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New Solutions of Nonlinear Dispersive Equation in Higher-Dimensional Space with Three Types of Local Derivatives

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Abstract: The aim of this paper is to use the Nucci's reduction method to obtain some novel exact solutions to the s-dimensional generalized nonlinear dispersive mK(m,n) equation. To the best of the authors' knowledge, this paper is the first work on the study of differential equations with local derivatives using the reduction technique. This higher-dimensional equation is considered with three types of local derivatives in the temporal sense. Different types of exact solutions in five cases are reported. Furthermore, with the help of the Maple package, the solutions found in this study are verified. Finally, several interesting 3D, 2D and density plots are demonstrated to visualize the nonlinear wave structures more efficiently.

Keywords: Nucci's reduction method; M-derivative; beta derivative; hyperbolic local derivative; s-dimensional generalized nonlinear dispersive mK(m,n) equation

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1. Introduction

Nonlinear partial differential equations (NPDEs) play a significant role in almost all branches of science and technology [1–6]. Solutions to these problems can describe many natural phenomena in engineering, chemistry, physics, etc. Therefore, exact solutions to Nonlinear partial differential equations is an interesting field for many researchers and there are various types of methods to find exact solutions to these problems. Additionally, there are some studies about the practical investigation of natural models. For example, in [7], significant chaotic features for different experimental conditions that are useful for the initial understanding of two-phase flow patterns in complex micro-channels, are considered. An optical system is presented to provide an innovative solution for distributed detection in microfluidics as a bridge between point-wise and full-field off-line monitoring systems [8].

Many studies have been carried out in recent years to find new solutions to these equations, using various techniques. For example, the Lie symmetry method [9–12], invariant subspace method [13–15], the exponential rational function method [16–18], the modified simple equation method [19–21], the Exp function method [22,23], the modified extended tanh-function method [24,25], and the Kudryashov method [26,27]. Different types of exact solutions are reported using these approaches. Among them, soliton-type solutions play an important role in science and engineering. N-soliton solutions for the coupling of differential equations and higher-dimensional differential equations are investigated in the literature [28,29,29–34].

One of the interesting NPDEs, which was first reported by Rosenau and Hyman [35], is the K(m,n) equation:

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \;\; m > 0, \;\; 1 < n \leq 3. \tag{1}$$

Indeed, this equation is the Korteweg–de Vries-like equation with nonlinear dispersion. The role of nonlinear dispersion in the formation of patterns in liquid drops (nuclear physics) is interpreted by the aforementioned K(m,n) equation. The very closed behavior and stability of solitary waves with compact support (compactons) to completely integrable systems were found.

A natural generalization of the K(m,n) equation is the generalized nonlinear dispersive mK(m,n) equations in a higher dimension [36,37]:

$$u^{n-1}u_t + a(u^m)_x + \sum_{i=1}^{s} \alpha_i (u^n)_{x_i x_i x_i} = 0, \;\; n \geq 1, \tag{2}$$

where α_i are constants. In [38], the bifurcation behavior of travelling wave solutions of Equation (2) by $s = 1$, along with all possible exact explicit parametric representations for periodic travelling wave solutions, solitary wave solutions, kink and anti-kink wave solutions and periodic cusp wave solutions are investigated. Moreover, a new version of Equation (2), that is, the modified K(m,n,k), is discussed in [39]. Some compacton solutions and solitary pattern solutions of mK(m,n, k) equations are reported in this paper.

In this work, we investigate analytical solutions to the $s-$dimensional mK(m,n) equation with a recently defined local derivative [40]:

$$u^{n-1}\mathfrak{D}_t u + a(u^m)_x + \sum_{i=1}^{s} \alpha_i (u^n)_{x_i x_i x_i} = 0, \;\; a, \alpha_i \in \mathbb{R}, \tag{3}$$

where the operator $\mathfrak{D}_t \in \{ {}_0^A\mathcal{D}_t^\beta, \, {}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta}, \, \mathcal{D}_h^\alpha \}$, and ${}_0^A\mathcal{D}_t^\beta$, ${}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta}$, $\mathcal{D}_h^\alpha$, are beta derivative, M-derivative and recently defined hyperbolic derivative, respectively. Moreover, for the order of fractional derivatives in Equation (3), we have $0 < \alpha \leq 1, \; \beta \in \mathbb{R}^+$, and non-linear powers m and n are non-negative constants.

The plan of the paper is organized as follows.

In Section 2, we provide some preliminaries and discussions about the definitions and basic properties of the utilized local derivatives. Section 3, which contains the main body of this research, deals with the exact solutions to the $s-$dimensional mK(m,n) equation with local derivatives in temporal direction using a novel reduction method. Finally, Section 4 contains the conclusions.

2. Preliminaries

In this section, we provided a brief discussion on three local derivatives, which are utilized in the current work. Recently, the local fractional-order derivatives absorbed the attention of many researchers in science and technology. The concept of local fractional calculus, which also is known as fractal calculus, was first proposed in [41,42]. Indeed, the proposed fractals, defined based on the Riemann–Liouville fractional derivative [43–45], were utilized to deal with the non-differentiable equations raised by science and engineering [46–49].

Firstly, we require the definition of an applicable function, namely, the Mittag–Leffler function, which plays a significant role in the fractional calculus. One- and two-parameter kinds of this function are introduced in the literature. In the current work, we need the one-parameter version, as defined by

$$E_\gamma(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + 1)},$$

where $\Gamma(.)$ is the Euler Gamma function.

The local M-derivative of order $0 < \alpha \leq 1,\ \beta \in \mathbb{R}^+$ for a real valued function ϖ, is a developed version of a traditional first-order derivative, which is defined by [50]

$$ {}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta}\varpi(t) := \lim_{\varepsilon \to 0} \frac{\varpi\big(tE_\beta(\varepsilon\, t^{-\alpha})\big) - \varpi(t)}{\varepsilon}. $$

Moreover, when the limit exists, we have

$$ {}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta}\varpi(0) := \lim_{t \to 0} {}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta}\varpi(t), $$

and the function ϖ is called $\alpha-$differentiable(w.r.t. M-derivative) on $(0, \infty)$, whenever ${}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta}\varpi(t)$ exists and is finite.

Our other utilized local derivative is the beta fractional derivatives defined in [51]:

$$ {}_0^A\mathcal{D}_t^{\beta}\varpi(t) = \lim_{\varepsilon \to 0} \frac{\varpi\Big(t + \varepsilon\,(t + \frac{1}{\Gamma(\beta)})^{1-\beta}\Big) - \varpi(t)}{\varepsilon}. $$

where $\beta \in (0,1)$ and $t > 0$. It is notable that a real function ϖ defined on $[x_0, x_f]$ is said to be $\beta-$differentiable if

$$ \lim_{t \to x_0^+} {}_0^A\mathcal{D}_t^{\beta}\varpi(t) = {}_0^A\mathcal{D}_t^{\beta}\varpi(x_0^+), $$

provided that $\lim_{t \to x_0^+} {}_0^A\mathcal{D}_t^{\beta}\varpi(t)$ exists.

Moreover, a new type of local fractional derivative was recently defined in [40]:

$$ \mathcal{D}_h^{\alpha}\varpi(t) = \lim_{\varepsilon \to 0} \frac{\varpi\Big(t + \varepsilon\, t^{\frac{1-\alpha}{2}}\, Sech((1-\alpha)t^{\frac{1+\alpha}{2}})\Big) - \varpi(t)}{\varepsilon}. $$

where $\alpha \in (0,1)$ and $t > 0$. We call this type of derivative a *hyperbolic local derivative*. It is notable that a real function ϖ defined on $[x_0, x_f]$ is said to be $\alpha-$differentiable if

$$ \lim_{t \to x_0^+} \mathcal{D}_h^{\alpha}\varpi(t) = \mathcal{D}_h^{\alpha}\varpi(x_0^+), $$

provided that $\lim_{t \to x_0^+} \mathcal{D}_h^{\alpha}\varpi(t)$ exists.

The following theorem shows some properties of these three local derivatives [40,52,53].

Theorem 1. *Let* $0 < \alpha \leq 1$, $\beta \in \mathbb{R}^+$ *and* ϖ_1, ϖ_2 *are* α*-differentiable functions. If* $\mathfrak{D}_t \in \{ {}_0^A\mathcal{D}_t^{\beta},\ {}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta},\ \mathcal{D}_h^{\alpha}\}$*, then*

- $\mathfrak{D}_t(c_1\varpi_1 + c_2\varpi_2)(t) = c_1\mathfrak{D}_t\varpi_1(t) + c_2\mathfrak{D}_t\varpi_2(t),\ \ c_1, c_2 \in \mathbb{R},$
- $\mathfrak{D}_t(\varpi_1\varpi_2)(t) = \varpi_1(t)\mathfrak{D}_t\varpi_2(t) + \varpi_2(t)\mathfrak{D}_t\varpi_1(t),$
- $\mathfrak{D}_t\Big(\frac{\varpi_1}{\varpi_2}\Big)(t) = \frac{\varpi_2(t)\mathfrak{D}_t\varpi_1(t) - \varpi_1(t)\mathfrak{D}_t\varpi_2(t)}{\varpi_2^2(t)},$
- $\mathfrak{D}_t(\lambda) = 0,\ \ \lambda \in \mathbb{R},$
- ${}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta}\varpi(t) = \frac{t^{1-\alpha}}{\Gamma(1+\beta)}\varpi'(t),\ \ {}_0^A\mathcal{D}_t^{\beta}\varpi(t) = \Big(t + \frac{1}{\Gamma(\beta)}\Big)^{1-\beta}\varpi'(t),$
  $\mathcal{D}_h^{\alpha}\varpi(t) = t^{\frac{1-\alpha}{2}}\, Sech\Big((1-\alpha)t^{\frac{1+\alpha}{2}}\Big)\varpi'(t),\ \ \varpi \in C^1.$
- ${}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta}t^{\mu} = \frac{\mu t^{\mu-\alpha}}{\Gamma(1+\beta)},\ \ {}_0^A\mathcal{D}_t^{\beta}(t^{\mu}) = \mu t^{\mu-1}\Big(t + \frac{1}{\Gamma(\beta)}\Big)^{1-\beta},$
  $\mathcal{D}_h^{\alpha}(t^{\mu}) = \mu t^{\frac{2\mu-\alpha-1}{2}}\, Sech\Big((1-\alpha)t^{\frac{1+\alpha}{2}}\Big),\ \ \mu \in \mathbb{R}.$

