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Stability Analysis of the Nabla Distributed-Order Nonlinear Systems

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Abstract: The stability of the nabla discrete distributed-order nonlinear dynamic systems is investigated in this paper. Firstly, a sufficient condition for the asymptotic stability of the nabla discrete distributed-order nonlinear systems is proposed based on Lyapunov direct method. In addition, some properties of the nabla distributed-order operators are derived. Based on these properties, a simpler criterion is provided to determine the stability of such systems. Finally, two examples are given to illustrate the validity of these results.

Keywords: stability; Lyapunov direct method; Nabla fractional calculus; distributed-order

MSC: 93C10; 93D05; 93D20

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Citation: Wang, C.; Zhu, T.; Chen, Y. Stability Analysis of the Nabla Distributed-Order Nonlinear Systems. *Fractal Fract.* **2022**, *6*, 228. https://doi.org/10.3390/fractalfract6050228

Academic Editor: Luis Vázquez

Received: 1 February 2022
Accepted: 21 March 2022
Published: 20 April 2022

1. Introduction

Fractional calculus [1] is a generalization of integer-order calculus, which includes integration and differentiation of non-integer order. It is well known that the dynamic properties of more and more real systems can be characterized by fractional differential equations, which is due to the memory and hereditary properties of fractional calculus. Therefore, fractional calculus has been widely used in biology [2], economy [3], physics [4–7] and other fields [8–12].

At first, the study of fractional calculus mainly focused on the fractional operator with a constant and single-valued order. Although a constant value fractional order system can better and more accurately describe the actual physical system than an integer-order system, the constant and single-valued nature of the order still limits its ability to accurately capture certain complex phenomena whose underlying physics could either evolve in time or emerge as the result of the interplay of multiple orders. In relatively recent years, many scholars began to pay attention to the variable-order fractional operators [13,14] and the distribution-order operators [15,16]. The variable-order means that the derivative order can be a function of either dependent (e.g., state variables of the system) or independent (e.g., space or time) variables and can change value following the evolution of the system. The distributed-order definition of the operator allows considering an accumulation of orders and accounting for, as an example, physical phenomena such as memory effects in composite materials [17] or multi-scale effects [18]. A typical example that illustrates the capabilities of this class of operators is the mechanical behavior of viscoelastic materials having spatially varying properties [19,20]. Therefore distributed-order fractional calculus is a natural generalization of fractional calculus, which provides a tool to model more complex systems since it both retains the non-local and memory properties of the constant and single-valued order fractional calculus, and allows the case with multiple coexisting orders. Therefore, distributed-order systems can be applied in viscoelasticity, transportation process, robots and so on [21–29].

Discrete fractional calculus has also attracted great interest of researchers since most practical work in terms of discrete time series. There are many kinds of definitions for discrete fractional calculus, such as delta operator [30], nabla operator [31], etc. Among them, nabla discrete fractional calculus is especially noteworthy. The time–domain response and the infinite-dimensional nature of nabla discrete fractional-order systems are investigated in [32] and in [33,34], respectively. Some Lyapunov inequalities about nabla discrete fractional difference are presented in [35]. However, to our best knowledge, there are few results related to discrete distributed-order systems.

Stability analysis of dynamic systems is the most important and basic problem. The Lyapunov direct method is a powerful tool for analyzing the stability of a system since it does not need to solve explicit solutions of differential systems. Since then, many scholars have continuously improved and promoted the Lyapunov direct method. Li et al. [36] propose the fractional Lyapunov direct method to study the Mittag–Leffler stability of fractional-order nonlinear systems. Analogously, the definition of discrete Mittag–Leffler stability is given and the Lyapunov direct method is extended to study nabla discrete fractional-order systems in [37]. Recently, Fernández-Anaya et al. [38] generalize some properties of the Caputo fractional derivative to distributed-order cases and establish some stability conditions for such systems based on the distributed-order Lyapunov direct method. It may be noted that there are few references about the stability analysis of nabla discrete distributed-order nonlinear systems.

Motivated by the previous discussions, this paper mainly investigates the stability of the nabla discrete distributed-order nonlinear dynamic systems. We will establish a stability condition for the nabla discrete distributed-order nonlinear systems based on Lyapunov direct method and comparison principle. Additionally, in virtue of some important inequalities, we will propose a simpler stability criterion.

The rest of the paper is organized as follows. In Section 2, some basic definitions and preliminary knowledge are introduced. In Section 3, we propose some stability criteria for the nabla discrete distributed-order nonlinear systems. Moreover, some properties of the nabla distributed-order operators are derived. In Section 4, we illustrate the effectiveness of the results proposed in this paper via two examples. Finally, the main conclusions are drawn in Section 5.

