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Boundary Value Problem for Fractional q-Difference Equations with Integral Conditions in Banach Spaces

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Abstract: The authors investigate the existence of solutions to a class of boundary value problems for fractional q-difference equations in a Banach space that involves a q-derivative of the Caputo type and nonlinear integral boundary conditions. Their result is based on Mönch's fixed point theorem and the technique of measures of noncompactness. This approach has proved to be an interesting and useful approach to studying such problems. Some basic concepts from the fractional q-calculus are introduced, including q-derivatives and q-integrals. An example of the main result is included as well as some suggestions for future research.

Keywords: boundary value problems; fractional q-difference equations; Caputo fractional q-difference derivative; measure of noncompactness; Mönch's fixed point theorem

MSC: 26A33; 34A37

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1. Introduction

Fractional differential equations play an essential role when attempting to model phenomena in a number of areas and have recently been studied by researchers in engineering, physics, chemistry, biology, economics, and control theory. For additional details see, for example, the monographs of Hilfer [1], Kilbas et al. [2], Miller and Ross [3], Podlubny [4], Samko et al. [5], and Tarasov [6] as well as the references they contain. The existence of solutions to fractional boundary value problems is currently a very active area of research as can be seen, for example, from the recent papers of Ahmad et al. [7], Agarwal et al. [8], Benchohra et al. [9], Benhamida et al. [10], Hamini et al. [11], and Zahed et al. [12].

Considerable attention has been given to the problem of existence of solutions to boundary value problems for fractional differential equations in Banach spaces, and we refer the reader to the recent contributions in [13–15].

The q-difference calculus, or quantum calculus, was first introduced by Jackson in 1910 [16,17]. The basic definitions and properties of the q-difference calculus can be found in [18,19]. Later, Al-Salam [20] and Agarwal [21] proposed the study of the fractional q-difference calculus. Fractional q-difference calculus by itself and nonlinear fractional q-difference boundary value problems have appeared as the object of study for a number of researchers. Recent developments on the fractional q-difference calculus and boundary value problems for such can be found in [7,22–25] and the references therein.

In this paper, we study the existence of solutions to the boundary value problem (BVP for short) for fractional q-difference equations with nonlinear integral conditions

$$({}^C D_q^{\alpha} y)(t) = f(t, y(t)), \text{ for a.e. } t \in J = [0, T], \quad 1 < \alpha \leq 2, \tag{1}$$

$$y(0) - y'(0) = \int_0^T g(s, y(s))ds, \tag{2}$$

$$y(T) + y'(T) = \int_0^T h(s, y(s))ds, \tag{3}$$

where $T > 0$, $q \in (0,1)$, ${}^C D_q^\alpha$ is the Caputo fractional q-difference derivative of order $1 < \alpha \leq 2$, and $f, g, h : J \times E \to E$ are given functions and g and h are continuous.

In our investigation of the existence of solutions to the problem above, we utilize the method associated with the technique of measures of noncompactness and Mönch's fixed point theorem. This approach turns out to be very useful in proving the existence of solutions for several different types of equations. The method of using measures of noncompactness was mainly initiated in the monograph of Banas and Goebel [26], and subsequently developed and used in many papers; see, for example, Banas et al. [27], Guo et al. [28], Akhmerov et al. [29], Mönch [30], Mönch and Von Harten [31], and Szufla [32].

This paper is structured as follows. In Section 2, we introduce some preliminary concepts including basic definitions and properties from fractional q-calculus and some properties of the Kuratowski measure of noncompactness. In Section 3, the existence of solutions to problem (1)–(3) is proved by using Mönch's fixed point theorem. Section 4 contains an example to illustrate our main results. The final section contains some concluding remarks and suggestions for future research.

2. Materials and Methods

We begin by introducing definitions, notations, and some preliminary facts that are used in the remainder of this paper.

