fractal and fractional

MDPI

Article

Topological Structure of the Solution Sets for Impulsive Fractional Neutral Differential Inclusions with Delay and Generated by a Non-Compact Demi Group

Zainab Alsheekhhussain [1,*], Ahmed Gamal Ibrahim [2] and Akbar Ali [1]

[1] Department of Mathematics, Faculty of Science, University of Ha'il, Hail 55476, Saudi Arabia; ak.ali@uoh.edu.sa
[2] Department of Mathematics, College of Sciences, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia; agamal@kfu.edu.sa
[*] Correspondence: za.hussain@uoh.edu.sa

Abstract: In this paper, we give an affirmative answer to a question about the sufficient conditions which ensure that the set of mild solutions for a fractional impulsive neutral differential inclusion with state-dependent delay, generated by a non-compact semi-group, are not empty compact and an R_δ-set. This means that the solution set may not be a singleton, but it has the same homology group as a one-point space from the point of view of algebraic topology. In fact, we demonstrate that the solution set is an intersection of a decreasing sequence of non-empty compact and contractible sets. Up to now, proving that the solution set for fractional impulsive neutral semilinear differential inclusions in the presence of impulses and delay and generated by a non-compact semigroup is an R_δ-set has not been considered in the literature. Since fractional differential equations have many applications in various fields such as physics and engineering, the aim of our work is important. Two illustrative examples are given to clarify the wide applicability of our results.

Keywords: impulsive fractional differential inclusions; neutral differential inclusions; mild solutions; contractible sets; R_δ-set

Citation: Alsheekhhussain, Z.; Ibrahim, A.G.; Ali, A. Topological Structure of the Solution Sets for Impulsive Fractional Neutral Differential Inclusions with Delay and Generated by a Non-Compact Demi Group. *Fractal Fract.* **2022**, *6*, 188. https://doi.org/10.3390/fractalfract6040188

Academic Editor: Rodica Luca

Received: 12 March 2022
Accepted: 24 March 2022
Published: 28 March 2022

1. Introduction

Impulsive differential equations and inclusions describe phenomena in which states are changing rapidly at certain moments. In [1–8], the authors examined whether a mild solution for different types of impulsive differential inclusions exist.

The study of neutral differential equations appears in many applied mathematical sciences, such as viscoelasticity and equations that describe the distribution of heat. The structure of neutral equations involve derivatives related to delay beside the function. Neutral differential equations and inclusions were studied in [9–12]. These papers examined the mild solutions and controllability of the system.

Because the set of mild solutions for a differential inclusion having the same initial point may not be a singleton, many authors are interested in investigating the structure of this set in a topological point of view. An important aspect of such structure is the R_δ-property, which means that the homology group of the set of mild solutions is the same as a one-point space. We list some studies in which the authors demonstrated the solution sets satisfying R_δ-property: Gabor [13] considered impulsive semilinear differential inclusions with finite delay on the half-line of order one generated by a non-compact semi-group; Djebali et al. [14] worked on impulsive differential inclusions on unbounded domains; Zhou et al. [15] studied the neutral evolution inclusions of order one generated by a non-compact semi-group; Zhou et al. [16] considered fractional stochastic evolution inclusions generated by a compact semi-group; Zhao et al. [17] studied a stochastic differential equation of Sobolev-type which is semilinear with Poisson jumps of order

$\alpha \in (1,2)$; Beddani [18] examined a differential inclusion involving Riemann–Liouville fractional derivatives; Wang et al. [19] worked on semilinear fractional differential inclusions with non-instantaneous impulses; Ouahab et al. [20] considered fractional inclusions that are non-local and have impulses at different times; Zaine [21] studied weighted fractional differential equations. Recently, Zhang et al. [22] proved that the set of C^0-solutions for impulsive evolution inclusions of order one is an R_δ-set and generated by m–dissipative operator. Wang et al. [23] proved that the solution for evolution equations that have nonlinear delay and multivalued perturbation on a non-compact interval is an R_δ-set.

In [6,24–26], the authors studied different kinds of fractional differential inclusions, and, in all cases, they showed that the set of solutions is a compact set. For more work related to this, the reader can consult the book in [27] about the topological properties for evolution inclusions.

However, up to now, proving that the solution set for fractional impulsive neutral semilinear differential inclusions involving delay and generated by a non-compact semigroup is an R_δ-set has not been considered in the literature. Thus, this topic is new and interesting and, hence, the question whether there exists a solution set carrying an R_δ-structure remains unsolved for fractional differential inclusions when there are impulses, delay (finite or infinite) and the operator families generated by the linear part lack compactness. Therefore, our main goal is to give an affirmative answer to this question. In fact, we study a neutral fractional impulsive differential inclusion with delay which is generated by a non-compact semigroup, and we show that the set of solutions is non-empty and equal to an intersection of a decreasing sequence of sets each of which is non-empty compact and has a homotopy equivalent to a point.

Let $\alpha \in (0,1)$, $r > 0$, $J = [0,b]$, $T = \{\Upsilon(\eta) : \eta \geq 0\}$ a semigroup on E, which is Banach space, and A the infinitesimal generator of T. Let $F : J \times \Theta \to 2^E - \{\phi\}$ be a multifunction, $h : J \times \Theta \to E$, $0 = \eta_0 < \eta_1 < \cdots < \eta_m < \eta_{m+1} = b$, and $\psi \in \Theta$ be given. For every $\eta \in J$, let $\varkappa(\eta) : \mathcal{H} \to \Theta$, $(\varkappa(\eta)x)(\theta) = x(\eta + \theta); \theta \in [-r,0]$; where Θ and $\mathcal{H}$ are defined later.

The present paper shows the solution set of a fractional neutral impulsive semilinear differential inclusion with delay having details as follows:

$$\begin{cases} {}^cD^{\alpha}_{0,\eta}[x(\eta) - h(\eta,\varkappa(\eta)x)] \in Ax(\eta) + F(\eta,\varkappa(\eta)x),\ a.e.\ \eta \in [0,b] - \{\eta_1,\ldots,\eta_m\}, \\ I_i x(\eta_i^-) = x(\eta_i^-) - x(\eta_i^+), i = 1,\ldots,m, \\ x(\eta) = \psi(\eta), \eta \in [-r,0], \end{cases} \tag{1}$$

is not empty, compact and an R_δ-set, where $I_i : E \longrightarrow E, i = 1,\ldots,m$, and $x(\eta_i^+)$, $x(\eta_i^-)$ are the limits of the function x evaluated at η_i from the right and the left. Furthermore, ${}^cD^{\alpha}_{0,\eta}$ denotes the Caputo derivative that has order $\alpha \in (0,1)$ and lower limit at zero [28].

In the following points, we clarify the originality, importance and the main contributions of this article:

1. Up to now, proving that the solution set is an R_δ-set for fractional impulsive neutral semilinear differential inclusions involving delay and generated by a non- compact semigroup has not been considered in the literature.
2. Demonstrating that the set of solutions is an R_δ-set for fractional neutral differential inclusions involving impulses and delay has not been considered yet.
3. We do not assume that the semi-group which generates the linear part is compact.
4. Proving that the set of solutions is an R_δ-set for neutral differential inclusions (without impulses) with a finite delay, $\alpha = 1$, and generated by a non-compact semigroup, has been investigated in [15], while stochastic neutral differential inclusions (without impulsive effects) with finite delay of order $\alpha \in (0,1)$ and generated by a compact semigroup has been examined in [16].
5. Gabor [13] considered Problem 1 on the half-line when $\alpha = 1$ and $h \equiv 0$.
6. Problem 1 is investigated in [19] when $h \equiv 0$ and in the absence of delay.

7. Our technique can be used to derive suitable conditions, which implies that the solution set is an R_δ-set for the problems studied in [13–23] when they contain impulses and delay.

In order to clarify the difficulties encountered to achieve our aim, we point to the normed space $\mathcal{PC}([-r,b],E]$, which consists of piecewise continuous bounded functions defined on $[-r,b]$ with a finite number of discontinuity points and is left continuous at the discontinuity points, and is not necessarily complete. Moreover, unlike the Banach spaces $C([-r,b],E)$ and $PC(J,E)$, the Hausdorff measure of noncompactness on $\mathcal{PC}([-r,b],E]$ is not specific. Thus, when the problem involves delay and impulses, we cannot consider $\mathcal{PC}([-r,b],E]$ as the space of solutions. To overcome these difficulties, a complete metric space H is introduced as the space of mild solutions (see the next section). In addition, the function $\eta \to \varkappa(\eta)\overline{x}; \overline{x} \in H$ is not necessarily measurable (see Remark 1, and so, a norm different from the uniform convergence norm is introduced (see Equation (2) below).

For recent contributions on neutral differential inclusions of fractional order, Burqan et al. [29] give a numerical approach in solving fractional neutral pantograph equations via the ARA integral transform. Ma et al. [30] studied the controllability for a neutral differential inclusion with Hilfer derivative, and Etmad et al. [31] investigated a neutral fractional differential inclusion of Katugampola-type involving both retarded and advanced arguments.

For more recent papers we cite [32–34].

The sections of the paper are organized as follows: We include some background materials in Section 2 as we need them in the main sections. Section 3 is assigned for proving that the solution set of Problem (1) is non-empty and compact. In Section 4, we show that this set is an R_δ-set in the complete metric space H. In Section 5, e give an example as an application of the obtained results. Sections 6 and 7 are the discussion and conclusion sections.

2. Preliminaries and Notation

In all the text we denote for the set of mild solutions for Problem 1 by $\Sigma_{\psi}^{F}[-r,b]$ and by $L^1(J,E)$ to the quotient space consisting of $E-$valued Bohner integrable functions defined on J having the norm $\|f\|_{L^1(J,E)} = \int_0^b \|f(\theta)\| d\theta$. Let $P_{ck}(E) = \{B \subseteq E : B$ be non-empty, convex and compact$\}$.

Definition 1. *(Ref. [35]) Let $h : J \to E$, $\{\Upsilon(\eta) : \eta \geq 0\}$ a C_0-semigroup and A be the infinitesimal generator of it. A continuous function $x : J \to E$ is called a mild solution for the problem:*

$$\begin{cases} {}^cD^{\alpha} z(\eta) = Az(\eta) + h(\eta), \eta \in J, \\ z(0) = z_0 \in E, \end{cases}$$

if

$$z(\eta) = \mathfrak{K}_1(\eta) z_0 + \int_0^{\eta} (\eta - \tau)^{\alpha-1} \mathfrak{K}_2(\eta - \tau) h(\tau) d\tau, \eta \in J,$$

where $\mathfrak{K}_1(\eta) = \int_0^{\infty} \xi_\alpha(\theta) \Upsilon(\eta^{\alpha}\theta) d\theta, \mathfrak{K}_2(\eta) = \alpha \int_0^{\infty} \theta \xi_\alpha(\theta) \Upsilon(\eta^{\alpha}\theta) d\theta,$
$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}} w_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0, w_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\,\alpha+1)}{n!}\sin(n\pi\alpha), \theta \in (0,\infty)$ *and* $\int_0^{\infty} \xi_\alpha(\theta) d\theta = 1.$

Lemma 1. *(Ref. [35] (lemma 3.1)) The properties stated below are held:*

(i) *For every fixed $\eta \geq 0$, $\mathfrak{K}_1(\eta), \mathfrak{K}_2(\eta)$ are linear and bounded.*
(ii) *Assuming $||\eta(\eta)|| \leq M, \eta \geq 0$, we have that for any $x \in E$, $||\mathfrak{K}_1(\eta)x|| \leq M||x||$ and $||\mathfrak{K}_2(\eta)x|| \leq \frac{M}{\Gamma(\alpha)}||x||$.*
(iii) *If $\eta, \tau \geq 0$; then for any $x \in E$,*

$$\lim_{\eta \to \tau} ||\mathfrak{K}_1(\eta)x - \mathfrak{K}_1(\tau)x|| = 0, \text{ and } \lim_{\eta \to \tau} ||\mathfrak{K}_2(\eta)x - \mathfrak{K}_2(\tau)x|| = 0.$$

Consider the spaces:

1. The normed space

$$\Theta \quad : \quad = \{x : [-r,0] \to E, \text{ where } x \text{ is discontinuous at finite number of points } \tau \neq 0, \text{ and all the limits } x(\tau^+) \text{ and } x(\tau^-) \text{ are less than } \infty\}$$

endowed with the norm:

$$||x||_\Theta := \int_{-r}^{0} ||x(\tau)||d\tau. \tag{2}$$

2. The Banach space

$$PC(J,E) \quad : \quad = \{u : J \to E : u_{|J_i} \in C(J_i,E), i = 0,1,2,\ldots,m, \text{ and } u(\eta_i^+), \quad u(\eta_i) = u(\eta_i^-) \text{ are finite for every } i = 1,2,\ldots.m\},$$

where $J_0 = [0,\eta_1], J_i = (\eta_i,\eta_{i+1}], i = 1,2,\ldots,m,$ and $||v||_{PC(J;E)} = \tau up_{\eta\in J}||v(\eta)||$.

3. The complete metric space

$$H = \{x : [-r,b] \to E : \text{ where } x \text{ is continuous at } \eta = 0,\ x_{|_{[-r,0]}} = \psi, x_{|_{J_i}} \in PC\ (J,E)\},$$

where the metric function is given by:

$$d_H(x,y) = \tau up_{\eta\in J}||x(\eta) - y(\eta)||.$$

4. The Banach space

$$\mathcal{H} := \{x : [-r,b] \to E \text{ where } x(\eta) = 0, \forall\eta \in [-r,0], x_{|_{J_i}} \in PC\ (J,E)\}$$

together with the norm $||x||_{\mathcal{H}} = \tau up_{\eta\in J}||x(\eta)|| + ||x_{|_{[-r,0]}}||_\Theta = \tau up_{\eta\in J}||x(\eta)||$.

The Hausdorff measure of noncompactness on a Banach space $PC(J,E)$ is given by

$$\chi_{PC}(B) := \max_{i=0,1,2,\ldots,m} \chi_i(B_{|\ \overline{J_i}}),$$

where B is a bounded subset of $PC(J,E)$ and χ_i is the Hausdorff measure of noncompactness on the Banach space $C(\overline{J_i},E)$ and

$$B_{|\overline{J_i}} := \{x^* : \overline{J_i} \to E : x^*(\eta) = x(\eta), \eta \in J_i \text{ and } x^*(\eta_i) = x(\eta_i^+), x \in B\}.$$

The Hausdorff measure of noncompactness on $\mathcal{H}$ is defined by:

$$\chi_{\mathcal{H}}(B) = \max_{i=0,1,2,\ldots,m} \chi_i(B_{|\ \overline{J_i}}),$$

where B is a bounded subset of $\mathcal{H}$.

Remark 1. *Since the function $\eta \to \varkappa(\eta)x; \overline{x} \in H$ is not necessarily measurable, we do not consider the uniform convergence norm to be the norm defined on the space Θ (see Example 3.1, [36]). Therefore, the multivalued superposition operator*

$$x \to S^1_{F(.,\varkappa(.)x)} = \{f \in L^1(J,E) : f(\eta) \in F(\eta,\varkappa(\eta)x), a.e., \eta \in J\}$$

would not be well defined. Therefore, we consider a norm defined by Equation (2).

