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Article

Abundant Exact Travelling Wave Solutions for a Fractional Massive Thirring Model Using Extended Jacobi Elliptic Function Method

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Abstract: The fractional massive Thirring model is a coupled system of nonlinear PDEs emerging in the study of the complex ultrashort pulse propagation analysis of nonlinear wave functions. This article considers the NFMT model in terms of a modified Riemann–Liouville fractional derivative. The novel travelling wave solutions of the considered model are investigated by employing an effective analytic approach based on a complex fractional transformation and Jacobi elliptic functions. The extended Jacobi elliptic function method is a systematic tool for restoring many of the well-known results of complex fractional systems by identifying suitable options for arbitrary elliptic functions. To understand the physical characteristics of NFMT, the 3D graphical representations of the obtained propagation wave solutions for some free physical parameters are randomly drawn for a different order of the fractional derivatives. The results indicate that the proposed method is reliable, simple, and powerful enough to handle more complicated nonlinear fractional partial differential equations in quantum mechanics.

Keywords: fractional massive Thirring model; Jacobi expansion method; nonlinear partial differential equation; travelling wave solution; quantum field theory

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1. Introduction

Physics can be typically classified into two branches: classical and modern physics. Modern physics can be distinguished by considering spatiotemporal requirements for joint interaction, whereas, in classical physics, we can consider time and space separately because they are independent and absolute. Furthermore, classical physics usually deals with the macroscopic scale, while modern physics deals with microscopic or sub-microscopic scales. Although classical physics has different applications in science and engineering, modern physics can be considered a revolution in applied physics, as it can elucidate many essential phenomena, such as black body radiation, photoelectric effect, Compton's effect and stability of atoms that cannot be explained from a classical physics point of view. However, modern physics focuses on quantum mechanics and the theory of relativity; quantum mechanics considers the physical quantities restricted to be discrete values, where the thinking of the probability is dominant instead of certain measurements, which is represented mathematically by the Schrödinger wave equation. The theory of relativity studies the physical quantities moving at a speed near the speed of light, the time dilation, and the dimensions contraction started to be important concepts, and Einstein's mass-energy equation makes a revolution in the science [1–4].

The contemporary revolution in theoretical and applied physics combines quantum mechanics with the theory of relativity in a multi-body system, which establishes quantum field theory. Quantum field equations represent a general form of the Schrödinger wave equation, where the wavefunction is generalized to an infinite-dimensional space of field configurations [2]. Motivated by this, in this work, we consider the massive Thirring model (MTM) as an important application of the quantum field theory, which was derived by W. Thirring in 1958 [3]. Thereafter, many theoretical and applied studies of such a complex system were conducted. For example, but not limited to, Kondo 1995 studied the bosonization and duality of the MTM with a four-fermion interaction of the current type [4], and the Thirring model was also considered in a separate work as a gauge theory [5]. In 2018, Joshi et al. introduced an integrable semi-discretization of the MTM for the first time in laboratory coordinates [6].

Nevertheless, to find out an alternative methodology for the Schrodinger equation, Dirac discovered the integral path approach, similar to Lagrangian's least-action principle technique in classical mechanics; this approach was developed by Feynman to create Feynman diagrams. Feynman diagrams were modified to Wiener's path integral, which is equivalent to the Brownian path integral in classical mechanics. Recently, the Levy flight random process has been introduced to understand difficult classical and quantum physics phenomena, where the Levy index α is introduced. Now, the consequences of the path integral for the Levy flight paths' studies are an essential issue in fractional quantum mechanics and consequently in fractional quantum field theory [7]. Examining research to obtain novel and additional exact traveling-wave solutions for fractional models is prospering. Indeed, this is not an easy task and is one of the pivotal challenging problems in mathematics and physics. Hence, resorting to sophisticated analytical and digital methods is inevitable. In this direction, many effective and accurate analytical methods for solving these equations have been considered thus far, for example, the Bäcklund transformation method, the Riccati sub-equation method, the extended tanh-function method, the G′/G-expansion method, the Kudryashov method [8–11] and so forth.

The analysis in this paper highlights the complex behavior of nonlinear wavefunction, which is notably dependent on the genetic properties and temporal memory that can be explored with great skill using fractional calculus.

