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Average Process of Fractional Navier–Stokes Equations with Singularly Oscillating Force

Chunjiao Han, Yi Cheng *, Ranzhuo Ma and Zhenhua Zhao

School of Mathematical Sciences, Bohai University, Jinzhou 121013, China; h1961971968@126.com (C.H.); ranzhuoma1973@126.com (R.M.); dana2200@163.com (Z.Z.)
* Correspondence: chengyi407@126.com

Abstract: The averaging process between two-dimensional fractional Navier–Stokes equations driven by a singularly oscillating external force and the averaged equations corresponding to the limiting case are investigated. The uniform boundedness of the global attractors for a fractional Navier–Stokes equation with a singularly external force is established. Furthermore, these global attractors converge uniformly to the attractor of the averaged equations under suitable assumptions on the singularly external force, and the explicit convergence rate of the global attractors is guaranteed.

Keywords: fractional Navier–Stokes equations; global attractors; external force; time averaging

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1. Introduction

Navier–Stokes (N–S) equations have been investigated so extensively all the time mainly because of the wide range of applications in many important physical phenomena and theoretical studies, such as aeronautical sciences, meteorology, thermohydraulics, etc. It is known that the following N–S equations

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \ \ (t,x)\in(0,T]\times\Omega, \\ \nabla u = 0, \ \ (t,x)\in(0,T]\times\Omega, \end{cases} \tag{1}$$

controlled by external forces have attracted a lot of attention in recent years, where u denotes the velocity field, the symbols Δ and ∇ stand for the Laplace operator and the gradient acting in the x-space, respectively, the parameter $\nu > 0$ is the kinematic viscosity with the assumption that the density of the fluid is constant, $p := p(x,t)$ represents the associated pressure and f is an external force.

Among the many notable results regarding (1), it is worth noting that the controllability of N–S Equation (1) with periodic boundary conditions was established in [1], where the external force of the system was degenerating. Moreover, the investigation on the uniform global attractor of this N–S Equation (1) is also an important subject in many papers and monographs (see [2–5] and other references). When the external force $f = f(x)$ depends only on spatial variable x, Vishik and Chepyzhov [6] established that the trajectory attractor of the three-dimensional (3D) N–S equations, where the trajectory attractor was composed of a type of solutions to system (1) on the positive semi-interval of the time axis, was bounded, which can be extended to the entire time axis, and further obtained the convergence of the uniform global attractor. When the external force $f = f(x,t)$ depends not only on time but also on spatial variables, the structure of the uniform global attractor of the nonautonomous 2D N–S Equation (1) was examined in [7], where the different structural features of the uniform global attractors were discussed with quasi-periodic and oscillating external force. In [8], Vishik and Chepyzhov later also discussed the related properties of the uniform global attractors of the nonautonomous 2D N–S equations with singular oscillatory external forces $f_0(x,t) + \varepsilon^{-\rho} f_1(\frac{x}{\varepsilon}, t)$, for $x \in \Omega \subseteq \mathbb{R}^2$, $t \in \mathbb{R}$, and $0 \le \rho \le 1$.

In [9], they further considered the attractors of the following two nonautonomous 2D N–S equations:

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u = -\nabla p + f_0(t) + \varepsilon^{-\rho} f_1(\frac{t}{\varepsilon}), \\ \nabla u = 0, \end{cases} \tag{2}$$

where $\rho \in (0,1)$, $\varepsilon > 0$, and f is a singular oscillating external force, and subject to the average equations (with respect to the case $\varepsilon = 0$)

$$\begin{cases} u_t - \nu\triangle u + (u\cdot\nabla)u = -\nabla p + f_0(t), \\ \nabla u = 0. \end{cases} \tag{3}$$

It has been demonstrated that the uniform global attractor set $\mathcal{A}^\varepsilon$ of Equation (2) is uniformly bounded and $\mathcal{A}^\varepsilon$ is convergent to the uniform global attractor set $\mathcal{A}^0$ of Equation (3) as $\varepsilon \to 0^+$. In addition, some problems associated with the averaging and the homogenization of uniform global attractors for such systems have been investigated in [10–12].

On the other hand, fractional calculus sourced from the beginning of calculus has become an important subject discussed by many mathematicians (such as Leibniz, Fourier, Abel, L'Hopital, Euler, Riemann, and Liouville). Fractional calculus has long been considered as a purely mathematical tool with no practical applications. In recent decades, however, it has been discovered that fractional calculus can be used in the most diverse fields of science, due mainly to the nonlocal character of fractional differentiation. Among the numerous applications of fractional calculus, it is worth noting some works on stochastic processes motivated by fractional Brownian motion [13] and on physical phenomena such as electromagnetism [14] and viscoelasticity [15–17]. For more detail, we refer to the survey [18] and references therein. Therefore, a natural and interesting question is now to study the relevant dynamical characteristics of the solutions of fractional N–S equations. Some investigation on the suitability of spatial fractional N–S equations have attracted the attention of many authors. In [19], the authors mainly discussed 3D N–S equations with Coriolis forces in homogeneous Besov spaces, where the existence and uniqueness of global solutions of systems at high rotational speeds were obtained and the asymptotic behavior of solutions was analyzed when the rotational speed tended to infinity. The Cauchy problem for incompressible fractional N–S equations in critical variable-exponential Fourier–Besov–Morrey spaces was investigated in [20], in which the global well-posedness of incompressible fractional N–S equations in the frequency space of a variable exponential was provided. Furthermore, introducing a Besov-type function space represented by a time-evolving semigroup, [21] mainly established the unique existence of global mild solutions for small initial data belonging to the semigroup function space under scaled subcritical and critical conditions. In addition, the authors demonstrated the existence of a global attractor for the 2D incompressible Boussinesq equation with subcritical dissipation in [22], which revealed the relationship between the Laplace exponent and the regularity in velocity and temperature. However, there are few studies on the dynamics of time fractional N–S equations.

In this paper, we consider the following time-fractional 2D N–S equations

$$\begin{cases} {}_\tau D_t^\alpha u - \nu\Delta u + u^1\partial_{x_1}u + u^2\partial_{x_2}u = -\nabla p + f^\varepsilon(x,t), \quad \forall x := (x_1,x_2) \in \Omega, \\ \partial_{x_1}u^1 + \partial_{x_2}u^2 = 0, \;\; u|_{\partial\Omega} = 0, \end{cases} \tag{4}$$

where the sign ${}_\tau D_t^\alpha$ denotes the modified fractional Riemann–Liouville derivative with respect to t defined in Section 2,

$$f^\varepsilon(x,t) = \begin{cases} f_0(x,t) + \varepsilon^{-\rho} f_1(x,t/\varepsilon), \quad \varepsilon > 0, \\ f_0(x,t), \quad \varepsilon = 0, \end{cases} \tag{5}$$

represents the external force, $\rho \in (0, \frac{\alpha}{2})$ is a fixed parameter with $\alpha \in (0,1)$, and $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$ of class C^1, $u = (u^1(x,t), u^2(x,t))$ is the velocity vector field. In fact, we call System (4) the averaged equation when $\varepsilon = 0$. The investigation of the solutions of fractional N–S equations has received a lot of attention in recent years, for instance in [23,24].

It is worth pointing out that when $f = f(x,t)$ is a general external force, Zhou and Peng [25] established the existence and uniqueness of local and global mild solutions, and the regularity of classical solutions for System (4) in a fractional abstract space. Later, they continued to consider the existence, uniqueness and Hölder continuity of weak solutions by using iterative methods in [26]. In addition, Carvalho, Neto and Planas [24] demonstrated the existence and uniqueness of weak solutions to ND time-fractional N–S equations in R^N under the external force $f = (f_1(x,t), \cdots, f_N(x,t))$. However, to the author's knowledge, there is no related result on the averaging process for System (4).

The main contribution of this article is to address the averaging process of 2D time-fractional N–S equations with a singularly external force. By using a fractional inequality and the Fadeo–Galerkin method, the well-posedness of these fractional Navier–Stokes equations is completed. Then, the uniform global attractor family $\{\mathcal{A}^\varepsilon\}$ of the dynamic processes generated by the shift semigroups theory is demonstrated. Finally we further obtain the convergence of the global attractors as the parameter approaches zero and guarantee the explicit convergence rate of the global attractors.

The content of this paper is mainly divided into the following parts: In Section 2, some basic symbols, assumptions and lemmas are introduced. In Section 3, the existence of a uniformly global attractor to System (4) is obtained, and the dynamic process of the system and the structure of attractors are presented. The uniform global attractor set $\mathcal{A}^\varepsilon$ is uniformly bounded and the convergence of the uniform global attractors is provided in Section 4.