Let $J = [0, T]$, $T > 0$, and consider the Banach space $C(J, E)$ of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| \ : \ t \in J\}.$$

We let $C^2(J, E)$ be the space of differentiable functions $y : J \to E$, whose first and second derivatives are continuous, and let $L^1(J, E)$ be the Banach space of measurable functions $y : J \to E$ that are Bochner integrable with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

Let $L^\infty(J, E)$ be the Banach space of bounded measurable functions $y : J \to E$ equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 \ : \ \|y(t)\| \leq c, \ a.e \ t \in J\}.$$

We now recall some definitions and properties from the fractional q-calculus [18,19]. For $a \in \mathbb{R}$ and $0 < q < 1$, we set

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q-analogue of the power $(a - b)^{(n)}$ is given by

$$(a - b)^{(0)} = 1, \ (a - b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k), \ a, b \in \mathbb{R}, \ n \in \mathbb{N}.$$

In general,

$$(a - b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \left(\frac{a - bq^k}{a - bq^{k+\alpha}} \right), \ a, b, \alpha \in \mathbb{R}.$$

Note that if $b = 0$, then $a^{(\alpha)} = a^\alpha$.

Definition 1 ([19])**.** *The q-gamma function is defined by*

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}},\ \alpha \in \mathbb{R} - \{0, -1, -2, \dots\}.$$

We wish to point out that the q-gamma function satisfies the relation $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$.

Definition 2 ([19])**.** *The q-derivative of order $n \in \mathbb{N}$ of a function $f : J \to \mathbb{R}$ is defined by $(D_q^0 f)(t) = f(t)$,*

$$(D_q f)(t) = (D_q^1 f)(t) = \frac{f(t) - f(qt)}{(1-q)t},\ t \neq 0,\ (D_q f)(0) = \lim_{t \to 0} (D_q f)(t),$$

and

$$(D_q^n f)(t) = (D_q^1 D_q^{n-1} f)(t),\ t \in J,\ n \in \{1, 2, \dots\}.$$

Now set $J_t = \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 3 ([19])**.** *The q-integral of a function $f : J_t \to \mathbb{R}$ is defined by*

$$(I_q f)(t) = \int_0^t f(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q f)(t) = f(t)$, while if f is continuous at 0, then

$$(I_q D_q f)(t) = f(t) - f(0).$$

Definition 4 ([21])**.** *The Riemann–Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$ is defined by $(I_q^0 f)(t) = f(t)$, and*

$$(I_q^\alpha f)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s,\ t \in J.$$

Note that for $\alpha = 1$, we have $(I_q^1 f)(t) = (I_q f)(t)$.

Lemma 1 ([33])**.** *For $\alpha \in \mathbb{R}_+$ and $\beta \in (-1, +\infty)$, we have*

$$(I_q^\alpha (t-a)^{(\beta)})(t) = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)} (t-a)^{(\alpha+\beta)},\ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)}.$$

In what follows, we let $[\alpha]$ denote the integer part of α.

Definition 5 ([34])**.** *The Riemann–Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$ is defined by $(D_q^0 f)(t) = f(t)$, and*

$$(D_q^\alpha f)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(t),\ t \in J.$$

Definition 6 ([34])**.** *The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$ is defined by $(D_q^0 f)(t) = f(t)$, and*

$$({}^C D_q^\alpha f)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f)(t),\ t \in J.$$

Lemma 2 ([34])**.** *Let $\alpha, \beta \in \mathbb{R}_+$ and let f be a function defined on J. Then:*

(1) $(I_q^\alpha I_q^\beta f)(t) = (I_q^{\alpha+\beta} f)(t)$;
(2) $(D_q^\alpha I_q^\alpha f)(t) = f(t)$.

Lemma 3 ([34])**.** *Let $\alpha \in \mathbb{R}_+$ and let f be a function defined on J. Then:*

$$(I_q^\alpha\, {}^C D_q^\alpha f)(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha\, {}^C D_q^\alpha f)(t) = f(t) - f(0).$$

Next, we recall the definition of the Kuratowski measure of noncompactness and summarize some of the main properties of this measure.