Moreover, since the considered mK(m,n) equation in this work is s-dimensional, we define the corresponding local derivatives as follows:

$$ {}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta} u(t, x_1, \ldots, x_s) = \lim_{\varepsilon \to 0} \frac{u\big(tE_\beta(\varepsilon\, t^{-\alpha}), x_1, \ldots, x_s\big) - u(t, x_1, \ldots, x_s)}{\varepsilon}, $$

$$ {}_0^A\mathcal{D}_t^{\beta} u(t, x_1, \ldots, x_s) = \lim_{\varepsilon \to 0} \frac{u\Big(t + \varepsilon\,(t + \frac{1}{\Gamma(\beta)})^{1-\beta}, x_1, \ldots, x_s\Big) - u(t, x_1, \ldots, x_s)}{\varepsilon}, $$

and

$$ \mathcal{D}_{h,t}^{\alpha} u(t, x_1, \ldots, x_s) = \lim_{\varepsilon \to 0} \frac{u\Big(t + \varepsilon\, t^{\frac{1-\alpha}{2}}\, Sech((1-\alpha)t^{\frac{1+\alpha}{2}}), x_1, \ldots, x_s\Big) - u(t, x_1, \ldots, x_s)}{\varepsilon}. $$

Besides, from the chain rule of Theorem 1, one can write

$$ \begin{aligned} {}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta} u(t, x_1, \ldots, x_s) &= \frac{t^{1-\alpha}}{\Gamma(1+\beta)} \frac{\partial u(t, x_1, \ldots, x_s)}{\partial t}, \\ {}_0^A\mathcal{D}_t^{\beta} u(t, x_1, \ldots, x_s) &= \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{\partial u(t, x_1, \ldots, x_s)}{\partial t}, \\ \mathcal{D}_h^{\alpha} u(t, x_1, \ldots, x_s) &= t^{\frac{1-\alpha}{2}}\, Sech\Big((1-\alpha)t^{\frac{1+\alpha}{2}}\Big) \frac{\partial u(t, x_1, \ldots, x_s)}{\partial t}. \end{aligned} $$

3. Nucci's Reduction Method

In this section, we consider the nonlinear $s-$dimensional mK(m,n) equation with the mentioned temporal local derivative. By using some forthcoming transformations, this equation can be converted into a nonlinear ordinary differential equation. Then, using Nucci's reduction technique, different types of exact solution can be extracted. All computations are accomplished by the Maple software.

Among the existence methods used to obtain exact solutions to differential equations, most of them extract special solutions such as hyperbolic solutions, soliton solutions, exponential solutions, etcetera. However, there are some analytical approaches, which can obtain different types of exact solutions, such as Lie symmetry method, invariant subspace method and the one utilized by us, the Nucci's reduction method [54–57]. This point motivated our use of the reduction method to obtain exact solutions to Equation (3) with different differential operators.

Let us assume the $s-$dimensional mK(m,n) Equation (3) with three local derivatives and the following corresponding transformations:

$$ \mathfrak{W}(\theta) = u(t, x_1, \cdots, x_s), \;\; \theta = \sum_{i=1}^{s} k_i x_i - \frac{c}{\alpha}\Gamma(\beta+1)t^{\alpha}, \tag{4} $$

$$ \mathfrak{W}(\theta) = u(t, x_1, \cdots, x_s), \;\; \theta = \sum_{i=1}^{s} k_i x_i - \frac{1}{\beta}\left(ct + \frac{1}{\Gamma(\beta)}\right)^{\beta}. \tag{5} $$

and

$$ \mathfrak{W}(\theta) = u(t, x_1, \cdots, x_s), \;\; \theta = \frac{2}{1-\alpha^2} Sinh\left((1-\alpha)\left(\sum_{i=1}^{s} k_i x_i^{\frac{1+\alpha}{2}} - ct^{\frac{1+\alpha}{2}}\right)\right), \tag{6} $$

for the M-derivative, beta-derivative and hyperbolic derivative, respectively. These transformations can convert the Equation (3) with fractional derivatives, into an ordinary differential equation with integer differential operator. Transformations (4)–(6) are developed in [40,50,58].

Let us first consider the $s-$dimensional mK(n,n) equation, that is, $m = n$.

Applying transformations (4)–(6), we obtain the following single non-linear third-order ODE:

$$-(c+ak_1n)\mathfrak{W}^{n-1}\mathfrak{W}' + \left[n\mathfrak{W}^{n-1}\mathfrak{W}''' + 3n(n-1)\mathfrak{W}^{n-2}\mathfrak{W}'\mathfrak{W}'' + n(n-1)(n-2)\mathfrak{W}^{n-3}(\mathfrak{W}')^3\right] \times \sum_{i=1}^{s}\alpha_i k_i^3 = 0. \tag{7}$$

If we assume the change of variables [43,55,59,60]:

$$\psi_1(\theta) = \mathfrak{W}(\theta),\ \ \psi_2(\theta) = \mathfrak{W}'(\theta),\ \ \psi_3(\theta) = \mathfrak{W}''(\theta), \tag{8}$$

then Equation (7) reduces into the following autonomous system of equations:

$$\begin{cases} \dfrac{d\psi_1}{d\theta} = \psi_2, \\ \dfrac{d\psi_2}{d\theta} = \psi_3, \\ \dfrac{d\psi_3}{d\theta} = \dfrac{\psi_2}{n\sum_{i=1}^{s}\alpha_i k_i^3}\left[n(1-n)(n-2)(\sum\limits_{i=1}^{s}\alpha_i k_i^3)\dfrac{\psi_2^2}{\psi_1^2} + 3n(1-n)(\sum\limits_{i=1}^{s}\alpha_i k_i^3)\dfrac{\psi_3}{\psi_1} + (c+ak_1n)\right]. \end{cases} \tag{9}$$

Selecting ψ_1 as a new independent variable converts the system (9) into

$$\begin{cases} \dfrac{d\psi_2}{d\psi_1} = \dfrac{\psi_3}{\psi_2}, \\ \dfrac{d\psi_3}{d\psi_1} = \dfrac{1}{n\sum_{i=1}^{s}\alpha_i k_i^3}\left[n(1-n)(n-2)(\sum\limits_{i=1}^{s}\alpha_i k_i^3)\dfrac{\psi_2^2}{\psi_1^2} + 3n(1-n)(\sum\limits_{i=1}^{s}\alpha_i k_i^3)\dfrac{\psi_3}{\psi_1} + (c+ak_1n)\right]. \end{cases} \tag{10}$$

From the first equation in (10), we have

$$\psi_3 = \psi_2\frac{d\psi_2}{d\psi_1}.$$

Therefore, the second equation of (10) can be written as:

$$\left(\frac{d\psi_2}{d\psi_1}\right)^2 + \psi_2\frac{d^2\psi_2}{d\psi_1^2} = \frac{1}{n\sum_{i=1}^{s}\alpha_i k_i^3}\left[n(1-n)(n-2)(\sum_{i=1}^{s}\alpha_i k_i^3)\frac{\psi_2^2}{\psi_1^2} + 3n(1-n)(\sum_{i=1}^{s}\alpha_i k_i^3)\frac{\psi_2}{\psi_1}\frac{d\psi_2}{d\psi_1} + (c+ak_1n)\right]. \tag{11}$$

Solving Equation (11) concludes

$$\psi_2(\psi_1) = \pm\frac{\sqrt{{\psi_1}^{n-2}n(\sum_{i=1}^{s}\alpha_i k_i^3)\left(2\,{\psi_1}^{-n}\lambda_1\,n^2(\sum_{i=1}^{s}\alpha_i k_i^3) - 2\,\lambda_2\,n^2(\sum_{i=1}^{s}\alpha_i k_i^3) + {\psi_1}^{n}(c+ak_1n)\right)}}{{\psi_1}^{n-2}n^2(\sum_{i=1}^{s}\alpha_i k_i^3)}, \tag{12}$$

with λ_1 and λ_2 arbitrary constants. Hence, the first equation of (9) has the following form:

$$\frac{d\psi_1}{d\theta} = \pm\frac{\sqrt{{\psi_1}^{n-2}n(\sum_{i=1}^{s}\alpha_i k_i^3)\left(2\,{\psi_1}^{-n}\lambda_1\,n^2(\sum_{i=1}^{s}\alpha_i k_i^3) - 2\,\lambda_2\,n^2(\sum_{i=1}^{s}\alpha_i k_i^3) + {\psi_1}^{n}(c+ak_1n)\right)}}{{\psi_1}^{n-2}n^2(\sum_{i=1}^{s}\alpha_i k_i^3)}. \tag{13}$$

This equation is a separable ODE with an implicit general solution

$$\theta \mp \int \frac{n^2\psi_1^{n-2}(\theta)}{\sqrt{-\frac{n\left(-\psi_1^{2n}(\theta)(c+ank_1)+2\,\psi_1^{n}(\theta)(\sum_{i=1}^{s}\alpha_i k_i^3)\lambda_2\,n^2-2\,(\sum_{i=1}^{s}\alpha_i k_i^3)\lambda_1\,n^2\right)}{\psi_1^{2}(\theta)(\sum_{i=1}^{s}\alpha_i k_i^3)}}}d\psi_1(\theta) + \lambda_3 = 0, \tag{14}$$

where λ_3 is an arbitrary constant. To extract explicit solutions, we consider some special cases.

