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Article

Hermite-Hadamard Inequalities in Fractional Calculus for Left and Right Harmonically Convex Functions via Interval-Valued Settings

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Abstract: The purpose of this study is to define a new class of harmonically convex functions, which is known as left and right harmonically convex interval-valued function (LR-$\mathcal{H}$-convex *IV-F*), and to establish novel inclusions for a newly defined class of interval-valued functions (*IV-Fs)* linked to Hermite–Hadamard (*H-H*) and Hermite–Hadamard–Fejér (*H-H*-Fejér) type inequalities via interval-valued Riemann–Liouville fractional integrals (*IV-RL*-fractional integrals). We also attain some related inequalities for the product of two LR-$\mathcal{H}$-convex *IV-Fs.* These findings enable us to identify a new class of inclusions that may be seen as significant generalizations of results proved by Iscan and Chen. Some examples are included in our findings that may be used to determine the validity of the results. The findings in this work can be seen as a considerable advance over previously published findings.

Keywords: interval-valued function; LR-Harmonically convexity; fractional integral operator; Hermite–Hadamard type inequalities

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1. Introduction

The concept of convexity of functions is a useful instrument that is used to solve a wide range of pure and applied scientific issues. Many researchers have recently committed themselves to investigate the attributes and inequalities of convexity in various directions, as evidenced by [1–6] and the references therein. The Hermite–Hadamard inequality (*H-H* inequality), which is also used frequently in many other parts of practical mathematics, notably in optimization and probability, is one of the most important mathematical inequalities relevant to convex maps. Let us elicit it as follows:

Suppose that the mapping: $[\mathrm{t},\ \upsilon] \to \mathbb{R}$. For every for all $\varkappa,\ \mu \in [\mathrm{t},\ \upsilon]$ and $\mathrm{s} \in [0,\ 1]$, if the successive inequality

$$\mathfrak{A}(\ (1-\mathrm{s})\varkappa+\mathrm{s}\mu) \leq (1-\mathrm{s})\mathfrak{A}(\varkappa)+\mathrm{s}\mathfrak{A}(\mu) \tag{1}$$

Then, $\mathfrak{A}$ is named as convex function on the convex interval $[\mathrm{t},\ \upsilon]$. If (1) is reversed, then, $\mathfrak{A}$ is named as a concave function on $[\mathrm{t},\ \upsilon]$.

This famous inequality gives error bounds for the mean value of a continuous convex mapping: $[\mathrm{t},\ \upsilon] \to \mathbb{R}$, which has gotten a lot of attention from a lot of authors. Many investigations have been conducted on the *H-H* type inequalities for additional forms of convex mappings. For example, s-convex mappings may be found in Kórus [7], N-quasi-convex mappings in Abramovich and Persson [8], h-convex mappings in Delavar and De La Sen [9], etc. Kadakal and Bekar [10], Işcan [11], Marinescu and Monea [12], Kadakal et al. [13], and the references therein provide new developments on this important issue.

Fractional calculus has shown to be an important cornerstone in mathematics and applied sciences as a very valuable tool. As a result of this fruitful interaction of various approaches to fractional calculus, many authors have studied some prominent integral inequalities, including [14] in the study of the *H-H* inequality for Riemann–Liouville fractional integrals, [15] in the *H-H* Fejér type inequality for Katugampola fractional integrals, and [16] in the extensions of trapezium inequalities for k-fractional integrals. We recommend interested readers to [17,18] and the references therein for other significant conclusions relating to fractional integral operators.

Set-valued analysis is a subset of interval analysis. There is no denying that interval analysis is important in both pure and practical research. The error limits of numerical solutions of finite state machines were one of the first applications of interval analysis. However, interval analysis, as one of the strategies for resolving interval uncertainty, has been a key component of mathematical and computer models for the past fifty years. Several applications in automated error analysis [19], computer graphics [20], and neural network output optimization [21] have been described. Furthermore, Refs. [22,23] has several optimization theory applications involving *IV-Fs*. The interested reader is recommended to Zhao et al. [24] and Román-Flores et al. [25] and their references for current developments in the area of *IV-Fs*. We recommend interested readers to [26–34] and the references therein for other significant conclusions relating to inequalities and fractional integral inequalities.

We structured the article in the following manner in response to the aforementioned tendency and invigorated by ongoing research activity in this fascinating topic. To prove fractional integral inclusions, firstly, we have generalized the class of $\mathcal{H}$-convex functions in terms of LR-$\mathcal{H}$-convex *IV-Fs*. Then, a class of *IV-RL*-fractional integrals inequalities is presented to achieve this aim. Some inclusion relations for convex *IV-Fs* in connection with the renowned *H-H*, *H-H*-Fejér type inequalities are found in this paper utilizing the newly presented class of $\mathcal{H}$-convex functions.

2. Preliminaries

Let us begin the rest of this part by outlining the theory of interval analysis, which is mostly due to [28]. The sets of all closed intervals of $\mathbb{R}$, the sets of all negative closed intervals of $\mathbb{R}$, and the sets of all positive closed intervals of $\mathbb{R}$ are denoted by $\mathcal{K}_C$, $\mathcal{K}_C^-$, and $\mathcal{K}_C^+$, respectively. For more conceptions on *IV·Fs*, see [24]. Moreover, we have:

Remark 1 ([29])**.** *(i) The relation "* $\leq_p$ *" defined on* $\mathcal{K}_C$ *by*

$$[\mathcal{Q}_*,\ \mathcal{Q}^*] \leq_p [\mathcal{Z}_*,\ \mathcal{Z}^*] \text{ if and only if } \mathcal{Q}_* \leq \mathcal{Z}_*,\ \mathcal{Q}^* \leq \mathcal{Z}^*, \tag{2}$$

for all $[\mathcal{Q}_*,\ \mathcal{Q}^*],\ [\mathcal{Z}_*,\ \mathcal{Z}^*] \in \mathcal{K}_C$, *it is a pseudo order relation. For given* $[\mathcal{Q}_*,\ \mathcal{Q}^*],\ [\mathcal{Z}_*,\ \mathcal{Z}^*] \in \mathcal{K}_C$, *we say that* $[\mathcal{Q}_*,\ \mathcal{Q}^*] \leq_p [\mathcal{Z}_*,\ \mathcal{Z}^*]$ *if and only if* $\mathcal{Q}_* \leq \mathcal{Z}_*,\ \mathcal{Q}^* \leq \mathcal{Z}^*$ or $\mathcal{Q}_* \leq \mathcal{Z}_*,\ \mathcal{Q}^* < \mathcal{Z}^*$. *The relation* $[\mathcal{Q}_*,\ \mathcal{Q}^*] \leq_p [\mathcal{Z}_*,\ \mathcal{Z}^*]$ *coincident to* $[\mathcal{Q}_*,\ \mathcal{Q}^*] \leq [\mathcal{Z}_*,\ \mathcal{Z}^*]$ on $\mathcal{K}_C$.

(ii) It can be easily seen that " $\leq_p$ *" looks like "left and right" on the real line* $\mathbb{R}$, *so we call "* $\leq_p$ *" is "left and right" (or "LR" order, in short).*

Theorem 1 ([28])**.** *If* $\mathfrak{A} : [\mathrm{t}, v] \subset \mathbb{R} \to \mathcal{K}_C$ *is an I-V·F on such that* $\mathfrak{A}(\varkappa) = [\mathfrak{A}_*(\varkappa),\ \mathfrak{A}^*(\varkappa)]$, *then,* $\mathfrak{A}$ *is Riemann integrable over* $[\mathrm{t}, v]$ *if and only if,* $\mathfrak{A}_*$ *and* $\mathfrak{A}^*$ *both are Riemann integrable over* $[\mathrm{t}, v]$ *such that*

$$(IR)\int_{\mathrm{t}}^{v} \mathfrak{A}(\varkappa)d\varkappa = \left[(R)\int_{\mathrm{t}}^{v} \mathfrak{A}_*(\varkappa)d\varkappa,\ (R)\int_{\mathrm{t}}^{v} \mathfrak{A}^*(\varkappa)d\varkappa\right].$$

The following interval-valued Riemann–Liouville fractional integral (IV-RL-fractional integral) operators were presented by Buduk et al. [1]:

Let $\beta > 0$ and $L([\mathrm{t},\ \upsilon], \mathcal{K}_C^+)$ be the collection of all Lebesgue measurable I-V-Fs on $[\mathrm{t}, \upsilon]$. Then, the IV-RL-fractional integrals of $\mathfrak{A} \in L([\mathrm{t},\ \upsilon], \mathcal{K}_C^+)$ with order $\beta > 0$ are defined by

$$\mathfrak{T}_{\mathrm{t}^+}^{\beta}\ \mathfrak{A}(\varkappa) = \frac{1}{\Gamma(\beta)} \int_{\mathrm{t}}^{\varkappa} (\varkappa - \mathrm{s})^{\beta-1} \mathfrak{A}(\mathrm{s}) d\mathrm{s},\ (\varkappa > \mathrm{t}), \tag{3}$$

and

$$\mathfrak{T}_{\upsilon^-}^{\beta}\ \mathfrak{A}(\varkappa) = \frac{1}{\Gamma(\beta)} \int_{\varkappa}^{\upsilon} (\mathrm{s} - \varkappa)^{\beta-1} \mathfrak{A}(\mathrm{s}) d\mathrm{s},\ (\varkappa < \upsilon), \tag{4}$$

respectively, where $\Gamma(\beta) = \int_0^\infty \mathrm{s}^{\varkappa-1} e^{-\mathrm{s}} d\mathrm{s}$ is the Euler gamma function.