Notations: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{Z}_+$ denotes the set of positive integers. Given a real number a, we define set $\mathbb{N}_{a+1}$ as $\mathbb{N}_{a+1} := \{a+1, a+2, a+3, ...\}$. Function $\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt$ is the Gamma function.

2. Preliminaries

In this section, we firstly introduce some definitions, corresponding properties and lemmas on nabla fractional calculus.

Definition 1 ([31])**.** *The n-th order nabla backward difference of a function $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$ is defined by*

$$\nabla^n f(k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(k-j),$$

where $n \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$ and $\binom{n}{j} = \frac{\Gamma(n+1)}{\Gamma(j+1)\Gamma(n-j+1)}$.

Definition 2 ([31])**.** *The α-th order nabla fractional sum of a function $f : \mathbb{N}_{a+1} \to \mathbb{R}$ is defined by*

$${}_a\nabla_k^{-\alpha} f(k) = \sum_{j=0}^{k-a-1} (-1)^j \binom{-\alpha}{j} f(k-j),$$

where $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$ and $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$, and $\binom{-\alpha}{j} = (-1)^j \frac{\Gamma(\alpha+j)}{\Gamma(j+1)\Gamma(\alpha)}$.

Definition 3 ([31])**.** *The nabla Caputo and Riemann–Liouville fractional differences of a function* $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$ *are defined, respectively, by*

$$ {}_a^C\nabla_k^{\alpha} f(k) = {}_a\nabla_k^{-(n-\alpha)} \nabla^n f(k) = {}_a\nabla_k^{\alpha-n} \nabla^n f(k), $$

and

$$ {}_a^{RL}\nabla_k^{\alpha} f(k) = \nabla^n {}_a\nabla_k^{-(n-\alpha)} f(k) = \nabla^n {}_a\nabla_k^{\alpha-n} f(k), $$

where $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$ *and* $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$.

In this paper, we mainly adopt the Caputo definition method.

Definition 4. *The nabla distributed-order difference of a function* $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$ *is defined by*

$$ {}_a^C\nabla_k^{c(\alpha)} f(k) = \int_{n-1}^{n} c(\alpha) {}_a^C\nabla_k^{\alpha} f(k)\, \mathrm{d}\alpha, $$

where $c(\alpha)$ *denotes the weight function,* $c(\alpha) \geq 0$ *and* $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$.

Based on Definition 4, the nabla distributed-order difference of a constant function can be easily obtained as follows.

Property 1. *For* $0 < \alpha < 1$, *the nabla distributed-order difference of a constant function* $f(k) = d$ $(k \in \mathbb{N}_{a+1-n})$ *is*

$$ {}_a^C\nabla_k^{c(\alpha)} d = 0. $$

Proof. Using the definition of the nabla Caputo fractional difference and since d is a constant, one has

$$ {}_a^C\nabla_k^{\alpha} d = {}_a\nabla_k^{-(1-\alpha)} \nabla^1 d = {}_a\nabla_k^{-(1-\alpha)} 0. $$

Then, according to Definition 4, it is easily known that the nabla distributed-order difference of a constant function is also 0. □

Like Laplace transform in the standard calculus, the $\mathcal{N}$-transform is a powerful tool to analyze properties of the nabla fractional calculus, so the definition and key properties on $\mathcal{N}$-transform of a function are given as follows.

Definition 5 ([31])**.** *The* $\mathcal{N}$*-transform of a function* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *is defined by*

$$ \mathcal{N}_a\{f\}(s) = \sum_{k=1}^{+\infty} (1-s)^{k-1} f(k+a). $$

Lemma 1 ([33])**.** *Let* $\alpha \in (n-1, n)$ *and* $n \in \mathbb{Z}_+$. *If the* $\mathcal{N}$*-transform of a function* $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$ *converges for* $|s-1| < \rho$ *for some* $\rho > 0$, *then*

$$ \mathcal{N}_a\left\{ {}_a^C\nabla_k^{\alpha} f \right\}(s) = s^{\alpha} \mathcal{N}_a\{f\}(s) - \sum_{i=0}^{n-1} s^{\alpha-i-1} \nabla^i f(k)|_{k=a}, $$

for $|s-1| < \rho$.