Definition 2. *A function $\overline{x} \in H$ is said to be a mild solution for (1) if*

$$\overline{x}(\eta) = \begin{cases} \psi(\eta), \eta \in [-r, 0], \\ \mathfrak{K}_1(\eta)[\psi(0) - h(0, \psi)] + h(\eta, \varkappa(\eta)\overline{x}) \\ + \int_0^{\eta} (\eta - \tau)^{\alpha - 1} A \mathfrak{K}_2(\eta - \tau) h(\tau, \varkappa(\tau)\overline{x}) d\tau \\ + \int_0^{\eta} (\eta - \tau)^{\alpha - 1} \mathfrak{K}_2(\eta - \tau) f(\tau) d\tau \\ + \sum_{0 < \eta_i < \eta} \mathfrak{K}_1(\eta - \eta_i) I_i(\overline{x}(\eta_i^-)), \eta \in J = [0, b], \end{cases} \tag{3}$$

where $f \in S^1_{F(.,\varkappa(.)x)}$.

We assume the following conditions:

(HA) A is the infinitesimal generator of T, 0 is an element of the resolvent of $A, \rho(A)$ and $\sup_{\eta \geq 0} ||\Upsilon(\eta)|| \leq M$, where $M \geq 1$.

(HF) $F : J \times \Theta \to P_{ck}(E)$ where:

(HF_1) For any $z \in \Theta$, the multifunction $\eta \longrightarrow F(\eta, z)$ has a measurable selection, and for $\eta \in J, a.e.$, the multifunction $z \longrightarrow F(\eta, z)$ is upper semicontinuous.

(HF_2) There exists a $\varphi \in L^P(I, \mathbb{R}^+)(P > \frac{1}{\alpha})$ satisfying

$$\|F(\eta, z)\| \leq \varphi(\eta)\ (1 + \|z\|_{\Theta}), \forall z \in \Theta \text{ and for } a.e.\ \eta \in J.$$

(HF_3) There is a $\beta \in L^P([0, b], E), p > \frac{1}{\alpha}$ such that, for any $D \subset \Theta$ that is bounded, we have

$$\chi_E(F(\eta, D)) \leq \beta(\eta) \sup_{\theta \in [-r, 0]} \chi_E\{z(\theta) : z \in D\}, a.e. \text{ for } \eta \in J. \tag{4}$$

(HI) For any $i = 1, \ldots, m$, the function $I_i : E \to E$ is continuous, and there are $\sigma_i > 0$ and $\varsigma_i > 0$ satisfying $||I_i(x)|| \leq \sigma_i ||x||$, and for any bounded subset $D \subseteq E$,

$$\chi_E(I_i(D)) \leq \varsigma_i \chi_E(I_i(D)).$$

Lemma 2. *(Ref. [37]) Under condition (HA), for any $\gamma \in (0, 1)$, the fractional power A^γ can be defined, and it is linear and closed on its domain $D(A^\gamma)$. In addition, the following properties are satisfied:*

(i) *$D(A^\gamma)$ is a Banach space with the norm*

$$||x||_\gamma = ||A^\gamma x||.$$

(ii) *For any $\eta > 0$, $x \in E$, we have $\Upsilon(\eta)x \in D(A^\gamma)$ and, assuming $x \in D(A^\gamma)$, we get $A^\gamma \Upsilon(\eta)x = \Upsilon(\eta)A^\gamma x$.*

(iii) *For every $\eta > 0$, $A^\gamma \Upsilon(\eta)$ is bounded on E, and there is a constant $C_\gamma > 0$ such that*

$$||A^\gamma \Upsilon(\eta)|| \leq \frac{C_\gamma}{\eta^\gamma}. \tag{5}$$

(iv) *$A^{-\gamma}$ is a bounded linear operator on E.*

(v) *For every $x \in E$,*

$$A\mathfrak{K}_2(\eta)x = A^{1-\gamma}\mathfrak{K}_2(\eta)A^\gamma x, \eta \in J, \tag{6}$$

and

$$||A^\gamma \mathfrak{K}_2(\eta)|| \leq \frac{\alpha C_\gamma \Gamma(2 - \gamma)}{\eta^{\alpha\gamma}\Gamma(1 + \alpha(1 - \gamma))}, \eta \in (0, b]. \tag{7}$$

We need the next lemmas in order to prove our main results.

Lemma 3. *Assume $W \subseteq E$ to be bounded, closed and convex, $\Phi_1 : W \to E$ is a single-valued function, $\Phi_2 : W \to P_{ck}(E)$ is a multifunction, and for any $x \in W$, $\Phi_1(x) + y \in W$, $\forall y \in \Phi_2(x)$. Suppose that*

(a) *Φ_1 is a contraction with the contraction constant $k < \frac{1}{2}$;*

(b) *Φ_2 is a closed and completely continuous multifunction.*

Then, the fixed point set of $\Phi_1 + \Phi_2$ is not empty. Moreover, the set of fixed points for $\Phi_1 + \Phi_2$ is compact if it is bounded.

Proof. Φ_1 is continuous on W since it is a contraction and, hence, it follows by the closeness of Φ_2, that the multifunction $R = \Phi_1 + \Phi_2$ is closed. We show that R is χ_E-condensing, where χ_E is the Hausdorff measure of noncompactness on E. Let Z be a bounded set of W. Since Φ_1 is a contraction with the contraction constant k, we get $\mu_E(\Phi_1(Z)) \le k\mu_E(Z) \le 2k\chi_E(Z) < \chi_E(Z)$, where μ_E is the Kuratowski measure of noncompactness on E. Because Φ_2 is compact, $\chi_E(\Phi_2(Z)) = 0$. Therefore,

$$\begin{aligned} \chi_E(R(Z)) &= \chi_E(\Phi_1(Z)) + \chi_E(\Phi_2(Z) \\ &= \chi_E(\Phi_1(Z)) \le \mu_E(\Phi_1(Z)) \\ &< \chi_E(Z). \end{aligned}$$

This means that R is χ_E-condensing. By Proposition 3.5.1 in [38], the fixed point set of $\Phi_1 + \Phi_2$ is not empty. The second part follows from Proposition 3.5.1 in [38]. □

3. The Compactness of $\Sigma_\psi^F[-r,b]$

In this section, we show that the set of mild solutions for Problem 1 is nonempty and compact.

For any $x \in \mathcal{H}$ with $x(0) = \psi(0)$, let $\overline{x} \in H$ be defined by

$$\overline{x}(\eta) := \begin{cases} \psi(\eta), \eta \in [-r,0], \\ x(\eta), \eta \in (0,b]. \end{cases} \tag{8}$$

Lemma 4. *For any $\overline{x} \in H$, the function $\eta \to \varkappa(\eta)\overline{x}$ is continuous from J to Θ.*

Proof. Assume $\eta, \tau \in J, \eta \le \tau$. Then,

$$||\varkappa(\eta)\overline{x} - \varkappa(\tau)\overline{x}||_\Theta = \int_{-r}^{0} ||\overline{x}(\eta+\theta) - \overline{x}(\tau+\theta)||d\theta.$$

Because $\overline{x}$ is continuous on $[-r,b]$ except for a finite number of points, it follows that $\lim_{\eta\to\tau} ||\overline{x}(\eta+\theta) - \overline{x}(\tau+\theta)|| = 0$, a.e. Since $\overline{x} \in H$, $lim_{\eta\to\tau} \int_{-r}^{0} ||\overline{x}(\eta+\theta) - \overline{x}(\tau+\theta)||d\theta = 0$, and the proof is completed. □

Theorem 1. *Assume that (HA) and (HF) are held and that $\{\Upsilon(\eta) : \eta \ge 0\}$ is equicontinuous. Assume also that the following conditions are satisfied.*

(Hh) The function $h : J \times \Theta \to E$ is continuous and there exists a $\gamma \in (0,1)$ satisfying $h(\eta,u) \in D(A^\gamma), \forall(\eta,u) \in J \times \Theta$ and

(i) For any $\eta \in J$, $A^\gamma h(\eta,.)$ is strongly measurable.
(ii) There are $d_1 > 0$ and $d_2 > 0$ with

$$d_1||A^{-\gamma}|| + \frac{d_1 b^{\alpha\gamma} C_{1-\gamma}\Gamma(1+\gamma)}{\gamma\Gamma(1+\alpha\gamma)} < \frac{1}{2r}, \tag{9}$$

$$||A^\gamma h(\eta,u)|| \le d_2(1 + ||u||_\Theta), \forall(\eta,u) \in J \times \Theta, \tag{10}$$

and

$$||A^\gamma h(\eta,u_1) - A^\gamma h(\eta,u_2)|| \le d_1||u_1 - u_2||_\Theta, \forall\eta \in J. \tag{11}$$

Then, $\Sigma_\psi^F[-r,b]$ is not empty and a compact subset of H provided that

$$||A^{-\gamma}||d_2 r + d_2\frac{C_{1-\gamma}\Gamma(1+\gamma)b^{\alpha\gamma}}{\Gamma(1+\alpha\gamma)\gamma} r + \frac{M}{\Gamma(\alpha)}\Delta\|\varphi\|_{L^P_{(J,,\mathbb{R}^+)}} r + \sigma M < 1, \tag{12}$$

and

$$\frac{4\Delta M}{\Gamma(\alpha)}\,||\beta||_{L^P(J,\,\mathbb{R}^+)}\;+2M\sum_{i=1}^{i=m}\varsigma_i<\frac{1}{2}, \tag{13}$$

where $\sigma=\sum_{i=1}^{i=m}\sigma_i$ and $\Delta=(\frac{P-1}{\alpha P-1})^{\frac{P-1}{P}}b^{\alpha-\frac{1}{P}}$.

Proof. A multioperator $\Phi:\mathcal{H}\to P(\mathcal{H})$ is defined as the following: let $x\in\mathcal{H}$, hence, as a consequence of (HF_1), the multifunction $\eta\longrightarrow F(\eta,\varkappa(\eta)\overline{x})$ admits a measurable selection which, by (HF_2), belongs to $S^1_{F(.,\varkappa(.)\overline{x})}$, and, therefore, $y\in\Phi(x)$ can be defined by

$$y(\eta)=\begin{cases}0,\eta\in[-r,0],\\ \mathfrak{K}_1(\eta)[\psi(0)-h(0,\psi)]+h(\eta,\varkappa(\eta)\overline{x})\\ +\int_0^\eta(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)h(\tau,\varkappa(\tau)\overline{x})d\tau\\ +\int_0^\eta(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f(\tau)d\tau\\ +\sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}(\eta_i^-)),\eta\in J,\end{cases} \tag{14}$$

where $f\in S^1_{F(.,\varkappa(.)\overline{x})}$ and $\overline{x}$ is defined by (8).

We show that a point x is a fixed point for Φ if and only if $\overline{x}\in\Sigma^F_\psi[-r,b]$. Assume x is a fixed point to Φ. Hence,

$$x(\eta)=\begin{cases}0,\eta\in[-r,0],\\ \mathfrak{K}_1(\eta)[\psi(0)-h(0,\psi)]+h(\eta,\varkappa(\eta)\overline{x})\\ +\int_0^\eta(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)h(\tau,\varkappa(\tau)\overline{x})d\tau\\ +\int_0^\eta(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f(\tau)d\tau\\ +\sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}(\eta_i^-)),\eta\in J.\end{cases}$$

Therefore,

$$\overline{x}(\eta)=\begin{cases}\psi(\eta),\eta\in[-r,0],\\ \mathfrak{K}_1(\eta)[\psi(0)-h(0,\psi)]+h(\eta,\varkappa(\eta)\overline{x})\\ +\int_0^\eta(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)h(\tau,\varkappa(\tau)\overline{x})d\tau\\ +\int_0^\eta(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f(\tau)d\tau\\ +\sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}(\eta_i^-)),\eta\in J,\end{cases}$$

which means that $\overline{x}$ satisfies (3), and, thus, it is a mild solution for problem (1). In a similar way, it can be seen that if $\overline{x}$ satisfies (3), then x is a fixed point for Φ. Let $\Phi_1:\mathcal{H}\to\mathcal{H}$ and $\Phi_2:\Phi_2\to P(\mathcal{H})$ be such that

$$\Phi_1(x)(\eta)=\begin{cases}0,\eta\in[-r,0],\\ \mathfrak{K}_1(\eta)[\psi(0)-h(0,\psi)]+h(\eta,\varkappa(\eta)\overline{x})\\ +\int_0^\eta(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)h(\tau,\varkappa(\tau)\overline{x})d\tau,\eta\in J,\end{cases} \tag{15}$$

and a function $y\in\Phi_2(x)$ if and only if

$$y(\eta)=\begin{cases}0,\eta\in[-r,0],\\ +\int_0^\eta(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f(\tau)d\tau\\ +\sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}(\eta_i^-)),\eta\in J,\end{cases} \tag{16}$$

where $f\in S^1_{F(.,\varkappa(.)\overline{x})}$. Notice that $\Phi=\Phi_1+\Phi_2$. Let $\xi=\sup_{\theta\in[-r,0]}||\psi(\theta)||$,

$$\begin{aligned}\omega\;&=\;M\,[\xi+||A^{-\gamma}||d_2(1+r\xi)]\\ &+(1+r\xi)[||A^{-\gamma}||d_2+d_2\frac{C_{1-\gamma}\Gamma(1+\gamma)b^{\alpha\gamma}}{\Gamma(1+\alpha\gamma)\gamma}+\frac{M}{\Gamma(\alpha)}\Delta\|\varphi\|_{L^P_{(J,,\mathbb{R}^+)}}]\end{aligned}$$

and v be a positive real number satisfying

$$v > \frac{\omega}{1 - [||A^{-\gamma}||d_2 r + d_2 \frac{C_{1-\gamma}\Gamma(1+\gamma)b^{\alpha\gamma}}{\Gamma(1+\alpha\gamma)\gamma} r + \frac{M}{\Gamma(\alpha)}\Delta\|\varphi\|_{L^p_{(J,,\mathbb{R}^+)}} r + \sigma M]}. \quad (17)$$

Put $B_v = \{u \in \mathcal{H} : ||u||_{\mathcal{H}} \leq v\}$. Due to (12), v is well defined. The rest of the proof is divided in the following steps:

Step 1. This step shows that $\Phi(B_v) \subseteq B_v$. Let $x \in B_v$ and $y \in \Phi(x)$. There exists $f \in S^1_{F(.,\varkappa(.)x)}$ where

$$y(\eta) = \begin{cases} 0, \eta \in [-r, 0], \\ \mathfrak{K}_1(\eta)[\psi(0) - h(0,\psi)] + h(\eta, \varkappa(\eta)\overline{x}) \\ + \int_0^\eta (\eta-\tau)^{\alpha-1} A\mathfrak{K}_2(\eta-\tau)h(\tau, \varkappa(\tau)\overline{x})d\tau \\ + \int_0^\eta (\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f(\tau)d\tau \\ + \sum_{0<\eta_i<\eta} \mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}(\eta_i^-)), \eta \in J. \end{cases}$$