Definition 1 ([27]). *A set $K = [\mathrm{t},\ \upsilon] \subset \mathbb{R}^+ = (0, \infty)$ is said to be harmonically convex set, if, for all $\varkappa,\ \mu \in K,\ \mathrm{s} \in [0,\ 1]$, we have:*

$$\frac{\varkappa\mu}{\mathrm{s}\varkappa + (1-\mathrm{s})\mu} \in K. \tag{5}$$

Definition 2 ([27]). *Suppose that the mapping: $[\mathrm{t},\ \upsilon] \to \mathbb{R}$. For every $\varkappa,\ \mu \in [\mathrm{t},\ \upsilon]$ and $\mathrm{s} \in [0,\ 1]$, if the successive inequality*

$$\mathfrak{A}\left(\frac{\varkappa\mu}{\mathrm{s}\varkappa + (1-\mathrm{s})\mu}\right) \leq (1-\mathrm{s})\mathfrak{A}(\varkappa) + \mathrm{s}\mathfrak{A}(\mu), \tag{6}$$

Then, $\mathfrak{A}$ is named as harmonically convex function ($\boldsymbol{\mathcal{H}}$-convex function) on interval $[\mathrm{t},\ \upsilon]$. If *(6) is reversed, then, $\mathfrak{A}$ is named as a $\boldsymbol{\mathcal{H}}$-concave function on $[\mathrm{t},\ \upsilon]$.*

Definition 3 ([29]). *Suppose that the mapping: $[\mathrm{t},\ \upsilon] \to \mathcal{K}_C$. For every $\varkappa,\ \mu \in [\mathrm{t},\ \upsilon]$ and $\mathrm{s} \in [0,\ 1]$, if the successive inequality*

$$\mathfrak{A}(\ (1-\mathrm{s})\varkappa + \mathrm{s}\mu) \leq_{\mathrm{p}} (1-\mathrm{s})\mathfrak{A}(\varkappa) + \mathrm{s}\mathfrak{A}(\mu), \tag{7}$$

Then, $\mathfrak{A}$ is named as LR-convex IV-F on the convex interval $[\mathrm{t},\ \upsilon]$. If (7) is reversed, then, $\mathfrak{A}$ is named as a concave function on $[\mathrm{t},\ \upsilon]$.

Definition 4. *Suppose that the mapping $\mathfrak{A} : [\mathrm{t},\ \upsilon] \to \mathcal{K}_C$. For all $\varkappa,\ \mu \in [\mathrm{t},\ \upsilon]$ and $\mathrm{s} \in [0,\ 1]$, if the successive inequality*

$$\mathfrak{A}\left(\frac{\varkappa\mu}{\mathrm{s}\varkappa + (1-\mathrm{s})\mu}\right) \leq_{\mathrm{p}} (1-\mathrm{s})\mathfrak{A}(\varkappa) + \mathrm{s}\mathfrak{A}(\mu), \tag{8}$$

is valid, then, $\mathfrak{A}$ is named as LR-harmonically convex IV-F (LR-$\boldsymbol{\mathcal{H}}$-convex IV-F) defined on interval $[\mathrm{t},\ \upsilon]$. If (8) is reversed, then, $\mathfrak{A}$ is called LR-$\boldsymbol{\mathcal{H}}$-concave IV-F on $[\mathrm{t},\ \upsilon]$. The set of all LR-$\boldsymbol{\mathcal{H}}$-convex (LR-$\boldsymbol{\mathcal{H}}$-concave IV-F) is denoted

$$LRHSX([\mathrm{t},\ \upsilon],\ \mathcal{K}_C)(LRHSV([\mathrm{t},\ \upsilon],\ \mathcal{K}_C)).$$

Theorem 2. *Let K be harmonically convex set, and let $\mathfrak{A} : K \to \mathcal{K}_C$ be an IV-F is given by*

$$\mathfrak{A}(\varkappa) = [\mathfrak{A}_*(\varkappa),\ \mathfrak{A}^*(\varkappa)],\ \forall\ \varkappa, \tag{9}$$

for all $\varkappa \in K$. Then, $\mathfrak{A}$ is LR-$\boldsymbol{\mathcal{H}}$-convex function on K, if and only if, $\mathfrak{A}_*(\varkappa)$ and $\mathfrak{A}^*(\varkappa)$are $\boldsymbol{\mathcal{H}}$-convex functions.

Proof. Assume that $\mathfrak{A}_*(\varkappa)$ and $\mathfrak{A}^*(\varkappa)$ are $\boldsymbol{\mathcal{H}}$-convex on K. Then, from (6), we have

$$\mathfrak{A}_*\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) \leq (1-s)\mathfrak{A}_*(\varkappa)+s\mathfrak{A}_*(\mu),$$

and

$$\mathfrak{A}^*\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) \leq (1-s)\mathfrak{A}^*(\varkappa)+s\mathfrak{A}^*(\mu).$$

Then, by (9), we obtain

$$\mathfrak{A}\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) = [\mathfrak{A}_*(s\varkappa+(1-s)\mu),\ \mathfrak{A}^*(s\varkappa+(1-s)\mu)] \leq_p (1-s)[\mathfrak{A}_*(\varkappa),\ \mathfrak{A}^*(\varkappa)]+s[\mathfrak{A}_*(\mu),\ \mathfrak{A}^*(\mu)],$$

that is

$$\mathfrak{A}\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) \leq_p (1-s)\mathfrak{A}(\varkappa)+s\mathfrak{A}(\mu), \forall\ \varkappa,\mu\in K,\ s\in[0,\ 1].$$

Hence, $\mathfrak{A}$ is *LR-$\boldsymbol{\mathcal{H}}$-*convex *IV-F* on K.
Conversely, let $\mathfrak{A}$ be LR-$\boldsymbol{\mathcal{H}}$-convex IV-F on K. Then, for all $\varkappa,\mu\in K$, $s\in[0,\ 1]$, we have

$$\mathfrak{A}\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) \leq_p (1-s)\mathfrak{A}(\varkappa)+s\mathfrak{A}(\mu).$$

Therefore, from (9), left side of above inequality, we have

$$\mathfrak{A}\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) = \left[\mathfrak{A}_*\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right),\ \mathfrak{A}^*\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right)\right].$$

Again, from (9), we obtain

$$(1-s)\mathfrak{A}(\varkappa)+s\mathfrak{A}(\varkappa)= (1-s)[\mathfrak{A}_*(\varkappa),\ \mathfrak{A}^*(\varkappa)]+s[\mathfrak{A}_*(\mu),\ \mathfrak{A}^*(\mu)],$$

for all $\varkappa,\mu\in K$, $s\in[0,\ 1]$. Then, by $\boldsymbol{\mathcal{H}}$-convexity of $\mathfrak{A}$, we have for all $\varkappa,\mu\in K$, $s\in[0,\ 1]$ such that

$$\mathfrak{A}_*\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) \leq (1-s)\mathfrak{A}_*(\varkappa)+s\mathfrak{A}_*(\mu),$$

and

$$\mathfrak{A}^*\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) \leq (1-s)\mathfrak{A}^*(\varkappa)+s\mathfrak{A}^*(\mu),$$

this concludes the proof. □

Remark 2. *If one attempts to take $\mathfrak{A}_*(\varkappa)=\mathfrak{A}^*(\varkappa)$, then, from Definition 3, we achieve Definition 2.*

Example 1. *We consider the IV-Fs $\mathfrak{A}:[1,\ 2]\to\mathcal{K}_C$ defined by $\mathfrak{A}(\varkappa)=\left[ln(\varkappa),\ 2\sqrt{\varkappa}\right]$. Since end point functions $\mathfrak{A}_*(\varkappa)$, $\mathfrak{A}^*(\varkappa)$ are $\boldsymbol{\mathcal{H}}$-convex functions. Hence, $\mathfrak{A}(\varkappa)$ is LR-$\boldsymbol{\mathcal{H}}$-convex IV-F.*

In next result, we will establish a relation between *LR*-convex *IV-F* and LR-$\boldsymbol{\mathcal{H}}$-convex *IV-F*.

Theorem 3. *Let $\mathfrak{A}:K\to\mathcal{K}_C$ be an IV-F such that $\mathfrak{A}(\varkappa)=[\mathfrak{A}_*(\varkappa),\ \mathfrak{A}^*(\varkappa)]$, for all $\varkappa\in K$. Then, $\mathfrak{A}(\varkappa)$ is LR-$\boldsymbol{\mathcal{H}}$-convex IV-F on K, if and only if, $\mathfrak{A}\left(\frac{1}{\varkappa}\right)$ is LR-convex IV-F on K.*

Proof. Since $\mathfrak{A}(\varkappa)$ is a LR-$\boldsymbol{\mathcal{H}}$-convex *IV-F*, then, for $\varkappa,\ \mu\in[t,\ \upsilon]$, $s\in[0,\ 1]$, we have

$$\mathfrak{A}\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) \leq_p (1-s)\mathfrak{A}(\varkappa)+s\mathfrak{A}(\mu).$$

Therefore, we have

$$\begin{aligned}\mathfrak{A}_*\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) &\leq (1-s)\mathfrak{A}_*(\varkappa)+s\mathfrak{A}_*(\mu),\\ \mathfrak{A}^*\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) &\leq (1-s)\mathfrak{A}^*(\varkappa)+s\mathfrak{A}^*(\mu).\end{aligned} \tag{10}$$

Consider $\theta(\varkappa)=\mathfrak{A}\left(\frac{1}{\varkappa}\right)$. Taking $m=\frac{1}{\varkappa}$ and $n=\frac{1}{\mu}$ to replace $\varkappa$ and μ, respectively. Then, applying (10)

$$\begin{aligned}\mathfrak{A}_*\left(\frac{\frac{1}{\varkappa\mu}}{s\frac{1}{\varkappa}+(1-s)\frac{1}{\mu}}\right) &= \mathfrak{A}_*\left(\frac{1}{(1-s)\varkappa+s\mu}\right)\\ &= \theta_*((1-s)\varkappa+s\mu)\\ &\leq s\mathfrak{A}_*\left(\frac{1}{\mu}\right)+(1-s)\mathfrak{A}_*\left(\frac{1}{\varkappa}\right)\\ &= s\theta_*(\mu)+(1-s)\theta_*(\varkappa),\\ \mathfrak{A}^*\left(\frac{\frac{1}{\varkappa\mu}}{s\frac{1}{\varkappa}+(1-s)\frac{1}{\mu}}\right) &= \mathfrak{A}^*\left(\frac{1}{(1-s)\varkappa+s\mu}\right)\\ &= \theta^*((1-s)\varkappa+s\mu)\\ &\leq s\mathfrak{A}^*\left(\frac{1}{\mu}\right)+(1-s)\mathfrak{A}^*\left(\frac{1}{\varkappa}\right)\\ &= s\theta^*(\mu)+(1-s)\theta^*(\varkappa)\end{aligned}$$

It follows that

$$\left[\mathfrak{A}_*\left(\frac{\frac{1}{\varkappa\mu}}{s\frac{1}{\varkappa}+(1-s)\frac{1}{\mu}}\right),\ \mathfrak{A}^*\left(\frac{\frac{1}{\varkappa\mu}}{s\frac{1}{\varkappa}+(1-s)\frac{1}{\mu}}\right)\right] = [\theta_*((1-s)\varkappa+s\mu),\ \theta^*((1-s)\varkappa+s\mu)] \leq_p s[\theta_*(\mu),\ \theta^*(\mu)]+(1-s)[\theta_*(\varkappa),\ \theta^*(\varkappa)].$$

which implies that

$$\theta((1-s)\varkappa+s\mu) \leq_p s\theta(\mu)+(1-s)\theta(\varkappa).$$

This concludes that $\theta(\varkappa)$ is a LR-convex IV-F.

