



Article

Certain Hybrid Matrix Polynomials Related to the Laguerre-Sheffer Family

Tabinda Nahid ¹ and Junesang Choi ^{2,*} ¹ Department of Mathematics, Aligarh Muslim University, Aligarh 202001, India; tabindanahid@gmail.com² Department of Mathematics, Dongguk University, Gyeongju 38066, Korea

* Correspondence: junesang@dongguk.ac.kr; Tel.: +82-010-6525-2262

Abstract: The main goal of this article is to explore a new type of polynomials, specifically the Gould-Hopper-Laguerre-Sheffer matrix polynomials, through operational techniques. The generating function and operational representations for this new family of polynomials will be established. In addition, these specific matrix polynomials are interpreted in terms of quasi-monomiality. The extended versions of the Gould-Hopper-Laguerre-Sheffer matrix polynomials are introduced, and their characteristics are explored using the integral transform. Further, examples of how these results apply to specific members of the matrix polynomial family are shown.

Keywords: Gould-Hopper-Laguerre-Sheffer matrix polynomials; quasi-monomiality; umbral calculus; fractional calculus; Euler's integral of gamma functions; beta function; generalized hypergeometric series; operational methods

MSC: 05A40; 08A40; 26A33; 33B10; 33C45; 33C50; 44A20



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1. Introduction and Preliminaries

Significant discoveries in the theory of group representation, statistics, quadrature and interpolation, scattering theory, imaging of medicine, and splines have led to the development of matrix polynomials and special matrix functions. Numerous disciplines of mathematics and engineering make use of special matrix polynomials (see, for example, [1,2], and the citations included therein). For instance, many mathematicians investigate and explore special matrix polynomials.

The Sheffer sequences [3] are used extensively in mathematics, theoretical physics, theory of approximation, and various different mathematical disciplines. Roman [4] naturally discusses the Sheffer polynomials' properties in the context of contemporary classical umbral calculus. The Sheffer polynomials are given as follows (see [4], p. 17): Set $p(\tau)$ and $q(\tau)$ power series, which are formally given as follows:

$$p(\tau) = \sum_{\ell=0}^{\infty} p_{\ell} \frac{\tau^{\ell}}{\ell!} \quad (p_{\ell} \in \mathbb{C}, \ell \in \mathbb{Z}_{\geq 0}; p_0 = 0, p_1 \neq 0), \quad (1a)$$

and

$$q(\tau) = \sum_{\ell=0}^{\infty} q_{\ell} \frac{\tau^{\ell}}{\ell!} \quad (q_{\ell} \in \mathbb{C}, \ell \in \mathbb{Z}_{\geq 0}; q_0 \neq 0), \quad (1b)$$

which are referred to as delta series and invertible series, respectively. Here and elsewhere, let \mathbb{C} , \mathbb{R} , and \mathbb{Z} be, respectively, the sets of complex numbers, real numbers, and integers. Let

$$\mathbb{E}_{\leq \zeta}, \quad \mathbb{E}_{< \zeta}, \quad \mathbb{E}_{\geq \zeta}, \quad \text{and} \quad \mathbb{E}_{> \zeta}$$

be the sets of numbers in \mathbb{E} less than or equal to ζ , less than ζ , greater than or equal to ζ , and greater than ζ , respectively, for some $\zeta \in \mathbb{R}$, where \mathbb{E} is either \mathbb{Z} or \mathbb{R} .

With each pairing of an invertible series $q(\tau)$ and a delta series $p(\tau)$, there is a unique sequence $s_\ell(\mathbf{x})$ of polynomials that satisfies the conditions of orthogonality (consult [4], p. 17):

$$\langle q(\tau) p(\tau)^k | s_\ell(\mathbf{x}) \rangle = \ell! \delta_{\ell,k} \quad (\ell, k \in \mathbb{Z}_{\geq 0}), \tag{2}$$

where $\delta_{\ell,k}$ is the Kronecker delta function defined by $\delta_{\ell,k} = 1$ ($\ell = k$) and $\delta_{\ell,k} = 0$ ($\ell \neq k$). The operator $\langle \cdot | \cdot \rangle$ is unchanged from [4], Chapter 2.

Remark 1. The sequence $s_\ell(\mathbf{x})$ satisfying (2) is called the Sheffer sequence for $(q(\tau), p(\tau))$, or $s_\ell(\mathbf{x})$ is Sheffer for $(g(\tau), p(\tau))$, which is usually denoted as $s_\ell(\mathbf{x}) \sim (q(\tau), p(\tau))$. Remain aware that $q(\tau)$ and $p(\tau)$ should be an invertible series and a delta series, respectively.

There are two forms of Sheffer sequences worth noting:

- (i) If $s_\ell(\mathbf{x}) \sim (1, p(\tau))$, the $s_\ell(\mathbf{x})$ is said to be the associated sequence for $p(\tau)$, or $s_\ell(\mathbf{x})$ is associated with $p(\tau)$;
- (ii) If $s_\ell(\mathbf{x}) \sim (q(\tau), \tau)$, the $s_\ell(\mathbf{x})$ is said to be the Appell sequence for $q(\tau)$, or $s_\ell(\mathbf{x})$ is Appell for $q(\tau)$ (see [4], p. 17; see also [5]).

If $s_\ell(\mathbf{x})$ is Sheffer for $(q(\tau), p(\tau))$, the Sheffer sequence $s_\ell(\mathbf{x})$ is generated by depending solely on the series $q(\tau)$ and $p(\tau)$. To emphasize this dependence, in [5], the $s_\ell(\mathbf{x})$ was represented by ${}_{[q,p]}s_\ell(\mathbf{x})$.

Amid various Sheffer sequences' characterizations, the following generating function is recalled (consult, for instance, [4], p. 18): The sequence $s_\ell(\mathbf{x})$ is Sheffer for $(q(\tau), p(\tau))$ if and only if:

$$\frac{1}{q(\bar{p}(\tau))} e^{\mathbf{x}\bar{p}(\tau)} = \sum_{k=0}^{\infty} \frac{s_k(\mathbf{x})}{k!} t^k \tag{3}$$

for every \mathbf{x} in \mathbb{C} , where $\bar{p}(\tau) = p^{-1}(\tau)$ is the inverse of composition of $p(\tau)$.

The particular polynomials of two variables are significant in view of an application. In addition, these polynomials facilitate the derivation of numerous valuable identities and aid in the introduction of new families of particular polynomials; see, for instance, [6–9]. The Laguerre-Sheffer polynomials ${}_{LS}s_\ell(\mathbf{x}, y)$ are generated by the following function (consult [10]):

$$\frac{1}{q(p^{-1}(\tau))} \exp(y p^{-1}(\tau)) C_0(x p^{-1}(\tau)) = \sum_{n=0}^{\infty} {}_{LS}s_n(\mathbf{x}, y) \frac{\tau^n}{n!}, \tag{4}$$

for all \mathbf{x}, y in \mathbb{C} , where $C_0(x\tau)$ denotes the 0th-order Bessel-Tricomi function, which possesses the subsequent operational law:

$$C_0(\xi \mathbf{x}) := \sum_{k=0}^{\infty} \frac{(-1)^k (\xi \mathbf{x})^k}{(k!)^2} = \exp(-\xi \hat{D}_{\mathbf{x}}^{-1})\{1\}, \tag{5}$$

where

$$\hat{D}_{\mathbf{x}}^{-n}\{1\} := \frac{\mathbf{x}^n}{n!} \quad (n \in \mathbb{Z}_{\geq 0}). \tag{6}$$

Generally,

$$\hat{D}_{\mathbf{x}}^{-\xi}\{p(\mathbf{x})\} = \frac{1}{\Gamma(\xi)} \int_0^{\mathbf{x}} (\mathbf{x} - \eta)^{\xi-1} p(\eta) d\eta, \tag{7}$$

where Γ is the well-known Gamma function (consult, for example, [11], Section 1.1), which is a left-sided Riemann-Liouville fractional integral of order $\xi \in \mathbb{C}$ ($\Re(\xi) > 0$) (see, for example, [12], Chapter 2). For some recent applications for geometric analysis, one may consult, for example, [13,14].

As in Remark 1, the case $q(\tau) = 1$ and the case $p(\tau) = \tau$ of the Laguerre-Sheffer polynomials ${}_{LS}s_\ell(\mathbf{x}, y)$ in (4) are called, respectively, the Laguerre-associated Sheffer sequence

and the Laguerre-Appell sequence, and denoted, respectively, by ${}_L s_\ell(x, y)$ and ${}_L A_\ell(x, y)$ (consult [15]).

Remark 2. For $\kappa \in \mathbb{Z}_{>0}$, let $\mathbb{C}^{\kappa \times \kappa}$ indicate the set of all κ by κ matrices whose entries are in \mathbb{C} . Let $\sigma(B)$ be the set of all eigenvalues of $B \in \mathbb{C}^{\kappa \times \kappa}$, which is said to be the spectrum of B . For $B \in \mathbb{C}^{\kappa \times \kappa}$, let $\alpha(B) := \max\{\Re(w) \mid w \in \sigma(B)\}$ and $\beta(B) := \min\{\Re(w) \mid w \in \sigma(B)\}$. If $\beta(B) > 0$, that is, $\Re(w) > 0$ for all $w \in \sigma(B)$, the matrix B is referred to as positive stable.

For $B \in \mathbb{C}^{\kappa \times \kappa}$, its 2-norm is denoted by:

$$\|B\| = \sup_{\rho \neq 0} \frac{\|B\rho\|_2}{\|\rho\|_2},$$

where for any vector $\rho \in \mathbb{C}^\kappa$, $\|\rho\|_2 = (\rho^H \rho)^{1/2}$ is the Euclidean norm of ρ . Here ρ^H indicates the Hermitian matrix of ρ .