Based on the definition of the nabla distributed-order difference and Lemma 1, we can deduce that

$$\begin{aligned}\mathcal{N}_a\left\{{}_a^C\nabla_k^{c(\alpha)} f\right\}(s) &= \int_{n-1}^{n} c(\alpha)[s^\alpha \mathcal{N}_a\{f\}(s) - \sum_{i=0}^{n-1} s^{\alpha-i-1}\nabla^i f(k)|_{k=a}]\mathrm{d}\alpha \\ &= \int_{n-1}^{n} c(\alpha)s^\alpha \mathcal{N}_a\{f\}(s)\mathrm{d}\alpha - \int_{n-1}^{n} c(\alpha)[\sum_{i=0}^{n-1} s^{\alpha-i-1}\nabla^i f(k)|_{k=a}]\mathrm{d}\alpha \\ &= C(s)F(s) - \sum_{i=0}^{n-1} \frac{1}{s^{i+1}} C(s)\nabla^i f(k)|_{k=a}],\end{aligned}$$

where $C(s) = \int_{n-1}^{n} c(\alpha)s^\alpha\,\mathrm{d}\alpha$ and $F(s) = \mathcal{N}_a\{f\}(s)$.

Lemma 2 ([32])**.** *For the discrete Mittag–Leffler function defined by* $\mathcal{F}_{\alpha,\beta}(\lambda,k,a) = \sum_{j=0}^{+\infty} \frac{\lambda^j}{\Gamma(j\alpha+\beta)} (k-a)^{\overline{j\alpha+\beta-1}}$*, its* $\mathcal{N}$*-transform is*

$$\mathcal{N}_a\{\mathcal{F}_{\alpha,\beta}(\lambda,k,a)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda},$$

for $|s-1| < 1$ *and* $|s^\alpha| > |\lambda|$*, where* $(k-a)^{\overline{j\alpha+\beta-1}} = \frac{\Gamma(k-a+j\alpha+\beta-1)}{\Gamma(k-a)}$.

The Convolution Theorem builds the bridge between the time domain and the frequency domain, In the following, we will give the Convolution Theorem on nabla fractional calculus.

Lemma 3 ([31])**.** ***(Convolution Theorem)*** *Let functions* f, $g : \mathbb{N}_{a+1} \to \mathbb{R}$*. Then* $\mathcal{N}_a\{f * g\}(s) = \mathcal{N}_a\{f\}(s)\mathcal{N}_a\{g\}(s)$*, where* $*$ *denotes the convolution operation, i.e.,*

$$f * g = \sum_{j=a+1}^{k} f(k-j+a+1)g(j).$$

Next, we provide the Final Value Theorem on the $\mathcal{N}$-transform, which plays a key role in the stability analysis.

Lemma 4 ([32])**.** ***(Final Value Theorem)*** *Let a function* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *and* $a \in \mathbb{R}$*. If* $F(s) = \mathcal{N}_a\{f\}(s)$ *and the poles of* $sF(s)$ *satisfy* $|s-1| > 1$*, then*

$$\lim_{k\to+\infty} f(k) = \lim_{s\to 0} sF(s).$$

3. Stability Analysis

Consider the following nabla discrete distributed-order nonlinear systems

$${}_a^C\nabla_k^{c(\alpha)} x(k) = f(x(k),k), \tag{1}$$

where $\alpha \in (0,1)$, $k \in \mathbb{N}_{a+1}$, $x(k) \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is a domain containing the origin, $f : \Omega \times \mathbb{N}_{a+1} \to \mathbb{R}^n$ and $x(a)$ is initial condition.

Throughout this paper, we assumed that $\int_0^1 c(\alpha)s^\alpha\,\mathrm{d}\alpha \neq 0$ and the solution of system (1) is existent and unique. Accordingly, the form of the unique solution of system (1) is given as follows.

Theorem 1. *Consider the nabla discrete distributed-order nonlinear system (1), its unique solution has the following form:*

$$x(k) = x(a) + f * \mathcal{N}_a^{-1}\left\{ \frac{1}{\int_0^1 c(\alpha)s^\alpha \, \mathrm{d}\alpha} \right\}.$$

Proof. Taking the $\mathcal{N}$-transform on both sides of Equation (1) yields

$$C(s)X(s) - \frac{1}{s}C(s)x(a) = F(s),$$

then we have

$$X(s) = \frac{1}{s}x(a) + \frac{F(s)}{C(s)}. \tag{2}$$

Taking the inverse $\mathcal{N}$-transform to Equation (2) yields

$$x(k) = u(k-a-1)x(a) + f * \mathcal{N}_a^{-1}\left\{ \frac{1}{C(s)} \right\}.$$

Due to the discrete-time unit, step function is

$$u(n) = \begin{cases} 1 & n \geq 0, \\ 0 & n < 0, \end{cases}$$

and $k \in \mathbb{N}_{a+1}$, then $u(k-a-1) = 1$.