In this direction, consider the following semi-discrete nonlinear massive Thirring model (MTM) that can be typically provided by a complex triple system of difference equations:

$$\begin{aligned} &4i\frac{d\chi_n}{dt}+\phi_{n+1}+\phi_n+\frac{2i}{\nu}(\psi_{n+1}-\psi_n)+\chi_n^2(\check{\psi}_n+\check{\psi}_{n+1})-\chi_n\left(|\phi_{n+1}|^2+|\phi_n|^2+|\psi_{n+1}|^2+|\psi_n|^2\right)\\ &\qquad -\frac{i\nu}{2}\chi_n^2(\check{\phi}_n+\check{\phi}_{n+1})=0,\\ &\frac{2i}{\nu}(\phi_{n+1}-\phi_n)-2\chi_n+|\chi_n|^2(\phi_{n+1}+\phi_n)=0,\\ &\psi_{n+1}+\psi_n-2\chi_n+\frac{i\nu}{2}(\psi_{n+1}-\psi_n)=0, \end{aligned} \tag{1}$$

where n denotes the discrete lattice to index iterates, ν denotes the lattice-spacing parameter of aspace discretization, and the symbol i is an imaginary unit. The complex-conjugates of ψ_n and ϕ_n are denoted respectively by $\check{\psi}_n$ and $\check{\phi}_n$. The first equation refers to the case of temporal evolution, while the last two difference equations refer to the semi-discrete massive Thirring equations constrained with the components of $\{\psi_n\}_{n\in\mathbb{Z}}$ and $\{\phi_n\}_{n\in\mathbb{Z}}$, which can be defined in terms of $\{\chi_n\}_{n\in\mathbb{Z}}$ in the temporal and spatial coordinates [6]. With the continuity of $\nu\to 0$, the slowly changing solutions between the lattice nodes can be written as:

$$\chi_n(t)=\chi(x=\nu n,t),\psi_n(t)=\psi(x=\nu n,\ t),\ \phi_n(t)=\phi(x=\nu n,\ t), \tag{2}$$

where continuous variables fulfill the following system of partial equations:

$$\begin{array}{c} 2i\frac{\partial \chi}{\partial t} + i\frac{\partial \psi}{\partial x} + \phi + \chi^2 \check{\psi} - \chi\left(|\phi|^2 + |\psi|^2\right) = 0, \\ i\frac{\partial \phi}{\partial x} - \chi + |\chi|^2 \phi = 0, \\ \psi - \chi = 0, \end{array} \tag{3}$$

which leads to an MTM system of two semi-linear equations for $(F, G) \in \mathbb{C}^2$ in terms of the variables $\psi(x,t) = \mathrm{F}(x, t-x)$ and $\phi(x,t) = G(x, t-x)$ in the normalized form:

$$\begin{array}{c} i\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\right) + G = |G|^2 F, \\ i\left(\frac{\partial G}{\partial t} - \frac{\partial G}{\partial x}\right) + F = |F|^2 G. \end{array} \tag{4}$$

This paper deals with the fractional version of such a system. Therefore, we consider the following nonlinear space–time fractional MTM system:

$$\begin{array}{c} i\left(D_t^\alpha F + \frac{\partial F}{\partial x}\right) + G = |G|^2 F, \\ i\left(D_t^\alpha G - \frac{\partial G}{\partial x}\right) + F = |F|^2 G. \end{array} \tag{5}$$

Considerable analytical and numerical investigations of the MTM have been made in the literature using various techniques. The construction of the MTM using the functional integral scheme within quantum field theory was discussed in [12]. The physical states, as well as a solution of the MTM, by means of many-body wave functions, are presented in [13]. In [14], Bethe ansatz solutions of the MTM were tested numerically by solving periodic boundary value problems. Delepine et al. [15] demonstrated that the MTM is equivalent to the quantum sine-Gordon model in quantum field theories at a finite temperature. The white noise of the oscillator MTM was examined in [16] in terms of the phase–space displays. In [17], the non-thermal phase structure of the MTM was studied using ansatz matrix-product states. Using the N-fold Darboux transform, the rogue wave solutions of the MTM equations were derived in [18]. On the other side as well, the fractional versions of the nonlinear complex MTM were numerically solved using advanced semi-analytical and approximate methods; for example, the q-HAM was applied in [19] to solve the fractional massive Thirring model in Caputo sense. In [20], the fractional residual power series method was implemented to solve a class of the fractional MTM with conformable derivatives. For more details regarding the numerical and analytical solutions of different fractional models, we refer to [21–31].