If $p(w)$ and $q(w)$ are holomorphic functions of the variable $w \in \mathbb{C}$, which are defined in an open set Λ of the plane \mathbb{C} , and R is a matrix in $\mathbb{C}^{\kappa \times \kappa}$ such that $\sigma(R) \subset \Lambda$, then from the matrix functional calculus's characteristics ([16], p. 558), one finds that $f(R)g(R) = g(R)f(R)$. Therefore, if Q in $\mathbb{C}^{\kappa \times \kappa}$ is another matrix with $\sigma(Q) \subset \Lambda$, such that $RQ = QR$, then $f(R)g(Q) = g(Q)f(R)$ (consult, for instance, [17,18]).

As the reciprocal of the Gamma function indicated by $\Gamma^{-1}(w) = 1/\Gamma(w)$ is an entire function of the variable $w \in \mathbb{C}$, for any R in $\mathbb{C}^{\kappa \times \kappa}$, the functional calculus of Riesz-Dunford reveals that the image of $\Gamma^{-1}(w)$ acting on R , symbolized by $\Gamma^{-1}(R)$, is a well-defined matrix (consult [16], Chapter 7).

Recently, the matrix polynomials of Gould-Hopper (GHMaP) $g_n^\ell(x, y; C, E)$ were introduced by virtue of the subsequent generating function (consult [19]):

$$\sum_{n=0}^{\infty} g_n^\ell(x, y; C, E) \frac{\tau^n}{n!} = \exp(x\tau\sqrt{2C}) \exp(Ey\tau^\ell). \tag{8}$$

Here C, E are matrices in $\mathbb{C}^{\kappa \times \kappa}$ ($\kappa \in \mathbb{Z}_{>0}$) such that C is positive stable and an $\ell \in \mathbb{Z}_{>0}$. Consider the principal branch of $w^{1/2} = \exp\left(\frac{1}{2} \log w\right)$ defined on the domain $\Lambda := \mathbb{C} \setminus (-\infty, 0]$. Then, as in Remark 2, \sqrt{C} is well-defined if $\sigma(C) \subset \Lambda$.

The polynomials $g_n^\ell(x, y; C, E)$ are specified to be the series

$$g_n^\ell(x, y; C, E) = \sum_{k=0}^{\lfloor \frac{n}{\ell} \rfloor} \frac{n! (\sqrt{2C})^{n-\ell k} E^k}{(n-\ell k)! k!} x^{n-\ell k} y^k. \tag{9}$$

As a result of the idea of monomiality, the majority of the features of generalized and conventional polynomials have been demonstrated to be readily derivable within a framework of operations. The monomiality principle is underpinned by Steffensen's [20] introduction of the idea of poweroid. Following that, Dattoli [21] reconstructed and elaborated the idea of monomiality (consult, for instance, [22]).

As per the monomiality principle, there are two operators \hat{M} and \hat{P} that operate on a polynomial set $\{q_\ell(x)\}_{\ell \in \mathbb{Z}_{>0}}$, termed the multiplicative and derivative operators, respectively. Then the polynomial set $\{q_\ell(x)\}_{\ell \in \mathbb{Z}_{>0}}$ is said to be quasi-monomial if it satisfies:

$$\hat{M}\{q_\ell(x)\} = q_{\ell+1}(x), \quad \hat{P}\{q_\ell(x)\} = \ell q_{\ell-1}(x), \quad q_0(x) = 1. \tag{10}$$

One easily finds from (10) that

$$\hat{M}\hat{P}\{q_\ell(x)\} = \ell q_\ell(x), \tag{11}$$

and

$$\hat{P}\hat{M}\{q_\ell(x)\} = (\ell + 1) q_\ell(x). \tag{12}$$

A Weyl group structure of the operators \hat{M} and \hat{P} is shown by the relation of commutation:

$$[\hat{P}, \hat{M}] := \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1}, \tag{13}$$

where $\hat{1}$ is the identity operator.

As a result of \hat{M}^m acting on $q_0(x)$, we may deduce the $q_m(x)$:

$$q_m(x) = \hat{M}^m \{q_0(x)\}. \tag{14}$$

The matrix polynomials of Gould-Hopper $g_m^\ell(x, y; C, E)$ are quasi-monomial with regard to the subsequent derivative and multiplicative operators [23]:

$$\hat{P}_g = (\sqrt{2C})^{-1} D_x, \tag{15}$$

and

$$\hat{M}_g = x\sqrt{2C} + \ell E y (\sqrt{2C})^{-(\ell-1)} D_x^{\ell-1}, \tag{16}$$

respectively, where $D_x := \frac{\partial}{\partial x}$.

The generalization ${}_aF_\beta(\alpha, \beta \in \mathbb{Z}_{\geq 0})$ of the hypergeometric series is given by (consult, for instance, [11], Section 1.5):

$$\begin{aligned} {}_aF_\beta \left[\begin{matrix} \mu_1, \dots, \mu_\alpha; \\ \nu_1, \dots, \nu_\beta; \end{matrix} w \right] &= \sum_{n=0}^{\infty} \frac{(\mu_1)_n \cdots (\mu_\alpha)_n w^n}{(\nu_1)_n \cdots (\nu_\beta)_n n!} \\ &= {}_aF_\beta(\mu_1, \dots, \mu_\alpha; \nu_1, \dots, \nu_\beta; w), \end{aligned} \tag{17}$$

where $(\xi)_\eta$ indicates the Pochhammer symbol (for $\xi, \eta \in \mathbb{C}$) defined by

$$(\xi)_\eta := \frac{\Gamma(\xi + \eta)}{\Gamma(\xi)} = \begin{cases} 1 & (\eta = 0; \xi \in \mathbb{C} \setminus \{0\}), \\ \xi(\xi + 1) \cdots (\xi + n - 1) & (\eta = n \in \mathbb{Z}_{>0}; \xi \in \mathbb{C}). \end{cases} \tag{18}$$

Here it is assumed that $(0)_0 := 1$, an empty product as 1, and that the variable w , the parameters of numerators μ_1, \dots, μ_α , and the parameters of denominators ν_1, \dots, ν_β are supposed to get complex values, provided that

$$(v_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; j = 1, \dots, \beta). \tag{19}$$

Recall the well-known generalized binomial theorem (consult, for example, [24], p. 34):

$$(1 - z)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{k!} \quad (\alpha \in \mathbb{C}; |z| < 1). \tag{20}$$

Recall the familiar beta function (consult, for instance, [11], p. 8):

$$B(\xi, \eta) = \begin{cases} \int_0^1 u^{\xi-1} (1-u)^{\eta-1} du & (\min\{\Re(\xi), \Re(\eta)\} > 0) \\ \frac{\Gamma(\xi)\Gamma(\eta)}{\Gamma(\xi + \eta)} & (\xi, \eta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}). \end{cases} \tag{21}$$

Here we introduce the Gould-Hopper-Laguerre-Sheffer matrix polynomials (GHLSMaP), which are denoted by ${}_gLS_n^\ell(x, y, z; C, E)$, by convoluting the Laguerre-Sheffer polynomials ${}_LS_n(x, y)$ with the Gould-Hopper matrix polynomials $g_n^\ell(x, y; C, E)$. The polynomials ${}_gLS_n^\ell(x, y, z; C, E)$ are generated as in the following definition.

Definition 1. The Gould-Hopper-Laguerre-Sheffer matrix polynomials ${}_gLS_n^\ell(x, y, z; C, E)$ are generated by the following function:

$$\begin{aligned}
 F(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E)(\tau) &:= \frac{1}{q(p^{-1}(\tau))} \exp \left[\mathbf{x}\sqrt{2C}p^{-1}(\tau) + E\mathbf{y} \left(p^{-1}(\tau) \right)^\ell \right] C_0 \left(\mathbf{z}p^{-1}(\tau) \right) \\
 &= \sum_{n=0}^{\infty} gLs_n^\ell(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E) \frac{\tau^n}{n!}.
 \end{aligned}
 \tag{22}$$

Here and in the sequel, the functions p, q, C_0 are as in (4); the matrices C, E are as in (8), (9), or (16); the variables $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}$.

In addition, to emphasize the invertible series q and the delta series p , whenever necessary, the following notation is used:

$$gLs_n^\ell(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E) = {}_{[q,p]}gLs_n^\ell(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E).
 \tag{23}$$

Further,

$$g_s^\ell(\mathbf{x}, \mathbf{y}; C, E) := gLs_n^\ell(\mathbf{x}, \mathbf{y}, 0; C, E)
 \tag{24}$$

is called the Gould-Hopper-Sheffer matrix polynomials.