Conversely, let θ is LR-convex *IV-F* on K. Then, for all $\varkappa, \mu \in K$, $s \in [0,\ 1]$, we have

$$\theta(s\varkappa+(1-s)\mu\ ) \leq_p s\theta(\varkappa)+(1-s)\theta(\mu).$$

By using the same steps as above, we have

$$\begin{aligned}\theta_*\left(s\frac{1}{\varkappa}+(1-s)\frac{1}{\mu}\right) &= \mathfrak{A}_*\left(\frac{1}{s\frac{1}{\varkappa}+(1-s)\frac{1}{\mu}}\right)\\ &= \mathfrak{A}_*\left(\frac{\varkappa\mu}{(1-s)\varkappa+s\mu}\right)\\ &\leq s\theta_*\left(\frac{1}{\varkappa}\right)+(1-s)\theta_*\left(\frac{1}{\mu}\right)\\ &= s\mathfrak{A}_*(\varkappa)+(1-s)\mathfrak{A}_*(\mu)\\ \theta^*\left(s\frac{1}{\varkappa}+(1-s)\frac{1}{\mu}\right) &= \mathfrak{A}_*\left(\frac{1}{s\frac{1}{\varkappa}+(1-s)\frac{1}{\mu}}\right)\\ &= \mathfrak{A}_*\left(\frac{\varkappa\mu}{(1-s)\varkappa+s\mu}\right)\\ &\leq s\theta^*\left(\frac{1}{\varkappa}\right)+(1-s)\theta^*\left(\frac{1}{\mu}\right)\\ &= s\mathfrak{A}_*(\varkappa)+(1-s)\mathfrak{A}_*(\mu)\end{aligned}$$

It follows that

$$\mathfrak{A}\left(\frac{\varkappa\mu}{s\varkappa+(1-s)\mu}\right) \leq_p (1-s)\mathfrak{A}(\varkappa)+s\mathfrak{A}(\mu).$$

This completes the proof. □

Remark 3. If one attempts to take $\mathfrak{A}_*(\varkappa) = \mathfrak{A}^*(\varkappa)$, *then, from Theorem 3, we acquire the Lemma 2.1 of* [30].

3. Main Results

Budak et al. [1] introduced the notion of *IV-RL*-fractional integrals. As may be seen, fractional integral definitions and *IV-RL*-fractional integral definitions have comparable configurations. As a result of this observation, we may state the *H-H* inequality for LR-harmonically *IV-Fs* using *IV-RL*-fractional integrals.

Theorem 4. *Let* $\mathfrak{A} \in LRHSX([\mathrm{t},\, \upsilon],\, \mathcal{K}_C^+)$, *and defined on the interval* $[\mathrm{t},\, \upsilon]$ *such that* $\mathfrak{A}(\varkappa) = [\mathfrak{A}_*(\varkappa),\, \mathfrak{A}^*(\varkappa)]$ *for all* $\varkappa \in [\mathrm{t},\, \upsilon]$. *If* $\mathfrak{A} \in L([\mathrm{t},\, \upsilon], \mathcal{K}_C^+)$ *and fractional integral over* $[\mathrm{t},\, \upsilon]$, *then*

$$\mathfrak{A}\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \leq_{\mathrm{p}} \frac{\Gamma(\beta+1)}{2(\upsilon-\mathrm{t})^{\beta}}\left[\mathfrak{T}^{\beta}_{\frac{1}{\mathrm{t}}^-}(\mathfrak{A}\circ\Psi)\left(\frac{1}{\upsilon}\right) + \mathfrak{T}^{\beta}_{\frac{1}{\upsilon}^+}(\mathfrak{A}\circ\Psi)\left(\frac{1}{\mathrm{t}}\right)\right] \leq_{\mathrm{p}} \frac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(\upsilon)}{2}. \quad (11)$$

If $\mathfrak{A}(\varkappa)$ *is LR-*$\boldsymbol{\mathcal{H}}$*-concave IV-F, then*

$$\mathfrak{A}\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \geq_{p} \frac{\Gamma(\beta+1)}{2(\upsilon-\mathrm{t})^{\beta}}\left[\mathfrak{T}^{\beta}_{\frac{1}{\mathrm{t}}^-}(\mathfrak{A}\circ\Psi)\left(\frac{1}{\upsilon}\right) + \mathfrak{T}^{\beta}_{\frac{1}{\upsilon}^+}(\mathfrak{A}\circ\Psi)\left(\frac{1}{\mathrm{t}}\right)\right] \geq_{p} \frac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(\upsilon)}{2}. \quad (12)$$

where $\Psi(\varkappa) = \frac{1}{\varkappa}$.

Proof. Let $\mathfrak{A} \in LRHSX([\mathrm{t},\, \upsilon],\, \mathcal{K}_C^+)$. Then, by hypothesis, we have

$$2\mathfrak{A}\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \leq_{\mathrm{p}} \mathfrak{A}\left(\frac{\mathrm{t}\upsilon}{\mathrm{s}\mathrm{t}+(1-\mathrm{s})\upsilon}\right) + \mathfrak{A}\left(\frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right).$$

Therefore, we have

$$\begin{aligned} 2\mathfrak{A}_*\left(\tfrac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) &\leq \mathfrak{A}_*\left(\tfrac{\mathrm{t}\upsilon}{\mathrm{s}\mathrm{t}+(1-\mathrm{s})\upsilon}\right) + \mathfrak{A}_*\left(\tfrac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right),\\ 2\mathfrak{A}^*\left(\tfrac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) &\leq \mathfrak{A}^*\left(\tfrac{\mathrm{t}\upsilon}{\mathrm{s}\mathrm{t}+(1-\mathrm{s})\upsilon}\right) + \mathfrak{A}^*\left(\tfrac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right). \end{aligned}$$

Consider $\theta(\varkappa) = \mathfrak{A}\left(\frac{1}{\varkappa}\right)$. By Theorem 3, we have $\theta(\varkappa)$ is LR-convex *IV-F*. Then, above inequality, we have

$$2\theta_*\left(\frac{\mathrm{t}+\upsilon}{2\mathrm{t}\upsilon}\right) \leq \theta_*\left(\frac{\mathrm{s}\mathrm{t}+(1-\mathrm{s})\upsilon}{\mathrm{t}\upsilon}\right) + \theta_*\left(\frac{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}{\mathrm{t}\upsilon}\right).$$

Multiplying both sides by $\mathrm{s}^{\beta-1}$ and integrating the obtained result with respect to s over $(0,1)$, we have

$$\begin{aligned} &2\int_0^1 \mathrm{s}^{\beta-1}\theta_*\left(\tfrac{\mathrm{t}+\upsilon}{2\mathrm{t}\upsilon}\right)ds\\ &\leq \int_0^1 \mathrm{s}^{\beta-1}\theta_*\left(\tfrac{\mathrm{s}\mathrm{t}+(1-\mathrm{s})\upsilon}{\mathrm{t}\upsilon}\right)d\mathrm{s} + \int_0^1 \mathrm{s}^{\beta-1}\theta_*\left(\tfrac{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}{\mathrm{t}\upsilon}\right)ds. \end{aligned}$$

Let $\varkappa = \frac{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}{\mathrm{t}\upsilon}$ and $\mu = \frac{\mathrm{s}\mathrm{t}+(1-\mathrm{s})\upsilon}{\mathrm{t}\upsilon}$. Then, we have

$$\frac{2}{\beta}\theta_*\left(\tfrac{\mathrm{t}+\upsilon}{2\mathrm{t}\upsilon}\right) \leq \left(\tfrac{\mathrm{t}\upsilon}{\upsilon-\mathrm{t}}\right)^{\beta} \int_{\frac{1}{\upsilon}}^{\frac{1}{\mathrm{t}}}\left(\frac{1}{\mathrm{t}}-\mu\right)^{\beta-1}\theta_*(\mu)d\mu + \left(\tfrac{\mathrm{t}\upsilon}{\upsilon-\mathrm{t}}\right)^{\beta}\int_{\frac{1}{\upsilon}}^{\frac{1}{\mathrm{t}}}\left(\varkappa-\frac{1}{\upsilon}\right)^{\beta-1}\theta_*(\varkappa)d\varkappa = \Gamma(\beta)\left(\tfrac{\mathrm{t}\upsilon}{\upsilon-\mathrm{t}}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}\theta_*\left(\frac{1}{\upsilon}\right) + \mathfrak{T}^{\beta}_{\left(\frac{1}{\upsilon}\right)} + \theta_*\left(\frac{1}{\mathrm{t}}\right)\right].$$

Similarly, for $\theta^*(\varkappa)$, we have

$$\frac{2}{\beta}\theta^*\left(\frac{\mathrm{t}+\upsilon}{2\mathrm{t}\upsilon}\right) \leq \Gamma(\beta)\left(\frac{\mathrm{t}\upsilon}{\upsilon-\mathrm{t}}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}\theta^*\left(\frac{1}{\upsilon}\right) + \mathfrak{T}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\theta^*\left(\frac{1}{\mathrm{t}}\right)\right]$$

It follows that

$$2\left[\theta_*\left(\frac{t+v}{2tv}\right),\ \theta^*\left(\frac{t+v}{2tv}\right)\right] \leq_p \Gamma(\beta+1)\left(\frac{tv}{v-t}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^-}\ \theta_*\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\ \theta_*\left(\frac{1}{t}\right),\ \mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^-}\ \theta^*\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\ \theta^*\left(\frac{1}{t}\right)\right].$$

That is,

$$2\,\theta\left(\frac{t+v}{2tv}\right) \leq_p \Gamma(\beta+1)\left(\frac{tv}{v-t}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^-}\ \theta\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\ \theta\left(\frac{1}{t}\right)\right]. \tag{13}$$

In a similar way as above, we have

$$\Gamma(\beta)\left(\frac{tv}{v-t}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^-}\ \theta\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\ \theta\left(\frac{1}{t}\right)\right] \leq_p \frac{\theta\left(\frac{1}{t}\right)+\theta\left(\frac{1}{v}\right)}{\beta}. \tag{14}$$

Combining (31) and (32), we have

$$\theta\left(\frac{t+v}{2tv}\right) \leq_p \frac{\Gamma(\beta+1)\ \left(\frac{tv}{v-t}\right)^{\beta}}{2}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^-}\ \theta\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\ \theta\left(\frac{1}{t}\right)\right] \leq_p \frac{\theta\left(\frac{1}{t}\right)+\theta\left(\frac{1}{v}\right)}{2},$$

that is

$$\mathfrak{A}\left(\frac{2tv}{t+v}\right) \leq_p \frac{\Gamma(\beta+1)}{2(v-t)^{\beta}}\left[\mathfrak{T}^{\beta}_{\frac{1}{t}-}(\mathfrak{A}\circ\Psi)\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\frac{1}{v}+}\ (\mathfrak{A}\circ\Psi)\left(\frac{1}{t}\right)\right] \leq_p \frac{\mathfrak{A}(t)+\mathfrak{A}(v)}{2}.$$