2. Preliminary

Throughout the paper, for $\tau \in \mathbb{R}^+$, we set $\mathbb{R}_\tau = [\tau, +\infty)$ and assume $0 < \rho < \frac{\alpha}{2}$ with $\alpha \in (0,1)$. Some basic concepts from partial differential equations are borrowed, such as $L^2(\Omega)$, $H^1(\Omega)$, $H^2(\Omega)$, and $C_0^\infty(\Omega)$, etc. In what follows, the dependence on the space variable x is omitted for brevity. Let X be a normed space with the norm $\|\cdot\|_X$, and $\text{dist}_X(B_1, B_2) := \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \|b_1 - b_2\|_X$ be the Hausdorff semidistance in X from a set B_1 to a set B_2. We define

$$\mathcal{H} := \overline{\{u \in [C_0^\infty(\Omega)]^2 | \partial_{x_1} u^1 + \partial_{x_2} u^2 = 0\}}^{[L^2(\Omega)]^2},$$

$$\mathcal{V} := \overline{\{u \in [C_0^\infty(\Omega)]^2 | \partial_{x_1} u^1 + \partial_{x_2} u^2 = 0\}}^{[H^1(\Omega)]^2}.$$

Suppose $P : [L^2(\Omega)]^2 \to \mathcal{H}$ is the Leray–Helmholtz orthogonal projection. Consider the following positive self-adjoint operator

$$A := -P\Delta : D(A) \to \mathcal{H} \tag{6}$$

with $D(A) := [H^2(\Omega)]^2 \cap \mathcal{V}$. We also need to define the scale of Hilbert spaces

$$H^\sigma := D(A^{\frac{\sigma}{2}}), \quad \forall \sigma \in \mathbb{R},$$

equipped with the norm and inner products by

$$\langle u, v \rangle_{H^\sigma} := \langle A^{\frac{\sigma}{2}} u, A^{\frac{\sigma}{2}} v \rangle_{[L^2(\Omega)]^2}, \quad \|u\|_{H^\sigma} := \|A^{\frac{\sigma}{2}} u\|_{[L^2(\Omega)]^2}.$$

In particular, we set

$$H = \mathcal{H}, \ \ H^1 = \mathcal{V}, \ \ H^2 = D(A), \ \ H^{-1} = H^{-1}(\Omega),$$

where the norm of H is denoted by $\|\cdot\|$. In addition, the generalized Poincaré inequality given by

$$\|u\|^2_{H^{\sigma+1}} \geq \lambda\|u\|^2_{H^\sigma}, \quad \forall u \in H^{\sigma+1}, \tag{7}$$

is needed, where $\lambda > 0$ is the first eigenvalue of operator A.

The following basic facts related to Navier–Stokes equations should be provided for completeness. For more detail, please refer to [2,5,9]. We define the standard bilinear map $B(u,u) := P(u^1\partial_{x_1}u + u^2\partial_{x_2}u)$ and trilinear forms

$$b(u,v,w) := \langle B(u,v), w\rangle \tag{8}$$

for any $(u,v,w) \in H^1 \times H^1 \times H^1$. It is easy to verify that for all $(x,y,z) \in H^1 \times H^1 \times H^1$

$$b(x,y,y) = 0, \tag{9}$$

$$b(x,y,z) = -b(x,z,y), \tag{10}$$

and

$$|b(x,y,z)| \leq C_1\|x\|^{\frac{1}{2}}\|x\|^{\frac{1}{2}}_{H^1}\|y\|_{H^1}\|z\|^{\frac{1}{2}}\|z\|^{\frac{1}{2}}_{H^1}, \tag{11}$$

$$|b(x,y,z)| \leq C_1\|x\|^{\frac{1}{2}}\|x\|^{\frac{1}{2}}_{H^1}\|y\|^{\frac{1}{2}}\|y\|^{\frac{1}{2}}_{H^1}\|z\|_{H^1}, \tag{12}$$

where $C_1 > 0$ is a constant independent of Ω.

The related concepts of fractional derivative to the modified Riemann–Liouville integral are introduced below; for detail the interested readers can refer to [27–29].

Definition 1. *The modified Riemann–Liouville fractional integral of order α is defined on the interval $[0,t)$ by the expression*

$$_0I^\alpha_t f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \ \ \alpha > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. *The modified Riemann–Liouville derivative of order α is defined on the interval $[0,t)$ by the expression*

$$_0D^\alpha_t f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}[f(s)-f(0)]ds, \ \ 0 < \alpha < 1. \tag{13}$$

Remark 1. *Compared with the classical Riemann–Liouville derivative and the Caputo derivative, some advantages of the modified Riemann–Liouville derivative are summarized as follows.*

- *If $f(t) = K = constant$, then it is easy to check that $_0D^\alpha_t f(t) = 0$, which is beneficial in engineering applications. However, the α derivative of the Riemann–Liouville is $Kt^\alpha/\Gamma(1-\alpha)$.*
- *From Definition 2, it is easy to see that f only needs to be continuous, but must be differentiable in the Caputo derivative. That is to say, for the modified Riemann–Liouville derivative, the requirement for the regularity of the function f is lower.*

The modified Riemann–Liouville derivative retains the characteristics of fractional derivative and has some good properties given in the following.

Proposition 1 (see [27–30])**.** *Let $f : R \to R$ be a differentiable function and $u, v : R \to R$ be continuous functions, then it holds that*

$$_0D^\alpha_t[f(u(t))] = f'_u(u)\,{}_0D^\alpha_t u, \tag{14}$$

$$_0D^\alpha_t(u(t)v(t)) = v(t)\,{}_0D^\alpha_t u(t) + u(t)\,{}_0D^\alpha_t v(t), \tag{15}$$

$$_0I^\alpha_t\,{}_0D^\alpha_t f(t) = f(t) - f(0), \ {}_0D^\alpha_t\,{}_0I^\alpha_t f(t) = f(t). \tag{16}$$

Moreover, one has

$$ {}_0I_t^\alpha f(t) \leq {}_0I_t^\alpha g(t), \tag{17} $$

if $f(t) \leq g(t)$, *and*

$$ {}_0D_t^\alpha (Kf(t) + \hat{K}g(t)) = K \cdot {}_0D_t^\alpha f(t) + \hat{K} \cdot {}_0D_t^\alpha g(t), \tag{18} $$

for arbitrary constants K *and* $\hat{K}$.

Now, we present the space $L_\alpha^\infty(\mathbb{R}_\tau; H^\sigma)$ given in [31] with norm

$$ \|f\|^2_{\alpha,H^\sigma} = \sup_{t\in\mathbb{R}_\tau} (t-\tau)^\alpha \|f(s)\|^2_{H^\sigma}, \ \ \forall f \in L_\alpha^\infty(\mathbb{R}_\tau; H^\sigma), \tag{19} $$

which is a Banach space; for detail see also [32]. In particular, let $\|\cdot\|_\alpha = \|\cdot\|_{\alpha,H^0}$ if $\sigma = 0$. Assume the external force $f_0, f_1 \in L_\alpha^\infty(\mathbb{R}_\tau; H)$ and

$$ \|f_0\|^2_\alpha = M_0, \tag{20} $$

$$ \|f_1\|^2_\alpha = M_1. \tag{21} $$

This together with (5) implies $\|f^\varepsilon\|^2_\alpha \leq M_\varepsilon$, where

$$ M_\varepsilon = \begin{cases} M_0 + \sqrt{2}M_1\varepsilon^{\alpha-2\rho}, & \varepsilon > 0, \\ M_0, & \varepsilon = 0. \end{cases} \tag{22} $$

Notice that the order of M_ε is $\varepsilon^{\alpha-2\rho}$ as $\varepsilon \to 0^+$.

Next, we show several lemmas, which play a crucial role in analyzing the global attractors of the fractional N–S Equation (4).

Lemma 1 (Young's inequality with η)**.** *For any* $a, b > 0$ *and* $\eta > 0$, *it holds that*

$$ ab \leq \eta\frac{a^p}{p} + \eta^{-\frac{q}{p}}\frac{b^q}{q}, \tag{23} $$

where $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the general Young inequality (see [33]), one obtains

$$ ab = (\eta p)^{\frac{1}{p}}a \cdot \frac{b}{(\eta p)^{\frac{1}{p}}} \leq \frac{\eta p a^p}{p} + \frac{b^q}{(\eta p)^{\frac{q}{p}}q} \leq \eta\frac{a^p}{p} + \eta^{-\frac{q}{p}}\frac{b^q}{q}. $$

□

Lemma 2. *Let* $\psi_2 \in L_\alpha^\infty(\mathbb{R}_\tau)$. *Assume that* $y : \mathbb{R}_\tau \to \mathbb{R}^+$ *satisfies the fractional differential inequality*

$$ {}_\tau D_t^\alpha y(t) + \psi_1(t)y(t) \leq \psi_2^2(t), \quad \forall t \geq \tau, \ \alpha \in (0,1), \tag{24} $$

where the function ψ_1 *fulfills*

$$ \beta(t-\tau)^\alpha \geq \frac{1}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\psi_1(s)ds \geq \beta(t-\tau)^\alpha - r, \quad \forall t \geq \tau, \tag{25} $$

for some non-negative constants β, r. *Then, we have*

$$ y(t) \leq y(\tau)e^{-\beta(t-\tau)^\alpha + r} + e^r\Gamma(1-\alpha)||\psi_2||^2_{\alpha,R}, \quad \forall t \geq \tau. \tag{26} $$