Remark 3. First we show how to derive the generating function in (22). In (4), replacing \mathbf{y} by the multiplicative operator \hat{M}_g in (16), and \mathbf{x} by \mathbf{z} , we obtain

$$\begin{aligned}
 F(\tau) &:= \frac{1}{q(p^{-1}(\tau))} C_0 \left(\mathbf{z}p^{-1}(\tau) \right) \\
 &\quad \times \exp \left[\left(\mathbf{x}\sqrt{2C}p^{-1}(\tau) + \ell E\mathbf{y}(\sqrt{2C})^{-(\ell-1)} p^{-1}(\tau) D_{\mathbf{x}}^{\ell-1} \right) \{1\} \right].
 \end{aligned}
 \tag{25}$$

Recall the Crofton-type identity (see, for instance, [25], p. 12; see also [26]:

$$f \left(\mathbf{x} + \ell \lambda \frac{d^{\ell-1}}{d\mathbf{x}^{\ell-1}} \right) \{1\} = \exp \left(\lambda \frac{d^\ell}{d\mathbf{x}^\ell} \right) \{f(\mathbf{x})\},
 \tag{26}$$

with f usually being an analytic function. Setting $\ell = 1$ gives:

$$f(\mathbf{x} + \lambda) \{1\} = \exp \left(\lambda \frac{d}{d\mathbf{x}} \right) \{f(\mathbf{x})\}.
 \tag{27}$$

Using (25) in (26), we get

$$F(\tau) = \frac{1}{q(p^{-1}(\tau))} C_0(\mathbf{z}p^{-1}(\tau)) \exp \left(E\mathbf{y}(\sqrt{2C})^{-\ell} D_{\mathbf{x}}^\ell \right) \left\{ \exp \left(\mathbf{x}\sqrt{2C}p^{-1}(\tau) \right) \right\}.
 \tag{28}$$

By performing the operation in (28), with the aid of (32), we can readily find that $F(\tau)$ is identical to the $F(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E)(\tau)$ in (22).

Second, as in (ii), Remark 1, setting $p(\tau) = p^{-1}(\tau) = \tau$ in (22), we get the generating function for the Gould-Hopper-Laguerre-Appell matrix polynomials (GHLAMaP) $gLC_n^\ell(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E)$ in [27].

Using Euler’s integral for the Gamma function Γ (consult, for instance, Section 1.1 in [11], p. 218 in [24]), we get

$$b^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty u^{v-1} e^{-bu} du \quad (\min\{\Re(v), \Re(b)\} > 0).
 \tag{29}$$

Dattoli et al. [28] used (29) to obtain the following operator:

$$\begin{aligned}
 \left(\alpha - \frac{\partial}{\partial x} \right)^{-v} f(x) &= \frac{1}{\Gamma(v)} \int_0^\infty u^{v-1} e^{-\alpha u} e^{u \frac{\partial}{\partial x}} \{f(x)\} du \\
 &= \frac{1}{\Gamma(v)} \int_0^\infty u^{v-1} e^{-\alpha u} f(x + u) du,
 \end{aligned}
 \tag{30}$$

for the second equality of which (27) is employed.

The following definition introduces the extended matrix polynomials of Gould-Hopper-Laguerre-Sheffer (EGHLSMaP), which are indicated by ${}_{gLS}S_{n,\nu}^\ell(x, y, z; C, E; \eta)$.

Definition 2. Let $\Re(\eta) > 0$ and $\Re(\nu) > 0$. Then the extended Gould-Hopper-Laguerre-Sheffer matrix polynomials ${}_{gLS}S_{n,\nu}^\ell(x, y, z; C, E; \eta)$ are defined by

$${}_{gLS}S_{n,\nu}^\ell(x, y, z; C, E; \eta) := \left(\eta - yE(\sqrt{2C})^{-\ell} \frac{\partial^\ell}{\partial x^\ell} \right)^{-\nu} \left\{ {}_L S_n(z, x\sqrt{2C}) \right\}. \tag{31}$$

In this article, we aim to introduce the Gould-Hopper-Laguerre-Sheffer matrix polynomials via the use of a generating function. For these newly presented matrix polynomials, we investigate quasi-monomial features and related operational principles. We also explore the extended form of these novel hybrid special matrix polynomials and their properties using an integral transform. Finally, we provide many instances to demonstrate how the results presented here may be used.

2. Gould-Hopper-Laguerre-Sheffer Matrix Polynomials

The following lemma provides an easily-derivable operational identity.

Lemma 1. Let ξ and η be constants independent of x . Also let $\ell \in \mathbb{Z}_{\geq 0}$. Then:

$$\exp\left(\xi \frac{d^\ell}{dx^\ell}\right) \{e^{\eta x}\} = \exp(\eta x + \xi \eta^\ell). \tag{32}$$

In particular,

$$\exp\left(\xi \frac{d}{dx}\right) \{e^{\eta x}\} = \exp(\eta x + \xi \eta). \tag{33}$$

Proof.

$$\exp\left(\xi \frac{d^\ell}{dx^\ell}\right) \{e^{\eta x}\} = \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \frac{d^{\ell k}}{dx^{\ell k}} e^{\eta x} = e^{\eta x} \sum_{k=0}^{\infty} \frac{(\xi \eta^\ell)^k}{k!} = \exp(\eta x + \xi \eta^\ell).$$

□

The following theorem shows that the Gould-Hopper-Laguerre-Sheffer matrix polynomials ${}_{gLS}S_n^\ell(x, y, z; C, E)$ may be obtained by performing a suitable differential operation on the Laguerre-Sheffer polynomials ${}_L S_n(x, y)$ in (4) with some suitable substitutions of x and y .

Theorem 1. The following identity holds true:

$${}_{gLS}S_n^\ell(x, y, z; C, E) = \exp\left(yE(\sqrt{2C})^{-\ell} D_x^\ell\right) \left\{ {}_L S_n(z, x\sqrt{2C}) \right\}. \tag{34}$$

Proof. Replacing x and y by z and $x\sqrt{2C}$, respectively, in (4), we get

$$\frac{1}{q(p^{-1}(\tau))} C_0(z p^{-1}(\tau)) \exp(x\sqrt{2C} p^{-1}(\tau)) = \sum_{n=0}^{\infty} {}_L S_n(z, x\sqrt{2C}) \frac{\tau^n}{n!}. \tag{35}$$

Performing the operation $\exp \left[yE \left(\sqrt{2C} \right)^{-\ell} D_x^\ell \right]$ on both sides of (35), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \exp \left[yE \left(\sqrt{2C} \right)^{-\ell} D_x^\ell \right] \left\{ L^{S_n}(z, x\sqrt{2C}) \right\} \frac{\tau^n}{n!} \\ &= \frac{1}{q(p^{-1}(\tau))} C_0 \left(zp^{-1}(\tau) \right) \exp \left[yE \left(\sqrt{2C} \right)^{-\ell} D_x^\ell \right] \left\{ \exp \left(x\sqrt{2C}p^{-1}(\tau) \right) \right\} \\ &= \sum_{n=0}^{\infty} gLS_n^\ell(x, y, z; C, E) \frac{\tau^n}{n!}, \end{aligned} \tag{36}$$

for the second equality of which (22) and (32) are used. Finally, matching the coefficients of τ^n on the first and last power series in (36) gives the identity (34). \square

Theorem 2. The Gould-Hopper-Laguerre-Sheffer matrix polynomials $gLS_n^\ell(x, y, z; C, E)$ are operationally represented by the Gould-Hopper-Sheffer matrix polynomials $gS_n^\ell(x, y; C, E)$:

$$gLS_n^\ell(x, y, z; C, E) = \exp \left[-\hat{D}_z^{-1} \left(\sqrt{2C} \right)^{-1} D_x \right] \left\{ gS_n^\ell(x, y; C, E) \right\}. \tag{37}$$

Proof. From (22) and (24), we have

$$\begin{aligned} & \frac{1}{q(p^{-1}(\tau))} \exp \left[x\sqrt{2C}p^{-1}(\tau) + Ey \left(p^{-1}(\tau) \right)^\ell \right] \\ &= \sum_{n=0}^{\infty} gLS_n^\ell(x, y; C, E) \frac{\tau^n}{n!}. \end{aligned} \tag{38}$$

Performing the following operation $\exp \left[-\hat{D}_z^{-1} \left(\sqrt{2C} \right)^{-1} D_x \right]$ on each side of (38), and using (5) and (33), in the same way as in the argument of Theorem 1, one may find the desired identity (37). \square

The following theorem reveals the quasi-monomial principle of the matrix polynomials of Gould-Hopper-Laguerre-Sheffer $gLS_n^\ell(x, y, z; C, E)$.

Theorem 3. The matrix polynomials $gLS_n^\ell(x, y, z; C, E)$ gratify the following quasi-monomiality, with respect to the operators of multiplication and differentiation:

$$\begin{aligned} \hat{M}_{gLS} &= \left(x\sqrt{2C} - \hat{D}_z^{-1} + \ell Ey \left(\sqrt{2C} \right)^{-(\ell-1)} D_x^{\ell-1} - \frac{q' \left(\left(\sqrt{2C} \right)^{-1} D_x \right)}{q \left(\left(\sqrt{2C} \right)^{-1} D_x \right)} \right) \\ &\times \frac{1}{p' \left(\left(\sqrt{2C} \right)^{-1} D_x \right)} \end{aligned} \tag{39}$$

and

$$\hat{P}_{qLS} = p \left(\left(\sqrt{2C} \right)^{-1} D_x \right), \tag{40}$$

respectively.