Therefore,

$$x(k) = x(a) + f * \mathcal{N}_a^{-1}\left\{ \frac{1}{\int_0^1 c(\alpha)s^\alpha \, \mathrm{d}\alpha} \right\}.$$

□

Before discussing the stability problem, we need to present the definition of the equilibrium point of the nabla discrete distributed-order nonlinear system.

Definition 6. *For the nabla discrete distributed-order nonlinear system (1), the constant vector $\bar{x}$ is called its equilibrium point if $f(\bar{x}, k) = 0$.*

Remark 1. *For convenience, we often assume that the equilibrium point is at the origin, i.e., $\bar{x} = 0$. This is no loss of generality, since if the equilibrium point is $\bar{x} \neq 0$, we can take the change of variable $y(k) = x(k) - \bar{x}$, then*

$${}_a^C\nabla_k^{c(\alpha)} y(k) = {}_a^C\nabla_k^{c(\alpha)}(x(k) - \bar{x}) = f(x(k), k) = f(y(k) + \bar{x}, k) = g(y(k), k),$$

where $g(0, k) = 0$ and the equilibrium point of the system about the new variable $y(k)$ is at the origin.

In the following, we will investigate the stability of the nabla discrete distributed-order nonlinear system (1), the relevant stability definitions are firstly introduced.

Definition 7. *Let $\bar{x} = 0$ is an equilibrium point of the nabla discrete distributed-order nonlinear system (1) and $k = a$ is the initial time, the equilibrium point $\bar{x}$ is stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x(k)\| < \varepsilon$ holds for all $k \geq a$ $(k \in \mathbb{N}_{a+1})$, when $\|x(a)\| < \delta$.*

Definition 8. *The equilibrium point $\bar{x} = 0$ of the nabla discrete distributed-order nonlinear system (1) is asymptotically stable if it is stable and δ can be chosen such that if $\|x(a)\| < \delta$, then $\lim\limits_{k \to +\infty} \|x(k)\| = 0$.*

Remark 2. *The system stability we stated in the following refers to the stability of system at the equilibrium point* $\bar{x} = 0$.

The following Lemma extends the comparison principle for the nabla fractional difference in [37] to the nabla distributed-order case.

Lemma 5. *(**Nabla Distributed-order Comparison Principle**) Let two functions* $x,\ y : \mathbb{N}_a \to \mathbb{R}$. *If* ${}_a^C\nabla_k^{c(\alpha)} x(k) \geq {}_a^C\nabla_k^{c(\alpha)} y(k)$, *where* $0 < \alpha < 1$, $\mathcal{N}_a^{-1}\left\{\frac{1}{C(s)}\right\} \geq 0$ $(\forall k \in \mathbb{N}_{a+1})$, $C(s) = \int_0^1 c(\alpha)s^\alpha \,\mathrm{d}\alpha$ *and* $x(a) = y(a)$, *then for any* $k \in \mathbb{N}_{a+1}$, *one has* $x(k) \geq y(k)$.

Proof. Since ${}_a^C\nabla_k^{c(\alpha)} x(k) \geq {}_a^C\nabla_k^{c(\alpha)} y(k)$ $(k \in \mathbb{N}_{a+1})$, then there is a non-negative function $z(k)$ satisfying

$$ {}_a^C\nabla_k^{c(\alpha)} x(k) = {}_a^C\nabla_k^{c(\alpha)} y(k) + z(k). \tag{3}$$

Taking the $\mathcal{N}$-transform on the both sides of Equation (3), then

$$ C(s)X(s) - \frac{1}{s}C(s)x(a) = C(s)Y(s) - \frac{1}{s}C(s)y(a) + Z(s). $$

Since $x(a) = y(a)$, then

$$ X(s) = Y(s) + \frac{Z(s)}{C(s)}. \tag{4}$$

Applying the inverse $\mathcal{N}$-transform on both side of Equation (4) and using Lemma 3, then we have

$$ \begin{aligned} x(k) &= y(k) + z * \mathcal{N}_a^{-1}\left\{\frac{1}{C(s)}\right\} \\ &= y(k) + z * g \\ &= y(k) + \sum_{j=a+1}^{k} z(k-j+a+1)g(j), \end{aligned} $$

where $g \triangleq \mathcal{N}_a^{-1}\left\{\frac{1}{C(s)}\right\}$.