Proof. Multiplying $e^{{}_\tau I_t^\alpha \psi_1(t)}$ on both sides of inequality (24) and using (14) and (15) in Proposition 1 lead to

$$
{}_\tau D_t^\alpha \{e^{{}_\tau I_t^\alpha \psi_1(t)} y(t)\} \le e^{{}_\tau I_t^\alpha \psi_1(t)} \psi_2^2(t), \tag{27}
$$

which, together with (16), gives

$$
e^{{}_\tau I_t^\alpha \psi_1(t)} y(t) \le y(\tau) + \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} e^{{}_\tau I_s^\alpha \psi_1(s)} \psi_2^2(s) ds, \ \ \forall t \ge \tau. \tag{28}
$$

From (28), apply (25) to obtain

$$
\begin{aligned}
y(t) &\le y(\tau) e^{-{}_\tau I_t^\alpha \psi_1(t)} + e^{-{}_\tau I_t^\alpha \psi_1(t)} \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} e^{{}_\tau I_s^\alpha \psi_1(s)} \psi_2^2(s) ds \\
&\le y(\tau) e^{-\beta(t-\tau)^\alpha + r} + \frac{e^r}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \psi_2^2(s) ds \\
&\le y(\tau) e^{-\beta(t-\tau)^\alpha + r} + \frac{e^r}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{(1-\alpha)-1} (s-\tau)^\alpha \psi_2^2(s) ds.
\end{aligned} \tag{29}
$$

Since

$$
\begin{aligned}
\int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{(1-\alpha)-1} ds &= \int_0^1 \omega^{\alpha-1} (1-\omega)^{1-\alpha-1} d\omega \\
&= B(\alpha, 1-\alpha) \\
&= \Gamma(\alpha)\Gamma(1-\alpha),
\end{aligned} \tag{30}
$$

where B is the Beta function, it follows from (29) that

$$
y(t) \le y(\tau) e^{-\beta(t-\tau)^\alpha + r} + e^r \Gamma(1-\alpha) \|\psi_2\|_{\alpha,R}^2, \tag{31}
$$

which completes the proof. □

3. Attractors for Navier–Stokes Equations

3.1. Well-Posedness for the Fractional Navier–Stokes Equations

According to the definition of operators given in Section 2, Equation (4) can be written as the following abstract form

$$
{}_\tau D_t^\alpha u + \nu A u + B(u,u) = f^\varepsilon(t), \tag{32}
$$

where the pressure term p has disappeared due to the application of the Leray–Helmholtz projection P. Before discussing the attractor of Equation (32), now we complete the well-posedness of fractional N–S Equation (32). For this, the following Lemma is needed.

Lemma 3. *For any fixed* $\varepsilon \in [0,1]$*, let* u *be the solution of the Cauchy problem in* (32) *with the initial value* $u|_{t=\tau} = u_\tau \in H$*, then it holds that*

$$
\|u(t)\|^2 \le \|u(\tau)\|^2 e^{-\frac{\lambda\nu}{\alpha\Gamma(\alpha)}(t-\tau)^\alpha} + \mathcal{M}_\varepsilon, \tag{33}
$$

$$
\|u(t)\|^2 + \frac{\nu}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|u(s)\|_{H^1}^2 ds \le \|u(\tau)\|^2 + \mathcal{M}_\varepsilon, \tag{34}
$$

where $\mathcal{M}_\varepsilon = \frac{\Gamma(1-\alpha)M_\varepsilon}{\lambda\nu}$ *with the constant* $M_\varepsilon > 0$ *given by* (22)*, for any* $t \ge \tau$*. Moreover, we have*

$$
\|u(t)\|_{H^1}^2 \le \mathcal{W}(t-\tau, \|u(\tau)\|^2, M_\varepsilon), \ \forall t \ge \tau, \tag{35}
$$

where $\mathcal{W}$ *is a positive function.*

Proof. Taking the inner product of Equation (32) with u, we get

$$\langle {}_\tau D_t^\alpha u, u\rangle + \nu\langle Au, u\rangle + \langle B(u,u), u\rangle = \langle f^\varepsilon, u\rangle.$$

It is easy to check that $\langle B(u,u), u\rangle = b(u,u,u) = 0$ by (9), then using Proposition 1 and the Cauchy–Schwartz inequality gets

$$\frac{1}{2}{}_\tau D_t^\alpha \|u\|^2 + \nu\|u\|_{H^1}^2 \le \|f^\varepsilon\|\|u\|. \tag{36}$$

Following Young's inequality (23) ($p = 2, q = 2$), the estimate (36) becomes

$${}_\tau D_t^\alpha \|u\|^2 + 2\nu\|u\|_{H^1}^2 \quad \le \eta\|u\|^2 + \tfrac{1}{\eta}\|f^\varepsilon\|^2.$$

It follows from Poincaré's inequality (7) that

$${}_\tau D_t^\alpha \|u\|^2 + 2\nu\|u\|_{H^1}^2 \le \frac{\eta}{\lambda}\|u\|_{H^1}^2 + \frac{1}{\eta}\|f^\varepsilon\|^2. \tag{37}$$

Since the Young's parameter $\eta > 0$ is arbitrary, we choose $\eta = \lambda\nu$, and from (37) we obtain

$${}_\tau D_t^\alpha \|u\|^2 + \nu\|u\|_{H^1}^2 \le (\lambda\nu)^{-1}\|f^\varepsilon\|^2. \tag{38}$$

Applying Poincaré's inequality (7), one gets from (38)

$${}_\tau D_t^\alpha \|u\|^2 + \lambda\nu\|u\|^2 \le (\lambda\nu)^{-1}\|f^\varepsilon\|^2. \tag{39}$$

Set $\bar{\psi}_1(t) := \lambda\nu$ for any $t \ge \tau$, then it is easy to calculate

$$\frac{1}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\lambda\nu ds = \frac{\lambda\nu}{\alpha\Gamma(\alpha)}(t-\tau)^\alpha. \tag{40}$$

In view of $f^\varepsilon \in L_\alpha^\infty(\mathbb{R}_\tau; H)$ and the above equation, then Lemma 2 can be invoked to produce our desired estimate (33).

Integrating both sides of (38) and using Proposition 1, one can show that

$$\|u(t)\|^2 - \|u(\tau)\|^2 + \frac{\nu}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\|u(s)\|_{H^1}^2 ds \le \frac{(\lambda\nu)^{-1}}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\|f^\varepsilon(s)\|^2 ds,$$

which, together with (30) and $f^\varepsilon \in L_\alpha^\infty(\mathbb{R}_\tau; H)$, leads to

$$\|u(t)\|^2 + \frac{\nu}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\|u(s)\|_{H^1}^2 ds \le \|u(\tau)\|^2 + \mathcal{M}_\varepsilon.$$

where $\mathcal{M}_\varepsilon$ is a constant defined in (33). Hence, the estimate (34) holds.

The estimate (35) shall be proved by taking the inner product of Equation (32) with Au as follows:

$$\langle {}_\tau D_t^\alpha u, Au\rangle + \nu\langle Au, Au\rangle + \langle B(u,u), Au\rangle = \langle f^\varepsilon, Au\rangle. \tag{41}$$

Applying Cauchy-Schwartz's inequality and Young's inequality ($p, q = 2$, $\eta = \frac{2}{\nu}$) on (41), one obtains

$$\frac{1}{2}{}_\tau D_t^\alpha \|u\|_{H^1}^2 + \nu\|u\|_{H^2}^2 + \langle B(u,u), Au\rangle \le \frac{1}{\nu}\|f^\varepsilon\|^2 + \frac{\nu}{4}\|u\|_{H^2}^2,$$

which yields

$${}_\tau D_t^\alpha \|u\|_{H^1}^2 + \frac{3\nu}{2}\|u\|_{H^2}^2 \le \frac{2}{\nu}\|f^\varepsilon\|^2 + 2|\langle B(u,u), Au\rangle|. \tag{42}$$