Definition 7 ([26])**.** *Let E be a Banach space and let Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0, \infty)$ defined by*

$$\mu(B) = \inf\{\epsilon > 0 : B \subset \cup_{i=1}^m B_i \text{ and } diam(B_i) \le \epsilon\},\ \text{where } B \in \Omega_E.$$

Property 1 ([26])**.** *The Kuratowski measure of noncompactness satisfies:*

(1) $\mu(B) = 0$ *if and only if $\overline{B}$ is compact (B is relatively compact).*
(2) $\mu(B) = \mu(\overline{B})$.
(3) $A \subseteq B$ *implies* $\mu(A) \le \mu(B)$.
(4) $\mu(A + B) \le \mu(A) + \mu(B)$.
(5) $\mu(cB) = |c|\mu(B),\ c \in \mathbb{R}$.
(6) $\mu(conB) = \mu(B)$.
(7) $\mu(B + x_0) = \mu(B)$, *for all $x_0 \in E$.*

Here $\overline{B}$ and $conB$ denote the closure and the convex hull of the bounded set B, respectively.

Definition 8. *The map $f : J \times E \to E$ is Carathéodory if*

1. $t \to f(t, u)$ *is measurable for each $u \in E$, and*
2. $u \to f(t, u)$ *is continuous for almost each $t \in J$.*

For a given set V of functions $v : J \to E$, let

$$\begin{aligned} V(t) &= \{v(t) : v \in V\},\ t \in J, \\ V(J) &= \{v(t) : v \in V,\ t \in J\}. \end{aligned}$$

We next recall Mönch's fixed point theorem.

Theorem 1 ([30,35])**.** *Let D be a bounded, closed, and convex subset of a Banach space E such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{con} N(V) \text{ or } V = N(V) \cup \{0\} \text{ implies } \mu(V) = 0,$$

holds for every subset V of D, then N has a fixed point.

The next lemma is a useful result.

Lemma 4 ([28]). *If $V \subset C(J, E)$ is a bounded and equicontinuous set, then*

1. *The function $t \to \mu(V(t))$ is continuous on J.*
2. $\mu\left(\left\{\int_J y(t)dt \, : \, y \in V\right\}\right) \leq \int_J \mu(V(t))dt.$

3. Results

We now define what is meant by a solution of the problem (1)–(3).

Definition 9. *A function $y \in C^2(J, E)$ is said to be a solution of the problem (1)–(3) if y satisfies the equation $({}^C D_q^\alpha y)(t) = f(t, y(t))$ on J, and satisfies the boundary conditions (2) and (3).*

In order to prove the existence of solutions to the problem (1)–(3), we need the following lemma.

Lemma 5. *Let σ, ρ_1, $\rho_2 : J \to E$ be continuous functions. The solution of the boundary value problem*

$$({}^C D_q^\alpha y)(t) = \sigma(t),\ t \in J = [0, T], \quad 1 < \alpha \leq 2, \tag{4}$$

$$y(0) - y'(0) = \int_0^T \rho_1(s)ds, \tag{5}$$

$$y(T) + y'(T) = \int_0^T \rho_2(s)ds, \tag{6}$$

is given by

$$y(t) = K(t) + \int_0^T H(t, s)\sigma(s)d_q s, \tag{7}$$

where

$$K(t) = \frac{(1 + T - t)}{(2 + T)} \int_0^T \rho_1(s)ds + \frac{(1 + t)}{(2 + T)} \int_0^T \rho_2(s)ds, \tag{8}$$

and

$$H(t,s) = \begin{cases} \dfrac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \dfrac{(1 + t)(T - qs)^{(\alpha-1)}}{(2 + T)\Gamma_q(\alpha)} - \dfrac{(1 + t)(T - qs)^{(\alpha-2)}}{(2 + T)\Gamma_q(\alpha - 1)}, & 0 \leq s < t, \\ -\dfrac{(1 + t)(T - qs)^{(\alpha-1)}}{(2 + T)\Gamma_q(\alpha)} - \dfrac{(1 + t)(T - qs)^{(\alpha-2)}}{(2 + T)\Gamma_q(\alpha - 1)}, & t \leq s \leq T. \end{cases} \tag{9}$$