For all $k \in \mathbb{N}_{a+1}$, since $g(k) \geq 0$ and $z(k) \geq 0$, then $x(k) \geq y(k)$. □

Remark 3. *It is worth nothing that if we choose a Dirac function as the distribution function, the comparison principle on the nabla fractional difference can be obtained (see Lemma 3 in [37]).*

It is well known that Lyapunov direct method is the most effective tool for analyzing the stability of nonlinear systems. Therefore, in the following we will generalize Lyapunov direct method to establish the stability conditions for the nabla discrete distributed-order nonlinear system.

Theorem 2. *Consider the nabla discrete distributed-order nonlinear system* (1), *suppose there exists a Lyapunov function* $V(x(k), k)$: $\Omega \times \mathbb{N}_{a+1} \to \mathbb{R}$ *such that*

$$ \gamma_1\|x(k)\|^b \leq V(x(k), k) \leq \gamma_2\|x(k)\|^{bc}, \tag{5}$$

and

$$ {}_a^C\nabla_k^{c(\alpha)} V(x(k), k) \leq -\gamma_3\|x(k)\|^{bc}, \tag{6}$$

where $\alpha \in (0,1)$, $k \in \mathbb{N}_{a+1}$ *and* b, c, $\gamma_i > 0$ $(i = 1,2,3)$. *If the roots of* $C(s) + \frac{\gamma_3}{\gamma_2} = 0$ *satisfy* $|s-1| > 1$, $\mathcal{N}_a^{-1}\left\{\frac{1}{(C(s)+\frac{\gamma_3}{\gamma_2})}\right\} \geq 0$, $C(s) = \int_0^1 c(\alpha)s^\alpha\,\mathrm{d}\alpha$, *then this system is asymptotically stable.*

Proof. Given $\varepsilon > 0$, there must exist $\delta > 0$ such that $\gamma_2\delta^{bc} < \gamma_1\varepsilon^b$.

Based on the inequality (6), one has

$$ {}_a^C\nabla_k^{c(\alpha)} V(x(k),k) \leq 0. $$

Applying Property 1 and Lemma 5, we can obtain

$$ V(x(k),k) \leq V(x(a),a). $$

Therefore, when $\|x(a)\| < \delta$, we get

$$ \gamma_1\|x(k)\|^b \leq V(x(k),k) \leq V(x(a),a) \leq \gamma_2\|x(a)\|^{bc} \leq \gamma_2\delta^{bc} \leq \gamma_1\varepsilon^b, $$

and thus $\|x(k)\| < \varepsilon$, which implies the stability of system (1).

Based on inequalities (5) and (6), we have

$$ {}_a^C\nabla_k^{c(\alpha)} V(x(k),k) \leq -\frac{\gamma_3}{\gamma_2} V(x(k),k), $$

then there exists a non-negative function $z(k)$ such that

$$ {}_a^C\nabla_k^{c(\alpha)} V(x(k),k) + z(k) = -\frac{\gamma_3}{\gamma_2} V(x(k),k). \tag{7} $$

Taking the $\mathcal{N}$-transform on both sides of Equation (7) results in

$$ C(s)V(s) - \frac{C(s)}{s}V(a) + Z(s) = -\frac{\gamma_3}{\gamma_2}V(s), $$

then, we can find that

$$ V(s) = \left\{\frac{C(s)V(a)}{s(C(s)+\frac{\gamma_3}{\gamma_2})}\right\} - \left\{\frac{Z(s)}{(C(s)+\frac{\gamma_3}{\gamma_2})}\right\}. \tag{8} $$

Applying the inverse $\mathcal{N}$-transform to (8) gives

$$ V(x(k),k) = \mathcal{N}_a^{-1}\left\{\frac{C(s)V(a)}{s(C(s)+\frac{\gamma_3}{\gamma_2})}\right\} - \mathcal{N}_a^{-1}\left\{\frac{Z(s)}{(C(s)+\frac{\gamma_3}{\gamma_2})}\right\}. \tag{9} $$

The last term of Equation (9) can be rewritten as

$$ \begin{aligned} \mathcal{N}_a^{-1}\left\{\frac{Z(s)}{(C(s)+\frac{\gamma_3}{\gamma_2})}\right\} &= z * \mathcal{N}_a^{-1}\left\{\frac{1}{(C(s)+\frac{\gamma_3}{\gamma_2})}\right\} \\ &= z * g \\ &= \sum_{j=a+1}^{k} z(k-j+a+1)g(j), \end{aligned} $$

where $g \triangleq \mathcal{N}_a^{-1}\left\{\frac{1}{(C(s)+\frac{\gamma_3}{\gamma_2})}\right\}$.