Hence, the required result. □

Remark 4. *On the basic of the inequality (29), we consider certain special cases as below. If we attempt to take $\beta=1$, then, we achieve the coming inequality which is also new one:*

$$\mathfrak{A}\left(\frac{2tv}{t+v}\right) \leq_p \frac{tv}{v-t}\int_t^v \frac{\mathfrak{A}(\varkappa)}{\varkappa^2}d\varkappa \leq_p \frac{\mathfrak{A}(t)+\mathfrak{A}(v)}{2}. \tag{15}$$

If we attempt to take $\mathfrak{A}_*(\varkappa)=\mathfrak{A}^*(\varkappa)$, then, we achieve the coming inequality, see [30]:

$$\mathfrak{A}\left(\frac{2tv}{t+v}\right) \leq \frac{\Gamma(\beta+1)}{2(v-t)^{\beta}}\left[\mathfrak{T}^{\beta}_{\frac{1}{t}-}(\mathfrak{A}\circ\Psi)\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\frac{1}{v}+}\ (\mathfrak{A}\circ\Psi)\left(\frac{1}{t}\right)\right] \leq \frac{\mathfrak{A}(t)+\mathfrak{A}(v)}{2}. \tag{16}$$

If we attempt to take $\mathfrak{A}_*(\varkappa)=\mathfrak{A}^*(\varkappa)$ with $\beta=1$, then, we acquire the coming inequality, see [27].

$$\mathfrak{A}\left(\frac{2tv}{t+v}\right) \leq \frac{tv}{v-t}\int_t^v \frac{\mathfrak{A}(\varkappa)}{\varkappa^2}d\varkappa \leq \frac{\mathfrak{A}(t)+\mathfrak{A}(v)}{2} \tag{17}$$

Example 2. *If we consider taking the IV-Fs $\mathfrak{A}:[0,\ 2]\to\mathbb{F}_C(\mathbb{R})$ such that $[1,\ 2]\sqrt{\varkappa}$, then, all assumptions mentioned in Theorem 4 are met. Since $\mathfrak{A}_*(\varkappa)=\sqrt{\varkappa}$, $\mathfrak{A}^*(\varkappa,\ \theta)=2\sqrt{\varkappa}$. If $\beta=1$, then, we compute the following:*

$$\mathfrak{A}_*\left(\frac{2tv}{t+v}\right) \leq \frac{\Gamma(\beta+1)}{2(v-t)^{\beta}}\left[\mathfrak{T}^{\beta}_{\frac{1}{t}-}(\mathfrak{A}_*\circ\Psi)\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\frac{1}{v}+}\ (\mathfrak{A}_*\circ\Psi)\left(\frac{1}{t}\right)\right] \leq \frac{\mathfrak{A}_*(t)+\mathfrak{A}_*(v)}{2}.$$

$$\mathfrak{A}_*\left(\frac{2tv}{t+v}\right)=\mathfrak{A}_*(0)=0,$$

$$\frac{\Gamma(\beta+1)}{2(v-t)^{\beta}}\left[\mathfrak{T}^{\beta}_{\frac{1}{t}-}(\mathfrak{A}_*\circ\Psi)\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\frac{1}{v}+}\ (\mathfrak{A}_*\circ\Psi)\left(\frac{1}{t}\right)\right]=0,$$

$$\frac{tv}{v-t}\ \int_t^v \frac{\mathfrak{A}_*(\varkappa)}{\varkappa^2}d\varkappa=\frac{0}{2}\ \int_0^2 \frac{\sqrt{\varkappa}}{\varkappa^2}d\varkappa=0,$$

$$\frac{\mathfrak{A}_*(t)+\mathfrak{A}_*(v)}{2}=\frac{1}{\sqrt{2}}.$$

That means

$$0 \leq 0 \leq \frac{1}{\sqrt{2}}.$$

Similarly, it can be easily shown that

$$\mathfrak{A}^*\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right) \leq \frac{\Gamma(\beta+1)}{2(v-\mathrm{t})^{\beta}}\left[\mathfrak{T}^{\beta}_{\frac{1}{\mathrm{t}}^-}(\mathfrak{A}^*\circ\Psi)\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\frac{1}{v}^+}(\mathfrak{A}^*\circ\Psi)\left(\frac{1}{\mathrm{t}}\right)\right] \leq \frac{\mathfrak{A}^*(\mathrm{t})+\mathfrak{A}^*(v)}{2}.$$

Now

$$\mathfrak{A}^*\left(\tfrac{2\mathrm{t}v}{\mathrm{t}+v}\right) = \mathfrak{A}_*(0) = 0,$$

$$\tfrac{\Gamma(\beta+1)}{2(v-\mathrm{t})^{\beta}}\left[\mathfrak{T}^{\beta}_{\frac{1}{\mathrm{t}}^-}(\mathfrak{A}^*\circ\Psi)\left(\tfrac{1}{v}\right)+\mathfrak{T}^{\beta}_{\frac{1}{v}^+}(\mathfrak{A}^*\circ\Psi)\left(\tfrac{1}{\mathrm{t}}\right)\right] = 0,$$

$$\tfrac{\mathfrak{A}^*(\mathrm{t})+\mathfrak{A}^*(v)}{2} = \sqrt{2}.$$

From which, we have

$$0 \leq 0 \leq \sqrt{2},$$

that is

$$[0,\ 0] \leq_{\mathrm{p}} [0,\ 0] \leq_{\mathrm{p}} \left[\frac{1}{\sqrt{2}},\ \sqrt{2}\right].$$

Hence,

$$\mathfrak{A}\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right) \leq_{\mathrm{p}} \frac{\Gamma(\beta+1)}{2(v-\mathrm{t})^{\beta}}\left[\mathfrak{T}^{\beta}_{\frac{1}{\mathrm{t}}^-}(\mathfrak{A}\circ\Psi)\left(\frac{1}{v}\right)+\mathfrak{T}^{\beta}_{\frac{1}{v}^+}(\mathfrak{A}\circ\Psi)\left(\frac{1}{\mathrm{t}}\right)\right] \leq_{\mathrm{p}} \frac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(v)}{2}.$$

Based on the IV-RL-fractional integrals, our next main results in association with the H-H type inequalities for product of two LR-harmonically IV-Fs are presented as follows.

Theorem 5. *Let $\mathfrak{A}, \Psi \in LRHSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)$, and defined on the interval $[\mathrm{t},\ v]$ such that $\mathfrak{A}(\varkappa) = [\mathfrak{A}_*(\varkappa),\ \mathfrak{A}^*(\varkappa)]$ and $\Psi(\varkappa) = [\Psi_*(\varkappa),\ \Psi^*(\varkappa)]$ for all $\varkappa \in [\mathrm{t},\ v]$. If $\mathfrak{A} \times \Psi \in L([\mathrm{t},\ v], \mathcal{K}_C^+)$, and fractional integral over $[\mathrm{t},\ v]$, then*

$$\tfrac{\Gamma(\beta+1)}{2}\left(\tfrac{\mathrm{t}v}{v-\mathrm{t}}\right)^{\beta}\left[\ \mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\mathfrak{A}\circ\Psi\left(\tfrac{1}{\mathrm{t}}\right)\times\Psi\circ\Psi\left(\tfrac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}\mathfrak{A}\circ\Psi\left(\tfrac{1}{v}\right)\times\Psi\circ\Psi\left(\tfrac{1}{v}\right)\ \right] \leq_{\mathrm{p}} \left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}(\mathrm{t},v)+\left(\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}(\mathrm{t},v),$$

where $\mathfrak{D}(\mathrm{t},v) = \mathfrak{A}(\mathrm{t})\times\Psi(\mathrm{t})+\mathfrak{A}(v)\times\Psi(v)$, $\mathcal{Q}(\mathrm{t},v) = \mathfrak{A}(\mathrm{t})\times\Psi(v)+\mathfrak{A}(v)\times\Psi(\mathrm{t})$, and $\mathfrak{D}(\mathrm{t},v) = [\mathfrak{D}_*(\mathrm{t},v),\ \mathfrak{D}^*(\mathrm{t},v)]$ and $\mathcal{Q}(\mathrm{t},v) = [\mathcal{Q}_*(\mathrm{t},v),\ \mathcal{Q}^*(\mathrm{t},v)]$.

Proof. Since $\mathfrak{A}, \Psi \in LRHSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)$, then, we have

$$\mathfrak{A}_*\left(\frac{\mathrm{t}v}{\mathrm{s}\mathrm{t}+(1-\mathrm{s})v}\right) \leq (1-\mathrm{s})\mathfrak{A}_*(\mathrm{t})+\mathrm{s}\mathfrak{A}_*(v),$$

and

$$\Psi_*\left(\tfrac{\mathrm{t}v}{\mathrm{s}\mathrm{t}+(1-\mathrm{s})v}\right) \leq (1-\mathrm{s})\Psi_*(\mathrm{t})+\mathrm{s}\Psi_*(v)\ .$$

From the definition of LR-$\boldsymbol{\mathcal{H}}$-convex *IV-Fs* it follows that $0 \leq_{\mathrm{p}} \mathfrak{A}(\varkappa)$ and $0 \leq_{\mathrm{p}} \Psi(\varkappa)$, so

$$\begin{array}{c}\mathfrak{A}_*\left(\tfrac{\mathrm{t}v}{\mathrm{s}\mathrm{t}+(1-\mathrm{s})v}\right)\times\Psi_*\left(\tfrac{\mathrm{t}v}{\mathrm{s}\mathrm{t}+(1-\mathrm{s})v}\right)\\ \leq \left(\ (1-\mathrm{s})\mathfrak{A}_*(\mathrm{t})+\mathrm{s}\mathfrak{A}_*(v)\ \right)\left(\ (1-\mathrm{s})\Psi_*(\mathrm{t})+\mathrm{s}\Psi_*(v)\ \right)\\ = (1-\mathrm{s})^2\mathfrak{A}_*(\mathrm{t})\times\Psi_*(\mathrm{t})+\mathrm{s}^2\mathfrak{A}_*(v)\times\Psi_*(v)\\ +\mathrm{s}(1-\mathrm{s})\mathfrak{A}_*(\mathrm{t})\times\Psi_*(v)+\mathrm{s}(1-\mathrm{s})\mathfrak{A}_*(v)\times\Psi_*(\mathrm{t})\end{array}. \tag{18}$$