Proof. Performing derivatives on each side of the first and second members in (22) about x, k times, we derive

$$\left(\left(\sqrt{2C} \right)^{-1} D_x \right)^k \{ F(x, y, z; C, E)(\tau) \} = \left(p^{-1}(\tau) \right)^k F(x, y, z; C, E)(\tau) \quad (k \in \mathbb{Z}_{\geq 0}). \tag{41}$$

In particular,

$$\left((\sqrt{2C})^{-1} D_x \right) \{F(x, y, z; C, E)(\tau)\} = p^{-1}(\tau) F(x, y, z; C, E)(\tau). \tag{42}$$

Applying (41) to the series in (1a), we find

$$\sum_{k=0}^{\infty} \frac{p_k}{k!} \left((\sqrt{2C})^{-1} D_x \right)^k \{F(x, y, z; C, E)(\tau)\} = \sum_{k=0}^{\infty} \frac{p_k}{k!} \left(p^{-1}(\tau) \right)^k \{F(x, y, z; C, E)(\tau)\},$$

which implies

$$\begin{aligned} p \left((\sqrt{2C})^{-1} D_x \right) \{F(x, y, z; C, E)(\tau)\} &= p \left(p^{-1}(\tau) \right) \{F(x, y, z; C, E)(\tau)\} \\ &= \tau F(x, y, z; C, E)(\tau). \end{aligned} \tag{43}$$

Then, utilizing the identity (43) in (22), we get

$$\begin{aligned} \sum_{n=1}^{\infty} p \left((\sqrt{2C})^{-1} D_x \right) g_L s_n^\ell(x, y, z; C, E) \frac{\tau^n}{n!} \\ = \sum_{n=1}^{\infty} g_L s_{n-1}^\ell(x, y, z; C, E) \frac{\tau^n}{(n-1)!}. \end{aligned} \tag{44}$$

Now, identifying the coefficients of τ^n on each side of (44), in view of (10), may prove the derivative operator (40).

Next, in view of (5), we have

$$\frac{d}{d\tau} C_0(zp^{-1}(\tau)) = \frac{d}{d\tau} \exp(-p^{-1}(\tau) \hat{D}_z^{-1}) \{1\} = - \left(p^{-1}(\tau) \right)' \hat{D}_z^{-1} C_0(zp^{-1}(\tau)). \tag{45}$$

Then, taking (45) into account, differentiating (22) about τ , we get

$$\begin{aligned} \sum_{n=0}^{\infty} g_L s_{n+1}^\ell(x, y, z; C, E) \frac{\tau^n}{n!} \\ = \frac{1}{p'(p^{-1}(\tau))} \left(x\sqrt{2C} + \ell E y (\sqrt{2C})^{-(\ell-1)} D_x^{\ell-1} - \hat{D}_z^{-1} - \frac{q'(p^{-1}(\tau))}{q(p^{-1}(\tau))} \right) \\ \times \sum_{n=0}^{\infty} g_L s_n^\ell(x, y, z; C, E) \frac{\tau^n}{n!}. \end{aligned} \tag{46}$$

Finally, applying (42) to (46), in view of (10), we can prove the multiplicative operator (39). \square

Remark 4. If $p(\tau)$ is a delta series, then $p'(\tau)$ is an invertible series. Therefore, the reciprocal $1/p'(p^{-1}(\tau))$ is well-defined in (46). \square

Combining the multiplicative operator in (39) and the derivative operator in (40), such as (11)–(14), we can provide several matrix differential equations for the matrix polynomials of Gould-Hopper-Laguerre-Sheffer $g_L s_n^\ell(x, y, z; C, E)$. One uses (11) to illustrate one of them in the next theorem, whose proof is simple and overlooked.

Theorem 4. The following differential equation holds true:

$$\begin{aligned} \left\{ \left(x\sqrt{2C} - \hat{D}_z^{-1} + \ell E y (\sqrt{2C})^{-(\ell-1)} D_x^{\ell-1} - \frac{q'((\sqrt{2C})^{-1} D_x)}{q((\sqrt{2C})^{-1} D_x)} \right) \right. \\ \left. \times \frac{p((\sqrt{2C})^{-1} D_x)}{p'((\sqrt{2C})^{-1} D_x)} - n \right\} g_L s_n^\ell(x, y, z; C, E) = 0. \end{aligned} \tag{47}$$

The polynomials $g_L s_n^\ell(x, y, z; C, E)$ may yield numerous particular matrix polynomials as special cases, some of which are offered in Table 1.

Table 1. Particular cases of the polynomials $g_{LS_n}^\ell(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E)$.

S. No.	Values of the Indices and Variables	Relation between $g_{LS_n}^\ell(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E)$ and Its Special Case	Name of the Special Matrix Polynomials	Generating Functions
I.	$\ell = 2$	$g_{LS_n}^2(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E)$ $= {}_{HLS_n}(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E)$	3-Variable Hermite-Laguerre-Sheffer matrix polynomials (3VHLSMaP)	$\frac{1}{q(p^{-1}(\tau))} \exp\left((\mathbf{x}p^{-1}(\tau)\sqrt{2C} + E\mathbf{y}(p^{-1}(\tau))^2)\right) \times C_0(\mathbf{z}p^{-1}(\tau)) = \sum_{n=0}^{\infty} {}_{HLS_n}(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E) \frac{\tau^n}{n!}$
II.	$\mathbf{z} = 0$	$g_{LS_n}^\ell(\mathbf{x}, \mathbf{y}, 0; C, E)$ $= g_{S_n}^\ell(\mathbf{x}, \mathbf{y}; C, E)$	Gould-Hopper-Sheffer-matrix polynomials (GHSMaP)	$\frac{1}{q(p^{-1}(\tau))} \exp\left(\mathbf{x}p^{-1}(\tau)\sqrt{2C} + E\mathbf{y}(p^{-1}(\tau))^\ell\right) = \sum_{n=0}^{\infty} g_{S_n}^\ell(\mathbf{x}, \mathbf{y}; C, E) \frac{\tau^n}{n!}$
III.	$\ell = r - 1,$ $\mathbf{z} = 0$	$g_{LS_n}^{r-1}(\mathbf{x}, \mathbf{y}, 0; C, E)$ $= {}_{US_n^r}(\mathbf{x}, \mathbf{y}; C, E)$	Generalized Chebyshev-Sheffer matrix polynomials (GCSMaP)	$\frac{1}{q(p^{-1}(\tau))} \exp\left(\mathbf{x}p^{-1}(\tau)\sqrt{2C} + E\mathbf{y}(p^{-1}(\tau))^{r-1}\right) = \sum_{n=0}^{\infty} {}_{US_n^r}(\mathbf{x}, \mathbf{y}; C, E) \frac{\tau^n}{n!}$
IV.	$\ell = 2,$ $\mathbf{z} = 0$	$g_{LS_n}^2(\mathbf{x}, \mathbf{y}, 0; C, E)$ $= {}_{HS_n}(\mathbf{x}, \mathbf{y}; C, E)$	Hermite Kampé de Fériet-Sheffer matrix polynomials (HKdFSMaP)	$\frac{1}{q(p^{-1}(\tau))} \exp\left(\mathbf{x}p^{-1}(\tau)\sqrt{2C} + E\mathbf{y}(p^{-1}(\tau))^2\right) = \sum_{n=0}^{\infty} {}_{HS_n}(\mathbf{x}, \mathbf{y}; C, E) \frac{\tau^n}{n!}$
V.	$\mathbf{z} = 0, \mathbf{x} \rightarrow \mathbf{y}$ $\mathbf{y} \rightarrow D_{\mathbf{x}}^{-1}$	$LS_n^\ell(\mathbf{y}, D_{\mathbf{x}}^{-1}, 0; C, E)$ $= LS_n^\ell(\mathbf{x}, \mathbf{y}; C, E)$	Generalized Laguerre-Sheffer matrix polynomials (GLSMaP)	$\frac{1}{q(p^{-1}(\tau))} C_0(-E\mathbf{x}(p^{-1}(\tau))^\ell) \times \exp\left(\mathbf{y}p^{-1}(\tau)\sqrt{2C}\right) = \sum_{n=0}^{\infty} LS_n^\ell(\mathbf{x}, \mathbf{y}; C, E) \frac{\tau^n}{n!}$
VI.	$\mathbf{x} = -D_{\mathbf{x}}^{-1},$ $\mathbf{z} = 0$	$g_{LS_n}^\ell(-D_{\mathbf{x}}^{-1}, \mathbf{y}; C, E)$ $= {}_{[q]LS_n}^\ell(\mathbf{x}, \mathbf{y}; C, E)$	2-Variable generalized Laguerre type Sheffer matrix polynomials (2VgLtSMaP)	$\frac{1}{q(p^{-1}(\tau))} C_0\left(\mathbf{x}p^{-1}(\tau)\sqrt{2C}\right) \exp\left(E\mathbf{y}(p^{-1}(\tau))^\ell\right) = \sum_{n=0}^{\infty} {}_{[q]LS_n}^\ell(\mathbf{x}, \mathbf{y}; C, E) \frac{\tau^n}{n!}$
VII.	$\mathbf{y} = 0, \mathbf{z} \rightarrow \mathbf{x},$ $\mathbf{x} \rightarrow \mathbf{y}$	$g_{LS_n}^\ell(\mathbf{y}, 0, \mathbf{x}; C, E)$ $= {}_{LS_n}(\mathbf{x}, \mathbf{y}; C)$	Laguerre-Sheffer matrix polynomials (LSaMP)	$\frac{1}{q(p^{-1}(\tau))} C_0(\mathbf{x}p^{-1}(\tau)) \exp\left(\mathbf{y}p^{-1}(\tau)\sqrt{2C}\right) = \sum_{n=0}^{\infty} {}_{LS_n}(\mathbf{x}, \mathbf{y}; C) \frac{\tau^n}{n!}$

Remark 5. For the particular matrix polynomials demonstrated in Table 1, we may offer some properties corresponding to those in Theorems 1–4.

We may get a variety of outcomes that correspond to the above-presented results by varying the invertible series $q(\tau)$ and the delta series $p(\tau)$. As in Remark 1, the following corollaries give the corresponding results to those in Theorems 3 and 4 for the associated and Appell polynomials.