• **Case 1:** $\lambda_1 = 0,\ k_1 = -\frac{c}{an}$

In this case, the integral in Equation (14) is solvable and we obtain

$$\theta \mp \frac{2n\psi_1^{n-1}(\theta)}{\sqrt{-2\lambda_2 n^3 \psi_1^{n-2}(\theta)}} + \lambda_3 = 0,$$

which, after solving this equation regarding the dependent variable ψ_1, concludes

$$\mathfrak{W}(\theta) = \psi_1(\theta) = e^{\frac{\ln\left(-\frac{n\lambda_2}{2}(\theta+\lambda_3)^2\right)}{n}}.$$

Finally, from the obtained solution and transformations (4)–(6), we obtain the final solutions:

$$u(t, x_1, \ldots, x_s) = e^{\frac{\ln\left(-\frac{n\lambda_2}{2}(\sum_{i=1}^{s} k_i x_i - \frac{c}{\alpha}\Gamma(\beta+1)t^{\alpha} + \lambda_3)^2\right)}{n}}, \tag{15}$$

$$u(t, x_1, \ldots, x_s) = e^{\frac{\ln\left(-\frac{n\lambda_2}{2}(\sum_{i=1}^{s} k_i x_i - \frac{1}{\beta}\left(ct + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \lambda_3)^2\right)}{n}}, \tag{16}$$

$$u(t, x_1, \ldots, x_s) = e^{\frac{\ln\left(-\frac{n\lambda_2}{2}(\frac{2}{1-\alpha^2} Sinh\left((1-\alpha)\left(\sum_{i=1}^{s} k_i x_i^{\frac{1+\alpha}{2}} - ct^{\frac{1+\alpha}{2}}\right)\right) + \lambda_3)^2\right)}{n}}, \tag{17}$$

for $s-$dimensional mK(n,n) equation with M-derivative, beta-derivative and hyperbolic derivative, respectively.

In Figure 1, density plots of the obtained exact solutions (15)–(17) are plotted with the same parameters and derivative orders but different types of derivatives. This figure shows that the type of local derivative effects the final results and solution profiles. Variations in local derivative orders and a comparison of the final solutions with three types of derivatives in the fixed time direction $t = 1$, are plotted in Figure 2. We have to note that the figures are corresponding to the one-dimensional mK(n,n) equation, namely, $s = 1$. This is very easy to plot in higher-dimensional cases $s > 1$.

• **Case 2:** $\lambda_1 = 0,\ k_1 = \frac{\sum_{i=1}^{s} \alpha_i k_i^3 - c}{2a},\ n = 2$
In this case, the Equation (14) reduces into

$$\theta \mp \sqrt{2}\ln\left(\psi_1^2(\theta) + \sqrt{\psi_1^4(\theta) + 8\lambda_2}\right) + \lambda_3 = 0,$$

which solving this equation with respect to the dependent variable ψ_1, yields

$$\mathfrak{W}(\theta) = \psi_1(\theta) = \pm\frac{\sqrt{2e^{-\frac{\sqrt{2}}{2}(\theta+\lambda_3)}\left(\left(e^{-\frac{\sqrt{2}}{2}(\theta+\lambda_3)}\right)^2 - 8\,\lambda_2\right)}}{2e^{-\frac{\sqrt{2}}{2}(\theta+\lambda_3)}}.$$

Therefore, from the obtained solution and transformations (4)–(6), we can obtain the final solutions:

$$u(t, x_1, \ldots, x_s) = \pm\frac{\sqrt{2e^{-\frac{\sqrt{2}}{2}\left(\sum_{i=1}^{s} k_i x_i - \frac{c}{\alpha}\Gamma(\beta+1)t^{\alpha} + \lambda_3\right)}\left(\left(e^{-\frac{\sqrt{2}}{2}\left(\sum_{i=1}^{s} k_i x_i - \frac{c}{\alpha}\Gamma(\beta+1)t^{\alpha} + \lambda_3\right)}\right)^2 - 8\,\lambda_2\right)}}{2e^{-\frac{\sqrt{2}}{2}\left(\sum_{i=1}^{s} k_i x_i - \frac{c}{\alpha}\Gamma(\beta+1)t^{\alpha} + \lambda_3\right)}}, \tag{18}$$

$$u(t, x_1, \ldots, x_s) = \pm \frac{\sqrt{2e^{-\frac{\sqrt{2}}{2}\left(\sum_{i=1}^{s} k_i x_i - \frac{1}{\beta}\left(ct+\frac{1}{\Gamma(\beta)}\right)^{\beta}+\lambda_3\right)}\left(\left(e^{-\frac{\sqrt{2}}{2}\left(\sum_{i=1}^{s} k_i x_i - \frac{1}{\beta}\left(ct+\frac{1}{\Gamma(\beta)}\right)^{\beta}+\lambda_3\right)}\right)^2 - 8\,\lambda_2\right)}}{2e^{-\frac{\sqrt{2}}{2}\left(\sum_{i=1}^{s} k_i x_i - \frac{1}{\beta}\left(ct+\frac{1}{\Gamma(\beta)}\right)^{\beta}+\lambda_3\right)}}, \tag{19}$$

and

$$u(t, x_1, \ldots, x_s) = \pm \frac{\sqrt{2e^{-\frac{\sqrt{2}}{2}\left(\frac{2}{1-\alpha^2} Sinh\left((1-\alpha)\left(\sum_{i=1}^{s} k_i x_i^{\frac{1+\alpha}{2}} - ct^{\frac{1+\alpha}{2}}\right)\right)+\lambda_3\right)}\left(\left(e^{-\frac{\sqrt{2}}{2}\left(\frac{2}{1-\alpha^2} Sinh\left((1-\alpha)\left(\sum_{i=1}^{s} k_i x_i^{\frac{1+\alpha}{2}} - ct^{\frac{1+\alpha}{2}}\right)\right)+\lambda_3\right)}\right)^2 - 8\,\lambda_2\right)}}{2e^{-\frac{\sqrt{2}}{2}\left(\frac{2}{1-\alpha^2} Sinh\left((1-\alpha)\left(\sum_{i=1}^{s} k_i x_i^{\frac{1+\alpha}{2}} - ct^{\frac{1+\alpha}{2}}\right)\right)+\lambda_3\right)}}, \tag{20}$$

for $s-$dimensional mK(n,n) equation with M-derivative, beta-derivative and hyperbolic derivative, respectively.

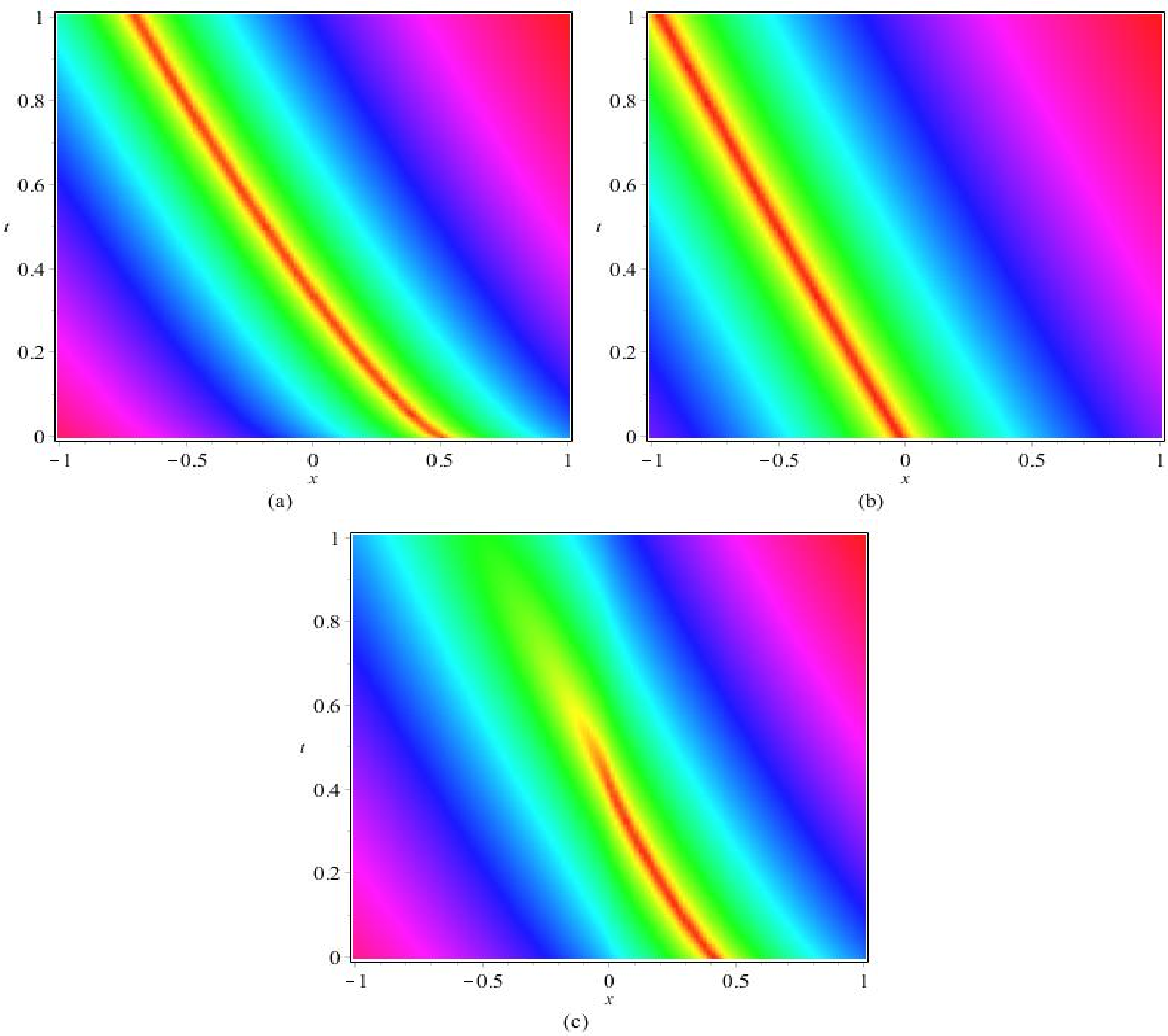

Figure 1. Exact solutions with $\alpha_1 = a = c = 2,\ \lambda_2 = \lambda_3 = 1,\ m = 3$, and $n = 5, \beta = \alpha = 0.9$ *w.r.t.* (**a**) M-derivative (15), (**b**) beta-derivative (16), (**c**) hyperbolic-derivative (17).