Proof. Applying the Riemann–Liouville fractional q-integral of order α to both sides of Equation (4), and by using Lemma 3, we have

$$y(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\sigma(s)d_q s + c_0 + c_1 t. \tag{10}$$

Using the boundary conditions (5) and (6), we obtain

$$c_0 - c_1 = \int_0^T \rho_1(s)ds, \tag{11}$$

and

$$c_0 + (1+T)c_1 + \int_0^T \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\sigma(s)d_qs + \int_0^T \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\sigma(s)d_qs = \int_0^T \rho_2(s)ds. \quad (12)$$

Equations (11) and (12) give

$$c_1 = \frac{1}{(2+T)}\left(\int_0^T \rho_2(s)ds - \int_0^T \rho_1(s)ds - \int_0^T \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\sigma(s)d_qs - \int_0^T \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\sigma(s)d_qs\right), \quad (13)$$

and

$$c_0 = \frac{(1+T)}{(2+T)}\int_0^T \rho_1(s)ds + \frac{1}{(2+T)}\left(\int_0^T \rho_2(s)ds - \int_0^T \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\sigma(s)d_qs - \int_0^T \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\sigma(s)d_qs\right). \quad (14)$$

From (10), (13), and (14) and using the fact that $\int_0^T = \int_0^t + \int_t^T$, we have

$$y(t) = K(t) + \int_0^T H(t,s)\sigma(s)d_qs,$$

where

$$K(t) = \frac{(1+T-t)}{(2+T)}\int_0^T \rho_1(s)ds + \frac{(1+t)}{(2+T)}\int_0^T \rho_2(s)ds,$$

and

$$H(t,s) = \begin{cases} \dfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \dfrac{(1+t)(T-qs)^{(\alpha-1)}}{(2+T)\Gamma_q(\alpha)} - \dfrac{(1+t)(T-qs)^{(\alpha-2)}}{(2+T)\Gamma_q(\alpha-1)}, & 0 \le s < t, \\[2ex] -\dfrac{(1+t)(T-qs)^{(\alpha-1)}}{(2+T)\Gamma_q(\alpha)} - \dfrac{(1+t)(T-qs)^{(\alpha-2)}}{(2+T)\Gamma_q(\alpha-1)}, & t \le s \le T, \end{cases}$$

which is what we wanted to show. □

We now prove an existence result for the problem (1)–(3) by applying Mönch's fixed point theorem (Theorem 1 above).

Let

$$H^* = \sup_{(t,s)\in J\times J} |H(t,s)|.$$

Theorem 2. *Assume that the following conditions hold.*

(P1) The functions $f, g, h : J \times E \to E$ satisfy Carathéodory conditions.

(P2) There exists $p_f, p_g, p_h \in L^\infty(J, \mathbb{R}_+)$ such that

$$\|f(t,y)\| \le p_f(t)\|y\|, \text{ for a.e. } t \in J \text{ and all } y \in E,$$

$$\|g(t,y)\| \le p_g(t)\|y\|, \text{ for a.e. } t \in J \text{ and all } y \in E,$$

$$\|h(t,y)\| \le p_h(t)\|y\|, \text{ for a.e. } t \in J \text{ and all } y \in E.$$