Almost all scientific problems can be solved using different fractional calculus techniques, where one or many suitable methods can be chosen for each problem; some problems that are solved using modified Riemann–Liouville fractional calculus techniques were noted as incorrect conditions [32–35]. Although these cases are not related to this work, this note must be mentioned here. These cases do not affect the Riemann–Liouville fractional calculus technique, which solves a huge number of problems successfully, as do other methods, such as the Mittag–Leffler function, the fractional Riccati method, the fractional double function method, and the fractional Y-function expansion method [36,37].

The novelty of this paper is to explore new travelling wave solutions for fractional MTM equations (5) by employing an effective analytic approach based on a complex fractional transformation and Jacobi elliptic functions. It is worth noting that the previous study of soliton for the fractional MTM equations was performed to provide an approximate solution for or study a special case of the MTM equations [4–6]. This paper introduces a general case exact solution for MTM equations for the first time; this study can be considered as a strong motivation to provide the obtained results.

The outline of this analysis has the following sections: In Section 2, some basic definitions and characteristics of the considered fractional operator are presented. In Section 3, the key idea of the proposed method is described. Then, in Section 4, we apply this method to create new sets of exact traveling wave solutions to the fractional massive Thirring model. Finally, a brief conclusion is also provided.

2. Preliminaries

Recently, many researchers have used various fractional operators to study several models associated with the functions of complex variables, and they proved that these fractional operators are more influential than the classical ones while analyzing the natural behavior of those models. Herein, we introduce the basic definition and some properties of Jumarie's modification of Riemann–Liouville derivative [38–41] that are very useful for displaying this work in a standardized way.

Definition 1. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, the modified Riemann–Liouville derivative of the order α is as follows*

$$D_t^\alpha \omega(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha-1} (\omega(\xi) - \omega(0)) d\xi, \ \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (\omega(\xi) - \omega(0)) d\xi, \ 0 < \alpha < 1, \\ \left[\omega^{(\alpha-n)}(t)\right]^{(n)}, \ n \le \alpha < n+1, \ n \ge 1. \end{cases}$$

In this work, if $\omega(t)$ has a modified Riemann–Liouville derivative of the order α, it will be defined as D_t^α-differentiable. Further, it is obvious that the operator D_t^α of Jumarie's modification satisfies the following interesting properties:

Theorem 1. *Let $\omega_1 : \mathbb{R} \to \mathbb{R}$ be D_t^α -differentiable function at a point $t > 0$ and $\omega_2 : \mathbb{R} \to \mathbb{R}$ be D_t^α -differentiable and defined in the range of ω_1. Then, we have:*

(I) *If $\omega_1(t) = t^\gamma$, then $D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}$ for $\gamma > 0$.*

(II) $D_t^\alpha(\omega_1(t)\omega_2(t)) = \omega_2(t) D_t^\alpha \omega_1(t) + \omega_1(t) D_t^\alpha \omega_2(t)$.

(III) $D_t^\alpha \omega_1(\omega_2(t)) = \frac{d}{d\omega_2} \omega_1(\omega_2(t)) D_t^\alpha \omega_2(t) = D_{\omega_2}^\alpha \omega_1(\omega_2(t)) \left(\frac{d}{dt}\omega_2(t)\right)^\alpha$.

3. The Extended Jacobi Elliptic Equation Method

This section presents the definition of Jacobi elliptic functions and reviews some important properties that we will use within the framework of this paper. Then, we introduce the algorithm of the proposed method.