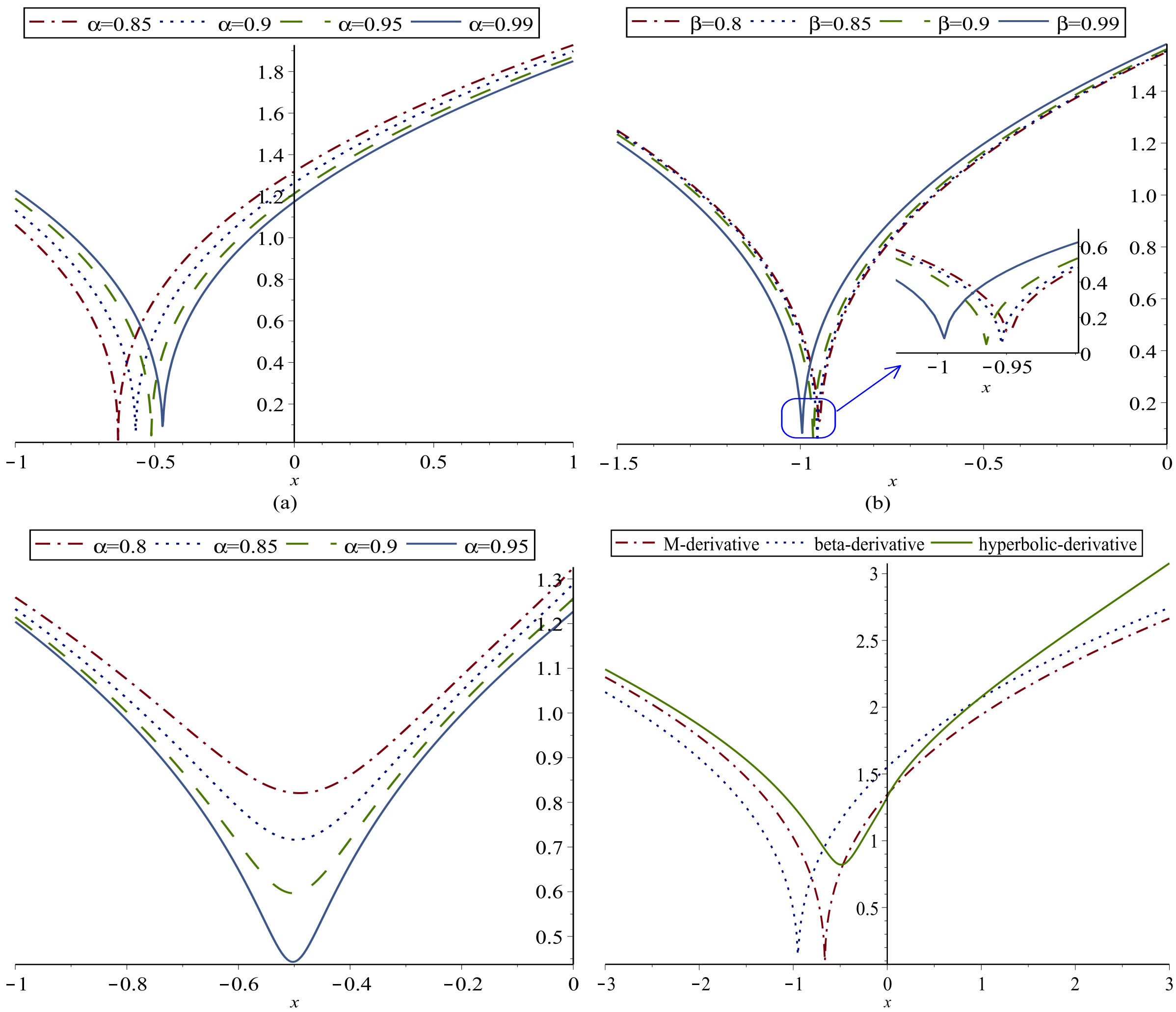

Figure 2. Exact solutions with $\alpha_1 = a = c = 2$, $\lambda_2 = \lambda_3 = t = 1$, $m = 3$, $n = 5$, and (**a**) $\beta = 0.9$, and various α *w.r.t.* M-derivative (15), (**b**) various β *w.r.t.* beta-derivative (16) (**c**) various α *w.r.t.* hyperbolic-derivative (17) (**d**) $\alpha = \beta = 0.8$, and various derivatives.

In Figure 3, density plots of the obtained exact solutions (18)–(20) are plotted with the same parameters and derivative orders but different type of derivatives. Variations in local derivative orders and a comparison of the final solutions with three types of derivatives in the fixed time direction $t = 1$, are plotted in Figure 4.

• **Case 3:** $n = 1$

In this case, the integral in Equation (14) is solvable, and we can obtain

$$\theta + \sqrt{\frac{-\sum_{i=1}^{s}\alpha_i k_i^3}{ak_1 + c}}\arctan\left(\frac{\lambda_1\sum_{i=1}^{s}\alpha_i k_i^3 - (ak_1 + c)\psi_1(\theta)}{\sqrt{(ak_1 + c)\left(2\lambda_1\sum_{i=1}^{s}\alpha_i k_i^3\psi_1(\theta) - (ak_1 + c)\psi_1^2(\theta) - 2\lambda_2\sum_{i=1}^{s}\alpha_i k_i^3\right)}}\right) + \lambda_3 = 0,$$

which, solving this equation with respect to the dependent variable ψ_1, yields

$$\mathfrak{W}(\theta) = \psi_1(\theta) = \frac{1}{ak_1 + c}\left[\sqrt{\sum_{i=1}^{s} \alpha_i k_i^3 (2(ak_1 + c)\lambda_2 - \lambda_1^2 \sum_{i=1}^{s} \alpha_i k_i^3)\left(\cos^2\left(\sqrt{-\frac{ak_1 + c}{\sum_{i=1}^{s} \alpha_i k_i^3}}(\theta + \lambda_3)\right) - 1\right)} + \lambda_1 \sum_{i=1}^{s} \alpha_i k_i^3\right].$$

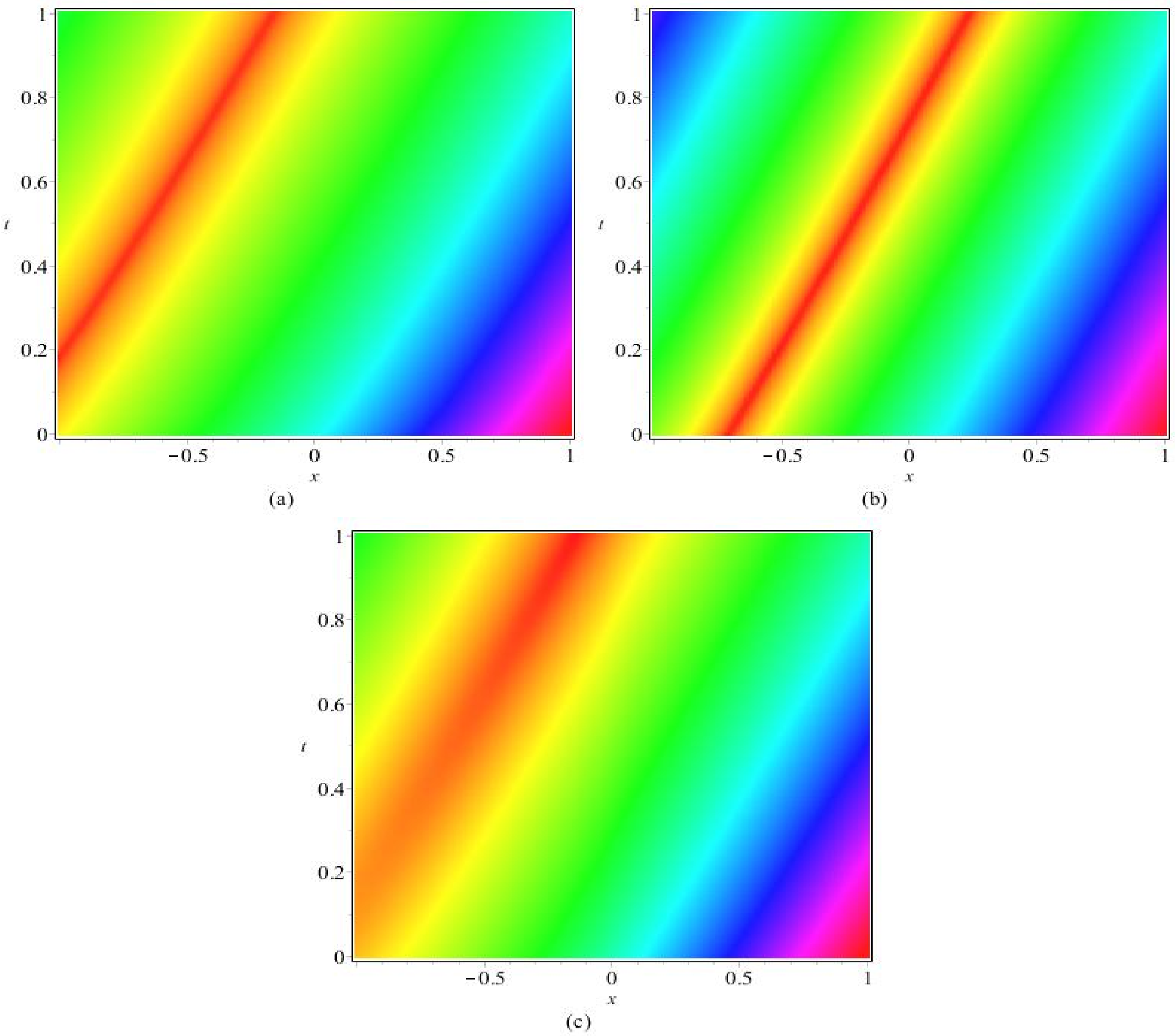

Figure 3. Density plots with $a = c = 2$, $\lambda_2 = \lambda_3 = 1$, $\alpha_1 = \frac{3}{2}\sqrt[3]{2}$, $m = 3$, and $\beta = \alpha = 0.9$ *w.r.t.* (**a**) M-derivative (18), (**b**) beta-derivative (19), (**c**) hyperbolic-derivative (20).