Due to the following estimate

$$\|B(u,u)\| \leq (\int_{\Omega} |u|^2|\nabla u|^2 dx)^{\frac{1}{2}} \leq \|u\|_{L^4}\|\nabla u\|_{L^4}, \tag{43}$$

then by Ladyzhenskaya's estimates, one can deduce

$$\|u\|_{L^4} \leq \hat{C}\|u\|_{H^1}^{\frac{1}{2}}\|u\|^{\frac{1}{2}}, \quad \|\nabla u\|_{L^4} \leq \tilde{C}\|u\|_{H^2}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}}, \tag{44}$$

where $\hat{C}$ and $\tilde{C}$ are positive constants. Plugging (44) into (43), we get

$$\|B(u,u)\| \leq C_2\|u\|^{\frac{1}{2}}\|u\|_{H^1}\|u\|_{H^2}^{\frac{1}{2}} \tag{45}$$

for some constant $C_2 > 0$. Thus, applying the Cauchy–Schwartz inequality, it holds that

$$|\langle B(u,u), Au\rangle| \leq C_2\|u\|^{\frac{1}{2}}\|u\|_{H^1}\|u\|_{H^2}^{\frac{3}{2}},$$

which, together with Young's inequality (23) ($p = \frac{4}{3}, q = 4$), shows

$$|\langle B(u,u), Au\rangle| \leq \frac{3\eta}{4}\|u\|_{H^2}^2 + \frac{C_2^4}{4\eta^3}\|u\|^2\|u\|_{H^1}^4. \tag{46}$$

Let the Young's parameter $\eta = \frac{\nu}{3}$ and substitute (46) into the estimate (42), after which (42) becomes

$${}_{\tau}D_t^{\alpha}\|u\|_{H^1}^2 + \nu\|u\|_{H^2}^2 \leq \frac{2}{\nu}\|f^{\varepsilon}\|^2 + C_3\|u\|^2\|u\|_{H^1}^4,$$

with $C_3 = \frac{27C_2^4}{2\nu^3} > 0$. Thanks to Poincaré's inequality (7), we also obtain

$${}_{\tau}D_t^{\alpha}\|u\|_{H^1}^2 + \lambda\nu\|u\|_{H^1}^2 - C_3\|u\|^2\|u\|_{H^1}^4 \leq \frac{2}{\nu}\|f^{\varepsilon}\|^2,$$

which means that

$${}_{\tau}D_t^{\alpha}\|u\|_{H^1}^2 + (\lambda\nu - C_3\|u\|^2\|u\|_{H^1}^2)\|u\|_{H^1}^2 \leq \frac{2}{\nu}\|f^{\varepsilon}\|^2. \tag{47}$$

In a similar fashion as (39), we set $\hat{\psi}_1(t) := \lambda\nu - C_3\|u(t)\|^2\|u(t)\|_{H^1}^2$. To simplify the process, we take $u(\tau) = 0$ in the estimate (33). Using (33), one can deduce

$$\begin{aligned} {}_{\tau}I_t^{\alpha}\hat{\psi}_1(t) &= \frac{1}{\Gamma(\alpha)}\int_{\tau}^{t}(t-s)^{\alpha-1}(\lambda\nu - C_3\|u(s)\|^2\|u(s)\|_{H^1}^2)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_{\tau}^{t}(t-s)^{\alpha-1}\lambda\nu ds - \frac{C_3}{\Gamma(\alpha)}\int_{\tau}^{t}(t-s)^{\alpha-1}\|u(s)\|^2\|u(s)\|_{H^1}^2 ds \\ &\geq \frac{\lambda\nu}{\alpha\Gamma(\alpha)}(t-\tau)^{\alpha} - \frac{C_3\mathcal{M}_{\varepsilon}}{\Gamma(\alpha)}\int_{\tau}^{t}(t-s)^{\alpha-1}\|u(s)\|_{H^1}^2 ds. \end{aligned} \tag{48}$$

Using the estimate (34), it follows from (48) that

$${}_{\tau}I_t^{\alpha}\hat{\psi}_1(t) \geq \frac{\lambda\nu}{\alpha\Gamma(\alpha)}(t-\tau)^{\alpha} - r_1, \tag{49}$$

where $r_1 = \frac{C_3\mathcal{M}_{\varepsilon}}{\nu}[\|u(\tau)\|^2 + \mathcal{M}_{\varepsilon}]$ for the constant $\mathcal{M}_{\varepsilon}$ given by (33). Therefore, we apply Lemma 2 on (47) to obtain

$$\|u(t)\|_{H^1}^2 \leq \mathcal{W}(t-\tau, \|u(\tau)\|^2, \mathcal{M}_{\varepsilon}) := \|u(\tau)\|_{H^1}^2 e^{-\hat{\mathcal{N}}(t)} + \mathcal{N}, \tag{50}$$

where

$$\hat{\mathcal{N}}(t) := \frac{\lambda\nu}{\alpha\Gamma(\alpha)}(t-\tau)^{\alpha} - \frac{C_3\mathcal{M}_{\varepsilon}}{\nu}[\|u(\tau)\|^2 + \mathcal{M}_{\varepsilon}],$$
$$\mathcal{N} := 2\lambda\mathcal{M}_{\varepsilon}e^{\frac{C_3\mathcal{M}_{\varepsilon}}{\nu}[\|u(\tau)\|^2+\mathcal{M}_{\varepsilon}]}.$$

The proof is thus complete. □

Next, the well-posedness of the problem (32) is proved by exploiting the Faedo–Galerkin method, given as follows.

Theorem 1. *For $u_\tau \in H$ and $f^{\varepsilon} \in L^{\infty}_{\alpha}([\tau, T); H)$, there exists a unique solution u of (32) such that for all $T > \tau$*

$$u \in C([\tau, T); H) \cap L^2([\tau, T); H^1).$$

Proof. We use the Fadeo–Galerkin method also adopted in [2,5] to establish the existence of a solution $u \in C([\tau, T); H) \cap L^2([\tau, T); H^1)$ of (32). Now, the approximation procedure satisfying (32) is provided by

$$u_m(t) = \sum_{j=1}^{m} c_{im}(t)w_j, \tag{51}$$

where the functions $w_j (j = 1, \cdots, m)$ representing the eigenvalues of the operator A are given by (6). Then, we substitute u_m into Equation (32) and take an inner product with w_j, to obtain

$$\begin{cases} \langle {}_\tau D^{\alpha}_t u_m, w_j\rangle + \nu\langle Au_m, w_j\rangle + \langle B(u_m, u_m), w_j\rangle = \langle f^{\varepsilon}, w_j\rangle, \\ u_m(\tau) = P_m u(\tau), \end{cases} \tag{52}$$

where $P_m : H \to \Delta_m$ is a projector with $\Delta_m := Span\{w_1, \cdots, w_m\}$. It is worth noting that Equation (52) is also equivalent to

$${}_\tau D^{\alpha}_t u_m + \nu Au_m + P_m B(u_m, u_m) = P_m f^{\varepsilon}, \tag{53}$$

and an ODE equation driven by $c_{im}(t)$ in (52) is known to have a local solution in the interval $[\tau, T_{max})$. According to Lemma 3, a priori estimate (34) implies $\int_\tau^T (T-s)^{\alpha-1}\|u_m(s)\|^2_{H^1}ds \le \infty$. Since $\int_0^T \|u_m(s)\|^2_{H^1}ds \le C_T \int_\tau^T (T-s)^{\alpha-1}\|u_m(s)\|_{H^1}ds$ with the constant C_T depending only on T, the a priori estimates (33) and (34) in Lemma 3 give $T_{max} = +\infty$ and

$$u_m \text{ is bounded in } C([\tau, T); H) \cap L^2([\tau, T); H^1). \tag{54}$$

On the basis of $u_m \in L^2([\tau, T); H^1)$, it is easy to know that $B(u_m, u_m)$ and $P_m B(u_m, u_m)$ are bounded in $L^2([\tau, T); H^{-1})$; this shows from (53)

$${}_\tau D^{\alpha}_t u_m \text{ is bounded in } L^2([\tau, T); H^{-1}). \tag{55}$$

In light of (54) and the weak compactness, there exists $u \in C([\tau, T); H) \cap L^2([\tau, T); H^1)$ and a subsequence still denoted by itself, such that

$$u_m \to u \text{ in } L^2([\tau, T); H^1) \text{ weakly},$$
$${}_\tau D^{\alpha}_t u_m \to {}_\tau D^{\alpha}_t u \text{ in } L^2([\tau, T); H^{-1}) \text{ weakly},$$

which, together with the classical compactness embedding theorem ($H^1 \subset H$), yields

$$u_m \to u \text{ in } L^2([\tau, T); H) \text{ strongly}.$$

Hence, passing to the limit as $m \to \infty$ for (52) and (53), we find that u satisfies Equation (32) and $u \in C([\tau, T); H) \cap L^2([\tau, T); H^1)$.