Let $\eta \in J$. For every $x \in \mathcal{H}$, we get

$$||\varkappa(\eta)\overline{x}||_\Theta = \int_{-r}^0 ||\overline{x}(\eta+\theta)||d\theta \leq r(\xi + v),$$

which implies that (HF_2), $||f(\tau)|| \leq \varphi(\tau)(1 + ||\varkappa(\eta)\overline{x}||_\Theta) \leq r(\xi + v); a.e.\tau \in J$. So, by (ii) of Lemma 1, and the Holder inequality, it follows that

$$\begin{aligned} & ||\int_0^\eta (\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f(\tau)d\tau|| \\ \leq & \frac{M}{\Gamma(\alpha)}(1 + r(\xi+v))\int_0^\eta (\eta-\tau)^{\alpha-1}\varphi(\tau)d\tau \\ \leq & \frac{M}{\Gamma(\alpha)}\Delta\|\varphi\|_{L^p_{(J,,\mathbb{R}^+)}}(1 + r(\xi+v)). \end{aligned}$$

Then, from (6), (7), (10) and (HI), one has, for $\eta \in J$,

$$\begin{aligned} ||y(\eta)|| \leq & M\,[\xi + ||A^{-\gamma}A^\gamma h(0,\psi)||] + ||A^{-\gamma}A^\gamma h(\eta, \varkappa(\eta)\overline{x})|| \\ & + \int_0^\eta (\eta-\tau)^{\alpha-1}||A^{1-\gamma}\mathfrak{K}_2(\eta-\tau)A^\gamma h(\tau,\varkappa(\tau)\overline{x})||d\tau \\ & + \frac{M}{\Gamma(\alpha)}\Delta\|\varphi\|_{L^p_{(J,,\mathbb{R}^+)}}(1 + r(\xi+v)) + Mv\sigma \\ \leq & M\,[\xi + ||A^{-\gamma}||d_2(1 + r\xi)] + ||A^{-\gamma}||d_2(1 + ||\varkappa(\eta)\overline{x}||_\Theta) \\ & + d_2(1 + r(\xi+v))\frac{\alpha C_{1-\gamma}\Gamma(2-(1-\gamma))}{\Gamma(1+\alpha(1-(1-\gamma))}\int_0^\eta \frac{(\eta-\tau)^{\alpha-1}}{(\eta-\tau)^{\alpha(1-\gamma)}}d\tau \\ & + \frac{M}{\Gamma(\alpha)}(1 + r(\xi+v))\Delta\|\varphi\|_{L^p_{(J,,\mathbb{R}^+)}} + Mv\sigma \\ \leq & M\,[\xi + ||A^{-\gamma}||d_2(1 + r\xi)] + ||A^{-\gamma}||d_2(1 + r(\xi+v)) \\ & + d_2(1 + r(\xi+v))\frac{C_{1-\gamma}\Gamma(1+\gamma)b^{\alpha\gamma}}{\Gamma(1+\alpha\gamma)\gamma} \\ & + \frac{M}{\Gamma(\alpha)}(1 + r(\xi+v))\Delta\|\varphi\|_{L^p_{(J,,\mathbb{R}^+)}} + Mv\sigma. \end{aligned}$$

This equation with (12) leads to

$$\begin{aligned}||y||_{\mathcal{H}} &\leq M\,[\xi+||A^{-\gamma}||d_2(1+r\xi)]\\ &\quad +(1+r\xi)[||A^{-\gamma}||d_2+d_2\frac{C_{1-\gamma}\Gamma(1+\gamma)b^{\alpha\gamma}}{\Gamma(1+\alpha\gamma)\gamma}+\frac{M}{\Gamma(\alpha)}\Delta\|\varphi\|_{L^P_{(J,,\mathbb{R}^+)}}]\\ &\quad +v[||A^{-\gamma}||d_2r+d_2\frac{C_{1-\gamma}\Gamma(1+\gamma)b^{\alpha\gamma}}{\Gamma(1+\alpha\gamma)\gamma}r+\frac{M}{\Gamma(\alpha)}\Delta\|\varphi\|_{L^P_{(J,,\mathbb{R}^+)}}r+\sigma M]\\ &< v.\end{aligned}$$

Then, $\Phi(B_v)\subseteq B_v$.

Step 2. Φ_1 is a contraction with a contraction constant $k<\frac{1}{2}$.

Let $u,v\in B_v$ and $\eta\in J$. Then, $||\varkappa(\eta)\overline{u}-\varkappa(\eta)\overline{v}||_\Theta=\int_{-r}^0||\overline{u}(\eta+\theta)-\overline{v}(\eta+\theta)||d\theta\leq r||u-v||_{\mathcal{H}}$. From (6), (7) and (11), for every $u,v\in B_v$ and any $\eta\in J$, we have that

$$\begin{aligned}&||\Phi_1(u)(\eta)-\Phi_1(v)(\eta)||\\ \leq\;&||h(\eta,\varkappa(\eta)\overline{u})-h(\eta,\varkappa(\eta)\overline{v})||\\ &+||\int_0^\eta(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)[h(\tau,\varkappa(\tau)\overline{u})-h(\tau,\varkappa(\tau)\overline{v})]d\tau\\ \leq\;&||A^{-\gamma}A^{\gamma}[h(\eta,\varkappa(\eta)\overline{u})-h(\eta,\varkappa(\eta)\overline{v})]||\\ &+||\int_0^\eta(\eta-\tau)^{\alpha-1}A^{1-\gamma}\mathfrak{K}_2(\eta-\tau)A^{\gamma}[h(\tau,\varkappa(\tau)\overline{u})-h(\tau,\varkappa(\tau)\overline{v})]d\tau\\ \leq\;&||A^{-\gamma}||\;||A^{\gamma}h(\eta,\varkappa(\eta)\overline{u})-A^{\gamma}h(\eta,\varkappa(\eta)v)||\\ &+\frac{\alpha C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}\int_0^\eta(\eta-\tau)^{\alpha\gamma-1}||A^{\gamma}h(\tau,\varkappa(\tau)\overline{u})-A^{\gamma}h(\tau,\varkappa(\tau)\overline{v})||d\tau\\ \leq\;&d_1||A^{-\gamma}||\;||\varkappa(\eta)\overline{u}-\varkappa(\eta)\overline{v}||_\Theta\\ &+\frac{d_1\alpha C_{1-\gamma}\Gamma(2-\gamma)}{\Gamma(1+\alpha\gamma)}\sup_{\tau\in[0,\eta]}||\varkappa(\tau)\overline{u}-\varkappa(\tau)\overline{v}||_\Theta\frac{b^{\alpha\gamma}}{\alpha\gamma}\\ \leq\;&||u-v||_{\mathcal{H}}[d_1||A^{-\gamma}||+\frac{d_1b^{\alpha\gamma}C_{1-\gamma}\Gamma(1+\gamma)}{\gamma\Gamma(1+\alpha\gamma)}]r,\end{aligned}$$

which yields with (9) that Φ_1 is a contraction with a contraction constant $k<\frac{1}{2}$.

Step 3. Φ_2 has a closed graph and $\Phi_2(x);x\in B_v$ is compact.

Assume $(x_n)_{n\geq1}$ and $(y_n)_{n\geq1}$ are sequences in B_v where $x_n\to x, y_n\to y$ and $y_n\in\Phi_2(x_n);n\geq1$. Then,

$$y_n(\eta)=\begin{cases}0,\eta\in[-r,0],\\ +\int_0^\eta(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f_n(\tau)d\tau\\ +\sum_{0<\eta_k<\eta}\mathfrak{K}_1(\eta-\eta_k)I_i(x_n(\eta_k^-)),\eta\in J,\end{cases}\tag{18}$$

where $f_n\in\tau^1_{F(.,\varkappa(.)\overline{x}_n)}$. Using (HF_2), it yields that

$$||f_n(\eta)||\leq\varphi(\eta)(1+r(v+\xi)),\text{a.e.}\eta\in J.$$

So, $(f_n)_{n\geq1}$ is bounded in $L^P(J,E)$ and, hence, there exists a subsequence of $\{f_n\}_{n=1}^\infty$. We denote them by $(f_n)_{n\geq1}$, where $f_n\longrightarrow f\in L^P(J,E)$. From Mazur's Lemma, there exists a sequence of convex combination, $\{z_n\}_{n=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ that converges almost everywhere to f. Note that by (HF_2), again, for any $\eta\in J,\tau\in(0,\eta]$ and any $n\geq1$,

$$||(\eta-\tau)^{\alpha-1}f_n(\tau)||\leq\;|\eta-\tau)|^{\alpha-1}\,\varphi(\tau)(1+r(v+\xi))\in L^P((0,\eta],\mathbb{R}^+).$$

Set

$$\widetilde{y}_n(\eta) = \begin{cases} 0, \eta \in [-r,0], \\ +\int_0^{\eta}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)z_n(\tau)d\tau \\ +\sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(x_n(\eta_i^-)), \eta \in J. \end{cases} \tag{19}$$

Note that by (18), $\widetilde{y}_n(\eta) \to y(\eta)$, $\eta \in J$. Moreover, since $\varkappa(\eta)\overline{x}_n \to \varkappa(\eta)\overline{x}; \eta \in J$, $F(\eta,.); a.e.\ \eta \in J$ is upper semicontinuous, it yields $f(\eta) \in F(\eta, \varkappa(\eta)x), a.e.$ Therefore, from the continuity of $\mathfrak{K}_2(\eta-\tau); \tau \in [0,\eta]$, I_i $(i = 1,2,\dots)$, and by taking the limit of (19) as $n \to \infty$, one gets $y \in \Phi_2(x)$.

To prove that the values of Φ_2 are compact, assume $x \in \mathcal{H}$ and $y_n \in \Phi_2(x), n \geq 1$. Using similar arguments to the above, we get that $\{y_n : n \geq 1\}$ has a convergent subsequence $(\widetilde{y})_{n\geq1}$. So, $\Phi_2(x)$ is relatively compact. Since the graph of Φ_2 is closed its values are closed and, hence, $\Phi_2(x)$ is relatively compact in $\mathcal{H}$.

Step 4. We claim that the subsets $Z_{|\overline{J_i}}$ $(i = 0,1,\dots,m)$ are equicontinuous, where

$$Z_{|\overline{J_i}} = \{y^* \in C(\overline{J_i}, E) : y^*(\eta) = y(\eta), \eta \in (\eta_i, \eta_{i+1}], y^*(\eta_i) = y(\eta_i^+), y \in \Phi_2(x), x \in B_v\}.$$

Assume $y^* \in Z_{|\overline{J_i}}$. Then, there exists $x \in B_v$ and $f \in S^1_{F(.,\varkappa(.)\overline{x})}$, where, for $\eta \in J_i$,

$$\begin{aligned} y^*(\eta) &= \int_0^{\eta}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f(\tau)d\tau \\ &\quad+\sum_{0<\eta_k<\eta}\mathfrak{K}_1(\eta-\eta_k)I_k(\overline{x}(\eta_k^-)), \end{aligned}$$

and $y^*(\eta_i) = y(\eta_i^+)$.

Case 1. Let η_1, η_2 $(\eta_1 < \eta_2)$ be two points in $(\eta_i, \eta_{i+1}]$. Then,

$$\begin{aligned} & ||y^*(\eta_2) - y^*(\eta_1)|| \\ \leq\ & ||\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}\mathfrak{K}_2(\eta_2-\tau)f(\tau)d\tau \\ & -\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}\mathfrak{K}_2(\eta_1-\tau)f(\tau)|| \\ & +||\sum_{0<\eta_k<\eta_2}\mathfrak{K}_1(\eta_2-\eta_k)I_k(\overline{x}(\eta_k^-)) - \sum_{0<\eta_i<\eta_1}\mathfrak{K}_1(\eta_1-\eta_k)I_k(\overline{x}(\eta_k^-))|| \\ \leq\ & ||\int_{\eta_1}^{\eta_2}(\eta_2-\tau)^{\alpha-1}\mathfrak{K}_2(\eta_2-\tau)f(\tau)d\tau|| \\ & +\int_0^{\eta_1}|(\eta_2-\tau)^{\alpha-1}-(\eta_1-\tau)^{\alpha-1}|\ ||\mathfrak{K}_2(\eta_2-\tau)f(\tau)||d\tau \\ & +||\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}||\mathfrak{K}_2(\eta_2-\tau)f(\tau)-\mathfrak{K}_2(\eta_1-\tau)f(\tau)||\ d\tau\ . \\ & +\sum_{0<\eta_k<\eta_2}||\mathfrak{K}_1(\eta_2-\eta_k)-\mathfrak{K}_1(\eta_1-\eta_k)||\ ||I_i(\overline{x}(\eta_i^-))|| \\ =\ & \sum_{i=1}^{i=4} I_i. \end{aligned}$$

The hypothesis (HF_2) implies $||f(\eta)|| \leq \varphi(\eta)\ (1 + r(v+\xi)), a.e.\eta \in J,$ and, hence, by Lemma 1, we get

$$\begin{aligned} \lim_{\eta_2\to\eta_1} I_1 &= \lim_{\eta_2\to\eta_1}||\int_{\eta_1}^{\eta_2}(\eta_2-\tau)^{\alpha-1}\mathfrak{K}_2(\eta_2-\tau)f(\tau)d\tau|| \\ &\leq \frac{M(1+r(v+\xi))}{\Gamma(\alpha)}\lim_{\eta_2\to\eta_1}\int_{\eta_1}^{\eta_2}(\eta_2-\tau)^{\alpha-1}\varphi(\tau)d\tau \\ &= \frac{M(1+r(v+\xi))}{\Gamma(\alpha)}||\varphi||_{L^P([J,\mathbb{R}^+)}\lim_{\eta_2\to\eta_1}(\int_{\eta_1}^{\eta_2}(\eta_2-\tau)^{\frac{P(\alpha-1)}{P-1}}d\tau)^{\frac{P-1}{P}} = 0. \end{aligned}$$

For I_2, we have

$$\begin{aligned}\lim_{\eta_2\to\eta_1} I_2 &\leq \lim_{\eta_2\to\eta_1}\int_0^{\eta_1} |(\eta_2-\tau)^{\alpha-1}-(\eta_1-\tau)^{\alpha-1}|\,||\mathfrak{K}_2(\eta_2-\tau)f(\tau)||d\tau \\ &= \frac{M(1+r(v+\xi))}{\Gamma(\alpha)}\lim_{\eta_2\to\eta_1}\int_0^{\eta_1} |(\eta_2-\tau)^{\alpha-1}-(\eta_1-\tau)^{\alpha-1}|\varphi(\tau)d\tau.\end{aligned}$$

Note that $\overline{\omega}=\frac{\alpha-1}{1-\frac{1}{P}}\in(-1,0)$, then, for $\tau<\eta_1$, we have $(\eta_1-\tau)^{\overline{\omega}}\geq(\eta_2-\tau)^{\overline{\omega}}$. As an application of Lemma 3 in [8] and considering $\frac{P-1}{P}\in(0,1)$, we get

$$|\Big[(\eta_1-\tau)^{\overline{\omega}}\Big]^{1-\frac{1}{P}}-\Big[(\eta_2-\tau)^{\overline{\omega}}\Big]^{\frac{P-1}{P}}|\leq\Big[(\eta_1-\tau)^{\overline{\omega}}-(\eta-\tau)^{\overline{\omega}}\Big]^{\frac{P-1}{P}}.$$