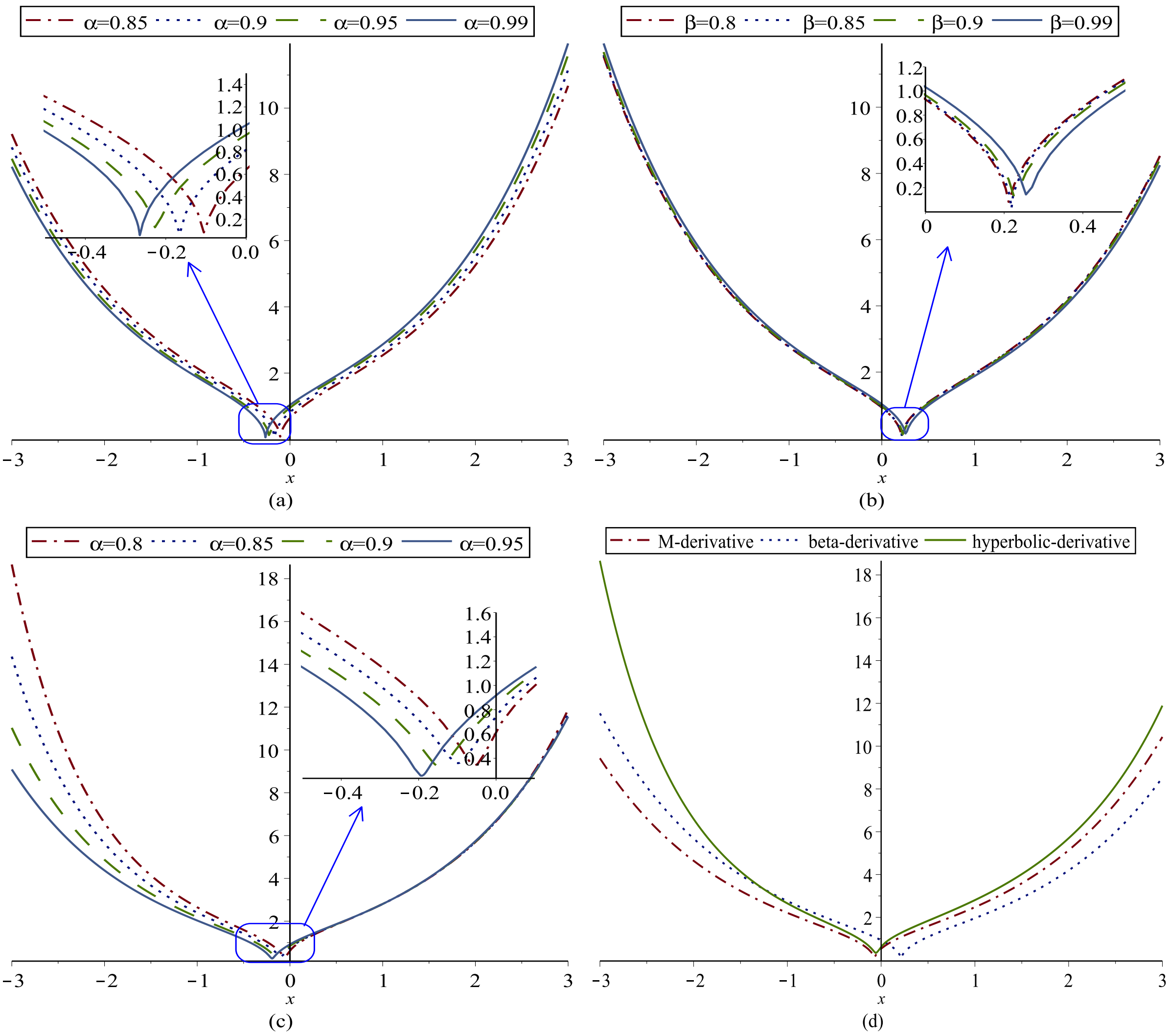

Figure 4. Exact solutions with $a = c = 2$, $\lambda_2 = \lambda_3 = t = 1$, $\alpha_1 = \frac{3}{2}\sqrt[3]{2}$, $m = 3$, and (**a**) $\beta = 0.9$, and various α *w.r.t.* M-derivative (18), (**b**) various β *w.r.t.* beta-derivative (19) (**c**) various α *w.r.t.* hyperbolic-derivative (20) (**d**) $\alpha = \beta = 0.8$, and various derivatives.

Lastly, from the obtained solution and transformations (4)–(6) we obtain the final solutions:

$$u_{\mathcal{K}}(t, x_1, \ldots, x_s) = \frac{1}{ak_1 + c}\left[\sqrt{(\sum_{i=1}^{s} \alpha_i k_i^3)\vartheta \sin^2\left(\sqrt{-\frac{ak_1 + c}{\sum_{i=1}^{s} \alpha_i k_i^3}}(\theta_{\mathcal{K}} + \lambda_3)\right)} + \lambda_1 \sum_{i=1}^{s} \alpha_i k_i^3\right], \tag{21}$$

where $\vartheta = \lambda_1^2 \sum_{i=1}^{s} \alpha_i k_i^3 - 2(ak_1 + c)\lambda_2$, $\theta_{\mathcal{K}} \in \{\theta_1, \theta_2, \theta_3\}$, and

$$\theta_1 = \sum_{i=1}^{s} k_i x_i - \frac{c}{\alpha}\Gamma(\beta+1)t^{\alpha},$$
$$\theta_2 = \sum_{i=1}^{s} k_i x_i - \frac{1}{\beta}\left(ct + \frac{1}{\Gamma(\beta)}\right)^{\beta},$$
$$\theta_3 = \frac{2}{1-\alpha^2}Sinh\left((1-\alpha)\left(\sum_{i=1}^{s} k_i x_i^{\frac{1+\alpha}{2}} - ct^{\frac{1+\alpha}{2}}\right)\right),$$

corresponding to M-derivative, beta-derivative and hyperbolic derivative, respectively.

Density plots and 2-D plots of (21) with three types of local derivatives are plotted in Figures 5 and 6, respectively.

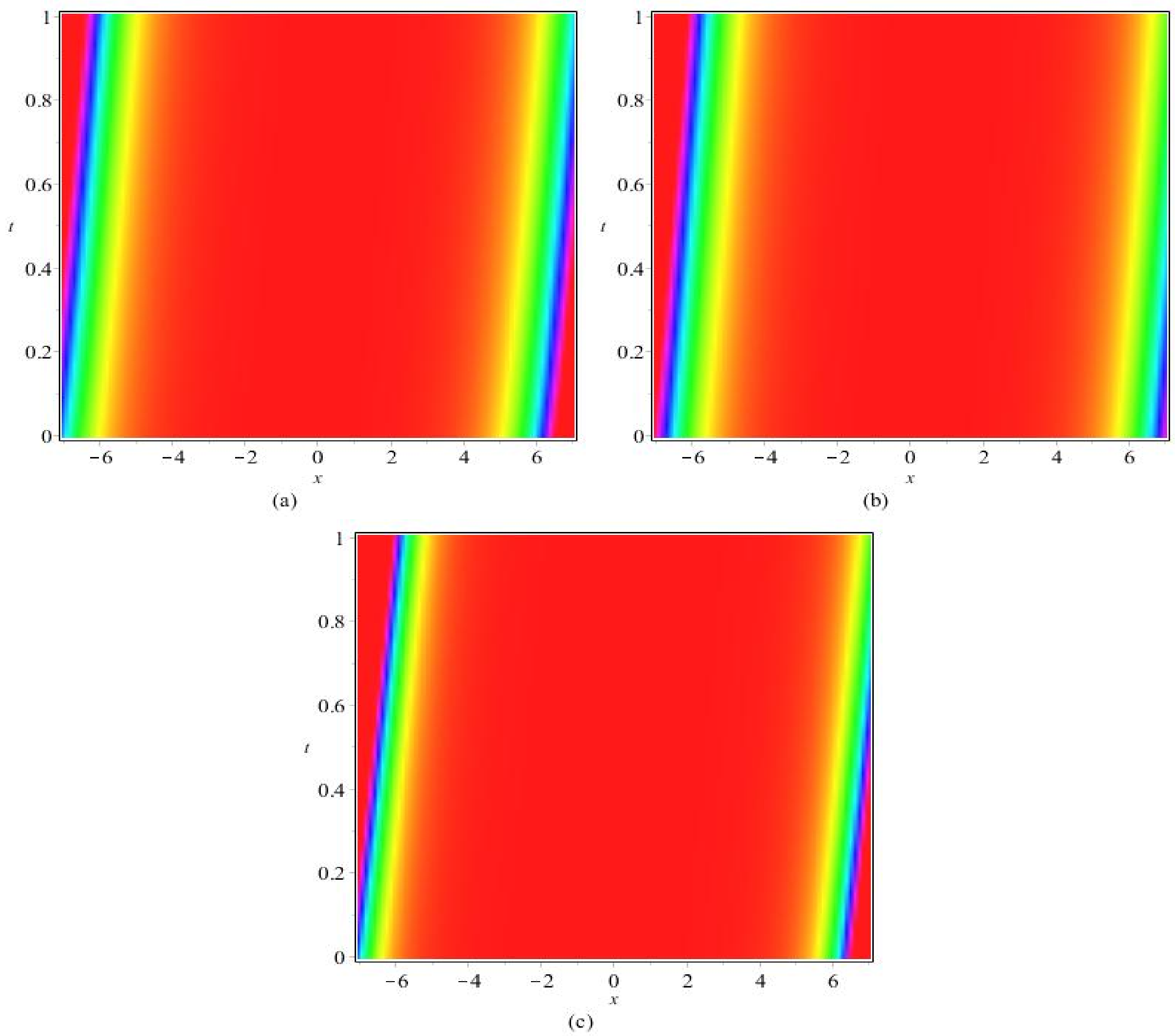

Figure 5. Exact solutions with $k_1 = \alpha_1 = a = c = 2,\ \lambda_1 = \lambda_2 = \lambda_3 = 1,\ m = 3$, and $n = 1.5, \beta = \alpha = 0.9$ *w.r.t.* (**a**) M-derivative (21), (**b**) beta-derivative (**c**) hyperbolic-derivative.