Analogously, we have

$$\begin{array}{c}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\Psi_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right) \\ \leq \mathrm{s}^2\mathfrak{A}_*(\mathrm{t})\times\Psi_*(\mathrm{t})+(1-\mathrm{s})^2\mathfrak{A}_*(v)\times\Psi_*(v) \\ +\mathrm{s}(1-\mathrm{s})\mathfrak{A}_*(\mathrm{t})\times\Psi_*(v)+\mathrm{s}(1-\mathrm{s})\mathfrak{A}_*(v)\times\Psi_*(\mathrm{t})\end{array}. \tag{19}$$

Adding (18) and (19), we have

$$\begin{array}{c}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)+\mathfrak{A}_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right) \\ \leq\left[\mathrm{s}^2+(1-\mathrm{s})^2\right]\left[\ \mathfrak{A}_*(\mathrm{t})\times\Psi_*(\mathrm{t})+\mathfrak{A}_*(v)\times\Psi_*(v)\ \right] \\ +2\mathrm{s}(1-\mathrm{s})\left[\ \mathfrak{A}_*(v)\times\Psi_*(\mathrm{t})+\mathfrak{A}_*(\mathrm{t})\times\Psi_*(v)\ \right]\end{array}. \tag{20}$$

Taking multiplication of (20) by $\mathrm{s}^{\beta-1}$ and integrating the obtained result with respect to s over (0, 1), we have

$$\begin{array}{l}\int_0^1 \mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)d\mathrm{s} \\ \quad+\int_0^1 \mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)d\mathrm{s} \\ \leq\mathfrak{D}_*(\mathrm{t},v)\int_0^1 \mathrm{s}^{\beta-1}\left[\mathrm{s}^2+(1-\mathrm{s})^2\right]d\mathrm{s}+2\mathcal{Q}_*(\mathrm{t},v)\int_0^1 \mathrm{s}^{\beta-1}\mathrm{s}(1-\mathrm{s})d\mathrm{s}.\end{array}$$

It follows that,

$$\begin{array}{l}\Gamma(\beta)\left(\frac{\mathrm{t}v}{v-\mathrm{t}}\right)^\beta\left[\ \mathfrak{T}^\beta_{\left(\frac{1}{v}\right)^+}\mathfrak{A}_*\left(\frac{1}{\mathrm{t}}\right)\times\Psi_*\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^\beta_{\left(\frac{1}{\mathrm{t}}\right)^-}\mathfrak{A}_*\left(\frac{1}{v}\right)\times\Psi_*\left(\frac{1}{v}\right)\ \right] \\ \qquad\leq\frac{2}{\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}_*(\mathrm{t},v)+\frac{2}{\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}_*(\mathrm{t},v)\end{array},$$

Similarly, for $\mathfrak{A}^*(\varkappa)$, we have

$$\begin{array}{l}\Gamma(\beta)\left(\frac{\mathrm{t}v}{v-\mathrm{t}}\right)^\beta\left[\ \mathfrak{T}^\beta_{\left(\frac{1}{v}\right)^+}\mathfrak{A}_*\left(\frac{1}{\mathrm{t}}\right)\times\Psi_*\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^\beta_{\left(\frac{1}{\mathrm{t}}\right)^-}\mathfrak{A}_*\left(\frac{1}{v}\right)\times\Psi_*\left(\frac{1}{v}\right)\ \right] \\ \qquad\leq\frac{2}{\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}_*(\mathrm{t},v)+\frac{2}{\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}_*(\mathrm{t},v)\end{array},$$

that is

$$\begin{array}{c}\Gamma(\beta)\left(\frac{\mathrm{t}v}{v-\mathrm{t}}\right)^\beta\left[\mathfrak{T}^\beta_{\left(\frac{1}{v}\right)^+}\mathfrak{A}_*\left(\frac{1}{\mathrm{t}}\right)\times\Psi_*\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^\beta_{\left(\frac{1}{\mathrm{t}}\right)^-}\mathfrak{A}_*\left(\frac{1}{v}\right)\times\Psi_*\left(\frac{1}{v}\right),\ \mathfrak{T}^\beta_{\left(\frac{1}{v}\right)^+}\mathfrak{A}^*\left(\frac{1}{\mathrm{t}}\right)\times\right. \\ \left.\Psi^*\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^\beta_{\left(\frac{1}{\mathrm{t}}\right)^-}\mathfrak{A}^*\left(\frac{1}{v}\right)\times\Psi^*\left(\frac{1}{v}\right)\right]\leq_{\mathrm{p}}\frac{2}{\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)[\mathcal{D}_*(\mathrm{t},v),\ \mathcal{D}^*(\mathrm{t},v)] \\ +\frac{2}{\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)[\mathcal{Q}_*(\mathrm{t},v),\ \mathcal{Q}^*(\mathrm{t},v)].\end{array}$$

Thus,

$$\begin{array}{c}\frac{\Gamma(\beta+1)}{2}\left(\frac{\mathrm{t}v}{v-\mathrm{t}}\right)^\beta\left[\ \mathfrak{T}^\beta_{\left(\frac{1}{v}\right)^+}\mathfrak{A}\circ\Psi\left(\frac{1}{\mathrm{t}}\right)\times\Psi\circ\Psi\left(\frac{1}{\mathrm{t}}\right)\ +\mathfrak{T}^\beta_{\left(\frac{1}{\mathrm{t}}\right)^-}\mathfrak{A}\circ\Psi\left(\frac{1}{v}\right)\times\Psi\circ\right. \\ \left.\Psi\left(\frac{1}{v}\right)\right]\ \leq_{\mathrm{p}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}(\mathrm{t},v)+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}(\mathrm{t},v).\end{array}$$

and the theorem has been established. □

Theorem 6. *Let* $\mathfrak{A},\Psi\in LRHSX\left([\mathrm{t},\ v],\ \mathcal{K}_C^+\right)$, *and defined on the interval* $[\mathrm{t},\ v]$ *such that* $\mathfrak{A}(\varkappa)=[\mathfrak{A}_*(\varkappa),\ \mathfrak{A}^*(\varkappa)]$ *and* $\Psi(\varkappa)=[\Psi_*(\varkappa),\ \Psi^*(\varkappa)]$ *for all* $\varkappa\in[\mathrm{t},\ v]$. *If* $\mathfrak{A}\times\Psi\in L\left([\mathrm{t},\ v],\mathcal{K}_C^+\right)$ *and fractional integral over* $[\mathrm{t},\ v]$, *then*

$$\mathfrak{A}\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\times\Psi\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\leq_{\mathrm{p}}\frac{\Gamma(\beta+1)}{4}\left(\frac{\mathrm{t}v}{v-\mathrm{t}}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^{+}}\mathfrak{A}\left(\frac{1}{\mathrm{t}}\right)\times\Psi\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^{-}}\mathfrak{A}\left(\frac{1}{v}\right)\times\Psi\left(\frac{1}{v}\right)\right]+\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}(\mathrm{t},v)+\frac{1}{2}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}(\mathrm{t},v),$$

where $\mathfrak{D}(\mathrm{t},v)=\mathfrak{A}(\mathrm{t})\times\Psi(\mathrm{t})+\mathfrak{A}(v)\times\Psi(v)$, $\mathcal{Q}(\mathrm{t},v)=\mathfrak{A}(\mathrm{t})\times\Psi(v)+\mathfrak{A}(v)\times\Psi(\mathrm{t})$, *and* $\mathfrak{D}(\mathrm{t},v)=[\mathfrak{D}_*(\mathrm{t},v),\ \mathfrak{D}^*(\mathrm{t},v)]$ *and* $\mathcal{Q}(\mathrm{t},v)=[\mathcal{Q}_*(\mathrm{t},v),\ \mathcal{Q}^*(\mathrm{t},v)]$.

Proof. Consider $\mathfrak{A},\Psi\in LRHSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)$. Then, by hypothesis, we have

$$\begin{aligned}
&\mathfrak{A}_*\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\times\Psi_*\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\\
&\leq\frac{1}{4}\begin{bmatrix}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\\+\mathfrak{A}_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\end{bmatrix}+\frac{1}{4}\begin{bmatrix}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\\+\mathfrak{A}_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\end{bmatrix},\\
&\leq\frac{1}{4}\begin{bmatrix}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\\+\mathfrak{A}_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\end{bmatrix}+\frac{1}{4}\begin{bmatrix}(\mathrm{s}\mathfrak{A}_*(\mathrm{t})+(1-\mathrm{s})\mathfrak{A}_*(v))\\\times((1-\mathrm{s})\Psi_*(\mathrm{t})+\mathrm{s}\Psi_*(v))\\+((1-\mathrm{s})\mathfrak{A}_*(\mathrm{t})+\mathrm{s}\mathfrak{A}_*(v))\\\times(\mathrm{s}\Psi_*(\mathrm{t})+(1-\mathrm{s})\Psi_*(v))\end{bmatrix},\\
&=\frac{1}{4}\begin{bmatrix}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\\+\mathfrak{A}_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\end{bmatrix}+\frac{1}{4}\begin{bmatrix}\left\{\mathrm{s}^2+(1-\mathrm{s})^2\right\}\mathcal{Q}_*(\mathrm{t},v)\\+\{\mathrm{s}(1-\mathrm{s})+(1-\mathrm{s})\mathrm{s}\}\mathfrak{D}_*(\mathrm{t},v)\end{bmatrix}.
\end{aligned}\tag{21}$$

Multiplying inequality (21) by $\mathrm{s}^{\beta-1}$ and integrating over $(0,\ 1)$,

$$\begin{aligned}
&\mathfrak{A}_*\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\times\Psi_*\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\\
&\leq\frac{1}{4}\begin{bmatrix}\int_0^1\mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)d\mathrm{s}\\+\int_0^1\mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\times\Psi_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\end{bmatrix}d\mathrm{s}+\begin{bmatrix}\frac{1}{4}\mathcal{Q}_*(\mathrm{t},v)\int_0^1\mathrm{s}^{\beta-1}\left[\mathrm{s}^2+(1-\mathrm{s})^2\right]d\mathrm{s}\\+2\mathfrak{D}_*(\mathrm{t},v)\int_0^1\mathrm{s}^{\beta-1}\mathrm{s}(1-\mathrm{s})d\mathrm{s}\end{bmatrix}.
\end{aligned}$$