In what follows, the uniqueness of the solution can be proved. For this, let u and u_1 be two solutions to (32) and $w(t) = u - u_1$. Then, replacing u by u_1 and subtracting (32), then taking an inner product with w, we get

$$\langle {}_\tau D_t^\alpha w, w\rangle + \nu\langle \nabla w, \nabla w\rangle + \langle B(u,u), w\rangle - \langle B(u_1,u_1), w\rangle = 0. \tag{56}$$

With the help of (8), (9), and Proposition 1, it holds that

$$\frac{1}{2}{}_\tau D_t^\alpha \|w\|^2 + \nu\|w\|_{H^1}^2 + b(w,u,w) = 0. \tag{57}$$

It follows by applying Young's inequality (23) and inequality (11) that

$$\begin{aligned} {}_\tau D_t^\alpha \|w\|^2 + 2\nu\|w\|_{H^1}^2 &\le 2|b(w,u,w)| \\ &\le 2C_1\|w\|\|w\|_{H^1}\|u\|_{H^1} \\ &\le 2\nu\|w\|_{H^1}^2 + \frac{C_1}{8\nu}\|w\|^2\|u\|_{H^1}^2, \end{aligned} \tag{58}$$

where Young's parameter $\eta = 2\nu$ is taken. Thus, it is easy to see that

$${}_\tau D_t^\alpha \|w(t)\|^2 \le C_5\|w(t)\|^2\|u(t)\|_{H^1}^2, \tag{59}$$

where $C_5 = \frac{C_1}{8\nu} > 0$ is a constant. Utilizing the method of (27) in Lemma 2, one has

$$\|w(t)\|^2 \le \|w(\tau)\|^2 e^{-C_5 {}_\tau I_t^\alpha \|u(t)\|_{H^1}^2}, \tag{60}$$

which, with $w(\tau) = 0$, implies $\|w(t)\| = 0$ for any $t \ge \tau$. Therefore, the uniqueness of the solution of (32) follows from (60). □

3.2. Dynamical Processes and Attractors

Compared with integer order differential systems, fractional differential systems have more uncertain long-time behavior due to the fact that the α-order fractional semigroup introduced in [34] does not preserve the properties of classical semigroup theory. In order to determine the dynamic characteristics of a fractional N–S system, we apply the shift semigroups theory, which is used to investigate the attractors of nonautonomous systems in [35] and nonlinear evolution equations in [36]. For this, we now introduce the shift semigroup of fractional order systems. Consider the following fractional evolution equation

$${}_\tau D_t^\alpha u = F(t,u), \quad u \in H, \quad t \in \mathbb{R}_\tau, \tag{61}$$

where it is assumed that a unique solution $u(t) = u(t;t_0,u_0)$ with initial value $u(t_0;t_0,u_0) = u_0$ at time t_0 exists for all $u_0 \in H$ and $t, t_0 \in \mathbb{R}_\tau$. It is easy to see that the counterpart of the semigroup property is shown by $u(t+s;t_0,u_0) = u(t+s;s,u(s;t_0,u_0))$. Let $u_\tau(t) := u(\tau+t)$, then it is easy to find $u_\tau(t)$ satisfies fractional differential Equation (61). Setting $u(t;u_0,F)$ for the solution of (61) with initial value u_0 at $t_0 = 0$, let $\mathcal{F}$ be a set of functions $f : \mathbb{R}^+ \to H$ such that $f_\tau := f(\tau + \cdot) \in \mathcal{F}$ for all $\tau \in \mathbb{R}^+$, and consider the group of shift operators $\theta_\tau : \mathcal{F} \to \mathcal{F}$ by $\theta_\tau f := f_\tau$ for each $\tau \in \mathbb{R}^+$. Finally, let $Y = H \times \mathcal{F}$ and for each $t \ge \tau$ define $U_t : Y \mapsto Y$ by $U_t(u_0,F) := (u(t;u_0,F), \theta_t F)$. Then, the family of mappings $U_t, t \in \mathbb{R}_\tau$ is a continuous-time semigroup on the state space Y. The asymptotic behavior of this semigroups is outlined in the following, which can be interpreted into the fractional N–S system. For this, it is easy to observe that the first component of the semigroup identity $U_{t+s}(u_0,F) = U_t \circ U_s(u_0,F)$ can be represented by

$$u(t+s;u_0,F) = u(t;u(s;u_0,F),\theta_s F). \tag{62}$$

Definition 3. *A compact subset B of H is called an absorbing set for a semigroup $U_t, t \in \mathbb{R}_\tau$ on H if, for every bounded subset D of H, there exists a $t_D \in \mathbb{R}_\tau$ such that $U_t(D) \subseteq B$ for all $t \geq t_D$ in $\mathbb{R}_\tau$.*

Let u be the solution of (32) corresponding to initial data u_τ. If external forces f_0, f_1 belong to $L^\infty_\alpha(\mathbb{R}_\tau; H)$, a dynamical process $\{U_t; t \geq \tau,\ \tau \in \mathbb{R}^+\}$ on H by the representation $u(t) = U_t(u_\tau, f^\varepsilon)$ is generated under the conditions (20) and (21). Inequality (33) reveals that the process $\{U_t\}$ has a uniform attractive set

$$\mathcal{B}_\varepsilon = \{u \in H \mid \|u\| \leq \mathcal{M}_\varepsilon\}$$

with respect to τ, which is bounded in H for any fixed $\varepsilon \in [0,1]$. Let $\mathcal{B} \subset H$ be a bounded set of initial data, therefore, there exists a time $\hat{T}$ depending on $\mathcal{B}$, ε such that $U_t(\mathcal{B}) \subseteq \mathcal{B}_\uparrow$, $\forall \ll \in \mathbb{R}^+$, for any $t \geq \tau + \hat{T}$. From inequality (35), we obtain that

$$\hat{\mathcal{B}}^\varepsilon = \bigcup_{\tau \in \mathbb{R}^+} U_t(\mathcal{B}_\varepsilon)$$

is a uniformly attractive set. Since the embedding $H^1 \to H$ is compact and $\hat{\mathcal{B}}^\varepsilon$ is a bounded set on H^1, it is easy to see that $\hat{\mathcal{B}}^\varepsilon$ is compact in H. Since the process $\{U_t\}$ is uniformly compact in H, the uniform global attractor is provided by

$$\mathcal{A}^\varepsilon := \bigcap_{\tau \geq 0} [\overline{\bigcup_{t \geq \tau} U_t(\hat{\mathcal{B}})}^H],$$

where $\hat{\mathcal{B}}$ is an arbitrary uniformly attractive set driven by the process $\{U_t\}$. For similar discussions on the uniform global attractor, please refer to the literature [9,37]. If $\hat{\mathcal{B}} = \mathcal{B}_\varepsilon$, it can be verified that

$$\|\mathcal{A}^\varepsilon\| \leq \mathcal{M}_\varepsilon, \ \ \forall \varepsilon \in [0,1], \tag{63}$$

where $\mathcal{M}_\varepsilon$ is the constant in (33). The hull of ϕ in $L^\infty_\alpha(\mathbb{R}^+; H)$ can be defined by

$$\mathcal{F}(\phi) := \overline{\{\phi(t+\tau) | \tau \in \mathbb{R}^+\}}^{L^\infty_\alpha(\mathbb{R}^+;H)}.$$

Then, it is easy to get $\|\hat{\phi}\|_\alpha \leq \|\phi\|_\alpha$ for every $\hat{\phi} \in \mathcal{F}(\phi)$. Note $f_0, f_1 \in L^\infty_\alpha(\mathbb{R}^+; H)$, then the external force $f^\varepsilon \in L^\infty_\alpha(\mathbb{R}^+; H)$. Furthermore, if $\hat{f}^\varepsilon \in \mathcal{F}(f^\varepsilon)$ for any $\varepsilon > 0$, it follows that

$$\hat{f}^\varepsilon(t) = \hat{f}_0(t) + \varepsilon^{-\rho} \hat{f}_1(\frac{t}{\varepsilon}),$$

for some $\hat{f}_0 \in \mathcal{F}(f_0)$ and $\hat{f}_1 \in \mathcal{F}(f_1)$. Therefore, to show the structure of the global attractor set $\mathcal{A}^\varepsilon$, the family of equations

$$D^\alpha_t \hat{u}(t) + \nu A \hat{u}(t) + B(\hat{u}(t), \hat{u}(t)) = \hat{f}^\varepsilon(t), \quad \hat{f}^\varepsilon(t) \in \mathcal{F}(f^\varepsilon(t)) \tag{64}$$

is considered. A process $\{U_t(u_\tau, \hat{f}^\varepsilon)\}$ on H generated by Equation (64) for every external force $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon)$ has similar properties as $\{U_t\}$ matching Equation (32) with an external force, which yields that the map $(u_\tau, \hat{f}^\varepsilon) \to U_t(u_\tau, \hat{f}^\varepsilon)$ is $(H \times \mathcal{H}(f^\varepsilon), H)$-continuous.

Remark 2. *Since attractors are completely determined by the limit points of the system, they are strictly invariant in the sense of identity, which is different from absorption sets or attraction sets. The attractors based on shift semigroups include many examples, such as equilibrium points, limit cycles, and geometrically more complex singular attractors.*

4. Uniform Boundedness and Convergence of Attractors

4.1. Stokes Evolution Equation with Oscillating External Force

The objective of this section is to show the uniform boundedness of the attractor of Equation (32). For this, we present a property of the fractional Stokes evolution equation with the initial time $\tau \in [0, +\infty)$ and external force g in this subsection. Consider the following fractional evolution equation.