Associated Polynomials

Corollary 1. The associated polynomials ${}_{[1,p]}g_{LS_n}^\ell(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E)$ satisfy the following quasi-monomiality with regard to the operators of multiplication and differentiation:

$${}_{[1,p]}\hat{M}_{g_{LS}} = \left(\mathbf{x}\sqrt{2C} - \hat{D}_{\mathbf{z}}^{-1} + \ell E\mathbf{y}(\sqrt{2C})^{-(\ell-1)}D_{\mathbf{x}}^{\ell-1}\right) \frac{1}{p'((\sqrt{2C})^{-1}D_{\mathbf{x}})} \tag{48}$$

and

$${}_{[1,p]}\hat{P}_{g_{LS}} = p\left(\left(\sqrt{2C}\right)^{-1}D_{\mathbf{x}}\right), \tag{49}$$

respectively.

Corollary 2. The associated polynomials ${}_{[1,p]}g_{LS_n}^\ell(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E)$ satisfy the following differential equation:

$$\left\{ \left(x\sqrt{2C} - \hat{D}_z^{-1} + \ell E y (\sqrt{2C})^{-(\ell-1)} D_x^{\ell-1} \right) \times \frac{p((\sqrt{2C})^{-1} D_x)}{p'((\sqrt{2C})^{-1} D_x)} - n \right\}_{[1,p] gLS_n^\ell(x, y, z; C, E)} = 0. \tag{50}$$

Appell Polynomials

Corollary 3. The Appell polynomials $_{[q(\tau),\tau]} gLS_n^\ell(x, y, z; C, E)$ gratify the following quasi- monominality with respect to the operators of multiplication and differentiation:

$$_{[q(\tau),\tau]} \hat{M}_{gLS} = \left(x\sqrt{2C} - \hat{D}_z^{-1} + \ell E y (\sqrt{2C})^{-(\ell-1)} D_x^{\ell-1} - \frac{q'((\sqrt{2C})^{-1} D_x)}{q((\sqrt{2C})^{-1} D_x)} \right) \tag{51}$$

and

$$_{[q(\tau),\tau]} \hat{P}_{gLS} = (\sqrt{2C})^{-1} D_x, \tag{52}$$

respectively.

Corollary 4. The Appell polynomials $_{[q(\tau),\tau]} gLS_n^\ell(x, y, z; C, E)$ gratify the following differential equation:

$$\left\{ \left(x\sqrt{2C} - \hat{D}_z^{-1} + \ell E y (\sqrt{2C})^{-(\ell-1)} D_x^{\ell-1} - \frac{q'((\sqrt{2C})^{-1} D_x)}{q((\sqrt{2C})^{-1} D_x)} \right) \times \sqrt{2C}^{-1} D_x - n \right\}_{[q(\tau),\tau] gLS_n^\ell(x, y, z; C, E)} = 0. \tag{53}$$

3. Extended Gould-Hopper-Laguerre-Sheffer Matrix Polynomials

Fractional calculus is a well-established theory that is extensively employed in a broad variety of fields of science, engineering, and mathematics today. The use of integral transforms and operational procedures to new families of special polynomials is a reasonably effective technique (consult, for instance, [28]).

This section provides some properties for the extended Gould-Hopper-Laguerre-Sheffer matrix polynomials in (31).

Theorem 5. Let $\Re(\eta) > 0$ and $\Re(\nu) > 0$. Then the following integral representation for the extended Gould-Hopper-Laguerre-Sheffer matrix polynomials $gLS_{n,\nu}^\ell(x, y, z; C, E; \eta)$ holds true:

$$gLS_{n,\nu}^\ell(x, y, z; C, E; \eta) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\eta t} t^{\nu-1} gLS_n^\ell(x, yt, z; C, E) dt. \tag{54}$$

Proof. Let \mathcal{L} be the left-sided member of (54). Using (29) and (31), we have

$$\begin{aligned} \mathcal{L} &= \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\eta t} t^{\nu-1} \exp\left(yEt (\sqrt{2C})^{-\ell} \frac{\partial^\ell}{\partial x^\ell} \right) \left\{ LS_n(z, x\sqrt{2C}) \right\} dt \\ &= \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\eta t} t^{\nu-1} gLS_n^\ell(x, yt, z; C, E) dt, \end{aligned} \tag{55}$$

the second equality of which follows from (34). □

The following theorem gives the generating function of the EGHLMaP.

Theorem 6. The following function generates the extended Gould-Hopper-Laguerre-Sheffer matrix polynomials ${}_{gLS}^{\ell}_{n,\nu}(x, y, z; C, E; \eta)$:

$$\frac{\exp(x\sqrt{2C}p^{-1}(u))C_0(zp^{-1}(u))}{q(p^{-1}(u))\{\eta - Ey(p^{-1}(u))^\ell\}^\nu} = \sum_{n=0}^{\infty} {}_{gLS}^{\ell}_{n,\nu}(x, y, z; C, E; \eta) \frac{u^n}{n!}. \tag{56}$$

Additionally, the following differential-recursive relation holds true:

$$\frac{\partial}{\partial \eta} {}_{gLS}^{\ell}_{n,\nu}(x, y, z; C, E; \eta) = -\nu {}_{gLS}^{\ell}_{n,\nu+1}(x, y, z; C, E; \eta). \tag{57}$$

Proof. Multiplying each member of (54) by $\frac{u^n}{n!}$ and adding over n , one derives

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_{gLS}^{\ell}_{n,\nu}(x, y, z; C, E; \eta) \frac{u^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\eta t} t^{\nu-1} {}_{gLS}^{\ell}_n(x, yt, z; C, E) \frac{u^n}{n!} dt. \end{aligned} \tag{58}$$

Using (22) in the integrand of the right-sided member of (58) gives

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_{gLS}^{\ell}_{n,\nu}(x, y, z; C, E; \eta) \frac{u^n}{n!} \\ &= \frac{C_0(z(p^{-1}(u))^\ell) \exp(x\sqrt{2C}p^{-1}(u))}{q(p^{-1}(u))\Gamma(\nu)} \int_0^\infty e^{-\{\eta - Ey(f^{-1}(u))^\ell\}t} t^{\nu-1} dt, \end{aligned}$$

the right member of which, upon using (29), leads to the left-sided member of (56).

Differentiating each member of (56) about η , one may get (57). \square

The following theorem reveals that the EGHLSMaP ${}_{gLS}^{\ell}_{n,\nu}(x, y, z; C, E; \eta)$ is an extension of the GHLSMaP ${}_{gLS}^{\ell}_n(x, y, z; C, E)$.

Theorem 7. The following identities hold true:

$$\begin{aligned} & \frac{\exp(x\sqrt{2C}p^{-1}(u))C_0(zp^{-1}(u))}{q(p^{-1}(u))} {}_1F_1\left(\nu; 1; Ey(p^{-1}(u))^\ell\right) \\ &= \sum_{n=0}^{\infty} {}_{gLS}^{\ell}_{n,\nu}(x, \hat{D}_y^{-1}, z; C, E; 1) \{1\} \frac{u^n}{n!}; \end{aligned} \tag{59}$$

$${}_{gLS}^{\ell}_n(x, y, z; C, E) = {}_{gLS}^{\ell}_{n,1}(x, \hat{D}_y^{-1}, z; C, E; 1) \{1\}. \tag{60}$$

Proof. Taking $\eta = 1$ and $y = \hat{D}_y^{-1}$ in (56), we get

$$G(\nu; t) := \frac{\exp(x\sqrt{2C}p^{-1}(u))C_0(zp^{-1}(u))}{q(p^{-1}(u))} \left(1 - E\hat{D}_y^{-1}(p^{-1}(u))^\ell\right)^{-\nu} \{1\}. \tag{61}$$

Using (20), we obtain

$$\begin{aligned} \left(1 - E\hat{D}_y^{-1}(p^{-1}(u))^\ell\right)^{-\nu} \{1\} &= \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} E^n \left(p^{-1}(u)\right)^{\ell n} \hat{D}_y^{-n} \{1\} \\ &= \sum_{n=0}^{\infty} \frac{(\nu)_n E^n y^n \left(p^{-1}(u)\right)^{\ell n}}{(1)_n n!} \\ &= {}_1F_1\left(\nu; 1; Ey(p^{-1}(u))^\ell\right), \end{aligned} \tag{62}$$

for the second and third equalities of which (6) and (17) are employed, respectively.

Now, setting the last expression of (62) in (61), in view of (56), we obtain (59).

Noting

$${}_1F_1\left(1; 1; Ey(p^{-1}(u))^\ell\right) = \exp\left(Ey\left(p^{-1}(u)\right)^\ell\right),$$

we find that the resulting $G(t; 1)$ is the generating function of the Gould-Hopper-Laguerre-Sheffer matrix polynomials ${}_gLS_n^\ell(x, y, z; C, E)$ in (22). We therefore have

$$\sum_{n=0}^\infty {}_gLS_{n,1}^\ell(x, \hat{D}_y^{-1}, z; C, E; 1)\{1\} \frac{u^n}{n!} = \sum_{n=0}^\infty {}_gLS_n^\ell(x, y, z; C, E) \frac{u^n}{n!},$$

which, upon equating the coefficients of u^n , yields (60).