3.1. The Jacobi Elliptic Functions

The Jacobi elliptic functions are the standard forms of elliptic functions. There are three double periodic functions, namely the Jacobian elliptic sine function, Jacobian elliptic cosine function, and Jacobian elliptic function of a third kind denoted by $sn(u, \delta) = sn(u)$, $cn(u, \delta) = cn(u)$ and $dn(u, \delta) = dn(u)$, respectively, where δ is the elliptic modulus. In the next segment, we provide the details of the derivation of these functions. To this end, we consider the following nonlinear partial differential equation (PDE):

$$\frac{\partial^2 \varphi}{\partial x \partial t} = \lambda \sin(\varphi). \tag{6}$$

By substituting the linear transformation $\eta = \theta(x - \mu t)$ into PDE (6), we get the following nonlinear ordinary differential equation (NODE):

$$\frac{d^2 \varphi}{d\eta^2} = \frac{-\lambda}{\theta^2 \mu} \sin(\varphi). \tag{7}$$

Then, some simplifications lead to the following equivalent NODE:

$$\left(\frac{1}{2}\frac{d\varphi}{d\eta}\right)^2 = \frac{-\lambda}{\theta^2\mu}\sin^2\frac{1}{2}(\varphi) + c. \tag{8}$$

Letting $c = 1, -\lambda/\theta^2\mu = -\delta^2$ and $\omega = \varphi/2$, then Equation (8) takes the form

$$\frac{d\omega}{d\eta} = \sqrt{1-\delta^2\sin^2\omega}, \tag{9}$$

which is equivalent to

$$\int \frac{1}{\sqrt{1-\delta^2\sin^2\omega}}d\omega = \int d\eta, \tag{10}$$

where the integral in Equation (10) is called the Legendre elliptic integral of the first kind. Now, we define

$$u = \int_0^{\varphi} \frac{1}{\sqrt{1-\delta^2\sin^2 y}}dy = \int_0^{t\ \equiv \sin\xi} \frac{1}{\sqrt{(1-x^2)(1-\delta^2x^2)}}dx. \tag{11}$$

Provided that $u = f(t)$ so that $t = f^{-1}(u) = sn(u)$, which is the Jacobi elliptic sine function. Nevertheless, the Jacobi elliptic cosine function can be defined by letting

$$u = \int_0^{\varphi} \frac{1}{\sqrt{1-\delta^2\cos^2 y}}dy = \int_0^{\sqrt{1-t^2}\ \equiv \cos y} \frac{1}{\sqrt{(1-x^2)(1-\delta^2x^2)}}dx. \tag{12}$$

Provided that $u = f\left(\sqrt{1-t^2}\right)$ so that $\sqrt{1-t^2} = f^{-1}(u) = cn(u)$. Consequently, one can write the following argument

$$t = sn(u),\ \sqrt{1-t^2} = cn(u),\ \sqrt{1-\delta^2t^2} = dn(u). \tag{13}$$

On the other side as well, Jacobi elliptic functions $sn(u),\ cn(u)$ and $dn(u)$ can be defined respectively as solutions to

$$\begin{aligned} y'' &= \left(2-\delta^2\right)y - 2y^3, \\ y'' &= -\left(1-2\delta^2\right)y - 2\delta^2y^3, \\ y'' &= -\left(1+\delta^2\right)y + 2\delta^2y^3, \end{aligned} \tag{14}$$

and possess the following properties in terms of their singular points:

$$\begin{aligned} &sc(u) = \tfrac{sn(u)}{cn(u)},\ sd(u) = \tfrac{sn(u)}{dn(u)},\ cd(u) = \tfrac{cn(u\ )}{dn(u)}, \\ &cs(u) = \tfrac{1}{sc(u)}, ds(u) = \tfrac{1}{sd(u)}, dc(u) = \tfrac{1}{cd(u)}, \\ &ns(u) = \tfrac{1}{sn(u)},\ nc(u) = \tfrac{1}{cn(u)}, nd(u) = \tfrac{1}{dn(u)}. \end{aligned} \tag{15}$$

when $\delta \to 1$, the Jacobi elliptic functions turn into hyperbolic functions as follows

$sn(u) \to \tanh u$,	$cn(u) \to \operatorname{sech} u$,	$dn(u) \to \operatorname{sech} u$,
$ns(u) \to \coth u$,	$nc(u) \to \cosh u$,	$nd(u) \to \cosh u$,
$sc(u) \to \sinh u$,	$sd(u) \to \sinh u$,	$cd(u) \to 1$,
$cs(u) \to \operatorname{csch} u$,	$ds(u) \to \operatorname{csch} u$,	$dc(u) \to 1$.

when $\delta \to 0$, they turn into trigonometric functions as follows

$sn(u) \to \sin u$,	$cn(u) \to \cos u$,	$dn(u) \to 1$,
$ns(u) \to \csc u$,	$nc(u) \to \sec u$,	$nd(u) \to 1$,
$sc(u) \to \tan u$,	$sd(u) \to \sin u$,	$cd(u) \to \cos u$,
$cs(u) \to \cot u$,	$ds(u) \to \csc u$,	$dc(u) \to \sec u$.