Taking $\varkappa=\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}$ and $\mu=\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}$

$$\begin{aligned}
&\frac{1}{\beta}\,\mathfrak{A}_*\left(\frac{2tv}{t+v}\right)\times\Psi_*\left(\frac{2tv}{t+v}\right)\\
&\leq\frac{\Gamma(\beta)}{4}\left(\frac{tv}{v-t}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^{+}}\mathfrak{A}_*\circ\Psi\left(\frac{1}{t}\right)\times\Psi_*\circ\Psi\left(\frac{1}{t}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^{-}}\mathfrak{A}_*\circ\Psi\left(\frac{1}{v}\right)\times\Psi_*\circ\Psi\left(\frac{1}{v}\right)\right]\\
&\quad+\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}_*(t,v)+\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}_*(t,v),\\
&\frac{1}{\beta}\,\mathfrak{A}^*\left(\frac{2tv}{t+v}\right)\times\Psi^*\left(\frac{2tv}{t+v}\right)\\
&\leq\frac{\Gamma(\beta)}{4}\left(\frac{tv}{v-t}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^{+}}\mathfrak{A}^*\circ\Psi\left(\frac{1}{t}\right)\times\Psi^*\circ\Psi\left(\frac{1}{t}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^{-}}\mathfrak{A}^*\circ\Psi\left(\frac{1}{v}\right)\times\Psi^*\circ\Psi\left(\frac{1}{v}\right)\right]\\
&\quad+\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}^*(t,v)+\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}^*(t,v),
\end{aligned}$$

Similarly, for $\mathfrak{A}^*(\varkappa)$, we have

$$\begin{aligned}
&\frac{1}{\beta}\,\mathfrak{A}^*\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\times\Psi^*\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\\
&\leq\frac{\Gamma(\beta)}{4}\left(\frac{\mathrm{t}v}{v-\mathrm{t}}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^{+}}\mathfrak{A}^*\circ\Psi\left(\frac{1}{\mathrm{t}}\right)\times\Psi^*\circ\Psi\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^{-}}\mathfrak{A}^*\circ\Psi\left(\frac{1}{v}\right)\times\Psi^*\circ\Psi\left(\frac{1}{v}\right)\right],\\
&\quad+\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}^*(\mathrm{t},v)+\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}^*(\mathrm{t},v),
\end{aligned}$$

that is

$$\mathfrak{A}\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\widetilde{\times}\Psi\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\leq_{\mathrm{p}}\frac{\Gamma(\beta+1)}{4}\left(\frac{\mathrm{t}v}{v-\mathrm{t}}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^{+}}\mathfrak{A}\left(\frac{1}{\mathrm{t}}\right)\times\Psi\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^{-}}\mathfrak{A}\left(\frac{1}{v}\right)\times\Psi\left(\frac{1}{v}\right)\right]+\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}(\mathrm{t},v)+\frac{1}{2}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}(\mathrm{t},v).$$

Hence, the required result. □

Theorem 7. *Let* $\mathfrak{A}, \Psi \in LRHSX([\mathrm{t},\ \upsilon],\ \mathcal{K}_C^+)$*, and defined on the interval* $[\mathrm{t},\ \upsilon]$ *such that* $\mathfrak{A}(\varkappa) = [\mathfrak{A}_*(\varkappa),\ \mathfrak{A}^*(\varkappa)]$ *and* $\Psi(\varkappa) = [\Psi_*(\varkappa),\ \Psi^*(\varkappa)]$ *for all* $\varkappa \in [\mathrm{t},\ \upsilon]$*. If* $\mathfrak{A} \times \Psi \in L([\mathrm{t},\ \upsilon], \mathcal{K}_C^+)$ *and fractional integral over* $[\mathrm{t},\ \upsilon]$*, then*

$$2\mathfrak{A}\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \times \Psi\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \leq_{\mathrm{p}} \frac{\Gamma(\beta+1)}{2^{1-\beta}}\left(\frac{\mathrm{t}\upsilon}{\upsilon-\mathrm{t}}\right)^{\beta}\left[\ \mathfrak{T}^{\beta}_{\left(\frac{\mathrm{t}+\upsilon}{2\mathrm{t}\upsilon}\right)^+}\ \mathfrak{A}\circ\Psi\left(\frac{1}{\mathrm{t}}\right) \times \Psi\circ\Psi\left(\frac{1}{\mathrm{t}}\right) + \mathfrak{T}^{\beta}_{\left(\frac{\mathrm{t}+\upsilon}{2\mathrm{t}\upsilon}\right)^-}\ \mathfrak{A}\circ\Psi\left(\frac{1}{\upsilon}\right) \times \Psi\circ\Psi\left(\frac{1}{\upsilon}\right)\ \right] + \left(\frac{1}{2} - \frac{\beta^2+3\beta}{4(\beta+1)(\beta+2)}\right)\mathcal{Q}(\mathrm{t},\upsilon) + \frac{\beta^2+3\beta}{4(\beta+1)(\beta+2)}\mathfrak{D}(\mathrm{t},\upsilon),$$

where $\mathfrak{D}(\mathrm{t},\upsilon) = \mathfrak{A}(\mathrm{t}) \times \Psi(\mathrm{t}) + \mathfrak{A}(\upsilon) \times \Psi(\upsilon)$, $\mathcal{Q}(\mathrm{t},\upsilon) = \mathfrak{A}(\mathrm{t}) \times \Psi(\upsilon) + \mathfrak{A}(\upsilon) \times \Psi(\mathrm{t})$, *and* $\mathfrak{D}(\mathrm{t},\upsilon) = [\mathfrak{D}_*(\mathrm{t},\upsilon),\ \mathfrak{D}^*(\mathrm{t},\upsilon)]$ *and* $\mathcal{Q}(\mathrm{t},\upsilon) = [\mathcal{Q}_*(\mathrm{t},\upsilon),\ \mathcal{Q}^*(\mathrm{t},\upsilon)]$.

Proof. Consider $\mathfrak{A}, \Psi \in LRHSX([\mathrm{t},\ \upsilon],\ \mathcal{K}_C^+)$. Then, by hypothesis, we have

$$\begin{aligned}
&\mathfrak{A}_*\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \times \Psi_*\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right)\\
&\leq \frac{1}{4}\left[\begin{array}{c} \mathfrak{A}_*\left(\frac{\mathrm{t}\upsilon}{\mathrm{st}+(1-\mathrm{s})\upsilon}\right) \times \Psi_*\left(\frac{\mathrm{t}\upsilon}{\mathrm{st}+(1-\mathrm{s})\upsilon}\right) \\ +\mathfrak{A}_*\left(\frac{\mathrm{t}\upsilon}{\mathrm{st}+(1-\mathrm{s})\upsilon}\right) \times \Psi_*\left(\frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right)\end{array}\right] + \frac{1}{4}\left[\begin{array}{c} \mathfrak{A}_*\left(\frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right) \times \Psi_*\left(\frac{\mathrm{t}\upsilon}{\mathrm{st}+(1-\mathrm{s})\upsilon}\right) \\ +\mathfrak{A}_*\left(\frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right) \times \Psi_*\left(\frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right)\end{array}\right],\\
&\leq \frac{1}{4}\left[\begin{array}{c} \mathfrak{A}_*\left(\frac{\mathrm{t}\upsilon}{\mathrm{st}+(1-\mathrm{s})\upsilon}\right) \times \Psi_*\left(\frac{\mathrm{t}\upsilon}{\mathrm{st}+(1-\mathrm{s})\upsilon}\right) \\ +\mathfrak{A}_*\left(\frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right) \times \Psi_*\left(\frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right)\end{array}\right] + \frac{1}{4}\left[\begin{array}{c} (\mathrm{s}\mathfrak{A}_*(\mathrm{t}) + (1-\mathrm{s})\mathfrak{A}_*(\upsilon)) \\ \times((1-\mathrm{s})\Psi_*(\mathrm{t}) + \mathrm{s}\Psi_*(\upsilon)) \\ +((1-\mathrm{s})\mathfrak{A}_*(\mathrm{t}) + \mathrm{s}\mathfrak{A}_*(\upsilon)) \\ \times(\mathrm{s}\Psi_*(\mathrm{t}) + (1-\mathrm{s})\Psi_*(\upsilon))\end{array}\right]\\
&= \frac{1}{4}\left[\begin{array}{c} \mathfrak{A}_*\left(\frac{\mathrm{t}\upsilon}{\mathrm{st}+(1-\mathrm{s})\upsilon}\right) \times \Psi_*\left(\frac{\mathrm{t}\upsilon}{\mathrm{st}+(1-\mathrm{s})\upsilon}\right) \\ +\mathfrak{A}_*\left(\frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right) \times \Psi_*\left(\frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}\right)\end{array}\right] + \frac{1}{4}\left[\begin{array}{c} \left\{\mathrm{s}^2 + (1-\mathrm{s})^2\right\}\mathcal{Q}_*(\mathrm{t},\upsilon) \\ +2\mathrm{s}(1-\mathrm{s})\mathfrak{D}_*(\mathrm{t},\upsilon)\end{array}\right].
\end{aligned} \tag{22}$$

Multiplying inequality (22) by $2^{1+\beta}\beta\mathrm{s}^{\beta-1}$ and then, integrating the obtain outcome over $\left[0,\ \frac{1}{2}\right]$,

$$\begin{aligned}
&\mathfrak{A}_*\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \times \Psi_*\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right)\\
&\leq \frac{1}{4}\int_0^{\frac{1}{2}} 2^{1+\beta}\beta\mathrm{s}^{\beta-1}\left[\ \mathfrak{A}_*\left(\frac{t\upsilon}{\mathrm{s}t+(1-\mathrm{s})\upsilon}\right) \times \Psi_*\left(\frac{t\upsilon}{\mathrm{s}t+(1-\mathrm{s})\upsilon}\right) + \mathfrak{A}_*\left(\frac{t\upsilon}{(1-\mathrm{s})t+\mathrm{s}\upsilon}\right) \times \Psi_*\left(\frac{t\upsilon}{(1-\mathrm{s})t+\mathrm{s}\upsilon}\right)\ \right]d\mathrm{s}\\
&+\frac{1}{4}\left[\ \mathcal{Q}_*(t,\upsilon)\int_0^{\frac{1}{2}} 2^{1+\beta}\beta\mathrm{s}^{\beta-1}\left[\mathrm{s}^2 + (1-\mathrm{s})^2\right]d\mathrm{s} + 2\mathfrak{D}_*(t,\upsilon)\int_0^{\frac{1}{2}} 2^{1+\beta}\beta\mathrm{s}^{\beta-1}\mathrm{s}(1-\mathrm{s})d\mathrm{s}\ \right]
\end{aligned}$$