The identity (60) may be obtained as follows: Combining (31) and (34) gives

$${}_gLS_n^\ell(x, y, z; C, E) = \left(1 - \hat{D}_y^{-1}E\left(\sqrt{2C}\right)^{-\ell} D_x^\ell\right) \exp\left(yE\left(\sqrt{2C}\right)^{-\ell} D_x^\ell\right) \times \left\{{}_gLS_{n,1}^\ell(x, \hat{D}_y^{-1}, z; C, E; 1)\right\}.$$

As in (62), we find

$$\exp\left(yE\left(\sqrt{2C}\right)^{-\ell} D_x^\ell\right) = \left(1 - \hat{D}_y^{-1}E\left(\sqrt{2C}\right)^{-\ell} D_x^\ell\right)^{-1} \{1\}.$$

□

Remark 6. As in (ii), Remark 1, the Laguerre-Sheffer polynomials ${}_LS_n(x, y)$ reduce to the Laguerre-Appell polynomials ${}_LA_n(x, y)$ (see [15]). Additionally, taking $p^{-1}(u) = u$ in the generating equation (56), we can get the generalized Gould-Hopper-Laguerre-Appell matrix polynomials ${}_gLA_{n,\nu}^\ell(x, y, z; C, E; \eta)$ (see [27]).

The following theorem reveals the quasi-monomial principle of the extended Gould-Hopper-Laguerre-Sheffer matrix polynomials ${}_gLS_{n,\nu}^\ell(x, y, z; C, E; \eta)$.

Theorem 8. The matrix polynomials ${}_gLS_{n,\nu}^\ell(x, y, z; C, E; \eta)$ satisfy the following quasi-monomiality with regard to the operators of multiplication and differentiation:

$$\hat{M}_{gLS_\nu} = \left(x\sqrt{2C} - \hat{D}_z^{-1} - \ell Ey\left(\sqrt{2C}\right)^{-(\ell-1)} D_\eta D_x^{\ell-1} - \frac{q'((\sqrt{2C})^{-1}D_x)}{q((\sqrt{2C})^{-1}D_x)}\right) \times \frac{1}{p'((\sqrt{2C})^{-1}D_x)} \tag{63}$$

and

$$\hat{P}_{gLS_\nu} = p\left(\left(\sqrt{2C}\right)^{-1} D_x\right), \tag{64}$$

respectively. Here $D_\eta := \frac{\partial}{\partial \eta}$.

Proof. From Theorem 3, we have

$$\left(x\sqrt{2C} - \hat{D}_z^{-1} + \ell Ey\left(\sqrt{2C}\right)^{-(\ell-1)} D_x^{\ell-1} - \frac{q'((\sqrt{2C})^{-1}D_x)}{q((\sqrt{2C})^{-1}D_x)}\right) \times \frac{1}{p'((\sqrt{2C})^{-1}D_x)} {}_gLS_n^\ell(x, y, z; C, E) = {}_gLS_{n+1}^\ell(x, y, z; C, E), \tag{65}$$

and

$$p\left(\left(\sqrt{2C}\right)^{-1} D_x\right) {}_gLS_n^\ell(x, y, z; C, E) = n {}_gLS_{n-1}^\ell(x, y, z; C, E). \tag{66}$$

Replacing y by yt in each member of (66), multiplying both members of the resultant identity by $\frac{1}{\Gamma(\nu)}e^{-\eta t}t^{\nu-1}$, and integrating each member of the last resultant identity with respect to t from 0 to ∞ , with the aid of (54), one obtains

$$p\left(\left(\sqrt{2C}\right)^{-1}D_x\right)\left\{g_{LS_{n,\nu}}^\ell(x,y,z;C,E;\eta)\right\}=n g_{LS_{n-1,\nu}}^\ell(x,y,z;C,E;\eta),$$

which proves (64).

Furthermore, replacing y by yt in both sides of (65), multiplying both members of the resultant identity by $\frac{1}{\Gamma(\nu)}e^{-\eta t}t^{\nu-1}$, and integrating both sides of the last resulting identity with respect to t from 0 to ∞ , with the help of (54) and (57), one can derive

$$\hat{M}_{g_{LS_{n,\nu}}^\ell}\left\{g_{LS_{n,\nu}}^\ell(x,y,z;C,E;\eta)\right\}=g_{LS_{n+1,\nu}}^\ell(x,y,z;C,E;\eta).$$

This proves (63). \square

As in Theorem 4, using the results in Theorem 8, a differential equation for the extended Gould-Hopper-Laguerre-Sheffer matrix polynomials $g_{LS_{n,\nu}}^\ell(x,y,z;C,E;\eta)$ can be given in Theorem 9.

Theorem 9. *The following differential equation holds true:*

$$\left\{\left(x\sqrt{2C}-\hat{D}_z^{-1}-\ell E y(\sqrt{2C})^{-(\ell-1)}D_z D_x^{\ell-1}-\frac{q'((\sqrt{2C})^{-1}D_x)}{q((\sqrt{2C})^{-1}D_x)}\right)\right. \\ \left.\times\frac{p((\sqrt{2C})^{-1}D_x)}{p'((\sqrt{2C})^{-1}D_x)}-n\right\}g_{LS_{n,\nu}}^\ell(x,y,z;C,E;\eta)=0. \tag{67}$$

As in Table 1, Table 2 includes certain particular cases of the extended Gould-Hopper-Laguerre-Sheffer matrix polynomials $g_{LS_{n,\nu}}^\ell(x,y,z;C,E;\eta)$, among numerous ones.

Table 2. Special cases of the EGHLSMaP $g_{LS_{n,\nu}}^\ell(x,y,z;C,E;\eta)$.

S. No.	Values of the Indices and Variables	Name of the Hybrid Special Polynomials	Generating Function
I.	$\ell = 2$	3-Variable extended Hermite-Laguerre-Sheffer matrix polynomials (3VEHLSMaP)	$\frac{\exp((xp^{-1}(u)\sqrt{2C})C_0(zp^{-1}(u)))}{q(p^{-1}(u))(\eta-Ey(p^{-1}(u))^2)^\nu}$ $= \sum_{n=0}^\infty H_{LS_{n,\nu}}^\ell(x,y,z;C,E,\eta)\frac{z^n}{n!}$
II.	$z = 0$	Extended Gould-Hopper-Sheffer-matrix polynomials (EGHSMaP)	$\frac{\exp(xp^{-1}(u)\sqrt{2C})}{q(p^{-1}(u))(\eta-Ey(p^{-1}(u))^\ell)^\nu}$ $= \sum_{n=0}^\infty g_{LS_{n,\nu}}^\ell(x,y;C,E,\eta)\frac{z^n}{n!}$
III.	$\ell = r - 1,$ $z = 0$	Extended generalized Chebyshev-Sheffer matrix polynomials (EGCSMaP)	$\frac{\exp((xp^{-1}(u)\sqrt{2C}))}{q(p^{-1}(u))(\eta-Ey(p^{-1}(u))^{r-1})^\nu}$ $= \sum_{n=0}^\infty U_{LS_{n,\nu}}^r(x,y;C,E,\eta)\frac{z^n}{n!}$
IV.	$\ell = 2,$ $z = 0$	Extended Hermite Kampé de Fériet-Sheffer matrix polynomials (EHKdFSMaP)	$\frac{\exp(xp^{-1}(u)\sqrt{2C})}{q(p^{-1}(u))(\eta-Ey(p^{-1}(u))^2)^\nu}$ $= \sum_{n=0}^\infty H_{LS_{n,\nu}}^\ell(x,y;C,E,\eta)\frac{z^n}{n!}$
V.	$z = 0, x \rightarrow y$ $y \rightarrow D_x^{-1}$	Extended generalized Laguerre-Sheffer matrix polynomials (EGLSMaP)	$\frac{C_0(-Ex(p^{-1}(u))^\ell)}{q(p^{-1}(u))(\eta-y\sqrt{2C}p^{-1}(u))^\nu}$ $= \sum_{n=0}^\infty L_{LS_{n,\nu}}^\ell(x,y;C,E,\eta)\frac{z^n}{n!}$

Table 2. Cont.

S. No.	Values of the Indices and Variables	Name of the Hybrid Special Polynomials	Generating Function
VI.	$x = -D_z^{-1},$ $z = 0$	2-Variable extended generalized Laguerre type Sheffer matrix polynomials (2VEgLtSMaP)	$\frac{C_0(xp^{-1}(u)\sqrt{2C})}{q(p^{-1}(u))(\eta - Ey(p^{-1}(u)))^\ell} = \sum_{n=0}^{\infty} [n]_{[p]} L_{S_{n,\nu}}(x, y; C, E, \eta) \frac{x^n}{n!}$
VII.	$y = 0, z \rightarrow x,$ $x \rightarrow y$	Extended Laguerre-Sheffer matrix polynomials (ELSaMP)	$\frac{C_0(xp^{-1}(u))}{q(p^{-1}(u))(\eta - y\sqrt{2C}p^{-1}(u))^\ell} = \sum_{n=0}^{\infty} L_{S_{n,\nu}}(x, y; C, \eta) \frac{x^n}{n!}$

Remark 7. As in (i), Remark 1, if $q(\tau) = 1$, the Laguerre-Sheffer polynomials $L_{S_n}(x, y)$ reduce to the Laguerre-associated Sheffer polynomials ${}_{[1,p]}L_{S_n}(x, y)$. The extended Gould-Hopper-Laguerre-Sheffer matrix polynomials $gLS_{n,\nu}^\ell(x, y, z; C, E; \eta)$ reduce to the extended Gould-Hopper-Laguerre-associated Sheffer matrix polynomials (EGHLASMaP) ${}_{[1,p]}gLS_{n,\nu}^\ell(x, y, z; C, E; \eta)$. The following corollary contains the results for EGHLASMaP corresponding to those in Theorems 5–9. □

Corollary 5. (i) Let $\Re(\eta) > 0$ and $\Re(\nu) > 0$.

$${}_{[1,p]}gLS_{n,\nu}^\ell(x, y, z; C, E; \eta) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\eta u} u^{\nu-1} {}_{[1,p]}gLS_n^\ell(x, y, u, z; C, E) du. \tag{68}$$