Considering that $g \geq 0$ and $z(k) \geq 0$, then

$$\mathcal{N}_a^{-1}\left\{\frac{Z(s)}{(C(s)+\frac{\gamma_3}{\gamma_2})}\right\} \geq 0.$$

Therefore, from Equation (9), we have

$$V(x(k),k) \leq \mathcal{N}_a^{-1}\left\{\frac{C(s)V(a)}{s(C(s)+\frac{\gamma_3}{\gamma_2})}\right\}. \tag{10}$$

Since we assume that the roots of $C(s)+\frac{\gamma_3}{\gamma_2}=0$ satisfy $|s-1|>1$, and according to Lemma 4, we can deduce that

$$\lim_{k\to+\infty}\mathcal{N}_a^{-1}\left\{\frac{C(s)V(a)}{s(C(s)+\frac{\gamma_3}{\gamma_2})}\right\} = \lim_{s\to 0}\left\{\frac{C(s)V(a)}{(C(s)+\frac{\gamma_3}{\gamma_2})}\right\} = 0.$$

Combining (10), we get

$$\lim_{k\to+\infty} V(x(k),k) \leq \lim_{k\to+\infty}\mathcal{N}_a^{-1}\left\{\frac{C(s)V(a)}{s(C(s)+\frac{\gamma_3}{\gamma_2})}\right\} = 0.$$

It follows from (5) that

$$\lim_{k\to+\infty}\gamma_1\|x(k)\|^b \leq 0$$

Since $b>0$ and $\gamma_1>0$, then $\lim_{k\to+\infty}\|x(k)\|=0$, which indicates that system (1) is asymptotically stable. □

Remark 4. *Note that if we choose an appropriate Dirac function as the distribution function, then we can get the asymptotic stability conditions of the nabla discrete fractional-order nonlinear systems (see Theorem 2 in [37]).*

An important inequality for stability analysis based on the Lyapunov method is stated in following lemma, which will be generalized to the nabla distributed-order case.

Lemma 6 ([35])**.** *The following inequality holds*

$${}_a^C\nabla_k^\alpha x^2(k) \leq 2x(k){}_a^C\nabla_k^\alpha x(k),$$

for $\alpha\in(0,1)$, $k\in\mathbb{N}_{a+1}$, $x(k)\in\mathbb{R}$ *and* $a\in\mathbb{R}$.

Lemma 7. *Let* $x:\mathbb{N}_{a+1}\to\mathbb{R}$. *Then, the following inequality holds*

$${}_a^C\nabla_k^{c(\alpha)} x^2(k) \leq 2x(k){}_a^C\nabla_k^{c(\alpha)} x(k),$$

where $\alpha\in(0,1)$, $k\in\mathbb{N}_{a+1}$ *and* $a\in\mathbb{R}$.

Proof. Based on Lemma 6, one obtains

$${}_a^C\nabla_k^\alpha x^2(k) \leq 2x(k){}_a^C\nabla_k^\alpha x(k).$$

Due to $c(\alpha)\geq 0$, then we have

$$c(\alpha){}_a^C\nabla_k^\alpha x^2(k) \leq 2c(\alpha)x(k){}_a^C\nabla_k^\alpha x(k). \tag{11}$$

Integrating both sides of the inequality (11) with respect to α from 0 to 1, we have

$$\int_0^1 c(\alpha){}_a^C\nabla_k^{\alpha}x^2(k)\mathrm{d}\alpha \leq 2\int_0^1 c(\alpha)x(k){}_a^C\nabla_k^{\alpha}x(k)\mathrm{d}\alpha.$$

This completes the proof. □

Lemma 8. *For $x:\mathbb{N}_{a+1}\to\mathbb{R}^n$, Lemma 7 still holds, that is,*

$$ {}_a^C\nabla_k^{c(\alpha)}x^T(k)x(k) \leq 2x^T(k){}_a^C\nabla_k^{c(\alpha)}x(k), \quad (12)$$

where $\alpha\in(0,1)$.

Proof. By decomposing the inequality (12) into the sum of scalar products and applying Lemma 7, the result is obvious. □

Based on Lemmas 5 and 8 and Property 1, we will provide a simpler method to analyze the stability of the nabla discrete distributed-order nonlinear system (1).