Then,

$$|(\eta_1-\tau)^{\alpha-1}-(\eta_2-\tau)^{\alpha-1}|\leq\Big[(\eta_1-\tau)^{\overline{\omega}}-(\eta_2-\tau)^{\overline{\omega}}\Big]^{\frac{P-1}{P}}.$$

This leads to

$$|(\eta-\tau)^{\alpha-1}-(\eta+\lambda-\tau)^{\alpha-1}|^{\frac{P-1}{P}}\leq\Big[(\eta-\tau)^{\overline{\omega}}-(\eta+\lambda-\tau)^{\overline{\omega}}\Big].$$

Therefore,

$$\begin{aligned}&\lim_{\eta_2\to\eta_1} I_2 \\ \leq\;& \frac{M(1+r(v+\xi))}{\Gamma(\alpha)}\lim_{\eta_2\to\eta_1}\int_0^{\eta_1} |(\eta_2-\tau)^{\alpha-1}-(\eta_1-\tau)^{\alpha-1}|\varphi(\tau)d\tau \\ \leq\;& \frac{M(1+r(v+\xi)))}{\Gamma(\alpha)}\lim_{\eta_2\to\eta_1}\left[\int_0^{\eta_1}|(\eta_2-\tau)^{\alpha-1}-(\eta_1-\tau)^{\alpha-1}|^{\frac{P}{P-1}}d\tau\right]^{\frac{P-1}{P}}\|\varphi\|_{L^P_{(J,\mathbb{R}^+)}} \\ \leq\;& \frac{M(1+r(v+\xi)))}{\Gamma(\alpha)}\lim_{\eta_2\to\eta_1}\left[\int_0^{\eta_1}[(\eta_2-\tau)^{\overline{\omega}}-(\eta_1-\tau)^{\overline{\omega}}]d\tau\right]^{\frac{P-1}{P}}\|\varphi\|_{L^P_{(J,\mathbb{R}^+)}} \\ \leq\;& \frac{M(1+r(v+\xi))}{\Gamma(\alpha)}\lim_{\eta_2\to\eta_1}\left[\frac{1}{\omega+1}[\eta_2^{\overline{\omega}+1}-(\eta_2-\eta_1)^{\overline{\omega}+1}-\eta_1^{\overline{\omega}+1}\right]^{\frac{P-1}{P}}\|\varphi\|_{L^P_{(J,\mathbb{R}^+)}} \\ =\;& 0.\end{aligned}$$

For I_3,

$$\lim_{\eta_2\to\eta_1} I_3\leq\lim_{\eta_2\to\eta_1}||\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}||\mathfrak{K}_2(\eta_2-\tau)f(\tau)-\mathfrak{K}_2(\eta_1-\tau)f(\tau)||\,d\tau.$$

Observe that for every $\tau\in[0,\eta]$,

$$\begin{aligned}&(\eta_1-\tau)^{\alpha-1}||K_\alpha(\eta_2-\tau)f(\tau)-K_\alpha(\eta_1-\tau)f(\tau)|| \\ \leq\;& \frac{2M(\nu+1)}{\Gamma(\alpha)}(\eta_1-\tau)^{\alpha-1}\varphi(\tau)\in L^P(J,\mathbb{R}^+).\end{aligned}$$

Moreover, since $\{\eta(\eta):\eta>0\}$ is equicontinuous, and, using the Lebesgue-dominated convergence theorem, one gets

$$
\begin{aligned}
\lim_{\eta_2\to\eta_1} I_3 &\le \frac{M(1+r(v+\xi))}{\Gamma(\alpha)} \lim_{\eta_2\to\eta_1}\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}||\mathfrak{K}_2(\eta_2-\tau)-\mathfrak{K}_2(\eta_1-\tau)||\varphi(\tau)d\tau \\
&= \frac{M(1+r(v+\xi))}{\Gamma(\alpha)}\int_0^{\eta_1}\int_0^{\infty}\theta(\eta_1-\tau)^{\alpha-1}\zeta_\alpha(\theta)\times \\
&\quad [\lim_{\eta_2\to\eta_1}\|(\Upsilon((\eta_2-\tau)^\alpha\theta)-\Upsilon(\eta_1-\tau)^\alpha\theta))\|]\, d\theta\varphi(\tau)d\tau \\
&= 0.
\end{aligned}
$$

For I_4,

$$\lim_{\eta_2\to\eta_1} I_4 \le \sigma v \lim_{\eta_2\to\eta_1}\sum_{0<\eta_k<\eta_2}||\mathfrak{K}_1(\eta_2-\eta_k)-\mathfrak{K}_1(\eta_1-\eta_k)||=0.$$

Case 2. $\eta=\eta_i$, $i=1,\dots,m$. Assume $\delta>0$, $\eta_i+\delta\in(\eta_i,\eta_{i+1}]$ and $\lambda>0$ where $\eta_i<\lambda<\eta_i+\delta\le\eta_{i+1}$. Hence, as above, it can be shown that

$$\|y^*(\eta_i+\delta)-y^*(\eta_i)\| = \lim_{\lambda\to\eta_i^+}\|y(\eta_i+\delta)-y(\lambda)\|=0.$$

Then, $Z_{|\overline{J_i}}$ $(i=0,1,\dots,m)$ are equicontinuous.

Step 5. Set $B_1=\overline{conv}\Phi(B_v)$ and $B_n=\overline{conv}\Phi(B_{n-1})$, $n\ge 2$. Then, the sequence (B_n), $n\ge1$ is a decreasing sequence of not empty, closed and bounded subsets of $\mathcal{H}$. So, the set $B=\bigcap_{n\ge1}B_n$ is bounded, closed, convex and $\Phi(B)\subset B$. Next, we show that B is compact. According to the generalized Cantor's intersection property, we only need to prove that

$$\lim_{n\to\infty}\chi_{\mathcal{H}}(B_n)=0, \tag{20}$$

where $\chi_{\mathcal{H}}$ is the Hausdorff measure of noncompactness on $\mathcal{H}$. Assume $n\in\mathbb{N}$ and $n\ge1$ are fixed. From the fact that Φ_1 is a contraction with a contraction constant $k<\frac{1}{2}$, it follows that

$$
\begin{aligned}
&\chi_{\mathcal{H}}\Phi(B_{n-1}) \\
\le\ &\chi_{\mathcal{H}}\Phi_1(B_{n-1})+\chi_{\mathcal{H}}\Phi_2(B_{n-1}) \\
\le\ &\frac{1}{2}\chi_{\mathcal{H}}(B_{n-1})+\chi_{\mathcal{H}}\Phi_2(B_{n-1}).
\end{aligned} \tag{21}
$$

Let $\varepsilon>0$. Using Lemma 5 in [39], there is a $(y_k)_{k\ge1}$ in $\Phi_2(B_{n-1})$ with

$$\chi_{\mathcal{H}}\Phi_2(B_{n-1})\le 2\chi_{\mathcal{H}}\{y_k:k\ge1\}+\varepsilon.$$

From the fact that the subsets $Z_{|\overline{J_i}}$ $(i=0,1,\dots,m)$ are equicontinuous, one obtains

$$
\begin{aligned}
&\chi_{\mathcal{H}}\Phi_2(B_{n-1}) \\
\le\ &2\chi_{\mathcal{H}}\{y_k:k\ge1\}+\varepsilon \\
\le\ &2\sup_{\eta\in[0,b]}\chi_E\{y_k(\eta):k\ge1\}+\varepsilon.
\end{aligned} \tag{22}
$$

Now, let $x_k\in B_{n-1}$ and $y_k\in\Phi_2(x_k)$, $k\ge1$. Then, for every $k\ge1$, there is a $f_k\in\tau^1_{F(.,\varkappa(\eta)\overline{x}_k)}$ such that, for any $\eta\in J$,

$$y_k(\eta)=\begin{cases} 0,\eta\in[-r,0], \\ \int_0^\eta(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f_k(\tau)d\tau \\ +\sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}_k(\eta_i^-)),\eta\in J.\end{cases}$$

Note that the assumption (HI) implies that for $\eta \in J$,

$$\begin{aligned}\chi_E\{\sum_{0<\eta_i<\eta} \mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}_k(\eta_i^-)) \quad &: \quad k \geq 1\} \\ &\leq M\sum_{i=1}^{i=m} \varsigma_i\, \chi_E\{\overline{x}_k(\eta_i^-)) : k \geq 1\} \\ &\leq M\sum_{i=1}^{i=m} \varsigma_i\, \chi_E\{x_k(\eta_i^-)) : k \geq 1\} \\ &\leq M\,\chi_{\mathcal{H}}(B_{n-1})\sum_{i=1}^{i=m} \varsigma_i\,. \qquad (23)\end{aligned}$$

Moreover, from (4) , we have that for $a.e.\tau \in J$,

$$\begin{aligned}\chi_E\{f_k(\tau) \quad &: \quad k \geq 1\} \leq \chi\{F(\tau, \varkappa(\tau)\overline{x}_k) : k \geq 1\} \\ &\leq \beta(\tau) \sup_{\theta\in[-r,0]} \chi\{\overline{x}_k(\tau+\theta) : k \geq 1\} \\ &\leq \beta(\tau) \sup_{\delta\in[-r,\tau]} \chi\{\overline{x}_k(\delta) : k \geq 1\} \\ &\leq \beta(\tau) \sup_{\delta\in[0,\tau]} \chi\{x_k(\delta) : k \geq 1\} \\ &\leq \beta(\tau)\chi_{\mathcal{H}}(B_{n-1}) = \gamma(\eta). \qquad (24)\end{aligned}$$

Again, by $(HF_2)^*$, for every $k \geq 1$, and for almost $\eta \in J$, $||f_k(\eta)|| \ \leq \varphi(\eta)\ (1 + r(v + \xi))$ and, hence, $\{f_k : k \geq 1\}$ is integrably bounded. As a consequence of Lemma 4 in [40], there is a compact set $K_\epsilon \subseteq E$, a measurable set $J_\epsilon \subset J$ having a measure less than ϵ and $\{z_k^\epsilon\} \subset L^P(J, E)$ such that for every $\tau \in J$, $\{z_k^\epsilon(\tau) : k \geq 1\} \subseteq K_\epsilon$ and

$$||f_k(\tau) - z_k^\epsilon(\tau)|| \ < 2\gamma(\tau) + \epsilon \text{ for all } k \geq 1 \text{ and all } \tau \in J - J_\epsilon. \qquad (25)$$

Then, by (24) and (25) and Minkowski's inequality, it follows that for $k \geq 1$,

$$\begin{aligned}&||\int_{J-J_\epsilon} (\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)(f_k(\tau) - z_k^\epsilon(\tau))d\tau|| \\ \leq\ & \frac{M}{\Gamma(\alpha)}||f_k - z_k^\epsilon||_{L^P(J_0-J_\epsilon,\mathbb{R}^+)}(\int_{J-\ J_\epsilon}(\eta-\tau)^{\frac{(\alpha-1)P}{P-1}}d\tau)^{\frac{P-1}{P}} \\ \leq\ & \frac{\Delta M}{\Gamma(\alpha)}||f_k - z_k^\epsilon||_{L^P(J_0-J_\epsilon,\mathbb{R}^+)} \\ \leq\ & \frac{\Delta M}{\Gamma(\alpha)}(2||\gamma||_{L^P(J-\ J_\epsilon,\ \mathbb{R}^+)} + \epsilon b^{\frac{1}{P}}) \\ =\ & \frac{\Delta M}{\Gamma(\alpha)}\ (2||\beta||_{L^P(J,\ \mathbb{R}^+)}\ \chi_{\mathcal{H}}(B_{n-1}) + \epsilon b^{\frac{1}{P}}), \qquad (26)\end{aligned}$$

and

$$\begin{aligned}&||\int_{J_\epsilon}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f_k(\tau)d\tau|| \\ \leq\ & \frac{M}{\Gamma(\alpha)}(1 + r(v+\xi))\int_{J_\epsilon}(\eta-\tau)^{\alpha-1}\varphi(\tau)d\tau \\ \leq\ & \frac{M}{\Gamma(\alpha)}(1 + r(v+\xi)||\varphi||_{L^P(J_\epsilon,\ \mathbb{R}^+)}(\int_{J_\epsilon}(\eta-\tau)^{\frac{(\alpha-1)P}{P-1}}d\tau)^{\frac{P-1}{P}}. \qquad (27)\end{aligned}$$

Moreover, from the fact that $\{z_k^\epsilon(\tau) : k \geq 1\}; \tau \in J$ is contained in a compact subset, we get

$$\chi\{\int_{J-J_\epsilon}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)z_k^\epsilon(\tau)d\tau : k\geq 1\}=0.$$

Combining this relation with (26) and (27), it follows that

$$\begin{aligned}\chi\{\int_0^\eta(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f_k(\tau)d\tau \quad &: \quad k\geq 1\} \\ &\leq \frac{\Delta M}{\Gamma(\alpha)}(2||\beta||_{L^P(J,\,\mathbb{R}^+)}\,\chi_{\mathcal{H}}(B_{n-1})+\epsilon b^{\frac{1}{P}}) \\ &\quad +\frac{(1+r(v+\xi)M}{\Gamma(\alpha)}||\varphi||_{L^P(J_\epsilon,\,\mathbb{R}^+)}\Delta_\epsilon, \qquad (28)\end{aligned}$$

where $\Delta_\epsilon=(\int_{J_\epsilon}(\eta-\tau)^{\frac{(\alpha-1)P}{P-1}}d\tau)^{\frac{P-1}{P}}$. Using the fact that ε is chosen arbitrary, relation (28) becomes

$$\begin{aligned}\chi\{\int_0^\eta(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f_k(\tau)d\tau \quad &: \quad k\geq 1\} \\ &\leq \frac{2\Delta M}{\Gamma(\alpha)}\;||\beta||_{L^P(J,\,\mathbb{R}^+)}\,\chi_{\mathcal{H}}(B_{n-1}).\end{aligned}$$

Using the above inequality and (21)–(23), in addition to the fact that ε is arbitrary, it follows that

$$\chi_{\mathcal{H}}(B_n)\leq(\frac{4\Delta M}{\Gamma(\alpha)}\;||\beta||_{L^P(J,\,\mathbb{R}^+)}\;+2M\sum_{i=1}^{i=m}\varsigma_i+\frac{1}{2})\chi_{\mathcal{H}}(B_{n-1}).$$

This leads to

$$\chi_{\mathcal{H}}(B_n)\leq(\frac{4\Delta M}{\Gamma(\alpha)}\;||\beta||_{L^P(J,\,\mathbb{R}^+)}+M\sum_{i=1}^{i=m}\varsigma_i+\frac{1}{2})^{n-1}\chi_{\mathcal{H}}(B_1),\ \forall n\geq 1.$$

The above inequality holds for any natural number n, and by (13) together with taking the limit as $n\to\infty$, we get (20). Then, B is not empty and a compact subset of $\mathcal{H}$. So, $\Phi : B\to P_{ck}(B)$ is completely continuous. By applying Lemma 3, we conclude that the fixed points set of Φ is not an empty subset of $\mathcal{H}$. Furthermore, by arguing as in Step 1, we can prove that the set of fixed points of Φ is bounded and, hence, by Lemma 3, it is compact in $\mathcal{H}$. Therefore, the set $\Sigma_\psi^F[-r,b]$ is not empty and a compact subset of H. □

4. The Structure Topological of $\Sigma_\psi^F[-r,b]$

In the section we prove that $\Sigma_\psi^F[-r,b]$ is an R_δ-set

Definition 3 ([41])**.** *A topological space X, which is homotopy equivalent to a point, is called contractible. In other words, there is a continuous map* $h : [0,1]\times X\to X$, $h(0,.x)=x$ *and* $h(1,x)=x_0\in X$.