(ii) The polynomials ${}_{[1,p]}gLS_{n,\nu}^\ell(x, y, z; C, E; \eta)$ are generated by means of the following function:

$$\frac{\exp(x\sqrt{2C}p^{-1}(u))C_0(zp^{-1}(u))}{p^{-1}(u) \{ \eta - Ey(p^{-1}(u))^\ell \}^\nu} = \sum_{n=0}^{\infty} {}_{[1,p]}gLS_{n,\nu}^\ell(x, y, z; C, E; \eta) \frac{u^n}{n!}. \tag{69}$$

Additionally, the following differential-recursive relation holds true:

$$\frac{\partial}{\partial \eta} {}_{[1,p]}gLS_{n,\nu}^\ell(x, y, z; C, E; \eta) = -\nu {}_{[1,p]}gLS_{n,\nu+1}^\ell(x, y, z; C, E; \eta). \tag{70}$$

(iii) The matrix polynomials ${}_{[1,p]}gLS_{n,\nu}^\ell(x, y, z; C, E; \eta)$ gratify quasi-monomiality with regard to the following operators of multiplication and differentiation:

$${}_{[1,p]}M_{gLS_\nu} = \left(x\sqrt{2C} - \hat{D}_z^{-1} - \ell Ey(\sqrt{2C})^{-(\ell-1)} D_\eta D_x^{\ell-1} \right) \frac{1}{p'((\sqrt{2C})^{-1} D_x)} \tag{71}$$

and

$${}_{[1,p]}P_{gLS_\nu} = p \left((\sqrt{2C})^{-1} D_x \right), \tag{72}$$

respectively.

(iv) The following differential equation holds true:

$$\left\{ \left(x\sqrt{2C} - \hat{D}_z^{-1} - \ell Ey(\sqrt{2C})^{-(\ell-1)} D_z D_x^{\ell-1} \right) \times \frac{p((\sqrt{2C})^{-1} D_x)}{p'((\sqrt{2C})^{-1} D_x)} - n \right\} {}_{[1,p]}gLS_{n,\nu}^\ell(x, y, z; C, E; \eta) = 0. \tag{73}$$

4. Remarks and Further Particular Cases

The ${}_1F_1$ in (59), which is called the confluent hypergeometric function or Kummer’s function, is an important and useful particular case of ${}_aF_b$ in (17). It also has various other notations (consult, for instance, [11], p. 70). For properties and identities of ${}_1F_1$, one may consult the monograph [29]. In this regard, in view of (59), one may offer a vari-

ety of identities for the ${}_g L S_{n,\nu}^\ell(x, \hat{D}_y^{-1}, z; C, E; 1)\{1\}$. In order to give a demonstration, the ${}_1 F_1$ in (59) has the following integral representation (consult, for instance, [11], p. 70, Equation (46)):

$$\begin{aligned}
 & {}_1 F_1\left(\nu; 1; Ey(p^{-1}(u))^\ell\right) \\
 &= \frac{1}{\Gamma(\nu)\Gamma(1-\nu)} \int_0^1 \eta^{\nu-1}(1-\eta)^{-\nu} \exp\left(Ey\left(p^{-1}(u)\right)^\ell \eta\right) d\eta \quad (0 < \Re(\nu) < 1).
 \end{aligned}
 \tag{74}$$

Further, using (35) and (59), with the aid of (21) and (74), one may readily get the following identity:

$$\begin{aligned}
 & {}_{[q(u),u]} g L S_{n,\nu}^\ell(x, \hat{D}_y^{-1}, z; C, E; 1)\{1\} \\
 &= \sum_{k=0}^{\lfloor \frac{n}{\ell} \rfloor} \frac{n! (\nu)_k}{(k!)^2 (n-\ell k)!} (Ey)^k {}_{[q(u),u]} L S_{n-\ell k}(z, x\sqrt{2C}).
 \end{aligned}
 \tag{75}$$

The hybrid matrix polynomials introduced in Sections 2 and 3, besides the demonstrated particular cases, may produce numerous other particular cases as well as corresponding properties. In this section, we combine the findings from Sections 2 and 3 with several well-known (or classical) polynomials to derive some related identities.

- (a) The Hermite polynomials $H_n(x)$, which are generated by the following function (consult, for example, [30]):

$$\exp(2x\tau - \tau^2) = \sum_{n=0}^{\infty} H_n(x) \frac{\tau^n}{n!}
 \tag{76}$$

belongs to the Sheffer family by choosing

$$q(\tau) = e^{\tau^2/4}, \quad p(\tau) = \frac{\tau}{2}, \quad \text{and} \quad p^{-1}(\tau) = 2\tau
 \tag{77}$$

in (3).

For these choices of $q(\tau)$ and $p(\tau)$ in (22) and (56), the GHLSMaP ${}_g L S_n^\ell(x, y, z; C, E)$ and the EGHLSMaP ${}_g L S_{n,\nu}^\ell(x, y, z; C, E; \eta)$ are called (denoted) as the matrix polynomials of Gould-Hopper-Laguerre-Hermite (GHLHMaP) ${}_g L H_n^\ell(x, y, z; C, E)$ and the extended matrix polynomials of Gould-Hopper-Laguerre-Hermite (EGHLHMaP) ${}_g L H_{n,\nu}^\ell(x, y, z; C, E; \eta)$, respectively.

Some identities corresponding to those in Sections 2 and 3 are recorded in Tables 3 and 4.

Table 3. Results for the GHLHMaP ${}_g L H_n^\ell(x, y, z; C, E)$.

Results	Expressions
Generating function:	$\exp\left(2x\tau\sqrt{2C} + Ey(2\tau)^\ell - \tau^2\right) C_0(2z\tau) = \sum_{n=0}^{\infty} {}_g L H_n^\ell(x, y, z; C, E) \frac{\tau^n}{n!}$.
Multiplicative and derivative operators:	$\hat{M}_{g L H} = \left(x\sqrt{2C} - \hat{D}_z^{-1} + \frac{\ell Ey}{(\sqrt{2C})^{\ell-1}} \frac{\partial^{\ell-1}}{\partial x^{\ell-1}} - \frac{(\sqrt{2C})^{-1} D_x}{2}\right) 2,$ $\hat{P}_{g L H} = \frac{(\sqrt{2C})^{-1} D_x}{2}.$
Differential equation:	$\left(\left(x\sqrt{2C} - \hat{D}_z^{-1} + \frac{\ell Ey}{(\sqrt{2C})^{\ell-1}} \frac{\partial^{\ell-1}}{\partial x^{\ell-1}} - \frac{(\sqrt{2C})^{-1} D_x}{2}\right) (\sqrt{2C})^{-1} D_x - n\right) \times {}_g L H_n^\ell(x, y, z; C, E) = 0.$

Table 4. Results for the EGHLHMaP ${}_{\mathbf{g}L}H_{n,\nu}^\ell(x, y, \mathbf{z}; C, E; \alpha)$.

Results	Expressions
Generating function:	$\frac{\exp(2x\tau\sqrt{2C})C_0(2z\tau)}{e^{\tau^2}(\alpha - Ey(2\tau)^\ell)^\nu} = \sum_{n=0}^\infty {}_{\mathbf{g}L}H_{n,\nu}^\ell(x, y, \mathbf{z}; C, E; \alpha) \frac{\tau^n}{n!}$.
Multiplicative and derivative operators:	$\hat{M}_{\mathbf{g}LH\nu} = \left(x\sqrt{2C} - \hat{D}_z^{-1} - \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^\ell}{\partial \alpha \partial x^{\ell-1}} - \frac{(\sqrt{2C})^{-1} D_x}{2} \right) 2,$ $\hat{P}_{\mathbf{g}LH\nu} = \frac{(\sqrt{2C})^{-1} D_x}{2}.$
Differential equation:	$\left(\left(x\sqrt{2C} - \hat{D}_z^{-1} + \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^\ell}{\partial \alpha \partial x^{\ell-1}} - \frac{(\sqrt{2C})^{-1} D_x}{2} \right) (\sqrt{2C})^{-1} D_x - n \right) \times {}_{\mathbf{g}L}H_{n,\nu}^\ell(x, y, \mathbf{z}; C, E; \alpha) = 0.$

(b) The truncated exponential polynomials $e_n(x)$, which are generated by the following function (consult, for example, [31], p. 596, Equation (4); see also [32]):

$$\frac{e^{x\tau}}{1 - \tau} = \sum_{n=0}^\infty e_n(x) \frac{\tau^n}{n!} \tag{78}$$

belong to the Sheffer family by choosing $q(\tau) = \frac{1}{1-\tau}$ and $p(\tau) = \tau$. As in (a), the GHLSMaP ${}_{\mathbf{g}L}S_n^\ell(x, y, \mathbf{z}; C, E)$ and EGHLMaP ${}_{\mathbf{g}L}S_{n,\nu}^\ell(x, y, \mathbf{z}; C, E; \eta)$ are called (denoted) as the Gould-Hopper-Laguerre-truncated exponential matrix polynomials (GHLTEMaP) ${}_{\mathbf{g}L}e_n^\ell(x, y, \mathbf{z}; C, E)$ and extended Gould-Hopper-Laguerre-truncated exponential matrix polynomials (EGHLEMaP) ${}_{\mathbf{g}L}e_{n,\nu}^\ell(x, y, \mathbf{z}; C, E; \eta)$, respectively. As in (a), their properties are recorded in Tables 5 and 6.

Table 5. Results for the GHLTEMaP ${}_{\mathbf{g}L}e_n^\ell(x, y, \mathbf{z}; C, E)$.