(P3) For almost all $t \in J$ and each bounded set $B \subset E$, we have

$$\mu(f(t,B)) \leq p_f(t)\mu(B), \text{ for a.e. } t \in J,$$

$$\mu(g(t,B)) \leq p_g(t)\mu(B), \text{ for a.e. } t \in J,$$

$$\mu(h(t,B)) \leq p_h(t)\mu(B), \text{ for a.e. } t \in J.$$

Then, the BVP (1)–(3) has at least one solution in $C^2(J,E)$, provided

$$\frac{T(1+T)}{(2+T)}(\|p_g\|_{L^\infty} + \|p_h\|_{L^\infty}) + H^*T\|p_f\|_{L^\infty} < 1. \tag{15}$$

Proof. In order to transform problem (1)–(3) into a fixed point type problem, consider the operator

$$N : C^2(J,E) \longrightarrow C^2(J,E)$$

defined by

$$(Ny)(t) = K(t) + \int_0^T H(t,s)f(s,y(s))d_qs, \tag{16}$$

where

$$K(t) = \frac{(1+T-t)}{(2+T)}\int_0^T g(s,y(s))ds + \frac{(1+t)}{(2+T)}\int_0^T h(s,y(s))ds,$$

and H(t,s) is given by (9). It is easy to see that the fixed points of N are solutions of (1)–(3).

Let $R > 0$ and consider

$$D_R = \{y \in C^2(J,E) : \|y\|_\infty \leq R\}. \tag{17}$$

Clearly, D_R is a closed, bounded, and convex subset of $C^2(J,E)$. We show that N satisfies the hypotheses of Mönch's fixed point theorem. We give the proof in three steps.

Step 1: *N is continuous.* Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence with $y_n \to y$ in $C^2(J,E)$. For each $t \in J$, we have

$$\begin{aligned}|(Ny_n)(t) - (Ny)(t)| \leq\ & \frac{(1+T-t)}{(2+T)}\int_0^T |g(s,y_n(s)) - g(s,y(s))|ds \\ &+ \frac{(1+t)}{(2+T)}\int_0^T |h(s,y_n(s)) - h(s,y(s))|ds \\ &+ \int_0^T |H(t,s)||f(s,y_n(s)) - f(s,y(s))|d_qs.\end{aligned}$$

Hence,

$$\begin{aligned}\|N(y_n) - N(y)\| \leq\ & \frac{T(1+T)}{(2+T)}\|g(s,y_n(s)) - g(s,y(s))\| \\ &+ \frac{T(1+T)}{(2+T)}\|h(s,y_n(s)) - h(s,y(s))\| \\ &+ H^*T\|f(s,y_n(s)) - f(s,y(s))\|.\end{aligned}$$

Let $\rho > 0$ be such that

$$\|y_n\|_\infty \leq \rho, \ \|y\|_\infty \leq \rho.$$

By (P2), we have

$$\|f(s,y_n(s)) - f(s,y(s))\| \leq 2\rho p_f(s) := \sigma_f(s),$$

$$\|g(s,y_n(s)) - g(s,y(s))\| \le 2\rho p_g(s) := \sigma_g(s),$$
$$\|h(s,y_n(s)) - h(s,y(s))\| \le 2\rho p_h(s) := \sigma_h(s),$$

and $\sigma_f(s)$, $\sigma_g(s)$, $\sigma_h(s) \in L^1(J,\mathbb{R}_+)$. Since the functions f, g, and h satisfy Carathéodory conditions, the Lebesgue-dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \to 0 \ \ as \ \ n \to \infty.$$

Consequently, N is continuous on $C^2(J,E)$.