Theorem 3. *Consider the nabla discrete distributed-order nonlinear system (1).*

(i) *If $x^T(k)f(x(k),k)\leq 0$ and $\mathcal{N}_a^{-1}\left\{\dfrac{1}{C(s)}\right\}\geq 0$, where $C(s)=\int_0^1 c(\alpha)s^{\alpha}\,\mathrm{d}\alpha$, then system (1) is stable.*

(ii) *Let $\xi>0$, if $x^T(k)f(x(k),k)\leq -\xi\|x(k)\|^2$, the roots of $C(s)+\xi=0$ satisfy $|s-1|>1$, $\mathcal{N}_a^{-1}\left\{\dfrac{1}{(C(s)+\xi}\right\}\geq 0$, then system (1) is asymptotically stable.*

Proof. Choose the Lyapunov function

$$V(x(k),k)=\frac{1}{2}x^T(k)x(k). \quad (13)$$

(i) Using Lemma 8 to Equation (13) yields that

$$ {}_a^C\nabla_k^{c(\alpha)}V(x(k),k)\leq x^T(k){}_a^C\nabla_k^{c(\alpha)}x(k). \quad (14)$$

Since $x^T(k)f(x(k),k)\leq 0$ and note that ${}_a^C\nabla_k^{c(\alpha)}x(k)=f(x(k),k)$, then

$$ {}_a^C\nabla_k^{c(\alpha)}V(x(k),k)\leq x^T(k)f(x(k),k)\leq 0.$$

Based on Property 1 and Lemma 5, one has

$$V(x(k),k)\leq V(x(a),a).$$

In terms of the definition of the function $V(x(k),k)$, we obtain

$$x^T(k)x(k)\leq x^T(a)x(a).$$

Given $\varepsilon>0$ and choose $\delta=\varepsilon$, then when $\|x(a)\|<\delta$, we can deduce that $\|x(k)\|<\delta$ for all $k\in\mathbb{N}_{a+1}$, which indicates $\|x(k)\|<\varepsilon$ for all $k\in\mathbb{N}_{a+1}$, this means that system (1) is stable.

(ii) If $x^T(k)f(x(k),k)\leq -\xi\|x(k)\|^2$, then from the inequality (14), we have

$$ {}_a^C\nabla_k^{c(\alpha)}V(x(k),k)\leq -\xi\|x(k)\|^2.$$

Since the Lyapunov function

$$V(x(k),k) = \frac{1}{2}x^T(k)x(k) = \frac{1}{2}\|x(k)\|^2,$$

then

$$\frac{1}{2}\|x(k)\|^2 \leq V(x(k),k) \leq \|x(k)\|^2.$$

According to Theorem 2, the asymptotic stability of system (1) can be obtained. □

4. Numerical Examples

Example 1. *Consider the following nabla discrete distributed-order nonlinear system*

$$\begin{cases} {}_a^C\nabla_k^{c(\alpha)} x_1(k) = -x_1(k) - x_1(k)x_2^2(k), \\ {}_a^C\nabla_k^{c(\alpha)} x_2(k) = -x_2(k) + x_1^2(k)x_2(k), \end{cases} \tag{15}$$

where $\alpha \in (0,1)$ *and* $c(\alpha) = \delta(\alpha - \frac{1}{3})$.

Choosing the Lyapunov function $V(x(k),k) = \frac{1}{2}(x_1^2(k) + x_2^2(k))$, then we have

$$\begin{aligned} {}_a^C\nabla_k^{c(\alpha)} V(x(k),k) &= \frac{1}{2} {}_a^C\nabla_k^{c(\alpha)} (x_1^2(k) + x_2^2(k)) \\ &\leq x_1(k) {}_a^C\nabla_k^{c(\alpha)} x_1(k) + x_2(k) {}_a^C\nabla_k^{c(\alpha)} x_2(k) \\ &= -(x_1^2(k) + x_2^2(k)) \\ &= -\|x(k)\|^2. \end{aligned}$$

Due to $\frac{1}{2}\|x(k)\|^2 \leq V(x(k),k) \leq \|x(k)\|^2$, then one has $\gamma_1 = \frac{1}{2}, \gamma_2 = 1, \gamma_3 = 1$. The root of

$$C(s) + \frac{\gamma_3}{\gamma_2} = s^{\frac{1}{3}} + 1 = 0$$

is $s = -1$, which satisfies $|s - 1| > 1$. It follows from Lemma 2 that

$$\mathcal{N}_a^{-1}\left\{ \frac{1}{(C(s) + \frac{\gamma_3}{\gamma_2})} \right\} = \mathcal{N}_a^{-1}\left\{ \frac{1}{s^{\frac{1}{3}} + 1} \right\} = \mathcal{F}_{\frac{1}{3},\frac{1}{3}}(-1,k,a).$$

Since $\mathcal{F}_{\frac{1}{3},\frac{1}{3}}(-1,k,a) \geq 0$, then $\mathcal{N}_a^{-1}\left\{ \frac{1}{(C(s) + \frac{\gamma_3}{\gamma_2})} \right\} \geq 0$. From Theorem 2, we conclude that system (15) is asymptotically stable.