Results	Expressions
Generating function:	$\frac{1}{1-t} \exp(xt\sqrt{2C} + Eyt^\ell) C_0(zt) = \sum_{n=0}^\infty {}_{\mathbf{g}L}e_n^\ell(x, y, \mathbf{z}; C, E) \frac{t^n}{n!}$.
Multiplicative and derivative operators:	$\hat{M}_{\mathbf{g}Le} = x\sqrt{2C} - \hat{D}_z^{-1} + \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^{\ell-1}}{\partial x^{\ell-1}} - \frac{1}{1 - (\sqrt{2C})^{-1} D_x},$ $\hat{P}_{\mathbf{g}Le} = \left(\sqrt{2C} \right)^{-1} D_x.$
Differential equation:	$\left(\left(x\sqrt{2C} - \hat{D}_z^{-1} + \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^{\ell-1}}{\partial x^{\ell-1}} - \frac{1}{1 - (\sqrt{2C})^{-1} D_x} \right) (\sqrt{2C})^{-1} D_x - n \right) \times {}_{\mathbf{g}L}e_n^\ell(x, y, \mathbf{z}; C, E) = 0.$

Table 6. Results for the EGHLEMaP ${}_{\mathbf{g}L}e_{n,\nu}^\ell(x, y, \mathbf{z}; C, E; \eta)$.

Results	Expressions
Generating function:	$\frac{1}{1-u} \frac{\exp(xu\sqrt{2C})C_0(zu)}{(\alpha - Eyu^\ell)^\nu} = \sum_{n=0}^\infty {}_{\mathbf{g}L}e_{n,\nu}^\ell(x, y, \mathbf{z}; C, E; \alpha) \frac{u^n}{n!}$.
Multiplicative and derivative operators:	$\hat{M}_{\mathbf{g}Le\nu} = x\sqrt{2C} - \hat{D}_z^{-1} - \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^\ell}{\partial \alpha \partial x^{\ell-1}} - \frac{1}{1 - (\sqrt{2C})^{-1} D_x},$ $\hat{P}_{\mathbf{g}Le\nu} = \left(\sqrt{2C} \right)^{-1} D_x.$
Differential equation:	$\left(\left(x\sqrt{2C} - \hat{D}_z^{-1} - \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^\ell}{\partial \alpha \partial x^{\ell-1}} - \frac{1}{1 - (\sqrt{2C})^{-1} D_x} \right) (\sqrt{2C})^{-1} D_x - n \right) \times {}_{\mathbf{g}L}e_{n,\nu}^\ell(x, y, \mathbf{z}; C, E; \alpha) = 0.$

(c) The Mittag-Leffler polynomials $M_n(x)$, which are the member of associated Sheffer family and defined as follows (see [4]):

$$\left(\frac{1+\tau}{1-\tau}\right)^x = \sum_{n=0}^{\infty} M_n(x) \frac{\tau^n}{n!} \tag{79}$$

by choosing $q(\tau) = 1$ and $p(\tau) = \frac{e^{\tau}-1}{e^{\tau}+1}$. As in (a), the GHLASMaP ${}_g L s_n^{\ell}(x, y, z; C, E)$ and the EGHLASMaP ${}_g L s_{n,\nu}^{\ell}(x, y, z; C, E; \eta)$ are called (denoted) as the Gould-Hopper-Laguerre-Mittag-Leffler matrix polynomials (GHLMLMaP) ${}_g L M_n^{\ell}(x, y, z; C, E)$ and the extended Gould-Hopper-Laguerre-Mittag-Leffler matrix polynomials (EGHLMLMaP) ${}_g L M_{n,\nu}^{\ell}(x, y, z; C, E; \eta)$, respectively. As in (a) or (b), their properties are recorded in Tables 7 and 8.

Table 7. Results for the GHLMLMaP ${}_g L M_n^{\ell}(x, y, z; C, E)$.

Results	Expressions
Generating function:	$\exp\left(x \ln\left(\frac{1+\tau}{1-\tau}\right) \sqrt{2C} + Ey \ln\left(\frac{1+\tau}{1-\tau}\right)^{\ell}\right) C_0\left(z \ln\left(\frac{1+\tau}{1-\tau}\right)\right) = \sum_{n=0}^{\infty} {}_g L M_n^{\ell}(x, y, z; C, E) \frac{\tau^n}{n!}.$
Multiplicative and derivative operators:	$\hat{M}_{gLM} = \left(x\sqrt{2C} - \hat{D}_z^{-1} + \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^{\ell-1}}{\partial x^{\ell-1}}\right) \frac{\left(e^{(\sqrt{2C})^{-1}D_x+1}\right)^2}{2 e^{(\sqrt{2C})^{-1}D_x}},$ $\hat{P}_{gLM} = \frac{e^{(\sqrt{2C})^{-1}D_x-1}}{e^{(\sqrt{2C})^{-1}D_x+1}}.$
Differential equation:	$\left(\left(x\sqrt{2C} - \hat{D}_z^{-1} + \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^{\ell-1}}{\partial x^{\ell-1}}\right) \frac{e^{2(\sqrt{2C})^{-1}D_x-1}}{2 e^{(\sqrt{2C})^{-1}D_x}} - n\right) \times {}_g L M_n^{\ell}(x, y, z; C, E) = 0.$

Table 8. Results for the EGHLMLMaP ${}_g L M_{n,\nu}^{\ell}(x, y, z; C, E; \eta)$.

Results	Expressions
Generating function:	$\frac{\exp\left(x \ln\left(\frac{1+\tau}{1-\tau}\right) \sqrt{2C}\right) C_0\left(z \ln\left(\frac{1+\tau}{1-\tau}\right)\right)}{\left(\alpha - Ey \ln\left(\frac{1+\tau}{1-\tau}\right)^{\ell}\right)^{\nu}} = \sum_{n=0}^{\infty} {}_g L M_{n,\nu}^{\ell}(x, y, z; C, E; \eta) \frac{\tau^n}{n!}.$
Multiplicative and derivative operators:	$\hat{M}_{gLM_{\nu}} = \left(x\sqrt{2C} - \hat{D}_z^{-1} - \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^{\ell}}{\partial \alpha \partial x^{\ell-1}}\right) \frac{\left(e^{(\sqrt{2C})^{-1}D_x+1}\right)^2}{2 e^{(\sqrt{2C})^{-1}D_x}},$ $\hat{P}_{gLM_{\nu}} = \frac{e^{(\sqrt{2C})^{-1}D_x-1}}{e^{(\sqrt{2C})^{-1}D_x+1}}.$
Differential equation:	$\left(\left(x\sqrt{2C} - \hat{D}_z^{-1} - \frac{\ell Ey}{(\sqrt{2C})^{(\ell-1)}} \frac{\partial^{\ell}}{\partial \alpha \partial x^{\ell-1}}\right) \frac{e^{2(\sqrt{2C})^{-1}D_x-1}}{2 e^{(\sqrt{2C})^{-1}D_x}} - n\right) \times {}_g L M_{n,\nu}^{\ell}(x, y, z; C, E; \eta) = 0.$

Numerous necessary and sufficient properties for Sheffer sequences, accordingly, associated sequences and Appell sequences have been developed (see [4], pp. 17–28). In addition to the identities in Corollaries 3 and 4, here, we record several identities for the Appell polynomials ${}_{[q(\tau),\tau]} g L s_n^{\ell}(x, y, z; C, E)$ in the following corollary, without their proofs (see [4], pp. 26–28).

Corollary 6. *The following identities hold true:*

(a)

$${}_{[q(\tau),\tau]} g L s_n^{\ell}(x, y, z; C, E) = q\left(\left(\sqrt{2C}\right)^{-1} D_x\right)^{-1} \{x^n\}. \tag{80}$$

(b)

$$\begin{aligned}
& {}_{[q(\tau), \tau]} \mathcal{G}L S_n^\ell(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}, \mathbf{z}; C, E) \\
&= \sum_{k=0}^n \binom{n}{k} {}_{[q(\tau), \tau]} \mathcal{G}L S_{n-k}^\ell(\mathbf{x}_1, \mathbf{y}, \mathbf{z}; C, E) \left(\sqrt{2C} \mathbf{x}_2\right)^k.
\end{aligned} \tag{81}$$

(c) (Conjugate representation)

$$\begin{aligned}
& {}_{[q(\tau), \tau]} \mathcal{G}L S_n^\ell(\mathbf{x}, \mathbf{y}, \mathbf{z}; C, E) \\
&= \sum_{k=0}^n \binom{n}{k} \left[q \left(\left(\sqrt{2C} \right)^{-1} D_{\mathbf{x}} \right)^{-1} \left\{ \mathbf{x}^{n-k} \right\} \right] \mathbf{x}^k.
\end{aligned} \tag{82}$$

5. Conclusions and Posing a Problem

The authors introduced a new class of polynomials, the Gould-Hopper-Laguerre-Sheffer matrix polynomials, using operational approaches. This new family's generating function and operational representations were then constructed. They are also understood in terms of quasi-monomiality. The authors also extended Gould-Hopper-Laguerre-Sheffer matrix polynomials and explored their characteristics using the integral transform. There were other instances for individual members of the aforementioned matrix polynomial family.

It should be highlighted that the polynomials presented and studied in this article are regarded to be novel, primarily because they cannot be obtained by modifying previously published findings and identities, as far as we have researched. Also, the new polynomials and their identities are potentially useful, particularly in light of the tables' demonstrations of some of their special instances.

Posing a problem: Provide some new instances (which are nonexistent from the literature) for those novel polynomials, such as Gould-Hopper matrix polynomials and Gould-Hopper-Laguerre-Sheffer matrix polynomials.

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