Proposition 2. *Let $g \in L^{\infty}_{\alpha}([\tau, T); H)$, then the solution v of the problem*

$$ {}_{\tau}D^{\alpha}_{t} v(t) + \nu A v(t) = g(t/\varepsilon), \quad v|_{t=\tau} = 0, \tag{65}$$

with $\varepsilon \in (0, 1]$, satisfies

$$ \|v(t)\|^2 + \frac{\nu}{\Gamma(\alpha)} \int_{\tau}^{t} (t-s)^{\alpha-1} \|v(s)\|^2_{H^1} ds \le \frac{\varepsilon^{\alpha}\Gamma(1-\alpha)}{\lambda\nu} \|g\|^2_{\alpha}, \quad \forall\, t \ge \tau, \tag{66}$$

with the constant λ provided by (7).

Proof. Since the proof process is similar to that of Lemma 3, we only show the differences. In a same fashion as (38), we have

$$ {}_{\tau}D^{\alpha}_{t} \|v(t)\|^2 + \nu \|v(t)\|^2_{H^1} \le (\lambda\nu)^{-1} \|g(t/\varepsilon)\|^2. \tag{67}$$

By performing the fractional integration of (67), we have

$$ \|v(t)\|^2 + \frac{\nu}{\Gamma(\alpha)} \int_{\tau}^{t} (t-s)^{\alpha-1} \|v(s)\|^2_{H^1} ds \le \frac{1}{\lambda\nu\Gamma(\alpha)} \int_{\tau}^{t} (t-s)^{\alpha-1} \|g(s/\varepsilon)\|^2 ds, \ \forall t \ge \tau. \tag{68}$$

Due to

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_{\tau}^{t} (t-s)^{\alpha-1} \|g(\frac{s}{\varepsilon})\|^2 ds \\ &\overset{s=\varepsilon\mu}{=} \frac{1}{\Gamma(\alpha)} \varepsilon \int_{\frac{\tau}{\varepsilon}}^{\frac{t}{\varepsilon}} (t-\varepsilon\mu)^{\alpha-1} \|g(\mu)\|^2 d\mu \\ &= \frac{\varepsilon^{\alpha}}{\Gamma(\alpha)} \int_{\frac{\tau}{\varepsilon}}^{\frac{t}{\varepsilon}} (\frac{t}{\varepsilon} - \mu)^{\alpha-1} \|g(\mu)\|^2 d\mu \\ &\le \varepsilon^{\alpha}\Gamma(1-\alpha) \|g\|^2_{\alpha}, \end{aligned} \tag{69}$$

our desired estimate (66) follows easily, which completes the proof. □

4.2. Uniform Boundedness of Attractors

According to the well-posedness of the system, an a priori estimation of a solution in Lemma 3 represents the uniform boundedness of attractors of Equation (32). In order to estimate the range of attractors accurately, this section adopts another idea to establish the uniform boundedness of attractors according to Lemma 2 and Proposition 2.

The main results of this subsection are summarized below.

Theorem 2. *Let $f_0, f_1 \in L^{\infty}_{\alpha}(R_{\tau}; H)$, then the attractor set $\mathcal{A}^{\varepsilon}$ of Equation (32) is uniformly bounded in H for any $\varepsilon \in [0, 1]$, such that $\sup_{\varepsilon \in [0,1]} \|\mathcal{A}^{\varepsilon}\| < \infty$.*

Proof. Suppose that $u(t) = U_t(u_{\tau}, f^{\varepsilon})$ is the solution of Equation (32) with initial value u_{τ}. For $\varepsilon > 0$, first consider the problem

$$ {}_{\tau}D^{\alpha}_{t} v(t) + \nu A v(t) = \varepsilon^{-\rho} f_1(\frac{t}{\varepsilon}), \quad v|_{t=\tau} = 0. \tag{70}$$

From Proposition 2, we have

$$\|v(t)\|^2 + \frac{\nu}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|v(s)\|_{H^1}^2 ds \le \frac{\Gamma(1-\alpha)\varepsilon^{\alpha-2\rho}\|f_1\|_\alpha^2}{\lambda\nu}, \quad \forall t \ge \tau. \tag{71}$$

Set $w(t) = u - v$, then it satisfies the following equation

$${}_\tau D_t^\alpha w + \nu A w + B(w+v, w+v) = f_0, \quad w|_{t=\tau} = u_\tau. \tag{72}$$

Taking the inner product with w on both sides of Equation (72), and from (8), one can show

$$\langle {}_\tau D_t^\alpha w, w\rangle + \nu\|w\|_{H^1}^2 + b(w+v, w+v, w) = \langle f_0, w\rangle, \tag{73}$$

where

$$\begin{aligned} &b(w+v, w+v, w) \\ &= b(w, w+v, w) + b(v, w+v, w) \\ &= b(w,w,w) + b(w,v,w) + b(v,w,w) + b(v,v,w). \end{aligned} \tag{74}$$

It is easy to verify $b(w+v, w, w) = 0$ by Equation (9). Thus, one can find from Equation (74) that

$$b(w+v, w+v, w) = b(w,v,w) + b(v,v,w). \tag{75}$$

By taking equalities (11), (12) and Young's inequality ($p, q = 2, \eta = \frac{\nu}{4}$) into account, we can deduce

$$|b(w,v,w)| \le C_1\|w\|\|w\|_{H^1}\|v\|_{H^1} \le \frac{\nu}{8}\|w\|_{H^1}^2 + \frac{2{C_1}^2}{\nu}\|w\|^2\|v\|_{H^1}^2, \tag{76}$$

and

$$|b(v,v,w)| \le C_1\|v\|\|v\|_{H^1}\|w\|_{H^1} \le \frac{\nu}{8}\|w\|_{H^1}^2 + \frac{2{C_1}^2}{\nu}\|v\|^2\|v\|_{H^1}^2. \tag{77}$$

Consequently, substituting (76) and (77) into (75) reveals

$$|b(w+v, w+v, w)| \le \frac{\nu}{4}\|w\|_{H^1}^2 + \frac{2{C_1}^2}{\nu}\|w\|^2\|v\|_{H^1}^2 + \frac{2{C_1}^2}{\nu}\|v\|^2\|v\|_{H^1}^2. \tag{78}$$

Moreover, it can be derived from the Cauchy–Schwartz inequality and Young's inequality (23) that

$$\langle f_0, w\rangle \le \frac{\lambda\nu}{4}\|w\|^2 + \frac{1}{\lambda\nu}\|f_0\|^2. \tag{79}$$

Collecting (78) and (79) with Proposition 1, it follows from (73) that

$$\begin{aligned} \frac{1}{2}{}_\tau D_t^\alpha\|w\|^2 + \frac{3\nu}{4}\|w\|_{H^1}^2 &\le \langle f_0, w\rangle + |b(w+v, w+v, w)| \\ &\le \frac{\lambda\nu}{4}\|w\|^2 + \frac{1}{\lambda\nu}\|f_0\|^2 + \frac{2C_1^2}{\nu}\|w\|^2\|v\|_{H^1}^2 + \frac{2C_1^2}{\nu}\|v\|^2\|v\|_{H^1}^2. \end{aligned}$$

Owing to (21) and (71), we have

$$\|v\|^2 \le \frac{M_1\Gamma(1-\alpha)}{\lambda\nu}. \tag{80}$$

Thus, based on (80) and Poincaré' inequality (7) ($\|w\|_{H^1}^2 \ge \lambda\|w\|^2$), the above estimate becomes

$${}_\tau D_t^\alpha\|w\|^2 + \lambda\nu\|w\|^2 \le \frac{4C_1^2}{\nu}\|w\|^2\|v\|_{H^1}^2 + \frac{4C_1^2 M_1\Gamma(1-\alpha)}{\lambda\nu^2}\|v\|_{H^1}^2 + \frac{2}{\lambda\nu}\|f_0\|^2,$$

which gives

$$ {}_\tau D_t^\alpha \|w\|^2 + (\lambda\nu - \frac{4C_1^2}{\nu}\|v\|_{H^1}^2)\|w\|^2 \le \frac{4C_1^2 M_1 \Gamma(1-\alpha)}{\nu^2}\|v\|_{H^1}^2 + \frac{2}{\lambda\nu}\|f_0\|^2. $$