Lemma 5 ([41])**.** *Let* $A\subseteq X$*, where A is not empty and X is a complete metric space. Then, A is said to be* R_δ*-set if and only if it is an intersection of a decreasing sequence* $\{A_n\}$ *of contractible sets and* $\chi_X(A_n)\to 0$*, as* $n\to\infty$.

Now, consider the multi-valued function $\widetilde{F} : J\times\Theta\to P_{ck}(E)$ that is given by:

$$\widetilde{F}(\eta,u):=\begin{cases}F(\eta,u),||u||<v,\\ F(\eta,\frac{vu}{||u||}),||u||\geq v,\end{cases}$$

where v is defined by (17). Since $\widetilde{F} = F$ on D_v, the set of solutions consisting of mild solutions for Problem (1) is equal to the set of solutions consisting of mild solutions for the problem:

$$\begin{cases} {}^cD^{\alpha}_{0,\eta}[x(\eta) - h(\eta, \varkappa(\eta)x)] \in Ax(\eta) + \widetilde{F}(\eta, \varkappa(\eta)x), \ a.e. \ \eta \in [0,b] - \{\eta_1, \ldots, \eta_m\}, \\ I_i(x(\eta_i^-)) = x(\eta_i^-) - x(\eta_i^+), i = 1, \ldots, m, \\ x(\eta) = \psi(\eta), \eta \in [-r, 0]. \end{cases}$$

Obviously, $\widetilde{F}$ verifies (HF_1) and, for $\eta \in J$, *a.e.*,

$$||\widetilde{F}\,(\eta, u)|| \leq \begin{cases} \varphi(\eta)(1 + ||u||) \leq \varphi(\eta)(1 + r(\xi + v)) = \zeta(\eta), ||u|| < v, \\ \varphi(\eta)(1 + ||\frac{vu}{||u||}||) = \varphi(\eta)(1 + r(\xi + v)) = \zeta(\eta), ||u|| \geq v. \end{cases}$$

Then, we can assume that F verifies the next condition:
$(HF_2)^*$ There exists a function $\xi \in L^P(I, \mathbb{R}^+)(P > \frac{1}{\alpha})$, where for every $z \in \Theta$,

$$\|F(\eta, z)\| \leq \zeta(\eta), \ a.e. \ \eta \in J.$$

We recall the next Lemma. For its proof, we refer the reader to the second step in the proof of Theorem 3.5 in [13].

Lemma 6. *Assume that (HF_1) and $(HF_2)^*$ are satisfied. Then, there exists a sequence of multi-functions $\{F_i\}_{i=1}^{\infty}$ with $F_i : J \times \Theta \to P_{ck}(E)$ such that:*

(i) *Every $F_i\ (\eta, .)$ is continuous for almost $\eta \in J$.*
(ii) *$F(\eta, x) \subseteq \ldots . \subseteq F_{i+1}(\eta, x) \subseteq F_i(\eta, \varkappa(\eta)x) \subseteq \cdots \subseteq \overline{co}F(\eta, \{y \in \Theta : ||y - x|| \leq 3^{1-i}\}), i \geq 1,$ for each $\eta \in J$ and $x \in \Theta$.*
(iii) *$F(\eta, z) = \cap_{i\ \geq 1} F_i(\eta, z)$.*
(iv) *For all $i \geq 1$, there is a selection $g_i : J \times \Theta \to E$ of F_i such that $g_i(., x)$ is measurable for each $x \in \Theta$ and for $g_i(\eta, .)$ is locally Lipschitz.*

Remark 2. *(Ref. [19]) The property (iv) in Lemma 6 implies that, for almost $\eta \in J$, $g_i(\eta, .)$, $i \geq 1$ is continuous.*

Assume $\Sigma_{\psi}^{F_i}[-r, b]$ is the mild solutions set of the following fractional neutral impulsive semilinear differential inclusions with delay:

$$\begin{cases} {}^cD^{\alpha}_{0,\eta}[x(\eta) - h(\eta, \varkappa(\eta)x)] \in Ax(\eta) + F_i(\eta, \varkappa(\eta)x), \ a.e. \ \eta \in [0,b] - \{\eta_1, \ldots, \eta_m\}, \\ I_i(x(\eta_i^-)) = x(\eta_i^-) - x(\eta_i^+), i = 1, \ldots, m, \\ x(\eta) = \psi(\eta), \eta \in [-r, 0]. \end{cases} \tag{29}$$

Theorem 2. *Assume that the conditions in Theorem 1 after substituting $(HF2)$ by $(HF2)^*$ are held. Then, there exists $N_0 \in \mathbb{N}$ such that, for $i \geq N_0$, the set $\Sigma_{\psi}^{F_i}[-r, b]$ is compact and not empty in H.*

Proof. Let i be a fixed natural number. We define a multioperator $\Phi_i : \mathcal{H} \to P(\mathcal{H})$ as the following : $y \in \Phi_i(x)$ if and only if

$$y(\eta) = \begin{cases} 0, \eta \in [-r, 0], \\ \mathfrak{K}_1(\eta)[\psi(0) - h(0, \psi)] + h(\eta, \varkappa(\eta)\overline{x}) \\ + \int_0^{\eta} (\eta - \tau)^{\alpha - 1} A\mathfrak{K}_2(\eta - \tau) h(\tau, \varkappa(\tau)\overline{x}) d\tau \\ + \int_0^{\eta} (\eta - \tau)^{\alpha - 1} \mathfrak{K}_2(\eta - \tau) f(\tau) d\tau \\ + \sum_{0 < \eta_i < \eta} \mathfrak{K}_1(\eta - \eta_i) I_i(\overline{x}(\eta_i^-)), \eta \in J, \end{cases}$$

where $f \in \tau^1_{F_i(.,\varkappa(.)\overline{x})}$. Due to Lemma 5, F_i verifies (F_1), $(F_2)^*$. As a result of Theorem 1, Φ_i is closed , $\Phi_i(B_v) \subseteq B_v$ and $\Phi_i(B_v)$ is equicontinuous. Set $B_{1,i} = \overline{conv}\Phi_i(B_v)$ and $B_{n,i} = \overline{conv}\Phi_i(B_{n-1,i})$, $n \geq 2$. As in Theorem 1, the sequence $(B_{n,i})$, $n \geq 1$ is a decreasing sequence of non-empty, closed and bounded subsets of $\mathcal{H}$. We show that

$$\lim_{n\to\infty} \chi_{C([-r,b],E)}(B_{n,i}) = 0. \tag{30}$$

Let $\varepsilon > 0$. Choose a natural number N_0 with $3^{1-N_0} < \frac{\varepsilon}{2||\beta||_{L^P(J,\,\mathbb{R}^+)}}$ and let $i > N_0$ be a fixed natural number. Using a similar argument as the one used in the proof of Theorem 1, one gets

$$\begin{aligned} &\chi_{\mathcal{H}}(B_{n,i}) \\ \leq\ & 2\sup_{\eta\in J} \chi_E\{y_k(\eta) : k \geq 1\} + \frac{\varepsilon}{2}, \end{aligned}$$

where

$$y_k(\eta) = \begin{cases} 0, \eta \in [-r,0] \\ \int_0^\eta (\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f_k(\tau)d\tau \\ +\sum_{0<\eta_i<\eta} \mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}(\eta_i^-)), \eta \in J, \end{cases}$$

and $f_k \in \tau^1_{F_i(.,\varkappa(\eta)x_k)}$. Next, due to Remark 4.2 in [7], it follows that for any bounded subset $D \subset \Theta$,

$$\chi_E(F_i(\eta, D)) \leq \beta(\eta)[\sup_{\theta\in[-r,\eta]} \chi_E\{z(\theta) : z \in D\} + 3^{1-i}]. \tag{31}$$

Then, it yields from (ii) in Lemma 5 and (31), for $a.e.\tau \in J$,

$$\begin{aligned} \chi_E(\{f_k(\tau) \ &: \ k \geq 1\} \\ &\leq \chi_E\{F_i(\tau, \varkappa(\tau)x_k) : k \geq 1\} \\ &\leq \beta(\tau)[\sup_{\theta\in[-r,0]} \chi_E\{x_k(\tau+\theta) : k \geq 1\} + 3^{1-N_0}] \\ &\leq \beta(\tau)[\sup_{\delta\in[-r,\tau]} \chi_E\{x_k(\delta) : k \geq 1\} + 3^{1-N_0}] \\ &\leq \beta(\tau)[\sup_{\theta\in[0,,\tau]} \chi_E\{x_k(\delta) : k \geq 1\} + 3^{1-N_0}] \\ &\leq \beta(\tau)\chi_{\mathcal{H}}(B_{n-1,i}) + \beta(\tau)3^{1-N_0} = \overline{\gamma}(\tau). \end{aligned} \tag{32}$$

As in (28) but by using (32) instead of (24), we get

$$\begin{aligned} \chi\{\int_0^\eta (\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f_k(\tau)d\tau \ &: \ k \geq 1\} \\ &\leq \frac{\Delta M}{\Gamma(\alpha)}(2||\beta||_{L^P(J,\,\mathbb{R}^+)}\,\chi_{\mathcal{H}}(B_{n-1}) + \epsilon b^{\frac{1}{P}}) + \frac{\epsilon}{2} \\ &\quad + \frac{M}{\Gamma(\alpha)}(1 + rv + r\xi) \times \\ &\quad ||\varphi||_{L^P(J_\epsilon,\,\mathbb{R}^+)}(\int_{J_\epsilon} (\eta-\tau)^{\frac{P}{P-1}} d\tau)^{\frac{P-1}{P}}. \end{aligned}$$

Similarly, as in the proof of Theorem 1, we confirm the validity of (30). Therefore, by the generalized Cantor's intersection property, the set B_i is not empty and compact in $\mathcal{H}$. As in Theorem 1, the fixed points set of the multivalued function $\Phi_i : B_i \to P_{ck}(B_i)$ is not empty and a compact subset in $\mathcal{H}$. Consequently, the set $\sum_{\psi}^{F_n}[-r,b]$ is not empty and a compact subset of H. □

Theorem 3. *Under the conditions of Theorem 2,* $\sum_{\psi}^{F}[-r,b] = \cap_{n=N_0}^{\infty} \sum_{\psi}^{F_n}[-r,b]$.

Proof. In view of (iii) in Lemma 8, it can be seen that $\sum_{\psi}^{F}[-r,b] \subseteq \cap_{n=N_0}^{\infty} \sum_{\psi}^{F_n}[-r,b]$. Let $\overline{x} \in \cap_{n=N_0}^{\infty} \sum_{\psi}^{F_n}[-r,b]$. Then, there is $\mathfrak{f}_n \in \tau^1_{F_n(.,\varkappa(.)\overline{x})}, n \geq N_0$ such that

$$\overline{x}(\eta) = \begin{cases} \psi(\eta), \eta \in [-r,0], \\ \mathfrak{K}_1(\eta)[\psi(0) - h(0,\psi)] + h(\eta,\varkappa(\eta)\overline{x}) \\ + \int_0^{\eta}(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)h(\tau,\varkappa(\tau)\overline{x})d\tau \\ + \int_0^{\eta}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)\mathfrak{f}_n(\tau)d\tau \\ + \sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}(\eta_i^-)), \eta \in J. \end{cases} \quad (33)$$

It follows from $(HF2)^*$ that

$$||\mathfrak{f}_n(\eta)|| \leq \zeta(\eta), \text{for } a.e. \eta \in J.$$

This means that the sequence $(\mathfrak{f}_n)_{n\geq 1}$ is weakly relatively compact in $L^P(J,E)$, so we can assume $\mathfrak{f}_n \rightharpoonup f$ weakly, where $f \in L^P(J,\mathbb{R}^+)$. As in the proof of Theorem 1, there is a sequence of convex combinations $(z_n)_{n\geq 1}$ of $(\mathfrak{f}_n)_{n\geq 1}$ that converges almost everywhere to f. Note that

$$\overline{x}(\eta) = \begin{cases} \psi(\eta), \eta \in [-r,0], \\ \mathfrak{K}_1(\eta)[\psi(0) - h(0,\psi)] + h(\eta,\varkappa(\eta)\overline{x}) \\ + \int_0^{\eta}(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)h(\tau,\varkappa(\tau)\overline{x})d\tau \\ + \int_0^{\eta}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)z_n(\tau)d\tau \\ + \sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}(\eta_i^-)), \eta \in J, \end{cases} \quad (34)$$

and $z_n(\eta) \in F_n\ (\eta,\varkappa(\eta)\overline{x}), n \geq 1$. It yields, from (ii) of Lemma 8, that for almost $\eta \in J$,

$$z_n(\eta) \in \overline{co}F(\eta,\{y \in \Theta : ||y - \varkappa(\eta)\overline{x}|| \leq 3^{1-n}\}), n \geq 1,$$

which implies that $f(\eta) \in F(\eta,\varkappa(\eta)\overline{x})$, for *a.e.* $\eta \in J$. Moreover, using the fact that $\mathfrak{K}_2(\eta)(\eta > 0)$ is continuous, and taking the limit as $n \to \infty$ in (34), one gets

$$\overline{x}(\eta) = \begin{cases} \psi(\eta), \eta \in [-r,0], \\ \mathfrak{K}_1(\eta)[\psi(0) - h(0,\psi)] + h(\eta,\varkappa(\eta)\overline{x}) \\ + \int_0^{\eta}(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)h(\tau,\varkappa(\tau)\overline{x})d\tau \\ + \int_0^{\eta}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f(\tau)d\tau \\ + \sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(\overline{x}(\eta_i^-)), \eta \in J. \end{cases}$$

This means that $\overline{x} \in \sum_{\psi}^{F}[-r,b]$. □

To prove our main results, we need the next lemma.