Step 2: *N maps D_R into itself.* Now, for any $y \in D_R$, (P2) and (15) imply that for each $t \in J$,

$$\begin{aligned}
\|(Ny)(t)\| &\le \frac{(1+T-t)}{(2+T)}\int_0^T \|g(s,y(s))\|ds + \frac{(1+t)}{(2+T)}\int_0^T \|h(s,y(s))\|\,ds \\
&\quad + \int_0^T |H(t,s)|\|f(s,y(s))\|d_qs, \\
&\le \frac{(1+T-t)}{(2+T)}\int_0^T p_g(s)\|y\|ds + \frac{(1+t)}{(2+T)}\int_0^T p_h(s)\|y\|ds \\
&\quad + \int_0^T |H(t,s)|p_f(s)\|y\|d_qs, \\
&\le R\left(\frac{T(1+T)}{(2+T)}\|p_g\|_{L^\infty} + \frac{T(1+T)}{(2+T)}\|p_h\|_{L^\infty} + H^*T\|p_f\|_{L^\infty}\right), \\
&\le R.
\end{aligned}$$

Step 3: *$N(D_R)$ is bounded and equicontinuous.* In view of Step 2, it is clear that $N(D_R)$ is bounded. To show the equicontinuity of $N(D_R)$, let $t_1,\ t_2 \in J,\ t_1 < t_2$, and $y \in D_R$. Then,

$$\begin{aligned}
\|(Ny)(t_2) - (Ny)(t_1)\| &= \Bigg\|\frac{(t_1-t_2)}{(2+T)}\int_0^T g(s,y(s))ds + \frac{(t_2-t_1)}{(2+T)}\int_0^T h(s,y(s))ds \\
&\quad + \int_0^T (H(t_2,s) - H(t_1,s))f(s,y(s))d_qs\Bigg\|, \\
&\le \frac{(t_1-t_2)}{(2+T)}\int_0^T \|g(s,y(s))\|ds + \frac{(t_2-t_1)}{(2+T)}\int_0^T \|h(s,y(s))\|\,ds \\
&\quad + \int_0^T |H(t_2,s) - H(t_1,s)|\|f(s,y(s))\|d_qs.
\end{aligned}$$

By (P2), we have

$$\begin{aligned}
\|(Ny)(t_2) - (Ny)(t_1)\| &\le \frac{(t_1-t_2)}{(2+T)}\int_0^T p_g(s)\|y\|ds + \frac{(t_2-t_1)}{(2+T)}\int_0^T p_h(s)\|y\|\,ds \\
&\quad + \int_0^T |H(t_2,s) - H(t_1,s)|p_f(s)\|y\|d_qs, \\
&\le RT\frac{(t_1-t_2)}{(2+T)}\|p_g\|_{L^\infty} + RT\frac{(t_2-t_1)}{(2+T)}\|p_h\|_{L^\infty} \\
&\quad + R\|p_f\|_{L^\infty}\int_0^T |H(t_2,s) - H(t_1,s)|\,d_qs.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero, which shows the equicontinuity of $N(D_R)$.

Now, let $V \subset D_R$ be such that $V \subset \overline{con}(N(V) \cup \{0\})$. Since V is bounded and equicontinuous, the function $v \to v(t) = \mu(V(t))$ is continuous on J. Moreover, (P3), Lemma 4, and properties of the measure μ imply that for each $t \in J$,

$$\begin{aligned}
v(t) &\leq \mu(N(V)(t) \cup \{0\}), \\
&\leq \mu(N(V)(t)), \\
&\leq \frac{(1+T-t)}{(2+T)} \int_0^T p_g(s)\mu(V(s))ds + \frac{(1+t)}{(2+T)} \int_0^T p_h(s)\mu(V(s))ds \\
&\quad + \int_0^T |H(t,s)| p_f(s)\mu(V(s))d_q s, \\
&\leq \|v\|_\infty \left[\frac{T(1+T)}{(2+T)} (\|p_g\|_{L^\infty} + \|p_h\|_{L^\infty}) + H^* T \|p_f\|_{L^\infty} \right].
\end{aligned}$$

This means that

$$\|v\|_\infty \left(1 - \left[\frac{T(1+T)}{(2+T)} (\|p_g\|_{L^\infty} + \|p_h\|_{L^\infty}) + H^* T \|p_f\|_{L^\infty} \right] \right) \leq 0.$$