Example 2. *Consider the following nabla discrete distributed-order nonlinear system*

$$\begin{cases} {}_a^C\nabla_k^{c(\alpha)} x_1(k) = -4(x_1(k) + x_2(k)h(x_1(k),x_2(k),k)), \\ {}_a^C\nabla_k^{c(\alpha)} x_2(k) = -4(x_2(k) - x_1(k)h(x_1(k),x_2(k),k)), \end{cases} \tag{16}$$

where $\alpha \in (0,1)$, $c(\alpha) = \delta(\alpha - \frac{2}{3}) + 4\delta(\alpha - \frac{1}{3})$, *and* $h(x_1(k),x_2(k),k)$ *is a differentiable function.*

Since

$$\begin{aligned} & x^T(k)f(x(k),k) \\ =& \begin{bmatrix} x_1(k) & x_2(k) \end{bmatrix} \begin{bmatrix} -4(x_1(k) + x_2(k)h(x_1(k),x_2(k),k)) \\ -4(x_2(k) - x_1(k)h(x_1(k),x_2(k),k)) \end{bmatrix} \\ =& -4(x_1^2(k) + x_2^2(k)) = -4\|x(k)\|^2, \end{aligned}$$

we can let $\xi = 4$ in Theorem 3.

Now, the distribution function $c(\alpha) = \delta(\alpha - \frac{2}{3}) + 4\delta(\alpha - \frac{1}{3})$, then $C(s) = \int_0^1 c(\alpha)s^\alpha \, \mathrm{d}\alpha = s^{\frac{2}{3}} + 4s^{\frac{1}{3}}$, the roots of

$$C(s) + \xi = C(s) + 4 = s^{\frac{2}{3}} + 4s^{\frac{1}{3}} + 4 = (s^{\frac{1}{3}} + 2)^2 = 0$$

are $s_1 = -8$, $s_2 = -8$, which satisfies that $|s - 1| > 1$.

It follows from Lemma 2 that

$$\mathcal{N}_a^{-1}\left\{\frac{1}{s^{\frac{1}{3}} + 2}\right\} = \mathcal{F}_{\frac{1}{3},\frac{1}{3}}(-2, k, a) \geq 0.$$

According to Lemma 3, we can obtain that

$$\mathcal{N}_a^{-1}\left\{\frac{1}{C(s) + \xi}\right\} = \mathcal{N}_a^{-1}\left\{\frac{1}{C(s) + 4}\right\} = \mathcal{N}_a^{-1}\left\{\frac{1}{s^{\frac{1}{3}} + 2}\right\} * \mathcal{N}_a^{-1}\left\{\frac{1}{s^{\frac{1}{3}} + 2}\right\} \geq 0.$$

Therefore, system (16) is asymptotically stable based on Theorem 3.

5. Conclusions

In this paper, the stability of the nabla discrete distributed-order nonlinear systems have been studied. The nabla distributed-order comparison principle is introduced. We generalize the Lyapunov direct method to establish the stability condition for the nabla discrete distributed-order systems. In addition, combined with some important inequalities, a simpler stability analysis method is provided. Finally, two examples are given to illustrate the validity of the obtained results. Based on the stability results established in this paper, one can investigate the controller design problem or the performance analysis problem of the the nabla distributed-order nonlinear systems.

Author Contributions: Conceptualization, C.W. and Y.C. methodology, C.W.; software, C.W.; validation, C.W., T.Z. and Y.C. formal analysis, C.W.; investigation, C.W.; resources, Y.C. writing—original draft preparation, T.Z.; writing—review and editing, C.W., T.Z. and Y.C. supervision, C.W.; funding acquisition, C.W. All authors have read and agreed to the published version of the manuscript.

Funding: The research was supported by Foundation of National Natural Science of China under Grant No. 61907027 and Natural Science Foundation of Shanxi Normal University under Grant ZR1601.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Acknowledgments: We would like to express our great appreciation to the editors and reviewers.

Conflicts of Interest: The authors declare that they have no conflict of interest regarding the publication of this paper.

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