Lemma 7 ([19], Lemma 4.5)**.** *Assume that (X,d) and (Y,ρ) are two metric spaces. Then, if $f:(M,d) \to (Y,\rho)$ is locally Lipschitz, then it is Lipschitz on all subsets of X that are compact.*

Theorem 4. *Under the assumptions of Theorem 2, the set $\sum_{\psi}^{F}[-r,b]$ is an R_δ-set in H provided that $rd_1||A^{-\gamma}|| < 1$.*

Proof. Using Lemma 4 and Theorems 1–3, we only need to prove that $\sum_{\psi}^{F_n}[-r,b]$, where $n \geq N_0$ is contractible. Assume that $n \in \mathbb{N}$ and $n \geq N_0$. Consider the following fractional neutral impulsive semilinear:

$$\begin{cases} {}^cD^{\alpha}_{0,\eta}[x(\eta) - h(\eta,\varkappa(\eta)x)] = Ax(\eta) + g_n(\eta,\varkappa(\eta)x),\ a.e.\ \eta \in [0,b] - \{\eta_1,\ldots,\eta_m\}, \\ I_i(x(\eta_i^-)) = x(\eta_i^-) - x(\eta_i^+), i = 1,\ldots,m, \\ x(\eta) = \psi(\eta), \eta \in [-r,0]. \end{cases} \quad (35)$$

Using Lemma 6 and Remark 3, $g_n(.,u)$ is measurable, and for $\eta \in J, a.e., g_n(\eta,.)$ is continuous. Since the multi-valued F satisfies $(F_2)^*$and (F_3), then, following the arguments employed in the proof of Theorem 2, the fractional differential Equation (35) has a mild solution $\overline{y} \in \sum_{\psi}^{F_n}[-r,b]$ satisfying the following integral equation:

$$\overline{y}(\eta) = \begin{cases} \psi(\eta), \eta \in [-r,0], \\ \mathfrak{K}_1(\eta)[\psi(0) - h(0,\psi)] + h(\eta, \varkappa(\eta)\overline{y}) \\ + \int_0^\eta (\eta-\tau)^{\alpha-1} A\mathfrak{K}_2(\eta-\tau) h(\tau, \varkappa(\tau)\overline{y}) d\tau \\ + \int_0^\eta (\eta-\tau)^{\alpha-1} \mathfrak{K}_2(\eta-\tau) g_n(\eta, \varkappa(\eta)\overline{y}) d\tau \\ + \sum_{0<\eta_i<\eta} \mathfrak{K}_1(\eta-\eta_i) I_i(\overline{y}(\eta_i^-)), \eta \in J. \end{cases} \tag{36}$$

Next, we show that the solution is unique. Assume that $\overline{x} \in \sum_{\psi}^{F_n}[-r,b]$ is another mild solution for (35). Then,

$$\overline{x}(\eta) = \begin{cases} \psi(\eta), \eta \in [-r,0], \\ \mathfrak{K}_1(\eta)[\psi(0) - h(0,\psi)] + h(\eta, \varkappa(\eta)\overline{x}) \\ + \int_0^\eta (\eta-\tau)^{\alpha-1} A\mathfrak{K}_2(\eta-\tau) h(\tau, \varkappa(\tau)\overline{x}) d\tau \\ + \int_0^\eta (\eta-\tau)^{\alpha-1} \mathfrak{K}_2(\eta-\tau) g_n(\eta, \varkappa(\eta)\overline{x}) d\tau \\ + \sum_{0<\eta_i<\eta} \mathfrak{K}_1(\eta-\eta_i) I_i(\overline{x}(\eta_i^-)), \eta \in J. \end{cases} \tag{37}$$

Let $\eta \in [0,\eta_1]$ be fixed. Due to (6), (7), (11) (36) and (37), it yields

$$\begin{aligned} & ||\overline{y}(\eta) - \overline{x}(\eta)|| \\ \leq\ & ||h(\eta, \varkappa(\eta)\overline{y}) - h(\eta, \varkappa(\eta)\overline{x})|| \\ & + || \int_0^\eta (\eta-\tau)^{\alpha-1} A\mathfrak{K}_2(\eta-\tau)(h(\tau,\varkappa(\tau)\overline{y}) - h(\tau,\varkappa(\tau)\overline{x}))d\tau|| \\ & + || \int_0^\eta (\eta-\tau)^{\alpha-1} \mathfrak{K}_2(\eta-\tau)(g_n(\tau,\varkappa(\tau)\overline{y}) - g_n(\tau,\varkappa(\tau)\overline{x}))d\tau|| \\ \leq\ & ||A^{-\gamma}|| ||A^{\gamma} h(\eta,\varkappa(\eta)\overline{y}) - A^{\gamma} h(\eta, \varkappa(\eta)\overline{x})|| \\ & + \int_0^\eta (\eta-\tau)^{\alpha-1} ||A^{1-\gamma}\mathfrak{K}_2(\eta-\tau)||\ ||A^{\gamma} h(\tau,\varkappa(\tau)\overline{y}) - A^{\gamma} h(\tau,\varkappa(\tau)\overline{x}))||d\tau \\ & + \frac{M}{\Gamma(\alpha)} \int_0^\eta (\eta-\tau)^{\alpha-1} ||g_n(\tau,\varkappa(\tau)\overline{y}) - g_n(\tau,\varkappa(\tau)\overline{x}))||d\tau. \\ \leq\ & d_1||A^{-\gamma}||\ ||\varkappa(\eta)\overline{y} - \varkappa(\eta)\overline{x}||_{\Theta} \\ & + d_1||A^{-\gamma}|| \frac{\alpha C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma))} \int_0^\eta (\eta-\tau)^{\alpha\gamma-1} ||\varkappa(\tau)\overline{y} - \varkappa(\tau)\overline{x}||_{\Theta} d\tau \\ & + \frac{M}{\Gamma(\alpha)} \int_0^\eta (\eta-\tau)^{\alpha-1} ||g_n(\tau,\varkappa(\tau)\overline{y}) - g_n(\tau,\varkappa(\tau)\overline{x}))||d\tau. \end{aligned} \tag{38}$$

Now, from Lemma 5, the function $\tau \to \varkappa(\tau)\overline{x}$ is continuous from $[0,\eta_1]$ to Θ and, hence, the subset $Z_{\overline{x}} = \{\varkappa(\tau)\overline{x} : \tau \in [0,\eta_1]\}$ is compact in Θ. Similarly, the set $Z_{\overline{y}} = \{\varkappa(\tau)\overline{y} : \tau \in [0,\eta_1]\}$ is compact in Θ and, therefore, the set $Z_{\overline{x},\overline{y}} = Z_{\overline{x}} \cup Z_{\overline{y}}$ is compact in Θ, and consequently, $[0,\eta_1] \times Z_{\overline{x},\overline{y}}$ is compact in $[0,\eta_1] \times \Theta$. Thus, by (iv) in Lemma 6 and Lemma 7, there exists $c_{\eta_1} > 0$, for which the estimate

$$||g_n(\tau,\varkappa(\tau)\overline{y}) - g_n(\tau,\varkappa(\tau)\overline{x})|| \leq c_{\eta_1}||\varkappa(\tau)\overline{y} - \varkappa(\tau)\overline{x}||_{\Theta},$$

holds for $\tau \in J$. Therefore, from (38), it yields

$$\begin{aligned} & ||\overline{x}(\eta)-\overline{y}(\eta)|| \\ \leq\ & d_1||A^{-\gamma}||\ ||\varkappa(\eta)\overline{y}-\varkappa(\eta)\overline{x}||_\Theta \\ & +d_1||A^{-\gamma}||\frac{\alpha C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma))}\int_0^\eta(\eta-\tau)^{\alpha\gamma-1}||\varkappa(\tau)\overline{y}-\varkappa(\tau)\overline{x}||_\Theta d\tau \\ & +\frac{Mc_{\eta_1}}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}||\varkappa(\tau)\overline{y}-\varkappa(\tau)\overline{x}||_\Theta d\tau. \end{aligned}$$

Note that when $\tau \in [0,\eta]$, we have

$$\begin{aligned} ||\varkappa(\tau)\overline{y}-\varkappa(\tau)\overline{x}||_\Theta &= \int_{-r}^0||\overline{y}(\tau+\theta)-\overline{x}(\tau+\theta)||d\theta \\ &\leq r\sup_{\delta\in[0,\tau]}||\overline{y}(\delta)-\overline{x}(\delta)||. \end{aligned}$$

It yields

$$\begin{aligned} & ||\overline{x}(\eta)-\overline{y}(\eta)|| \\ \leq\ & d_1||A^{-\gamma}||\ ||\varkappa(\eta)\overline{y}-\varkappa(\eta)\overline{x}||_\Theta \\ & +rd_1||A^{-\gamma}||\frac{\alpha C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma))}\int_0^\eta(\eta-\tau)^{\alpha\gamma-1}\sup_{\delta\in[a,\tau]}||\overline{y}(\delta)-\overline{x}(\delta)||d\tau \\ & +\frac{rMc_{\eta_1}}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}\sup_{\delta\in[0,\tau]}||\overline{y}(\delta)-\overline{x}(\delta)||d\tau. \end{aligned}$$

Since $\overline{x}$ and $\overline{y}$ are continuous on $[0,\eta]$, there is $\rho \in [0,\eta]$ with $||\overline{x}(\rho)-\overline{y}(\rho)|| = \sup_{\delta\in[0,\eta]}||\overline{x}(\delta)-\overline{y}(\delta)||$. Then,

$$\begin{aligned} & \sup_{\delta\in[0,\eta]}||\overline{x}(\delta)-\overline{y}(\delta)|| = ||\overline{x}(\rho)-\overline{y}(\rho)|| \\ \leq\ & d_1||A^{-\gamma}||\ ||\varkappa(\rho)\overline{y}-\varkappa(\rho)\overline{x}||_\Theta \\ & +rd_1||A^{-\gamma}||\frac{\alpha C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma))}\int_0^\rho(\rho-\tau)^{\alpha\gamma-1}\sup_{\delta\in[0,\tau]}||\overline{y}(\delta)-\overline{x}(\delta)||d\tau \\ & +\frac{rMc_{\eta_1}}{\Gamma(\alpha)}\int_0^\rho(\rho-\tau)^{\alpha-1}\sup_{\delta\in[0,\tau]}||\overline{y}(\delta)-\overline{x}(\delta)||d\tau \\ \leq\ & rd_1||A^{-\gamma}||\sup_{\delta\in[0,\eta]}||\overline{x}(\delta)-\overline{y}(\delta)|| \\ & +rd_1||A^{-\gamma}||\frac{\alpha C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma))}\int_a^\rho(\rho-\tau)^{\alpha\gamma-1}\sup_{\delta\in[a,\tau]}||\overline{y}(\delta)-\overline{x}(\delta)||d\tau \\ & +\frac{rMc_{\eta_1}}{\Gamma(\alpha)}\int_a^\rho(\rho-\tau)^{\alpha-1}\sup_{\delta\in[a,\tau]}||\overline{y}(\delta)-\overline{x}(\delta)||d\tau. \end{aligned}$$

Since $rd_1||A^{-\gamma}|| < 1$, the last relations lead to

$$\begin{aligned} & \sup_{\delta\in[0,\eta]}||\overline{x}(\delta)-\overline{y}(\delta)|| \\ \leq\ & \frac{1}{1-rd_1||A^{-\gamma}||}[\int_0^\rho(\rho-\tau)^{\alpha\gamma-1}d_1||A^{-\gamma}||\frac{r\alpha C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma))} \\ & +\int_0^\rho(\rho-\tau)^{\alpha-1}\frac{rMc_V}{\Gamma(\alpha)}]\sup_{\delta\in[0,\tau]}||\overline{y}(\delta)-\overline{x}(\delta)||d\tau. \end{aligned}$$

Using the generalized Gronwall inequality [42], one has $\sup_{\delta\in[0,\eta]}||\overline{x}(\delta)-\overline{y}(\delta)||=0$. Since $\eta\in[0,\eta_1]$ is arbitrary, we conclude that $\overline{x}=\overline{y}$ on $[0,\eta_1]$.

Next, let $\eta\in[\eta_1,\eta_2]$ be fixed. Note that $x(\eta_1^-)=y(\eta_1^-)$. Then,

$$\begin{aligned}
&||\overline{y}(\eta)-\overline{x}(\eta)|| \\
\leq\ & ||h(\eta,\varkappa(\eta)\overline{y})-h(\eta,\varkappa(\eta)\overline{x})||_\Theta \\
&+||\int_{\eta_1}^{\eta}(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)(h(\tau,\varkappa(\tau)\overline{y})-h(\tau,\varkappa(\tau)\overline{x}))d\tau|| \\
&+||\int_{\eta_1}^{\eta}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)(g_n(\tau,\varkappa(\tau)\overline{y})-g_n(\tau,\varkappa(\tau)\overline{x}))d\tau|| \\
\leq\ & d_1||A^{-\gamma}||\ ||\varkappa(\eta)\overline{y}-\varkappa(\eta)\overline{x}||_\Theta \\
&+d_1||A^{-\gamma}||\frac{\alpha C_{1-\gamma}\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma))}\int_a^{\eta}(\eta-\tau)^{\alpha\gamma-1}||\varkappa(\tau)\overline{y}-\varkappa(\tau)\overline{x}||_\Theta d\tau \\
&+\frac{M}{\Gamma(\alpha)}\int_{\eta_1}^{\eta}(\eta-\tau)^{\alpha-1}||g_n(\tau,\varkappa(\tau)\overline{y})-g_n(\tau,\varkappa(\tau)\overline{x}))||d\tau.
\end{aligned}$$

By repeating the arguments employed above, we get $\overline{x}=\overline{y}$ on $[\eta_1,\eta_2]$. Continuing with the same processes, we arrive to $\overline{x}=\overline{y}$ on J.

Next, we prove that $\sum_{\psi}^{F_n}[-r,b]$ is homotopically equivalent to $\overline{y}$. To this end, we define a continuous function $Z_n:[0,1]\times\sum_{\psi}^{F_n}[-r,b]\to\sum_{\psi}^{F_n}[-r,b]$, where $Z_n\ (0,\widetilde{x})=\widetilde{x}$ and $(1,\widetilde{x})=y$. Assume $(\lambda,\widetilde{x}\ )\in[0,1]\times\sum_{\psi}^{F_n}[-r,b]$ is fixed. Then, there exists a $f\in\tau^1_{F_n(.,\varkappa(.)\widetilde{x})}$ such that

$$\widetilde{x}(\eta)=\begin{cases}\psi(\eta),\eta\in[-r,0],\\ \mathfrak{K}_1(\eta)[\psi(0)-h(0,\psi)]+h(\eta,\varkappa(\eta)\widetilde{x})\\ +\int_0^{\eta}(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)h(\tau,\varkappa(\tau)\widetilde{x})d\tau\\ +\int_0^{\eta}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)f(\tau)d\tau\\ +\sum_{0<\eta_i<\eta}\mathfrak{K}_1(\eta-\eta_i)I_i(\widetilde{x}(\eta_i^-)),\eta\in J.\end{cases}\tag{39}$$

Consider the partition $\{\ 0,\frac{1}{m+1},\frac{2}{m+1},\dots,\frac{m+1}{m+1}\}$ for $J=[0,1]$. We consider the following cases:

(i) $\lambda\in[0,\frac{1}{m+1}]$. Put $a_\lambda^1=\eta_{m+1}-\lambda\ (m+1)(\eta_{m+1}-\eta_m)$. The following fractional neutral differential inclusion is a result of the above discussion:

$$\begin{cases}{}^cD^{\alpha}_{a_\lambda^1,\eta}[x(\eta)-h(\eta,\varkappa(\eta)x)]=Ax(\eta)+g_n(\eta,\varkappa(\eta)x),\ a.e.\ \eta\in[a_{\lambda,1},b],\\ x(\eta)=\widetilde{x}(\eta),\eta\in[-r,a_\lambda^1],\end{cases}$$

has a unique mild solution $x_\lambda^1\in\sum_{\psi}^{F_n}[-r,b]$ satisfying the next integral equation:

$$x_\lambda^1(\eta)=\begin{cases}\widetilde{x}(\eta),\ \eta\in[-r,a_\lambda^1],\\ \mathfrak{K}_1(\eta-a_\lambda^1)[\widetilde{x}(a_\lambda^1)-h(a_\lambda^1,\varkappa(a_\lambda^1)\widetilde{x}(a_\lambda^1)]\\ +h(\eta,\varkappa(\eta)x_\lambda^1(\eta))\\ +\int_{a_\lambda^1}^{\eta}(\eta-\tau)^{\alpha-1}A\mathfrak{K}_2(\eta-\tau)h(\tau,\varkappa(\tau)x_\lambda^1(\eta))d\tau\\ +\int_{a_\lambda^1}^{\eta}(\eta-\tau)^{\alpha-1}\mathfrak{K}_2(\eta-\tau)g_n(\eta,\varkappa(\eta)x_\lambda^1)d\tau,\eta\in[a_{\lambda,1},b].\end{cases}\tag{40}$$

Note that $x_0^1(\eta)=\widetilde{x}(\eta);\ \eta\in[-r,b]$.