Furthermore, one can obtain the following identities:

$$
\begin{aligned}
&cn^2(u)+sn^2(u)=1, && dn^2(u)=1-\delta^2 sn^2(u),\\
&ns^2(u)-cs^2(u)=1, && nd^2(u)=1+\delta^2 sd^2(u),\\
&nc^2(u)-sc^2(u)=1, && cd^2(u)+\left(1-\delta^2\right)sd^2(u)=1,\\
&ns^2(u)-ds^2(u)=\delta^2, && dc^2(u)-\left(1-\delta^2\right)sc^2(u)=1,\\
&ds^2(u)-cs^2(u)=1-\delta^2, && dc^2(u)-\left(1-\delta^2\right)nc^2(u)=\delta^2,\\
&\delta^2\left(cn^2(u)-1\right)-dn^2(u)=1, && \delta^2 cd^2(u)+\left(1-\delta^2\right)nd^2(u)=1.
\end{aligned}
$$

The derivatives of the Jacobi elliptic functions are as follows

$$
\begin{aligned}
&(sn\ u)'=cn(u)dn(u), && (cn\ u)'=-sn(u)dn(u), && (dn\ u)'=-\delta^2 sn(u)cn(u),\\
&(ns\ u)'=-cs(u)ds(u), && (nc\ u)'=sc(u)dc(u), && (nd\ u)'=\delta^2 cd(u)sd(u),\\
&(sc\ u)'=nc(u)dc(u), && (sd\ u)'=nd(u)cd(u), && (cd\ u)'=\left(\delta^2-1\right)sd(u)nd(u),\\
&(cs\ u)'=-ns(u)ds(u), && (ds\ u)'=-ns(u)cs(u), && (dc\ u)'=\left(1-\delta^2\right)nc(u)sc(u).
\end{aligned}
$$

3.2. Extended Jacobi Elliptic Function Expansion Method

Herein, the algorithm of the extended Jacobi elliptic function expansion method will be illustrated to obtain the exact travelling wave solutions of NFPDEs. To this end, let us consider FPDE in the the following form

$$P\left(u, D_t^{\alpha}u, D_x^{\beta}u,\ D_y^{\gamma}u, D_t^{2\alpha}u, D_x^{2\beta}u, \ldots\right)=0,\ t\geq 0,\ 0<\alpha,\beta,\gamma<1, \tag{16}$$

where $u=u(t,x,y),\ P$ is a polynomial in u, and its partial derivatives, including fractional derivatives, $D_t^{\alpha}, D_x^{\beta}$ and D_y^{γ}, are a modified Riemann–Liouville derivative of u with respect to the independent variables $t,\ x$ and y. In the following, the main steps of the proposed algorithm are presented to find out the exact travelling wave solutions of FPDE (16):

Step 1. Use the fractional wave transformation

$$u(t,x,y)=U(\xi), \xi=\frac{x^{\beta}}{\Gamma(\beta+1)}+\frac{y^{\gamma}}{\Gamma(\gamma+1)}+\frac{vt^{\alpha}}{\Gamma(\alpha+1)}, \tag{17}$$

where v is the wave velocity that will later be determined. This permits us to reduce FPDE (16) into the following ODE of integer order in terms of ξ:

$$\widetilde{P}\left(U, dU/d\xi, d^2U/d\xi^2, d^3U/d\xi^3,\ \ldots\right)=0, \tag{18}$$