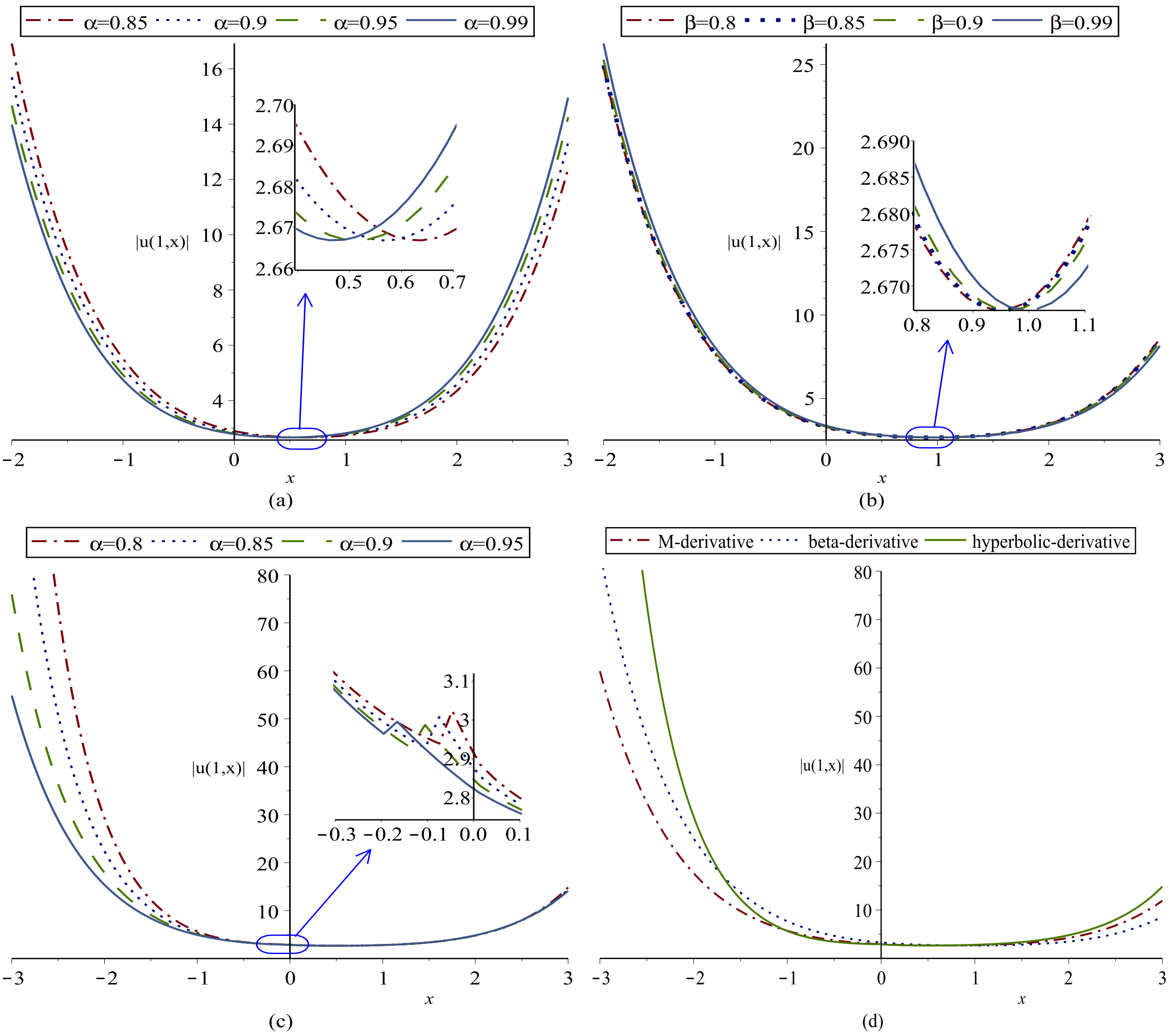

Figure 6. Exact solutions with $k_1 = \alpha_1 = a = c = 2$, $\lambda_1 = \lambda_2 = \lambda_3 = t = 1$, $m = 3$, and (**a**) $n = 1.5$, $\beta = 0.9$, and various α *w.r.t.* M-derivative (21), (**b**) $n = 2$, and various β *w.r.t.* beta-derivative (21) (**c**) $n = 2$, and various α *w.r.t.* hyperbolic-derivative (21) (**d**) $n = 2, \alpha = \beta = 0.8$, and various derivatives in (21).

- **Case 4:** $\lambda_1 \neq 0,\ \lambda_2 \neq 0,\ \ n \in \mathbb{R}^+$

In this case, the integral in Equation (14) is solvable and we can obtain

$$\theta \mp n^2 \sqrt{\frac{\gamma}{n(ak_1 n + c)}} \ln(\psi_1(\theta)) + \lambda_3 = 0,$$

which, solving this equation with respect to the dependent variable ψ_1, concludes the final solution

$$\mathfrak{W}(\theta) = \psi_1(\theta) = e^{\pm\sqrt{\frac{n(ak_1 n+c)}{\sum_{i=1}^{s} \alpha_i k_i^3}} \times \frac{\theta+\lambda_3}{n^2}}.$$

Lastly, from the obtained solution and transformations (4)–(6), we obtain the final solutions:

$$u(t, x_1, \ldots, x_s) = e^{\pm\sqrt{\frac{n(ak_1 n+c)}{\sum_{i=1}^{s} \alpha_i k_i^3}} \times \frac{\sum_{i=1}^{s} k_i x_i - \frac{c}{\alpha}\Gamma(\beta+1)t^{\alpha}+\lambda_3}{n^2}}, \tag{22}$$

$$u(t,x_1,\ldots,x_s) = e^{\pm\sqrt{\frac{n(ak_1n+c)}{\sum_{i=1}^{s}\alpha_i k_i^3}}\times\frac{\sum_{i=1}^{s}k_ix_i-\frac{1}{\beta}\left(ct+\frac{1}{\Gamma(\beta)}\right)^{\beta}+\lambda_3}{n^2}}, \tag{23}$$

and

$$u(t,x_1,\ldots,x_s) = e^{\pm\sqrt{\frac{n(ak_1n+c)}{\sum_{i=1}^{s}\alpha_i k_i^3}}\times\frac{\frac{2}{1-\alpha^2}Sinh\left((1-\alpha)\left(\sum_{i=1}^{s}k_ix_i^{\frac{1+\alpha}{2}}-ct^{\frac{1+\alpha}{2}}\right)\right)+\lambda_3}{n^2}}, \tag{24}$$

corresponding to M-derivative, beta-derivative and hyperbolic derivative, respectively.

Figure 7, shows the density plots of (22)–(24), and corresponding 2-D plots are demonstrated in Figure 8.

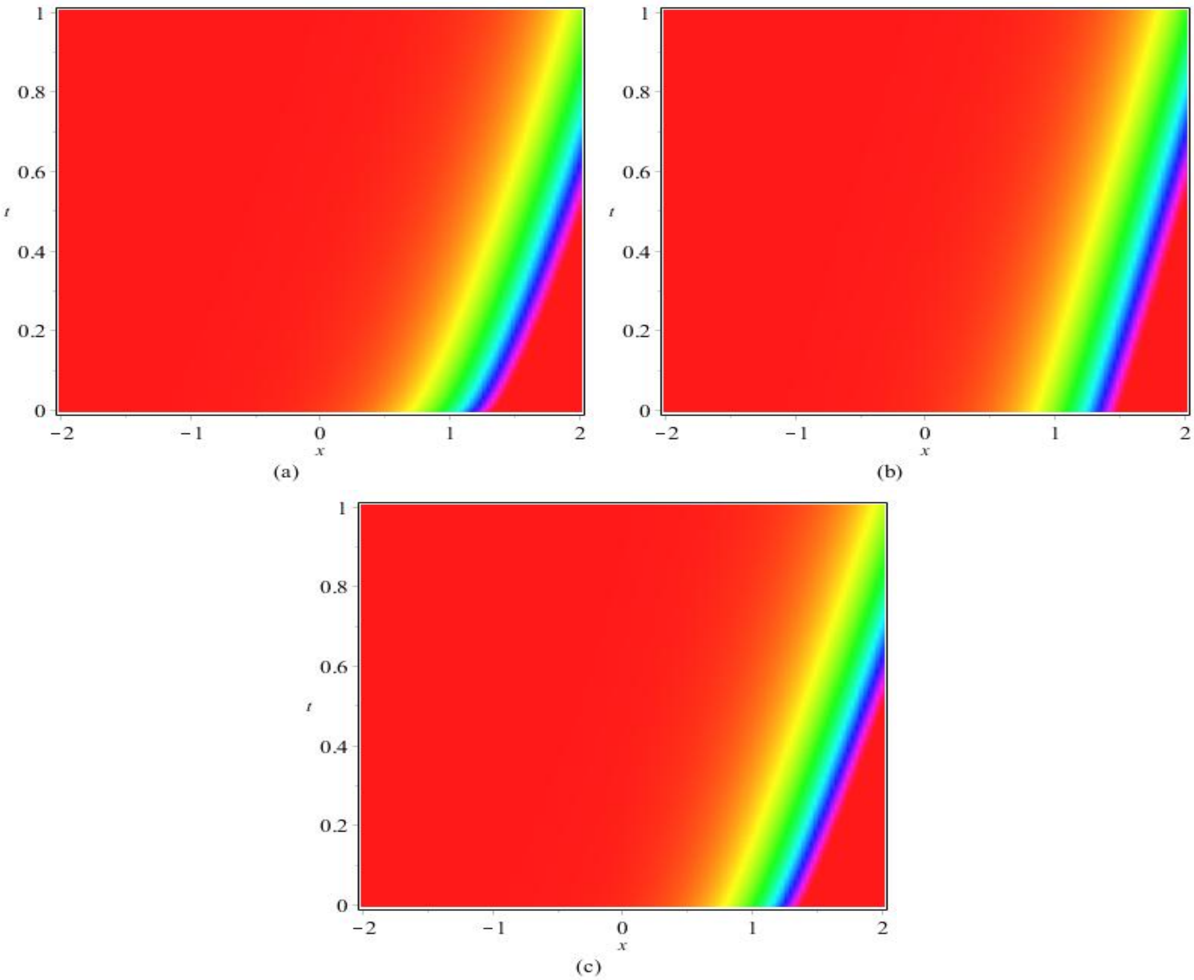

Figure 7. Exact solutions with $k_1 = \alpha_1 = a = c = 2$, $\lambda_3 = 1$, and (**a**) $n = 0.5$, $m = 2$, $\alpha = 0.8$, $\beta = 0.9$, *w.r.t.* M-derivative (22), (**b**) $n = 0.5$, $m = 2$, $\beta = 0.9$, *w.r.t.* beta-derivative (23), (**c**) $n = 0.5$, $m = 2$, $\alpha = 0.8$, *w.r.t.* hyperbolic-derivative (24).