(ii) $\lambda\in(\frac{1}{m+1},\frac{2}{m+1}]$.Put $a_\lambda^2=\eta_m-\ (m+1)(\lambda-\frac{1}{m+1})(\eta_m-\eta_{m-1})$. Again, the following fractional neutral differential inclusion:

$$\begin{cases}{}^cD^{\alpha}_{a_\lambda^2,\eta}[x(\eta)-h(\eta,\varkappa(\eta)x)]=Ax(\eta)+g_n(\eta,\varkappa(\eta)x),\ a.e.\ \eta\in[a_\lambda^2,b]-\{\eta_m\},\\ I_m(x(\eta_m^-))=x(\eta_m^-)-x(\eta_m^+),\\ x(\eta)=\widetilde{x}(\eta),\eta\in[-r,a_\lambda^2],\end{cases}$$

has a unique mild solution $x_\lambda^2 \in \sum_\psi^{F_n}[-r,b]$ and

$$x_\lambda^2(\eta) = \begin{cases} \widetilde{x}(\eta),\ \eta \in [-r, a_\lambda^2], \\ \mathfrak{K}_1(\eta - a_\lambda^2)[\widetilde{x}(a_\lambda^2) - h(a_{\lambda,1}, \varkappa(a_\lambda^2)\widetilde{x}(a_\lambda^2)] \\ +h(\eta, \varkappa(\eta)x_\lambda^2(\eta)) \\ + \int_{a_\lambda^2}^{\eta} (\eta-\tau)^{\alpha-1} A\mathfrak{K}_2(\eta-\tau) h(\tau, \varkappa(\tau)x_\lambda^2(\eta))d\tau \\ + \int_{a_\lambda^2}^{\eta} (\eta-\tau)^{\alpha-1} \mathfrak{K}_2(\eta-\tau) g_n(\eta, \varkappa(\eta)x_\lambda^2)d\tau \\ + \sum_{a_\lambda^2 < \eta_i < \eta} \mathfrak{K}_1(\eta-\eta_i) I_i(x_\lambda^2(\eta_i^-)), \eta \in [a_\lambda^2, b]. \end{cases}$$

We continue up to $m+1-$step. That is $\lambda \in (\frac{m}{m+1}, 1]$ and put $a_\lambda^{m+1} = \eta_1 - (m+1)(\lambda - \frac{m}{m+1})\eta_1$. Let $x_\lambda^{m+1} \in \sum_\psi^{F_n}[-r,b\ ]$ be the unique mild solution for the impulsive fractional neutral differential inclusion:

$$\begin{cases} {}^cD^{\alpha}_{a_\lambda^{m+1},\eta}[x(\eta) - h(\eta, \varkappa(\eta)x)] = Ax(\eta) + g_n(\eta, \varkappa(\eta)x),\ a.e.\ \eta \in [a_\lambda^{m+1}, b] - \{\eta_1, \eta_2, \dots \eta_m\}, \\ I_i(x(\eta_i^-)) = x(\eta_i^-) - x(\eta_i^+), i = 1, 2, \dots, m \\ x(\eta) = \widetilde{x}(\eta), \eta \in [-r, a_\lambda^{m+1}]. \end{cases}$$

Then,

$$x_\lambda^{m+1}(\eta) = \begin{cases} \widetilde{x}(\eta),\ \eta \in [-r, a_\lambda^{m+1}], \\ \mathfrak{K}_1(\eta)[\widetilde{x}(a_\lambda^{m+1}) - h(a_{\lambda,1}, \varkappa(a_\lambda^{m+1})\widetilde{x}(a_\lambda^{m+1})] \\ +h(\eta, \varkappa(\eta)x_\lambda^{m+1}) \\ + \int_{a_\lambda^{m+1}}^{\eta} (\eta-\tau)^{\alpha-1} A\mathfrak{K}_2(\eta-\tau) h(\tau, \varkappa(\tau)x_\lambda^{m+1}(\eta))d\tau \\ + \int_{a_\lambda^{m+1}}^{\eta} (\eta-\tau)^{\alpha-1} \mathfrak{K}_2(\eta-\tau) g_n(\eta, \varkappa(\eta)x_\lambda^{m+1})d\tau \\ + \sum_{a_\lambda^{m+1} < \eta_i < \eta} \mathfrak{K}_1(\eta-\eta_i) I_i(x_\lambda^{m+1}(\eta_i^-)), \eta \in [a_\lambda^{m+1}, b]. \end{cases} \tag{41}$$

Note that $a_1^{m+1} = 0$ and $x_1^{m+1} = y$. Now, we define Z_n at $(\lambda, \widetilde{x})$ as

$$Z_n(\lambda, \widetilde{x}) = \begin{cases} x_\lambda^1, \quad \text{if } \lambda \in [0, \frac{1}{m+1}], \\ x_\lambda^2, \text{ if } \lambda \in (\frac{1}{m+1}, \frac{2}{m+1}], \\ . \\ . \\ . \\ x_\lambda^{m+1}, \quad \text{if } \lambda \in (\frac{m}{m+1}, 1]. \end{cases} \tag{42}$$

Therefore, $Z_n(0, \widetilde{x}) = x_\lambda^1 = \widetilde{x}$ and $Z_n(1, \widetilde{x}) = x_1^{m+1} = y$.

It remains to clarify the continuity of Z_n. Let $(\lambda, u\ ), (\varrho, v) \in [0,1] \times \sum_\psi^{F_n}[-r,b]$. Let $\lambda = \varrho = 0$. Then, by (42), $\lim_{u \to v} Z_n(\lambda, u) = \lim_{u \to v} u = v = Z_n(\varrho, v)$. Let $\lambda, \varrho \in (0, \frac{1}{m+1}]$. So, $Z_n(\lambda, u) = \overline{u}_\lambda^1$ and $Z_n(\lambda, v) = \overline{v}_\mu^1$, where

$$\overline{u}_\lambda^1(\eta) = \begin{cases} \widetilde{x}(\eta),\ \eta \in [-r, a_\lambda^1], \\ \mathfrak{K}_1(\eta - a_\lambda^1)[\widetilde{x}(a_\lambda^1) - h(a_\lambda^1, \varkappa(a_\lambda^1)\widetilde{x}(a_\lambda^1)] \\ +h(\eta, \varkappa(\eta)\overline{u}_\lambda^1(\eta)) \\ + \int_{a_\lambda^1}^{\eta} (\eta-\tau)^{\alpha-1} A\mathfrak{K}_2(\eta-\tau) h(\tau, \varkappa(\tau)\overline{u}_\lambda^1(\eta))d\tau \\ + \int_{a_\lambda^1}^{\eta} (\eta-\tau)^{\alpha-1} \mathfrak{K}_2(\eta-\tau) g_n(\eta, \varkappa(\eta)\overline{u}_\lambda^1)d\tau, \eta \in [a_{\lambda,1}, b], \end{cases} \tag{43}$$

and

$$\overline{v}_{\mu}^{1}(\eta) = \begin{cases} \widetilde{x}(\eta),\ \eta \in [-r, a_{\mu}^{1}], \\ \mathfrak{K}_1(\eta - a_{\lambda}^{1})[\widetilde{x}(a_{\lambda}^{1}) - h(a_{\mu}^{1}, \varkappa(a_{\mu}^{1})\widetilde{x}(a_{\mu}^{1})] \\ +h(\eta, \varkappa(\eta)\overline{v}_{\mu}^{1}(\eta)) \\ + \int_{a_{\mu}^{1}}^{\eta} (\eta-\tau)^{\alpha-1} A\mathfrak{K}_2(\eta-\tau) h(\tau, \varkappa(\tau)\overline{v}_{\mu}^{1}(\eta)) d\tau \\ + \int_{a_{\mu}^{1}}^{\eta} (\eta-\tau)^{\alpha-1} \mathfrak{K}_2(\eta-\tau) g_n(\eta, \varkappa(\eta)\overline{v}_{\mu}^{1}) d\tau, \eta \in [a_{\mu}^{1}, b], \end{cases} \tag{44}$$

$a_{\lambda}^{1} = b - \mu(m+1)(b-\tau_m)$ and $a_{\mu}^{1} = b - \mu(m+1)(b-\tau_m)$. Obviously, $\lim_{\lambda \to \mu} a_{\lambda}^{1} = a_{\mu}^{1}$ and, hence, by (43) and (44), and by arguing as above, we get

$$\lim_{\substack{\lambda \to \mu \\ u \to v}} Z_n(\lambda, u) = Z_n(\mu, v),$$

which implies the continuity of $Z_n(.,.)$, when $\lambda \in [0, \frac{1}{m+1}]$. Similarly, we can show the continuity of Z_n and consequently, $\sum_{\psi}^{F_n}[-r, b]$ is contractible. This completes the proof. □

5. Example

Example 1. *Assume that* $E = L^2([0,\pi], \mathbb{R})$, $J = [0,1]$, $r = \frac{1}{2}$, $m = 1$, $\eta_0 = 0$ *and* $\eta_1 = \frac{1}{2}$, $\eta_2 = 1$. *For any* $x : J \to E = L^2([0,\pi], \mathbb{R})$, *we denote by* $x(\eta, \omega); \eta \in J, \omega \in [0,\pi]$ *the value of* $x(\eta)$ *at* ω. *Let* $A : D(A) \subseteq L^2[0,\pi] \to L^2[0,\pi]$, $Ax(\eta,\omega) := -\frac{\partial^2}{\partial\omega^2} x(\eta,\omega)$ *and domain* A *be defined as*

$$\begin{aligned} D(A) &= \{x \in L^2[0,\pi] : x, x' \text{ are absolutely continuous, } x'' \in L^2[0,1], \\ x(\eta,0) &= x(\eta,\pi) = 0\}. \end{aligned}$$

Using [37], there is a compact analytic semi-group $\{\Upsilon(\eta) : \eta \geq 0\}$ generated by A and

$$Ax = \sum_{n=1}^{\infty} n^2 < x,\ x_n > x_n, x \in D(A), \tag{45}$$

where $x_n(y) = \sqrt{2}\ \sin ny, n = 1, 2, \ldots$ is the orthonormal set of eigenvalues of A. In addition, for all $x \in L^2[0,1]$, one gets

$$\Upsilon(\eta)(x) = \sum_{n=1}^{\infty} e^{-n^2\eta} < x,\ x_n > x_n.$$

So, $M = \sup\{||\Upsilon(\eta)|| : \eta \geq 0\} = 1$. Furthermore, for each $x \in L^2([0,\pi], \mathbb{R})$,

$$A^{\frac{-1}{2}} x = \sum_{n=1}^{\infty} \frac{1}{n} < x,\ x_n > x_n.$$

$$A^{\frac{1}{2}} x = \sum_{n=1}^{\infty} n < x,\ x_n > x_n,$$

and $||A^{\frac{-1}{2}}|| = 1$. The domain of $A^{\frac{1}{2}}$ is defined as

$$D(A^{\frac{1}{2}}) = \{x \in L^2([0,\pi], \mathbb{R}) : \sum_{n=1}^{\infty} n < x,\ x_n > x_n \in L^2([0,\pi], \mathbb{R})\}.$$

Let $h : J \times \Theta \to E$ be such that

$$h(\eta, u) := A^{\frac{-1}{2}} (\int_{-r}^{0} \lambda u(\theta) d\theta), \tag{46}$$

where $\lambda > 0$. We have

$$\begin{aligned} ||A^{\frac{1}{2}}h(\eta,u_1) - A^{\frac{1}{2}}h(\eta,u_2)||_E &\leq \lambda \,||\int_{-r}^{0}(u_1(\theta)-u_2(\theta))d\theta|| \\ &\leq \lambda \int_{-r}^{0}||u_1(\theta)-u_2(\theta)||d\theta \\ &\leq \lambda||u_1-u_2||_\Theta, \end{aligned}$$

and

$$||A^\gamma h(\eta,u)|| \leq \lambda \,||\int_{-r}^{0}(u(\theta)d\theta|| \leq \lambda||u||_\Theta.$$

Then, (10) and (11) are satisfied with $d_1 = d_2 = \lambda$.

Let Λ be a convex compact subset in E, $\sup\{||z|| : z \in Z\} = \varrho$ and $\kappa > 0$. Define $F : J \times \Theta \to 2^{L^2[0,\pi]}$ by

$$F(\eta,u) := \frac{e^{-\kappa\eta}||u||}{\varrho}\Lambda. \tag{47}$$

We have

$$||F(\eta,u)|| = \sup\{||\frac{e^{\kappa\eta}||u||}{\varrho}z : z \in \Lambda\} \leq e^{\kappa\eta}; \eta \in J.$$

Moreover, for any bounded subset $D \subset \Theta$, we have $F(\eta,D) \subseteq \varsigma\frac{e^{\kappa\eta}}{\varrho}\Lambda$, where $\varsigma = \sup\{||u|| : u \in D\}$ and, hence, $\chi_E(F(\eta,D)) = 0$. Then, F satisfies $(HF1), (HF2)^*$and $(HF3)$ with $\xi(\eta) = e^{-\kappa\eta}, \beta(\eta) = 0; \eta \in J$.

Next, let

$$I : E \to E, I_i(x) := \sigma\, proj_\Lambda x, \tag{48}$$

where σ is a positive number. Obviously, I verifies (HI) with $\varsigma_i = 0\,; i = 1,2,\ldots$.