From (15), we see that $\|v\|_\infty = 0$, so $v(t) = 0$ for $t \in J$, and hence, $V(t)$ is relatively compact in E. The Ascoli–Arzelà theorem yields that V is relatively compact in D_R. Applying Theorem 1, we see that N has a fixed point that in turn is a solution of (1)–(3). □

4. Example

Let

$$E = l^1 = \{(y_1, y_2, \cdots, y_n, \cdots) \; : \; \sum_{n=1}^{\infty} y_n < \infty\},$$

be our Banach space with the norm

$$\|y\|_E = \sum_{n=1}^{\infty} |y_n|.$$

Consider the boundary value problem for fractional $\frac{1}{4}$-difference equations given by

$$({}^C D_{\frac{1}{4}}^{\frac{3}{2}} y)(t) = \frac{1}{(e^t + 5)} y_n(t), \;\; for\; a.e.\; t \in J = [0,1],\; 1 < \alpha \leq 2, \tag{18}$$

$$y(0) - y'(0) = \int_0^1 \frac{s^3 - 1}{9} y_n(s) ds, \tag{19}$$

$$y(1) + y'(1) = \int_0^1 \frac{s^3 + 1}{6} y_n(s) ds. \tag{20}$$

Here, $\alpha = \frac{3}{2}$, $q = \frac{1}{4}$, $T = 1$, and

$$f_n(t,y) = \frac{1}{e^t + 5} y_n, \; (t,y) \in J \times E,$$

$$g_n(t,y) = \frac{t^3 - 1}{9} y_n, \; (t,y) \in J \times E,$$

and

$$h_n(t,y) = \frac{t^3 + 1}{6} y_n, \; (t,y) \in J \times E,$$

where

$$\begin{aligned}
y &= (y_1, y_2, \ldots, y_n, \ldots), \\
f &= (f_1, f_2, \ldots, f_n, \ldots),
\end{aligned}$$

$$g = (g_1, g_2, \dots, g_n, \dots),$$

and

$$h = (h_1, h_2, \dots, h_n, \dots).$$

Clearly, conditions (P1) and (P2) hold with

$$p_f(t) = \frac{1}{e^t + 5}, \quad p_g(t) = \frac{t^3}{9}, \quad p_h(t) = \frac{t^3}{6}.$$

From (9), we have

$$H^* = \sup_{(t,s) \in J \times J} |H(t,s)| = \frac{5}{3\Gamma_{\frac{1}{4}}(\frac{3}{2})} + \frac{2}{3\Gamma_{\frac{1}{4}}(\frac{1}{2})}.$$

To see that condition (15) is satisfied with $T = 1$, notice that

$$\frac{T(1+T)}{(2+T)}(\|p_g\|_{L^\infty} + \|p_h\|_{L^\infty}) + H^* T \|p_f\|_{L^\infty}$$
$$= \frac{2}{3}\left(\frac{1}{9} + \frac{1}{6}\right) + \left(\frac{5}{3\Gamma_{\frac{1}{4}}(\frac{3}{2})} + \frac{2}{3\Gamma_{\frac{1}{4}}(\frac{1}{2})}\right)\frac{1}{6} \simeq 0.5564 < 1.$$

Then, by Theorem 2, the problem (18)–(20) has a solution on $[0, 1]$.

5. Discussion

In this work, we proved the existence of solutions to a fractional q-difference equation with nonlinear integral type boundary conditions in Banach spaces using a method involving the Kuratowski measure of noncompactness and Mönch's fixed point theorem. An example was presented to illustrate the effectiveness of the results.

An interesting direction for future research of course would be to consider fractional q-difference equations of order $0 < \alpha \leq 1$ and orders greater than the $1 < \alpha \leq 2$ considered here. Another direction would be to consider Riemann–Stieltjes integral-type boundary conditions. Adding impulsive effects to the problem would expand the ares of possible applications as well.

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