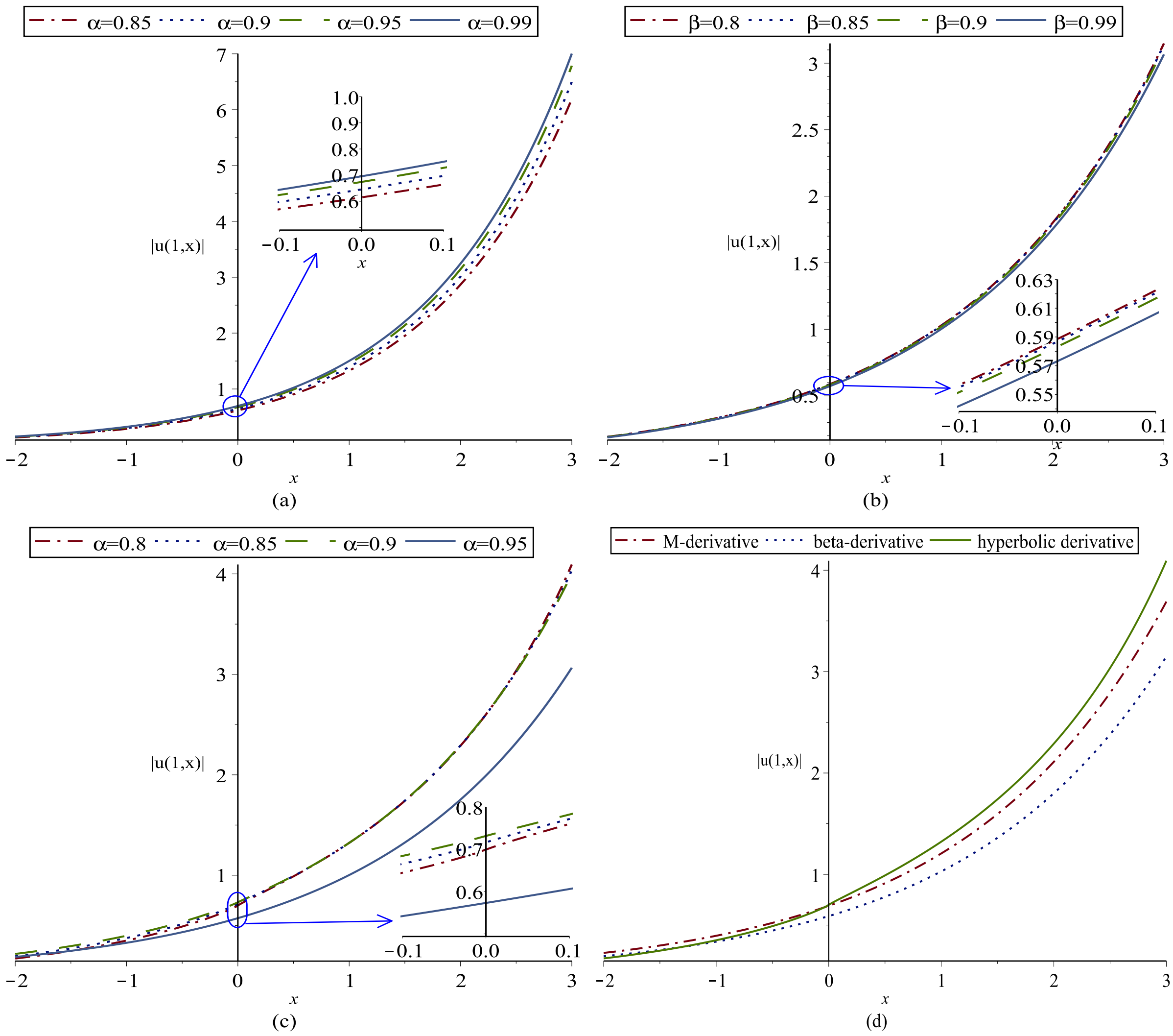

Figure 8. Exact solutions with $k_1 = \alpha_1 = a = c = 2$, $\lambda_3 = t = 1$, $m = 3$, and (**a**) $n = 1.5, \beta = 0.9$, and various α *w.r.t.* M-derivative (22), (**b**) $n = 2$, and various β *w.r.t.* beta-derivative (23) (**c**) $n = 2$, and various α *w.r.t.* hyperbolic-derivative (24) (**d**) $n = 2, \alpha = \beta = 0.8$, and various derivatives in (22)–(24).

• **Case 5:** $m \neq n$

In order to show the power of method, we tried to find exact solutions of the mK(m,n) equation with local derivatives as follows:

$$u^{n-1}\mathfrak{D}_t u + a(u^m)_x + (u^n)_{xxx} = 0, \;\; \mathfrak{D}_t \in \{ {}_0^A\mathcal{D}_t^{\beta}, \; {}^{\mathcal{M}}\mathcal{D}_t^{\alpha,\beta}, \; \mathcal{D}_h^{\alpha} \}, \tag{25}$$

whenever $m \neq n$.

Applying transformations (4)–(6), we obtain the following single nonlinear third-order ODE *w.r.t.* n and m:

$$-(c + ak_1 m)\mathfrak{W}^{m-1}\mathfrak{W}' + \left[n\mathfrak{W}^{n-1}\mathfrak{W}''' + 3n(n-1)\mathfrak{W}^{n-2}\mathfrak{W}'\mathfrak{W}'' + n(n-1)(n-2)\mathfrak{W}^{n-3}(\mathfrak{W}')^3 \right] \times \sum_{i=1}^{s} \alpha_i k_i^3 = 0. \tag{26}$$

Let us assume the change of variables

$$\psi_1(\theta) = \mathfrak{W}(\theta), \;\; \psi_2(\theta) = \mathfrak{W}'(\theta), \;\; \psi_3(\theta) = \mathfrak{W}''(\theta).$$

By assuming (8), the Equation (26) reduces into the following autonomous system of equations:

$$\begin{cases} \dfrac{d\psi_1}{d\theta} = \psi_2, \\ \dfrac{d\psi_2}{d\theta} = \psi_3, \\ \dfrac{d\psi_3}{d\theta} = \dfrac{\psi_2}{n\sum_{i=1}^{s}\alpha_i k_i^3}\Big[n(1-n)(n-2)(\sum_{i=1}^{s}\alpha_i k_i^3)\dfrac{\psi_2^2}{\psi_1^2} + 3n(1-n)(\sum_{i=1}^{s}\alpha_i k_i^3)\dfrac{\psi_3}{\psi_1} + (c+ak_1m)\psi_1^{m-n}\Big]. \end{cases} \tag{27}$$

Selecting ψ_1 as a new independent variable, converts the system (27) into

$$\begin{cases} \dfrac{d\psi_2}{d\psi_1} = \dfrac{\psi_3}{\psi_2}, \\ \dfrac{d\psi_3}{d\psi_1} = \dfrac{1}{n\sum_{i=1}^{s}\alpha_i k_i^3}\Big[n(1-n)(n-2)(\sum_{i=1}^{s}\alpha_i k_i^3)\dfrac{\psi_2^2}{\psi_1^2} + 3n(1-n)(\sum_{i=1}^{s}\alpha_i k_i^3)\dfrac{\psi_3}{\psi_1} + (c+ak_1m)\psi_1^{m-n}\Big]. \end{cases} \tag{28}$$

From the first equation in (28), we have

$$\psi_3 = \psi_2\frac{d\psi_2}{d\psi_1}. \tag{29}$$

Therefore, the second equation of (28) can be written as:

$$\begin{aligned} \left(\frac{d\psi_2}{d\psi_1}\right)^2 + \psi_2\frac{d^2\psi_2}{d\psi_1^2} &= \frac{1}{n\sum_{i=1}^{s}\alpha_i k_i^3} \\ &\times\Big[n(1-n)(n-2)(\sum_{i=1}^{s}\alpha_i k_i^3)\frac{\psi_2^2}{\psi_1^2} + 3n(1-n)(\sum_{i=1}^{s}\alpha_i k_i^3)\frac{\psi_2}{\psi_1}\frac{d\psi_2}{d\psi_1} + (c+ak_1m)\psi_1^{m-n}\Big]. \end{aligned} \tag{30}$$

Solving Equation (30) concludes

$$\psi_2(\psi_1) = \pm\frac{\sqrt{mn(m+n)\varrho{\psi_1}^{2(n-1)} + (ak_1m+c)\psi_1^{m+n} - \lambda_1 m(m+n)\varrho\psi_1^n + \lambda_2 m(m+n)\varrho}}{mn(m+n)\varrho{\psi_1}^{2(n-1)}}, \tag{31}$$

where $\varrho = \sum_{i=1}^{s}\alpha_i k_i^3$, and λ_1 and λ_2 are arbitrary constants. Hence, the first equation of (27) yields

$$\frac{d\psi_1}{d\theta} = \pm\frac{\sqrt{mn(m+n)\varrho{\psi_1}^{2(n-1)} + (ak_1m+c)\psi_1^{m+n} - \lambda_1 m(m+n)\varrho\psi_1^n + \lambda_2 m(m+n)\varrho}}{mn(m+n)\varrho{\psi_1}^{2(n-1)}}. \tag{32}$$

This equation is a separable ordinary differential equation. Therefore, we obtain

$$\theta \mp \frac{\sqrt{2}mn(m+n)\varrho\psi_1^{n-1}}{(m-n)\sqrt{mn(m+n)(c+ak_1m)\varrho\psi_1^{m+n-2}}} + \lambda_3 = 0, \tag{33}$$

where $\lambda_1 = \lambda_2 = 0$, and λ_3 is an arbitrary constant.