Set $\bar{\psi}_1(t) = \lambda\nu - \frac{4C_1^2}{\nu}\|v(t)\|_{H^1}^2$, $\bar{\psi}_2^2(t) = \frac{4C_1^2 M_1\Gamma(1-\alpha)}{\lambda\nu^2}\|v(t)\|_{H^1}^2 + \frac{2}{\lambda\nu}\|f_0(t)\|^2$. For $t \ge \tau$, it is easy to calculate from (71) that

$$ \begin{aligned} {}_\tau I_t^\alpha \bar{\psi}_1(t) &= \frac{\lambda\nu}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1} ds - \frac{4C_1^2}{\nu\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\|v(s)\|_{H^1}^2 ds \\ &\ge \frac{\lambda\nu}{\alpha\Gamma(\alpha)}(t-\tau)^\alpha - \frac{M_1\Gamma(1-\alpha)\varepsilon^{\alpha-2\rho}}{\lambda\nu^2} \\ &\ge \beta(t-\tau)^\alpha - r_2, \end{aligned} \tag{81} $$

with $\beta = \frac{\lambda\nu}{\alpha\Gamma(\alpha)}$, $r_2 = \frac{4C_1^2\Gamma(1-\alpha)M_1}{\lambda\nu^3}$. Similar to (31), it then follows from (81) that

$$ \begin{aligned} \|w(t)\|^2 &\le \|w(\tau)\|e^{-\beta(t-\tau)^\alpha + r_2} + \frac{e^{r_2}}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\bar{\psi}_2^2(s) ds \\ &\le \frac{e^{r_2}}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}[\frac{4C_1^2 M_1\Gamma(1-\alpha)}{\lambda\nu^2}\|v(t)\|_{H^1}^2 + \frac{2}{\lambda\nu}\|f_0(t)\|^2] ds \\ &\quad + \|w(\tau)\|e^{-\beta(t-\tau)^\alpha + r_2}, \end{aligned} \tag{82} $$

which, together with (18) and (71), leads to

$$ \|w(t)\|^2 \le \mathcal{M}_1 + \mathcal{M}_2 + \|w(\tau)\|e^{-\beta(t-\tau)^\alpha + r_2}, \ t \ge \tau, \tag{83} $$

where $\mathcal{M}_1 = \frac{4e^{r_2}C_1^2 M_1^2\Gamma^2(1-\alpha)}{\lambda^2\nu^4}$, $\mathcal{M}_2 = \frac{2e^{r_2}M_0\Gamma(1-\alpha)}{\lambda\nu}$ with the constants β, r_2 given by (81). Since $u(t) = w(t) + v(t)$, then using (71) gives

$$ \|u(t)\|^2 \le \|w(t)\|^2 + \|v(t)\|^2 \le \mathcal{M}_3, \ \ t \ge \tau $$

where $\mathcal{M}_3 = \mathcal{M}_1 + \mathcal{M}_2 + \|w(\tau)\|e^{-\beta(t-\tau)^\alpha + r_2} + \frac{M_1\Gamma(1-\alpha)}{\lambda\nu}$. Therefore, let $\widehat{\mathcal{M}} = \min\{\mathcal{M}_1 + \mathcal{M}_2 + \frac{M_1\Gamma(1-\alpha)}{\nu}, \mathcal{M}_\varepsilon\}$, for any $\varepsilon \in [0,1]$, the process $\{U_t\}$ has an attractor set

$$ \mathcal{B}_* = \{u \in H \mid \|u\|^2 \le \widehat{\mathcal{M}}\}. \tag{84} $$

which yields the attractor set $\mathcal{A}^\varepsilon \subset B_*$, which completes the proof of Theorem 2. □

4.3. Convergence of the Attractors

Let u^ε and u^0 be two solutions of Equation (32) with the same initial data corresponding to the cases of parameter $\varepsilon > 0$ and $\varepsilon = 0$, respectively. In addition, let $u^\varepsilon(t) := S_{f^\varepsilon}(t,\tau)u_\tau$ with $u_\tau \in \mathcal{B}_*$, where $\mathcal{B}_*$ is defined in (84). Since $u_\tau \in B_*$ and $\varepsilon = 0$, inequality (34) leads to

$$ \|u^0(t)\|^2 + \frac{\nu}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\|u^0(s)\|_{H^1}^2 ds \le \frac{\Gamma(1-\alpha)M_0}{\lambda\nu}, \tag{85} $$

where M_0 is a constant defined by (20). The aim of this subsection is to establish the convergence of the attractors. For this, we first investigate the estimation of the deviation between u^ε and u^0.

Lemma 4. *For any $\varepsilon \in (0,1]$ and all initial data $u_\tau \in B_*$, the deviation $w(t) = u^\varepsilon(t) - u^0(t)$ with $w(\tau) = 0$ satisfies*

$$ \|w(t)\| \le D\varepsilon^{\frac{\alpha}{2}-\rho}, \quad t \ge \tau, \tag{86} $$

for some positive constant D independent of ε.

Proof. Bring $u^\varepsilon(t)$ and $u^0(t)$ into Equation (32), respectively, and then take the difference to obtain

$$ {}_\tau D_t^\alpha w + \nu A w + B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0) = \varepsilon^{-\rho} f_1(t/\varepsilon), \quad w|_{t=\tau} = 0. $$

Let $q(t) := w(t) - v(t)$ for any $t \geq \tau$ where v is the solution of (70), it is easy to verify that

$$ {}_\tau D_t^\alpha q + \nu A q + B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0) = 0, \quad q|_{t=\tau} = 0. \tag{87} $$

Taking the inner product of Equation (87) with q, according to Proposition 1, one obtains

$$ \frac{1}{2} {}_\tau D_t^\alpha \|q\|^2 + \nu \|q\|_{H^1}^2 \leq \langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), q \rangle. \tag{88} $$

Recalling $w = u^\varepsilon - u^0$, it is easy to see $u^\varepsilon = q + u^0 - v$. It holds that

$$ \begin{aligned} & B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0) \\ & = B(u^0 + q + v, u^0 + q + v) - B(u^0, u^0) \\ & = B(u^0, q + v) + B(q + v, u^0) + B(q + v, q + v). \end{aligned} \tag{89} $$

Therefore, we get from (8) and (9) that

$$ \begin{aligned} & |\langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), q \rangle| \\ & = |\langle B(u^0, q + v) + B(q + v, u^0) + B(q + v, q + v), q \rangle| \\ & = |b(u^0, q + v, q) + b(q + v, u^0, q) + b(q + v, q + v, q)| \\ & = |b(u^0, v, q) + b(q, u^0, q) + b(v, u^0, q) + b(q, v, q) + b(v, v, q)|. \end{aligned} \tag{90} $$

By utilizing (11) and Young's inequality $(p, q = 2, \eta = \frac{\nu}{2})$, one can get

$$ |b(q, u^0, q)| \leq \frac{\nu}{4} \|q\|_{H^1}^2 + \frac{C_1^2}{\nu} \|q\|^2 \|u^0\|_{H^1}^2, \tag{91} $$

$$ |b(q, v, q)| \leq \frac{\nu}{4} \|q\|_{H^1}^2 + \frac{C_1^2}{\nu} \|q\|^2 \|v\|_{H^1}^2, \tag{92} $$

$$ |b(v, v, q)| \leq \frac{\nu}{4} \|q\|_{H^1}^2 + \frac{C_1^2}{\nu} \|v\|^2 \|v\|_{H^1}^2, \tag{93} $$

Similarly, by the meaning of (12), one gives

$$ |b(u^0, v, q)| + |b(v, u^0, q)| \leq \frac{\nu}{4} \|q\|_{H^1}^2 + \frac{2C_1^2}{\nu} \|u^0\| \|u^0\|_{H^1} \|v\| \|v\|_{H^1}, \tag{94} $$

In view of the estimates (91)–(94), it follows from Equation (90) that

$$ \begin{aligned} & |\langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), q \rangle| \\ & \leq \nu \|q\|_{H^1}^2 + \frac{C_1^2}{\nu} \|q\|^2 (\|u^0\|_{H^1}^2 + \|v\|_{H^1}^2) + \frac{C_1^2}{\nu} \|v\|^2 \|v\|_{H^1}^2 + \frac{2C_1^2}{\nu} \|u^0\| \|u^0\|_{H^1} \|v\| \|v\|_{H^1} \\ & \leq \nu \|q\|_{H^1}^2 + \mathcal{F}_1 \|q\|^2 + \mathcal{F}_2, \end{aligned} \tag{95} $$

with

$$ \begin{aligned} \mathcal{F}_1(t) &= \frac{C_1^2}{\nu} (\|u^0(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2), \\ \mathcal{F}_2(t) &= \frac{C_1^2}{\nu} \|v(t)\|^2 \|v(t)\|_{H^1}^2 + \frac{2C_1^2}{\nu} \|u^0(t)\| \|u^0(t)\|_{H^1} \|v(t)\| \|v(t)\|_{H^1}. \end{aligned} $$

Since $v(t)$ satisfies (71) and $u^0(t)$ satisfies (85), we substitute (95) into (88), to obtain $\frac{1}{2}{}_\tau D_t^\alpha \|q(t)\|^2 \leq \mathcal{F}_1(t)\|q(t)\|^2 + \mathcal{F}_2(t)$. Invoking $\|q(\tau)\| = 0$ and similar to (27) in Lemma 2, one has

$$\|q(t)\|^2 \leq \|q(\tau)\| e^{2{}_\tau I_t^\alpha \mathcal{F}_1(t)} + e^{2{}_\tau I_t^\alpha \mathcal{F}_1(t)} \frac{2}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \mathcal{F}_2(s) ds. \tag{96}$$