Step 2. Propose that Equation (18) has a solution in the following form

$$U(\xi)=b_0+b_1Q_{\jmath}(\xi)+b_2R_{\jmath}(\xi)+b_3S_{\jmath}(\xi)+\sum_{\hbar=2}^{L}Q_{\jmath}^{\hbar-2}(\xi)\left[p_{\hbar}Q_{\jmath}^2(\xi)+r_{\hbar}R_{\jmath}(\xi)S_{\jmath}(\xi)\right], \tag{19}$$

where $\jmath\ =\ 1,2,\ \ldots,\ 12$, in which L is a positive integer,$b_0,\ b_1,\ b_2,\ b_3$, and $p_{\hbar},\ r_{\hbar}$, $\hbar=2\ ,3,\ \ldots,\ L$ and are constants to be determined afterwards. The functions $Q_{\jmath}(\xi),\ R_{\jmath}(\xi)$ and $S_{\jmath}(\xi),\ \jmath\ =\ 1,2,\ldots,12$ can be expressed in terms of Jacobi elliptic functions (15) as follows

$$
\begin{aligned}
Q_1(\xi) &= \tfrac{1}{\rho+dn(\xi)}, R_1(\xi) = \tfrac{sn(\xi)}{\rho+dn(\xi)}, S_1(\xi) = \tfrac{cn(\xi)}{\rho+dn(\xi)},\\
Q_2(\xi) &= \tfrac{1}{\rho+sd(\xi)}, R_2(\xi) = \tfrac{cd(\xi)}{\rho+sd(\xi)}, S_2(\xi) = \tfrac{nd(\xi)}{\rho+sd(\xi)},\\
Q_3(\xi) &= \tfrac{1}{\rho+cd(\xi)}, R_3(\xi) = \tfrac{sd(\xi)}{\rho+cd(\xi)}, S_3(\xi) = \tfrac{nd(\xi)}{\rho+cd(\xi)},\\
Q_4(\xi) &= \tfrac{1}{\rho+ns(\xi)}, R_4(\xi) = \tfrac{cs(\xi)}{\rho+ns(\xi)}, S_4(\xi) = \tfrac{ds(\xi)}{\rho+ns(\xi)},\\
Q_5(\xi) &= \tfrac{1}{\rho+nd(\xi)}, R_5(\xi) = \tfrac{sd(\xi)}{\rho+nd(\xi)}, S_5(\xi) = \tfrac{cd(\xi)}{\rho+nd(\xi)},\\
Q_6(\xi) &= \tfrac{1}{\rho+sc(\xi)}, R_6(\xi) = \tfrac{nc(\xi)}{\rho+sc(\xi)}, S_6(\xi) = \tfrac{dc(\xi)}{\rho+sc(\xi)},\\
Q_7(\xi) &= \tfrac{1}{\rho+cn(\xi)}, R_7(\xi) = \tfrac{sn(\xi)}{\rho+cn(\xi)}, S_7(\xi) = \tfrac{dn(\xi)}{\rho+cn(\xi)},\\
Q_8(\xi) &= \tfrac{1}{\rho+dc(\xi)}, R_8(\xi) = \tfrac{sc(\xi)}{\rho+dc(\xi)}, S_8(\xi) = \tfrac{nc(\xi)}{\rho+dc(\xi)},\\
Q_9(\xi) &= \tfrac{1}{\rho+nc(\xi)}, R_9(\xi) = \tfrac{sc(\xi)}{\rho+nc(\xi)}, S_9(\xi) = \tfrac{dc(\xi)}{\rho+nc(\xi)},\\
Q_{10}(\xi) &= \tfrac{1}{\rho+sn(\xi)}, R_{10}(\xi) = \tfrac{cn(\xi)}{\rho+sn(\xi)}, S_{10}(\xi) = \tfrac{dn(\xi)}{\rho+sn(\xi)},\\
Q_{11}(\xi) &= \tfrac{1}{\rho+cs(\xi)}, R_{11}(\xi) = \tfrac{ds(\xi)}{\rho+cs(\xi)}, S_{11}(\xi) = \tfrac{ns(\xi)}{\rho+cs(\xi)},\\
Q_{12}(\xi) &= \tfrac{1}{\rho+ds(\xi)}, R_{12}(\xi) = \tfrac{cs(\xi)}{\rho+ds(\xi)}, S_{12}(\xi) = \tfrac{ns(\xi)}{\rho+ds(\xi)},
\end{aligned}
\tag{20}
$$

where ρ is an arbitrary constant.