Solving this equation concludes:

$$\mathfrak{W}(\theta) = \psi_1(\theta) = e^{\frac{\ln\left(\frac{2mn(m+n)\varrho}{(m-n)^2(c+ak_1m)(\theta+\lambda_3)^2}\right)}{m-n}}.$$

Hence, from the obtained solution and transformations (4)–(6), we obtain the final solutions:

$$u(t,x_1,\ldots,x_s)=e^{\frac{\ln\left(\frac{2mn(m+n)\varrho}{(m-n)^2(c+ak_1m)(\sum_{i=1}^{s}k_ix_i-\frac{c}{\alpha}\Gamma(\beta+1)t^{\alpha}+\lambda_3)^2}\right)}{m-n}}, \tag{34}$$

$$u(t,x_1,\ldots,x_s)=e^{\frac{\ln\left(\frac{2mn(m+n)\varrho}{(m-n)^2(c+ak_1m)(\sum_{i=1}^{s}k_ix_i-\frac{1}{\beta}\left(ct+\frac{1}{\Gamma(\beta)}\right)^{\beta}+\lambda_3)^2}\right)}{m-n}}, \tag{35}$$

and

$$u(t,x_1,\ldots,x_s)=e^{\frac{\ln\left(\frac{2mn(m+n)\varrho}{(m-n)^2(c+ak_1m)(\frac{2}{1-\alpha^2}Sinh\left((1-\alpha)\left(\sum_{i=1}^{s}k_ix_i^{\frac{1+\alpha}{2}}-ct^{\frac{1+\alpha}{2}}\right)\right)+\lambda_3)^2}\right)}{m-n}}, \tag{36}$$

corresponding to M-derivative, beta-derivative and hyperbolic derivative, respectively.

Soliton-type solutions (34)–(36) with different values of derivative orders and nonlinearity power are plotted in Figure 9. Corresponding 2-D plots are demonstrated in Figure 10.

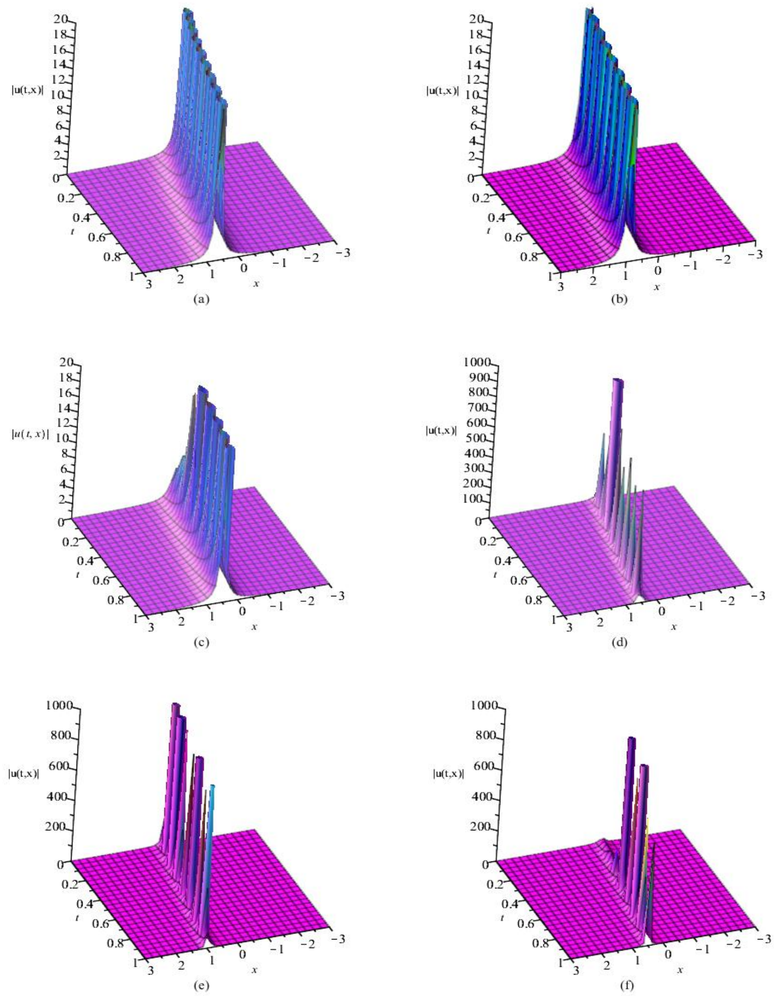

Figure 9. Exact solutions with $k_1=\alpha_1=a=c=2$, $\lambda_3=1$, and (**a**) $n=0.5$, $m=2$, $\alpha=0.8$, $\beta=0.9$, *w.r.t.* M-derivative (34), (**b**) $n=0.5$, $m=2$, $\beta=0.9$, *w.r.t.* beta-derivative (35), (**c**) $n=0.5$, $m=2$, $\alpha=0.8$, *w.r.t.* hyperbolic-derivative (36), (**d**) $n=1.5$, $m=3$, $\alpha=\beta=0.9$, *w.r.t.* M-derivative (34), (**e**) $n=1.5$, $m=3$, $\beta=0.9$, *w.r.t.* beta-derivative (35), (**f**) $n=1.5$, $m=3$, $\alpha=0.9$, *w.r.t.* hyperbolic-derivative (36).

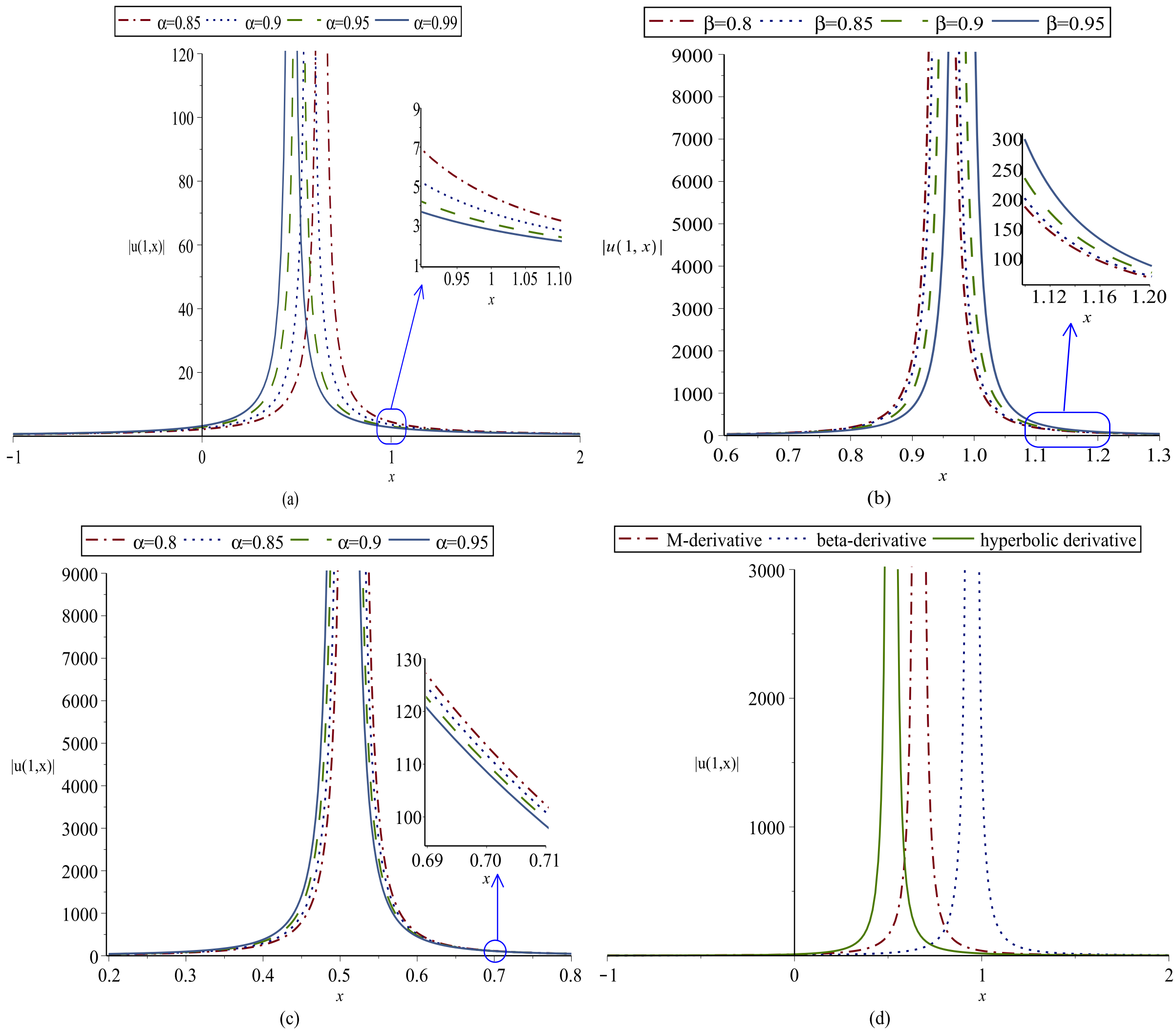

Figure 10. Exact solutions with $k_1 = \alpha_1 = a = c = 2,\ \lambda_3 = t = 1,\ m = 3$, and (**a**) $n = 1.5, \beta = 0.9$, and various α *w.r.t.* M-derivative (34), (**b**) $n = 2$, and various β *w.r.t.* beta-derivative (35) (**c**) $n = 2$, and various α *w.r.t.* hyperbolic-derivative (36) (**d**) $n = 2, \alpha = \beta = 0.8$, and various derivatives in (34)–(36).

4. Conclusions

In this paper, an important differential equation, namely, the higher-order generalized nonlinear dispersive mK(m,n) equation is considered with different values of m and n. The supposed derivatives in the time direction are M-derivative, beta-derivative and hyperbolic local derivative. Different types of soliton solutions in five cases, are extracted using Nucci's reduction method. A comparison of the obtained solutions with various local derivatives is graphically considered. In the literature, to the best knowledge of the author of this article, the reduction method is novel for the differential equations with local derivatives. Therefore, this paper can serve as a starting point for future works on local derivatives of other physical models.

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