In view of the estimates (71) and (85), we can find

$$\begin{aligned} {}_\tau I_t^\alpha \mathcal{F}_1(t) &= \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} [\frac{C_1^2}{\nu} \|u^0(s)\|_{H^1}^2 + \|v(s)\|_{H^1}^2] ds \\ &= \frac{C_1^2}{\nu\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|u^0(s)\|_{H^1}^2 ds + \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|v(s)\|_{H^1}^2 ds \\ &\leq \mathcal{K}. \end{aligned} \tag{97}$$

where $\mathcal{K} = \frac{C_1^2 \Gamma(1-\alpha) M_0}{\nu^3} + \frac{M_1 \Gamma(1-\alpha)}{\lambda \nu^2}$. In addition, we also get

$$\begin{aligned} {}_\tau I_t^\alpha \mathcal{F}_2(t) &= \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} [\frac{C_1^2}{\nu} \|v(s)\|^2 \|v(s)\|_{H^1}^2 \\ &\quad + \frac{2C_1^2}{\nu} \|u^0(s)\| \|u^0(s)\|_{H^1} \|v(s)\| \|v(s)\|_{H^1}] ds \\ &\leq \frac{2C_1^2}{\nu\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|u^0(s)\| \|u^0(s)\|_{H^1} \|v(s)\| \|v(s)\|_{H^1} ds \\ &\quad + \frac{C_1^2}{\nu\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|v(s)\|^2 \|v(s)\|_{H^1}^2 ds \\ &\leq \frac{2C_1^2 \Gamma(1-\alpha) M_0{}^{\frac{1}{2}} M_1{}^{\frac{1}{2}} \varepsilon^{\frac{\alpha}{2}-\rho}}{\lambda \nu^2 \Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|u^0(s)\|_{H^1} \|v(s)\|_{H^1} ds \\ &\quad + \frac{C_1^2 M_1 \Gamma(1-\alpha) \varepsilon^{\alpha-2\rho}}{\lambda \nu^2} \int_\tau^t (t-s)^{\alpha-1} \|v(s)\|_{H^1}^2. \end{aligned} \tag{98}$$

Combining this with (71), (85) and the Cauchy–Schwartz inequality (98) yield

$$\begin{aligned} {}_\tau I_t^\alpha \mathcal{F}_2(t) &\leq [\frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|u^0(s)\|_{H^1}^2 ds]^{\frac{1}{2}} [\frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|v(s)\|_{H^1}^2 ds]^{\frac{1}{2}} \\ &\quad \cdot \frac{2C_1^2 \Gamma(1-\alpha) M_0{}^{\frac{1}{2}} M_1{}^{\frac{1}{2}} \varepsilon^{\frac{\alpha}{2}-\rho}}{\lambda \nu^2} + \frac{C_1^2 M_1^2 \Gamma^2(1-\alpha) \varepsilon^{2(\alpha-2\rho)}}{\lambda^2 \nu^4} \\ &\leq \frac{2C_1^2 \Gamma^2(1-\alpha) M_0 M_1 \varepsilon^{\alpha-2\rho}}{\lambda^2 \nu^4} + \frac{C_1^2 \Gamma^2(1-\alpha) M_1^2 \varepsilon^{2(\alpha-2\rho)}}{\lambda^2 \nu^4} \\ &\leq \mathcal{L} \varepsilon^{\alpha-2\rho}, \end{aligned} \tag{99}$$

where $\mathcal{L} = (2M_0 M_1 + +M_1^2) \frac{C_1^2 \Gamma^2(1-\alpha)}{\lambda^2 \nu^4}$. Invoking (97) and (99) into (96) with $q(\tau) = 0$, the results show that

$$\|q(t)\|^2 \leq D_1^2 \varepsilon^{\alpha-2\rho}, \tag{100}$$

where $D_1^2 = 2\mathcal{L} e^{2\mathcal{K}}$, $\mathcal{L}$, and $\mathcal{K}$ are constants given in (97) and (99). Hence, our result (86) can be concluded by using $w(t) = q(t) + v(t)$ and a priori estimate (71), which ends the proof. □

To get the convergence of the uniform global attractors, it suffices to show that a generalized form of Lemma 4 is needed, which can be applied to the whole family of equations (64) under the external forces $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon)$. To this end, for $\varepsilon \in [0, 1]$, set $\hat{u}^\varepsilon(t) = U_t(\hat{u}_\tau, \hat{f}^\varepsilon)$ satisfying Equation (64) with external force $\hat{f}^\varepsilon = \hat{f}_0 + \varepsilon^{-\rho} \hat{f}_1(t/\varepsilon) \in \mathcal{H}(f^\varepsilon)$ and $\hat{u}_\tau \in \mathcal{B}_*$. For $\varepsilon > 0$, let

$$\hat{w}(t) = \hat{u}^\varepsilon(t) - \hat{u}^0(t).$$

Similar to the proof of Lemma 4, we can derive the following result.

Lemma 5. *For any $\varepsilon \in (0,1]$, let $\hat{w}(t) = \hat{u}^\varepsilon(t) - \hat{u}^0(t)$ with $\hat{w}(\tau) = 0$. Then, it holds that*

$$\|\hat{w}(t)\| \le D\varepsilon^{\frac{\alpha}{2}-\rho}, \ \forall t \ge \tau, \tag{101}$$

where D is the same as in Lemma 4.

Now we state our main results of this section.

Theorem 3. *Let $f_0, f_1 \in L^\infty_\alpha(R_\tau; H)$, then the uniform global attractor set $\mathcal{A}^\varepsilon \to \mathcal{A}^0$ as $\varepsilon \to 0^+$ in the following sense*

$$\lim_{\varepsilon\to 0^+} \{\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0)\} = 0. \tag{102}$$

Proof. Let $L > 0$ and $u^\varepsilon \in \mathcal{A}^\varepsilon$ for $\varepsilon > 0$, then there exists a bounded trajectory $\hat{u}^\varepsilon(t)$ of Equation (64) with $\hat{u}^\varepsilon(2L) = u^\varepsilon$ for any $L \ge \tau$. For every $L \ge \tau$, we know $\hat{u}^\varepsilon(L) \in \mathcal{A}^\varepsilon \subset B_*$ and $u^\varepsilon = U_{2L}(\hat{u}^\varepsilon(L), f^\varepsilon)$. In light of Lemma 5, using $\tau = L$ and $t = 2L$, one presents

$$\|u^\varepsilon - U_{2L}(\hat{u}^\varepsilon(L), \hat{f}_0)\| \le D\varepsilon^{\frac{\alpha}{2}-\rho},$$

which means that

$$\mathrm{dist}_H(u^\varepsilon, U_{2L}(\hat{u}^\varepsilon(L), \hat{f}_0)) \le D\varepsilon^{\frac{\alpha}{2}-\rho}. \tag{103}$$

Since the set $\mathcal{A}^0$ attracts $U_t(B_*, \hat{f}_0)$ uniformly with $\hat{f}_0 \in \mathcal{H}(f^0)$, for any $\sigma > 0$, there exists a constant $T = T(\sigma) \ge L$ depending on σ such that

$$\mathrm{dist}_H(U_{T+L}(\hat{u}^\varepsilon(L), \hat{f}_0), \mathcal{A}^0) \le \sigma. \tag{104}$$

Letting $T = L$, from (103) and (104) it is easy to check that

$$\mathrm{dist}_H(u^\varepsilon, \mathcal{A}^0) \le D\varepsilon^{\frac{\alpha}{2}-\rho} + \sigma.$$

According to the fact that $u^\varepsilon \in \mathcal{A}^\varepsilon$ and $\sigma > 0$ are arbitrary, the conclusion (102) follows easily as $\varepsilon \to 0^+$. □

Remark 3. *By taking the proof process of Theorem 3 into account, for any $\varepsilon \in (0,1]$ it is easy to see the estimate*

$$\|u^\varepsilon - U_t(\hat{u}^\varepsilon(L), \hat{f}_0)\| \le D\varepsilon^{\frac{\alpha}{2}-\rho}. \tag{105}$$

For $t = T$, the inequality (105) together with (104) reveals that

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \le D\varepsilon^{\frac{\alpha}{2}-\rho} + \sigma, \ \forall \sigma > 0,$$

which, with the arbitrariness of σ, implies the Hölder continuity property of $\mathcal{A}^\varepsilon$ at $\varepsilon = 0$, that is,

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \le D\varepsilon^{\frac{\alpha}{2}-\rho}.$$

5. Conclusions

In this paper, we investigated the averaging process of 2D time-fractional N–S equations with a singularly external force. By using a new fractional inequality (Lemma 2), the uniform boundedness of the global attractors for this fractional Navier–Stokes equations was demonstrated. Then, we further developed the convergence of global attractors as the parameter approached zero and guaranteed the explicit convergence rate of global attractors. The research on the averaging process of space-fractional N–S equations with a singularly external force is still an open problem which will be the focus of our future work.

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