Taking $\varkappa = \frac{\mathrm{t}\upsilon}{\mathrm{st}+(1-\mathrm{s})\upsilon}$ and $\mu = \frac{\mathrm{t}\upsilon}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}\upsilon}$, then, we get

$$\begin{aligned}
&2\ \mathfrak{A}_*\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \times \Psi_*\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right)\\
&\leq \frac{\Gamma(\beta+1)}{2^{1-\beta}}\left(\frac{\mathrm{t}\upsilon}{\upsilon-\mathrm{t}}\right)^{\beta}\left[\ \mathfrak{T}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\ \mathfrak{A}_*\circ\Psi\left(\frac{1}{\mathrm{t}}\right) \times \Psi_*\circ\Psi\left(\frac{1}{\mathrm{t}}\right) + \mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}\ \mathfrak{A}_*\circ\Psi\left(\frac{1}{\upsilon}\right) \times \Psi_*\circ\Psi\left(\frac{1}{\upsilon}\right)\ \right].\\
&+\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{Q}_*(\mathrm{t},\upsilon) + \left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathfrak{D}_*(\mathrm{t},\upsilon).
\end{aligned} \tag{23}$$

Similarly, for $\mathfrak{A}^*(\varkappa)$, we have

$$\begin{aligned}
&2\ \mathfrak{A}^*\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \times \Psi^*\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right)\\
&\leq \frac{\Gamma(\beta+1)}{2^{1-\beta}}\left(\frac{\mathrm{t}\upsilon}{\upsilon-\mathrm{t}}\right)^{\beta}\left[\ \mathfrak{T}^{\beta}_{\left(\frac{1}{\upsilon}\right)^+}\ \mathfrak{A}^*\circ\Psi\left(\frac{1}{\mathrm{t}}\right) \times \Psi^*\circ\Psi\left(\frac{1}{\mathrm{t}}\right) + \mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}\ \mathfrak{A}^*\circ\Psi\left(\frac{1}{\upsilon}\right) \times \Psi^*\circ\Psi\left(\frac{1}{\upsilon}\right)\ \right].\\
&+\left(\frac{1}{2} - \frac{\beta^2+3\beta}{4(\beta+1)(\beta+2)}\right)\mathcal{Q}^*(\mathrm{t},\upsilon) + \frac{\beta^2+3\beta}{4(\beta+1)(\beta+2)}\mathfrak{D}^*(\mathrm{t},\upsilon).
\end{aligned} \tag{24}$$

From (23) and (24), we have

$$2\mathfrak{A}\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \times \Psi\left(\frac{2\mathrm{t}\upsilon}{\mathrm{t}+\upsilon}\right) \leq_{\mathrm{p}} \frac{\Gamma(\beta+1)}{2^{1-\beta}}\left(\frac{\mathrm{t}\upsilon}{\upsilon-\mathrm{t}}\right)^{\beta}\left[\ \mathfrak{T}^{\beta}_{\left(\frac{\mathrm{t}+\upsilon}{2\mathrm{t}\upsilon}\right)^+}\ \mathfrak{A}\circ\Psi\left(\frac{1}{\mathrm{t}}\right) \times \Psi\circ\Psi\left(\frac{1}{\mathrm{t}}\right) + \mathfrak{T}^{\beta}_{\left(\frac{\mathrm{t}+\upsilon}{2\mathrm{t}\upsilon}\right)^-}\ \mathfrak{A}\circ\Psi\left(\frac{1}{\upsilon}\right) \times \Psi\circ\Psi\left(\frac{1}{\upsilon}\right)\ \right] + \left(\frac{1}{2} - \frac{\beta^2+3\beta}{4(\beta+1)(\beta+2)}\right)\mathcal{Q}(\mathrm{t},\upsilon) + \frac{\beta^2+3\beta}{4(\beta+1)(\beta+2)}\mathfrak{D}(\mathrm{t},\upsilon).$$

Now, we present the reformative version of the generalized *IV-RL*-fractional integral *H-H* Fejér inequality on convex interval. □

Theorem 8. *Let* $\mathfrak{A} \in LRHSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)$*, and defined on the interval* $[\mathrm{t},\ v]$ *such that* $\mathfrak{A}(\varkappa) = [\mathfrak{A}_*(\varkappa),\ \mathfrak{A}^*(\varkappa)]$ *for all* $\varkappa \in [\mathrm{t},\ v] \in [0,\ 1]$ *and let* $\mathfrak{A} \in L([\mathrm{t},\ v], \mathcal{K}_C^+)$ *and fractional integral over* $[\mathrm{t},\ v]$*. If* $\mathfrak{D} : [\mathrm{t},\ v] \to \mathbb{R}$, $\mathfrak{D}\left(\frac{1}{\frac{1}{\mathrm{t}}+\frac{1}{v}-\frac{1}{\varkappa}}\right) = \mathfrak{D}(\varkappa) \geq 0$*, then*

$$\mathfrak{A}\left(\tfrac{2\mathrm{t}v}{\mathrm{t}+v}\right)\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}(\mathfrak{D}\circ\Psi)\left(\tfrac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}(\mathfrak{D}\circ\Psi)\left(\tfrac{1}{v}\right)\right] \leq_{\mathrm{p}} \left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}(\mathfrak{A}\mathfrak{D}\circ\Psi)\left(\tfrac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}(\mathfrak{A}\mathfrak{D}\circ\Psi)\left(\tfrac{1}{v}\right)\right] \leq_{\mathrm{p}} \tfrac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(v)}{2}\left[\mathfrak{T}^{\beta}_{\frac{1}{v}^+}(\mathfrak{D}\circ\Psi)\left(\tfrac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\frac{1}{\mathrm{t}}^-}(\mathfrak{D}\circ\Psi)\left(\tfrac{1}{v}\right)\right]. \tag{25}$$

If $\mathfrak{A}$ *is LR-*$\boldsymbol{\mathcal{H}}$*-concave IV-F, then, inequality (25) is reversed.*

Proof. Since $\mathfrak{A} \in LRHSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)$, then, we have

$$\mathfrak{A}_*\left(\tfrac{2\mathrm{t}v}{\mathrm{t}+v}\right) \leq \tfrac{1}{2}\left(\mathfrak{A}_*\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)+\mathfrak{A}_*\left(\tfrac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\right). \tag{26}$$

Multiplying both sides by (26) by $\mathrm{s}^{\beta-1}\mathfrak{D}\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)$ and then, integrating the resultant with respect to s over $[0,\ 1]$, we obtain

$$\mathfrak{A}_*\left(\tfrac{2\mathrm{t}v}{\mathrm{t}+v}\right)\int_0^1 \mathrm{s}^{\beta-1}\mathfrak{D}\left(\tfrac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)d\mathrm{s} \leq \frac{1}{2}\begin{pmatrix}\int_0^1 \mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\mathfrak{D}\left(\tfrac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)d\mathrm{s}\\ +\int_0^1 \mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\tfrac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\mathfrak{D}\left(\tfrac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)ds\end{pmatrix}. \tag{27}$$

Let $\varkappa = \frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}$. Then, we have

$$\begin{aligned}&2\left(\tfrac{tv}{v-t}\right)^{\beta}\mathfrak{A}_*\left(\tfrac{2tv}{t+v}\right)\int_{\frac{1}{v}}^{\frac{1}{t}}\left(\varkappa-\tfrac{1}{v}\right)^{\beta-1}\mathfrak{D}\left(\tfrac{1}{\varkappa}\right)d\varkappa\\ &\leq \left(\tfrac{tv}{v-t}\right)^{\beta}\int_{\frac{1}{v}}^{\frac{1}{t}}\left(\varkappa-\tfrac{1}{v}\right)^{\beta-1}\mathfrak{A}_*\left(\tfrac{1}{\frac{1}{t}+\frac{1}{v}-\frac{1}{\varkappa}}\right)\mathfrak{D}\left(\tfrac{1}{\varkappa}\right)d\varkappa+\left(\tfrac{tv}{v-t}\right)^{\beta}\int_{t}^{\frac{1}{t}}\left(\varkappa-\tfrac{1}{v}\right)^{\beta-1}\mathfrak{A}_*\left(\tfrac{1}{\varkappa}\right)\mathfrak{D}\left(\tfrac{1}{\varkappa}\right)d\varkappa\\ &= \left(\tfrac{tv}{v-t}\right)^{\beta}\int_{\frac{1}{v}}^{\frac{1}{t}}\left(\tfrac{1}{t}-\varkappa\right)^{\beta-1}\mathfrak{A}_*(\varkappa)\mathfrak{D}\left(\tfrac{1}{\frac{1}{t}+\frac{1}{v}-\frac{1}{\varkappa}}\right)d\varkappa+\left(\tfrac{tv}{v-t}\right)^{\beta}\int_{\frac{1}{v}}^{\frac{1}{t}}\left(\varkappa-\tfrac{1}{v}\right)^{\beta-1}\mathfrak{A}_*\left(\tfrac{1}{\varkappa}\right)\mathfrak{D}\left(\tfrac{1}{\varkappa}\right)d\varkappa\\ &= \Gamma(\beta)\left(\tfrac{tv}{v-t}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\mathfrak{A}_*\mathfrak{D}\left(\tfrac{1}{t}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^-}\mathfrak{A}_*\mathfrak{D}\left(\tfrac{1}{v}\right)\right]\end{aligned}, \tag{28}$$

Similarly, for $\mathfrak{A}^*(\varkappa)$, we have

$$2\left(\frac{\mathrm{t}v}{v-\mathrm{t}}\right)^{\beta}\mathfrak{A}^*\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\int_{\frac{1}{v}}^{\frac{1}{\mathrm{t}}}\left(\varkappa-\frac{1}{v}\right)^{\beta-1}\mathfrak{D}\left(\frac{1}{\varkappa}\right)d\varkappa \leq \Gamma(\beta)\left(\frac{\mathrm{t}v}{v-\mathrm{t}}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\mathfrak{A}^*\mathfrak{D}\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}\mathfrak{A}^*\mathfrak{D}\left(\frac{1}{v}\right)\right]. \tag{29}$$

From (28) and (29), we have

$$\begin{aligned}&\Gamma(\beta)\left(\tfrac{tv}{v-t}\right)^{\beta}\left[\mathfrak{A}_*\left(\tfrac{2tv}{t+v}\right),\ \mathfrak{A}^*\left(\tfrac{2tv}{t+v}\right)\right].\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\mathfrak{D}\left(\tfrac{1}{t}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^-}\mathfrak{D}\left(\tfrac{1}{v}\right)\right]\\ &\leq\ {}_p\Gamma(\beta)\left(\tfrac{tv}{v-t}\right)^{\beta}\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\mathfrak{A}_*D\left(\tfrac{1}{t}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^-}\mathfrak{A}_*D\left(\tfrac{1}{v}\right),\ \mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}\mathfrak{A}^*D\left(\tfrac{1}{t}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{t}\right)^-}\mathfrak{A}^*D\left(\tfrac{1}{v}\right)\right]\end{aligned},$$

that is

$$\mathfrak{A}\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}(\mathfrak{D}\circ\Psi)\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}(\mathfrak{D}\circ\Psi)\left(\frac{1}{v}\right)\right] \leq_{\mathrm{p}} \left[\mathfrak{T}^{\beta}_{\left(\frac{1}{v}\right)^+}(\mathfrak{A}\mathfrak{D}\circ\Psi)\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^{\beta}_{\left(\frac{1}{\mathrm{t}}\right)^-}(\mathfrak{A}\mathfrak{D}\circ\Psi)\left(\frac{1}{v}\right)\right]. \tag{30}$$

