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)$$
$$\left.-cm(b^p-a^p)^2\left(\int_0^{\frac{1}{2}}\frac{t(1-t)((1-t)^{\mu}-t^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{t(1-t)(t^{\mu}-(1-t)^{\mu})}{(ta^p+m(1-t)b^p)^{1-\frac{1}{p}}}dt\right)\right].$$

**Proof.** If $h(t) = t$ in (38), then one can obtain the above inequality. □

**Corollary 12.** *The following inequality holds for strongly* $(\alpha, m)$*-HA-convex functions in second sense:*

$$\begin{aligned}
&\left|\frac{(f\circ\phi)(a^{-1})+(f\circ\phi)(mb^{-1})}{2}-\frac{\Gamma(\mu+1)(ab)^{\mu}}{2(b-am)^{\mu}}\left[I^{\mu}_{(\frac{1}{a})^{-}}(f\circ\phi)\left(\frac{m}{b}\right)+I^{\mu}_{(\frac{m}{b})^{+}}(f\circ\phi)\left(\frac{1}{a}\right)\right]\right|\\
&\leq\frac{am-b}{2ab}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{t((1-t)^{\mu}-t^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt+\int_{\frac{1}{2}}^{1}\frac{t(t^{\mu}-(1-t)^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt\right)\right.\\
&+m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{(1-t)((1-t)^{\mu}-t^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt+\int_{\frac{1}{2}}^{1}\frac{(1-t)(t^{\mu}-(1-t)^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt\right)\\
&\left.-\frac{cm(a-b)^2}{ab}\left(\int_0^{\frac{1}{2}}\frac{t(1-t)((1-t)^{\mu}-t^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt+\int_{\frac{1}{2}}^{1}\frac{t(1-t)(t^{\mu}-(1-t)^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt\right)\right].
\end{aligned}$$

**Proof.** If $p=-1$ and $h(t)=t$ in (38), then one can obtain the above inequality. □

**Corollary 13.** *The following inequality holds for strongly* $(s, m)$*-HA-convex functions in second sense:*

$$\begin{aligned}
&\left|\frac{(f\circ\phi)(a^{-1})+(f\circ\phi)(mb^{-1})}{2}-\frac{\Gamma(\mu+1)(ab)^{\mu}}{2(b-am)^{\mu}}\left[I^{\mu}_{(\frac{1}{a})^{-}}(f\circ\phi)\left(\frac{m}{b}\right)+I^{\mu}_{(\frac{m}{b})^{+}}(f\circ\phi)\left(\frac{1}{a}\right)\right]\right|\\
&\leq\frac{am-b}{2ab}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{t^s((1-t)^{\mu}-t^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt+\int_{\frac{1}{2}}^{1}\frac{t^s(t^{\mu}-(1-t)^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt\right)\right.\\
&+m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{(1-t)^s((1-t)^{\mu}-t^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt+\int_{\frac{1}{2}}^{1}\frac{(1-t)^s(t^{\mu}-(1-t)^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt\right)\\
&\left.-\frac{cm(a-b)^2}{ab}\left(\int_0^{\frac{1}{2}}\frac{t^s(1-t)^s((1-t)^{\mu}-t^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt+\int_{\frac{1}{2}}^{1}\frac{t^s(1-t)^s(t^{\mu}-(1-t)^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt\right)\right].
\end{aligned}$$

**Proof.** If $p=-1$, $\alpha=1$ and $h(t)=t^{-s}$ in (38), then one can obtain the above inequality. □

**Corollary 14.** *The following inequality holds for Godunova–Levin type of strongly* $(s, m)$*-HA-convex functions:*

$$\begin{aligned}
&\left|\frac{(f\circ\phi)(a^{-1})+(f\circ\phi)(mb^{-1})}{2}-\frac{\Gamma(\mu+1)(ab)^{\mu}}{2(b-am)^{\mu}}\left[I^{\mu}_{(\frac{1}{a})^{-}}(f\circ\phi)\left(\frac{m}{b}\right)+I^{\mu}_{(\frac{m}{b})^{+}}(f\circ\phi)\left(\frac{1}{a}\right)\right]\right|\\
&\leq\frac{am-b}{2ab}\left[|f'(a)|\left(\int_0^{\frac{1}{2}}\frac{t^{-s}((1-t)^{\mu}-t^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt+\int_{\frac{1}{2}}^{1}\frac{t^{-s}(t^{\mu}-(1-t)^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt\right)\right.\\
&+m|f'(b)|\left(\int_0^{\frac{1}{2}}\frac{(1-t)^{-s}((1-t)^{\mu}-t^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt+\int_{\frac{1}{2}}^{1}\frac{(1-t)^{-s}(t^{\mu}-(1-t)^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt\right)\\
&\left.-\frac{cm(a-b)^2}{ab}\left(\int_0^{\frac{1}{2}}\frac{t^{-s}(1-t)^{-s}((1-t)^{\mu}-t^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt+\int_{\frac{1}{2}}^{1}\frac{t^{-s}(1-t)^{-s}(t^{\mu}-(1-t)^{\mu})}{(ta^{-1}+m(1-t)b^{-1})^2}dt\right)\right].
\end{aligned}$$

**Proof.** If $p=-1$, $\alpha=1$ and $h(t)=t^{-s}$ in (38), then one can obtain the above inequality. □

## 4. Conclusions

The results obtained in this paper simultaneously produce the refinements of Hadamard inequalities for the Riemann–Liouville fractional integrals of $(p,h)$-convex, $(h-m)$-convex, $(\alpha,m)$-convex and $(s,m)$-convex functions. The Hadamard type inequalities published in recent articles [9,20,23,26,28,30,33–35,37,38] are special cases of the results of this paper. Several new Hadamard inequalities are deducible for new classes of functions defined in Section 2, some of them are given in Corollaries 1–14. The new definition is applicable to extend the classical inequalities for convex functions.

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