Therefore, by applying Theorems 1 and 4, the set of solutions for the following fractional neutral impulsive semilinear differential inclusions with delay:

$$\begin{cases} {}^cD^\alpha_{0,\eta}[x(\eta) - h(\eta,\varkappa(\eta)x)] \\ \in -\frac{\partial^2}{\partial\omega^2}x(\eta,\omega) + F(\eta,\varkappa(\eta)x),\ a.e.\ \eta \in [0,1] - \{\frac{1}{2},1\}, \\ I_i x(\eta_i^-,\omega) = x(\eta_i^-,\omega) - x(\eta_i^+,\omega), i = 1,2, \omega \in [0,\pi], \\ x(\eta,\omega) = \psi(\eta,\omega), \eta \in [-r,0], \eta \in [0,1] - \{\frac{1}{2},1\}, \end{cases} \tag{49}$$

is a not empty, compact and an R_δ-set provided that

$$\lambda(1 + \frac{C_{1-\gamma}\Gamma(\frac{3}{2})}{\Gamma(1+\frac{\alpha}{2})}) < 1, \tag{50}$$

and

$$\frac{\lambda}{2} + 2\lambda\frac{C_{1-\gamma}\Gamma(\frac{3}{2})}{\Gamma(1+\frac{\alpha}{2})} + \frac{1}{2\Gamma((\alpha)}(\frac{P-1}{\alpha P-1})^{\frac{P-1}{P}}\|\xi\|_{L^P_{(J,,\mathbb{R}^+)}} + \sigma < 1, \tag{51}$$

where $F, h\ I$ are given by (45)–(47). By choosing λ and σ small enough and κ large enough, we arrive to (50) and (51).

Example 2. *Let $J, E, A, r, \eta_0, \eta_1, \eta_2\ \Lambda$, and ϱ be as in Example (1) and $\theta \in [-r,0]$ be a fixed element.*

Let $h : J \times \Theta \to E$ be such that

$$h((\eta,\varkappa(\eta)x)(\omega) := \lambda\int_0^\pi U(\omega,y)x(\theta+\eta)(\omega)dy; \omega \in [0,\pi]; \eta \in [0,1], \tag{52}$$

where $\lambda > 0$, $U : [0, \pi] \times [0, \pi] \to \mathbb{R}$ is measurable, $\int_0^\pi \int_0^\pi U(\omega, y) dy d\omega < \infty$, $\frac{\partial U(\omega,\eta)}{\partial \omega}$ is measurable, $U(0, y) = U(\pi, y) = 0, \forall y \in [0, \pi]$ and $(\int_0^\pi \int_0^\pi (\frac{\partial U(\omega,\eta)}{\partial \omega})^2 dy d\omega)^{\frac{1}{2}} < \infty$.

Next, let $F : J \times \Theta \to 2^{L^2[0,\pi]}$, $F((\eta, \varkappa(\eta)x)(\omega) = \frac{\gamma G(\eta, x(\theta+\eta)(\omega))|}{\varrho} \Lambda$, where $\gamma > 0$, $G : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Then, by choosing λ and σ small enough, one can show that h and F satisfy all assumptions of Theorems 2 (see [15,43]) and, hence, the set of mild solutions for the partial differential inclusions of impulsive neutral type with delay:

$$\begin{cases} {}^c D^\alpha_{0,\eta}[x(\eta, \omega) - \int_0^\pi U(\omega, y) x(\theta + \eta)(\omega) dy,] \\ \in -\frac{\partial^2}{\partial \omega^2} x(\eta, \omega) + \frac{G(\eta, x(\theta+\eta)(\omega))|}{\varrho} \Lambda,\ a.e.\ \eta \in [0, 1] - \{\frac{1}{2}, 1\}, \\ I_i x(\eta_i^-, \omega) = x(\eta_i^-, \omega) - x(\eta_i^+, \omega), i = 1, 2, \omega \in [0, \pi], \\ x(\eta, \omega) = \psi(\eta, \omega), \eta \in [-r, 0], \eta \in [0, 1] - \{\frac{1}{2}, 1\}, \end{cases} \tag{53}$$

is an R_δ-set.

6. Discussion

The neutral differential equations and inclusions appear in many applied mathematical sciences such as viscoelasticity, and the equations describe the distribution of heat. Since the set of mild solutions for a differential inclusion having the same initial point may not be a singleton, many authors are interested to investigate the structure of this set in a topological point of view. An important aspect of such structure is the R_δ- property, which means that the homology group of the set of mild solutions is the same as a one-point space. In the literature, there are many results on this subject but no result about the topological properties of the set of mild solutions for a fractional neutral differential inclusion generated by a non-compact semigroup in the presence of impulses and delay. As cited in the introduction, when the problem involves delay and impulses, we cannot consider the space $\mathcal{PC}([-r, b], E]$ as the space of solutions. To overcome these difficulties, a complete metric space H is introduced as the space of mild solutions. In addition, the function $\eta \to \varkappa(\eta)\overline{x}; \overline{x} \in H$ is not necessarily measurable, therefore, a norm different from the uniform convergence norm is introduced on Θ (see Equation (2)).

7. Conclusions

During the past two decades, fractional differential equations and fractional differential inclusions have gained considerable importance due to their applications in various fields, such as physics, mechanics and engineering. For some of these applications, one can see [28] and the references therein. In this paper, we have given an affirmative answer for a basic question, which is whether there exists a solution set carrying an R_δ-structure when there are impulsive effects and delay on the system, the operator families generated by the linear part lack compactness and the order is fractional. More specifically,

1. By utilizing the properties of both multivalued functions, fraction powers of operators, measures of non-compactness and analytic semi-groups, we showed that the mild solutions set for a fractional impulsive neutral semilinear differential inclusions with delay and generated by a non-compact semi-group is not empty, compact and an R_δ-set. This means that, from an algebraic topological perspective, it is equivalent to a point.
2. Our work generalizes the obtained results in [19], where Problem 1 is investigated without delay and $h \equiv 0$.
3. Our work generalizes the obtained results in [15] to the case when there are impulsive effects on the system.
4. Our technique can be used to prove that the solutions set is an R_δ-set for problems considered in [13–23,30] when it is generated by a non-compact semi-group, the order is fractional and there are impulsive effects and delay.
5. As a future work, we suggest to extend the work conducted in [24–26] to find the sufficient conditions that guarantee that the solution set is an R_δ-set.

Author Contributions: Funding acquisition, Z.A. and A.G.I.; investigation, Z.A. and A.G.I.; methodology, Z.A., A.G.I. and A.A.; writing—original draft, Z.A. and A.G.I.; writing—-review and editing, Z.A., A.G.I. and A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: This research has been funded by the Scientific Research Deanship at University of Ha'il—Saudi Arabia through project number RG-21 101.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Aissani, K.; Benchohra, M. Impulsive fractional differential inclusions with state-dependent delay. *Math. Moravica* **2019**, *23*, 97–113. [CrossRef]
2. Chen, Y.; Wang, J.R. Continuous dependence of solutions of integer and fractional order non-instantaneous impulsive equations with random impulsive and junction points. *Mathematics* **2019**, *7*, 331. [CrossRef]
3. Ibrahim, A.G. Differential Equations and inclusions of fractional order with impulse effect in Banach spaces. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 69–109. [CrossRef]
4. Liu, S.; Wang, J.R.; Shen,D.; O'Regan, D. Iterative learning control for differential inclusions of parabolic type with non-instantaneous impulses. *Appl. Math. Comput.* **2019**, *350*, 48–59. [CrossRef]
5. Wang, J.R.; Li, M.; O'Regan, D. Robustness for linear evolution equation with non-instantaneous impulsive effects. *Bull. Sci. Math.* **2020**, *150*, 102827. [CrossRef]
6. Wang, J.R.; Ibrahim, A.G.; O'Regan, D. Nonempties and compactness of the solution set for fractional evolution inclusions with of non-instantaneous impulses. *Electron. J. Differ. Equ.* **2019**, *37*, 1–17.
7. Wang, J.R.; Ibrahim, A.G.; O'Regan, D. Global attracting solutions to Hilfer fractional differential inclusions of Sobolev type with non-instantaneous impulses and nonlocal conditions. *Nonlinear Anal. Model. Control* **2019**, *24*, 775–803.
8. Wang, J.R.; Ibrahim, A.G.; O'Regan, D. Controllability of Hilfer Fractional Noninstantaneous Impulsive Semilinear Differential Inclusions with Nonlocal Conditions. *Nonlinear Anal. Model. Control* **2019**, *24*, 958–984. [CrossRef]
9. Zhou, Y.; Vijayakumar, V.; Ravichandran, C.; Murugesu, R. Controllability results for fractional order neutral functional differential inclusions with infinite delay. *Fixed Point Theory* **2017**, *18*, 773–798. [CrossRef]
10. Yang, M.; Wang, Q. Approximate controllability of Caputo fractional neutral stochastic differential inclusions with state dependent delay. *IMA J. Math. Control Inf.* **2018**, *35*, 1061–1085. [CrossRef]
11. Yan, Z.; Jia, X. Approximate controllability of fractional impulsive partial neutral integro-differential inclusions with infinite delay in Hilbert spaces. *Adv. Differ. Equ.* **2015**, 1–31. [CrossRef]
12. Chalishajar, D.; Anguraj, A.; Malar, K.; Karthikeyan, K. Study of controllability of impulsive neutral evolution integro-differential equations with state-dependent delay in Banach Spaces. *Mathematics* **2016**, *4*, 1–16. [CrossRef]
13. Gabor, G.; Grudzka, A. Structure of the solution set to impulsive functional differential inclusions on the half-line. *Nonlinear Differ. Equ. Appl.* **2012**, *19*, 609–627. [CrossRef]
14. Djebali, S.; Gorniewicz, L.; Ouahab, A. Topological structure of solution sets for impulsive differential inclusions in Fré chet spaces. *Nonlinear Anal.* **2011**, *74*, 2141–2169. [CrossRef]
15. Zhou, Y.; Peng, L. Topological properties of solution sets for partial functional evolution inclusions. *C. R. Math.* **2017**, *1*, 45–64. [CrossRef]
16. Zhou, Y.; Peng, L.; Ahmed, B.; Alsaedi, A. Topological properties of solution sets of fractional stochastic evolution inclusions. *Adv. Differ. Equ.* **2017**, *90*, 1–20. [CrossRef]
17. Zhao, Z.H.; Chang, Y.-k. Topological properties of solution sets for Sobolev type fractional stochastic differential inclusions with Poisson jumps. *Appl. Anal.* **2020**, *99*, 1373–1401. [CrossRef]
18. Beddani, M.; Hedia, B. Solution sets for fractional differential inclusions. *J. Fract. Calc. Appl.* **2019**, *10*, 273–289.
19. Wang, J.R.; Ibrahim, A.G.; O'Regan, D. Topological structure of the solution set for fractional non-instantaneous impulsive evolution inclusions. *J. Fixed Point Theory Appl.* **2018**, *20*, 20–59. [CrossRef]
20. Ouahab, A.; Seghiri, S. Nonlocal fractional differential inclusions with impulses at variable times. *Surv. Math. Its Appl.* **2019**, *14*, 307–325.
21. Ziane, M. On the Solution Set for Weighted Fractional Differential Equations in Banach Spaces. *Differ Equ. Dyn. Syst.* **2020**, *28*, 419–430. [CrossRef]
22. Zhang, L.; Zhou, Y.; Ahmad, B. Topological properties of C_0-solution set for impulsive evolution inclusions. *Bound. Value Probl.* **2018**, *2018*, 182. [CrossRef]
23. Wang, R.N.; Ma, Z.X.; Miranville, A. Topological Structure of the Solution Sets for a Nonlinear Delay. *Int. Math. Res. Not.* **2021**, *2022*, 4801–4889. [CrossRef]
24. Castaing, C.; Godet-Thobie, C.; Phung, P.D.; Truong, L.X. On fractional differential inclusions with Nonlocal boundary conditions. *Fract. Calc. Appl. Anal.* **2019**, *22*, 444–478. [CrossRef]

25. Xiang, O.; Zhu, P. Some New Results for the Sobolev-Type Fractional Order Delay Systems with Noncompact Semigroup. *J. Funct. Spaces* **2020**, *2020*. [CrossRef]
26. Zhu, P.; Xiang, Q. Topological structure of solution sets for fractional evolution inclusions of Sobolev type. *Bound. Value Probl.* **2018**, *2018*, 1–3. [CrossRef]
27. Zhou, Y.; Wang, R-N.; Peng, L. *Topological Structure of the Solution Set for Evolution Inclusions, Developments in Mathematics*; Springer: Singapore, 2017.
28. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations, North Holland Mathematics Studies*; Elsevier Science: Amsterdam, The Netherlands, 2006.
29. Burqan, A.; Saadeh, R.; Qazza, A.A. Novel numerical approach in solving fractional neutral pantograph equations via the ARA integral transform. *Symmetry* **2022**, *14*, 50. [CrossRef]
30. Ma, Y.K.; Kavitha, K.; Albalawi, W.; Shukla, A.; Nisar, K.S.; Vijayakumar, V. An analysis on the approximate controllability of Hilfer fractional neutral differential systems in Hilbert spaces. *Alex. Eng. J.* 2022, *in press*. [CrossRef]
31. Etemad, S.; Souid, M.S.; Telli, B.; Kaabar, M.; Rezapour, S. Investigation of the neutral fractional differential inclusions of Katugampola-type involving both retarded and advanced arguments via Kuratowski MNC technique. *Adv. Differ. Equ.* **2021**, *2021*, 214. [CrossRef]
32. Sindhu, T.N.; Atangana, A. Reliability analysis incorporating exponentiated inverse Weibull distribution and inverse power law. *Qual. Reliab. Eng. Int.* **2021**, *37*, 2399–2422. [CrossRef]
33. Rahman, A.; Sindhu, T.N.; Lone, S.A.; Kamal, M. Statistical inference for Burr Type X distribution using geometric process in accelerated life testing design for time censored data. *Pak. J. Stat. Oper. Res.* **2020**, *16*, 577–586. [CrossRef]
34. Shafiq, A.; Sindhu, T.N.; Al-Mdallal, Q.M. A sensitivity study on carbon nanotubes significance in Darcy–Forchheimer flow towards a rotating disk by response surface methodology. *Sci. Rep.* **2021**, *11*, 1–26. [CrossRef] [PubMed]
35. Wang, J.R.; Zhou, Y. Existence and controllability results for fractional semilinear differential inclusions. *Nonlinear Anal. Real World Appl.* **2011**, *12*, 3642–3653. [CrossRef]
36. Guedda, L. Some remarks in the study of impulsive differential equations and inclusions with delay. *Fixed Point Theory* **2011**, *12*, 349–354.
37. Pazy, A. *Semigroup of Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983.
38. Kamenskii, M.; Obukhowskii, V.; Zecca, P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*; De Gruyter Series in Nonlinear Analysis and Applications; De Gruyter: Berlin, NY, USA, 2001; Volume 7.
39. Bothe, D. Multivalued perturbation of m-accerative differential inclusions. *Israel J. Math.* **1998**, *108*, 109–138. [CrossRef]
40. Bader, K.M.; Obukhowskii, V. On some class of operator inclusions with lower semicontinuous nonlinearity: Nonlinear Analysis. *J. Jul. Schauder Cent.* **2001**, *17*, 143–156.
41. Hyman, D.H. On decreasing sequence of compact absolute Retract. *Fund. Math.* **1969**, *64*, 91–97. [CrossRef]
42. Ye, H.; Gao, J.; Ding, J.Y. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 1075–1081. [CrossRef]
43. Zhou, Y.; Jiao, F. Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.* **2010,** *59*, 1063–1077. [CrossRef]