Similarly, if $\mathfrak{A}$ be a LR-$\boldsymbol{\mathcal{H}}$-convex *IV-F* and $\mathrm{s}^{\beta-1}\mathfrak{D}\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right) \geq 0$, then, we have

$$\mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\mathfrak{D}\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right) \leq \mathrm{s}^{\beta-1}((1-\mathrm{s})\mathfrak{A}_*(\mathrm{t})+\mathrm{s}\mathfrak{A}_*(v))\mathfrak{D}\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right). \tag{31}$$

And

$$\mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\frac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\mathfrak{D}\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right) \leq \mathrm{s}^{\beta-1}(\mathrm{s}\mathfrak{A}_*(\mathrm{t})+(1-\mathrm{s})\mathfrak{A}_*(v))\mathfrak{D}\left(\frac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right). \tag{32}$$

After adding (31) and (32), and integrating the resultant over $[0,\ 1]$, we get

$$\begin{aligned}
&\int_0^1 \mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\tfrac{tv}{\mathrm{s}t+(1-\mathrm{s})v}\right)\mathfrak{D}\left(\tfrac{tv}{\mathrm{s}t+(1-\mathrm{s})v}\right)ds+\int_0^1 \mathrm{s}^{\beta-1}\mathfrak{A}_*\left(\tfrac{tv}{(1-\mathrm{s})t+\mathrm{s}v}\right)\mathfrak{D}\left(\tfrac{tv}{\mathrm{s}t+(1-\mathrm{s})v}\right)ds\\
\leq &\int_0^1\left[\ \mathrm{s}^{\beta-1}\mathfrak{A}_*(t)\{\mathrm{s}+(1-\mathrm{s})\}\left(\tfrac{tv}{\mathrm{s}t+(1-\mathrm{s})v}\right)+\mathrm{s}^{\beta-1}\mathfrak{A}_*(v)\{(1-\mathrm{s})+\mathrm{s}\}\left(\tfrac{tv}{\mathrm{s}t+(1-\mathrm{s})v}\right)\ \right]ds,\\
&=\mathfrak{A}_*(t)\int_0^1\ \mathrm{s}^{\beta-1}D\left(\tfrac{tv}{\mathrm{s}t+(1-\mathrm{s})v}\right)\ ds+\mathfrak{A}_*(v)\int_0^1\ \mathrm{s}^{\beta-1}D\left(\tfrac{tv}{\mathrm{s}t+(1-\mathrm{s})v}\right)\ ds.
\end{aligned}$$

Similarly, for $\mathfrak{A}^*(\varkappa)$, we have

$$\begin{aligned}
&\int_0^1 \mathrm{s}^{\beta-1}\mathfrak{A}^*\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\mathfrak{D}\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)d\mathrm{s}+\int_0^1 \mathrm{s}^{\beta-1}\mathfrak{A}^*\left(\tfrac{\mathrm{t}v}{(1-\mathrm{s})\mathrm{t}+\mathrm{s}v}\right)\mathfrak{D}\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)d\mathrm{s}=\\
&\mathfrak{A}^*(\mathrm{t})\int_0^1\ \mathrm{s}^{\beta-1}D\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\ d\mathrm{s}+\mathfrak{A}^*(v)\int_0^1\ \mathrm{s}^{\beta-1}D\left(\tfrac{\mathrm{t}v}{\mathrm{st}+(1-\mathrm{s})v}\right)\ d\mathrm{s}.
\end{aligned}$$

From which, we have

$$\begin{aligned}
\Gamma(\beta)\left(\tfrac{\mathrm{t}v}{v-\mathrm{t}}\right)^\beta\left[\mathfrak{T}^\beta_{\frac{1}{v}^+}\ \mathfrak{A}\mathfrak{D}\circ\Psi(v)+\mathfrak{T}^\beta_{\left(\frac{1}{\mathrm{t}}\right)^-}\ \mathfrak{A}\mathfrak{D}\circ\Psi\left(\tfrac{1}{v}\right)\right]\leq_{\mathrm{p}}\Gamma(\beta)\left(\tfrac{\mathrm{t}v}{v-\mathrm{t}}\right)^\beta\tfrac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(v)}{2}\left[\mathfrak{T}^\beta_{\frac{1}{v}^+}\ (\mathfrak{D}\circ\Psi)\left(\tfrac{1}{\mathrm{t}}\right)+\right.\\
\left.\mathfrak{T}^\beta_{\left(\frac{1}{\mathrm{t}}\right)}(\mathfrak{D}\circ\Psi)\left(\tfrac{1}{v}\right)\right],
\end{aligned}$$

that is

$$\left[\mathfrak{T}^\beta_{\left(\frac{1}{v}\right)^+}\ \mathfrak{A}\mathfrak{D}\circ\Psi\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^\beta_{\left(\frac{1}{\mathrm{t}}\right)^-}\ \mathfrak{A}\mathfrak{D}\circ\Psi\left(\frac{1}{v}\right)\right]\leq_{\mathrm{p}}\frac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(v)}{2}\left[\mathfrak{T}^\beta_{\left(\frac{1}{v}\right)^+}\ (\mathfrak{D}\circ\Psi)\left(\frac{1}{\mathrm{t}}\right)+\mathfrak{T}^\beta_{\left(\frac{1}{\mathrm{t}}\right)^-}(\mathfrak{D}\circ\Psi)\left(\frac{1}{v}\right)\right].\quad(33)$$

By combining (30) and (33), we obtain the required inequality (25). □

Remark 5. *Let one attempt to take $\beta=1$. Then, from (25), we acquire the coming inequality, which is also new one:*

$$\mathfrak{A}\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\int_{\mathrm{t}}^v\frac{\mathfrak{D}(\varkappa)}{\varkappa^2}d\varkappa\leq_{\mathrm{p}}\int_{\mathrm{t}}^v\frac{\mathfrak{A}(\varkappa)}{\varkappa^2}\mathfrak{D}(\varkappa)d\varkappa\leq_{\mathrm{p}}\frac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(v)}{2}\int_{\mathrm{t}}^v\frac{\mathfrak{D}(\varkappa)}{\varkappa^2}d\varkappa$$

Let one attempt to take $\mathfrak{D}(\varkappa)=1$. Then, from (25), we obtain inequality (11).

Let one attempt to take $\mathfrak{D}(\varkappa)=1$ and $\beta=1$, then, from (25), we get H-H inequality for LR-$\boldsymbol{\mathcal{H}}$-convex IV-F.

$$\mathfrak{A}\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\leq_{\mathrm{p}}\frac{\mathrm{t}v}{v-\mathrm{t}}\int_{\mathrm{t}}^v\frac{\mathfrak{A}(\varkappa)}{\varkappa^2}d\varkappa\leq_{\mathrm{p}}\frac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(v)}{2}.$$

If one attempts to take $\mathfrak{A}_*(\varkappa)=\mathfrak{A}^*(\varkappa)$, then, from (40), we acquire the fractional H-H Fejér inequality, see [31].

Let one attempt to take $\mathfrak{A}_*(\varkappa)=\mathfrak{A}^*(\varkappa)$ with $\beta=1$. Then, from (25), we achieve the coming inequality, see [3].

$$\mathfrak{A}\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\int_{\mathrm{t}}^v\frac{\mathfrak{D}(\varkappa)}{\varkappa^2}d\varkappa\leq\int_{\mathrm{t}}^v\frac{\mathfrak{A}(\varkappa)}{\varkappa^2}\mathfrak{D}(\varkappa)d\varkappa\leq\frac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(v)}{2}\int_{\mathrm{t}}^v\frac{\mathfrak{D}(\varkappa)}{\varkappa^2}d\varkappa.$$

If one attempts to take $\mathfrak{A}_*(\varkappa)=\mathfrak{A}^*(\varkappa)$ with $\mathfrak{D}(\varkappa)=1$ then, from (25), we acquire the coming classical inequality for $\boldsymbol{\mathcal{H}}$-convex function.

$$\mathfrak{A}\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\leq\frac{\Gamma(\beta+1)}{2(v-\mathrm{t})^\beta}\left[\mathfrak{T}^\beta_{\frac{1}{\mathrm{t}}^-}(\mathfrak{A}\circ\Psi)\left(\frac{1}{v}\right)+\mathfrak{T}^\beta_{\frac{1}{v}^+}\ (\mathfrak{A}\circ\Psi)\left(\frac{1}{\mathrm{t}}\right)\right]\leq\frac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(v)}{2}.$$

If one attempts to take $\mathfrak{A}_*(\varkappa)=\mathfrak{A}^*(\varkappa)$ and $\mathfrak{D}(\varkappa)=\beta=1$ then, from (25), we acquire the coming classical inequality for $\boldsymbol{\mathcal{H}}$-convex function.

$$\mathfrak{A}\left(\frac{2\mathrm{t}v}{\mathrm{t}+v}\right)\leq\frac{\mathrm{t}v}{v-\mathrm{t}}\int_{\mathrm{t}}^v\frac{\mathfrak{A}(\varkappa)}{\varkappa^2}d\varkappa\leq\frac{\mathfrak{A}(\mathrm{t})+\mathfrak{A}(v)}{2}.$$

4. Conclusions

We use *IV-RL*-fractional integral operators to infer various inclusions in the *H-H*, *H-H*-Fejér type inequalities, and some related inequalities in this paper. We show the relationships between the examined results and previously published ones to show their generic properties. In addition, some nontrivial examples are given to demonstrate the accuracy of the results derived in the study. The point we wish to make here is that interval-valued analyses are commonly used in practical mathematics, particularly in the field of optimality analysis (see [22,23]). This important subject in interval-valued analysis using fractional integral operators deserves to be explored further.

In our final view, we believe that our work can be generalized to other models of fractional calculus, such as Atangana–Baleanu and Prabhakar fractional operators with Mittag–Liffler functions in their kernels. We have left this consideration as an open problem for the researchers who are interested in this field. The interested researchers can proceed as done in references [15,16].

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