Step 3. Determine the integer L in the predicted solution (19) by balancing the highest order nonlinear terms

$$O\left(U^{l_1}\frac{d^{l_2}}{d\xi^r}U\right) = (l_1+1)L + l_2 \text{ for } l_1,\ l_2 = 0,1,2,\ldots, \tag{21}$$

and the highest-order derivatives

$$O\left(\frac{d^{l_2}}{d\xi^{l_2}}U\right) = L + l_2 \text{for } l_2 = 0,1,2,\ldots. \tag{22}$$

Step 4. Substitute the predicted solution (19) back into ODE (18) to obtain an expression in terms of $sn^{\sigma_1}(\xi)cn^{\sigma_2}(\xi)dn^{\sigma_3}(\xi)(\sigma_1,\sigma_2,\sigma_3 = 0,\ 1,\ 2,\ \ldots)$ by means of reducing to a common denominator and setting the numerator to zero. Then, collect all terms with the same powers and put all the coefficients to zero leading to an over-determined system of nonlinear algebraic equations with respect to the unknown parameters $\rho,\ k,\ b_0,\ b_1,\ b_2,\ b_3$, and $p_\hbar,\ r_\hbar, \hbar = 2\ ,3,\ \ldots,\ L$.

Step 5. Solve the resulting algebraic system in Step 4 with the aid of Mathematica software to find out the values of $\rho,\ k,\ b_0,\ b_1,\ b_2,\ b_3$, and $p_\hbar,\ r_\hbar,\ \hbar = 2\ ,3,\ \ldots,\ L$.

Step 6. Substitute the obtained values in terms of $\rho,\ k,\ b_0,\ b_1,\ b_2,\ b_3$, and $p_\hbar,\ r_\hbar$ for $\hbar = 2,\ 3,\ \ldots,\ L$ in the predicted solution (19); new types of abundant traveling wave solutions are provided to FPDEs (16) involving the Jacobi elliptic functions.

4. Solving the Space–Time Fractional MTM

This section is designed to perform the steps of the extended Jacobi elliptic function expansion algorithm to construct wave solutions for the space–time fractional MTM system (5). To perform this, we propose a complex wave transformation in the following form

$$F(x,t) \to \mathscr{F}(\xi)e^{i\hbar},\ g(x,t) \to \mathscr{G}(\xi)e^{i\hbar},\ \text{which } \xi = k_1 x + k_2\frac{t^\alpha}{\Gamma(\alpha+1)},\ \hbar = r_1 x + r_2\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{23}$$

where k_1, k_2, r_1 and r_2 are constants to be determined afterwards.

This transformation leads to the following results

$$D_t^\alpha F = \left(k_2\frac{d\mathscr{F}}{d\xi} + ir_2\mathscr{F}\right)e^{i\hbar}, \frac{\partial F}{\partial x} = \left(k_1\frac{d\mathscr{F}}{d\xi} + ir_1\mathscr{F}\right)e^{i\hbar},\ |F|^2 = \mathscr{F}^2(\xi), \tag{24}$$

$$D_t^{\alpha} G = \left(k_2 \frac{d\mathscr{G}}{d\xi} + ir_2\mathscr{G}\right)e^{i\hbar}, \frac{\partial G}{\partial x} = \left(k_1 \frac{d\mathscr{G}}{d\xi} + ir_1\mathscr{G}\right)e^{i\hbar},\ |G|^2 = \mathscr{G}^2(\xi). \tag{25}$$

By substituting assumption (23) with relations (24) and (25) together into the space–time fractional MTM system (5), we obtain the corresponding system of nonlinear ODEs in the form,

$$\begin{aligned} i(k_1+k_2)\tfrac{d\mathscr{F}}{d\xi} - (r_1+r_2)\mathscr{F} + \mathscr{G} - \mathscr{G}^2\mathscr{F} = 0, \\ i(k_2-k_1)\tfrac{d\mathscr{G}}{d\xi} - (r_2-r_1)\mathscr{G} + \mathscr{F} - \mathscr{F}^2\mathscr{G} = 0. \end{aligned} \tag{26}$$

Now, by balancing the highest order nonlinear term and highest order derivatives, we have $L = 1$. Then, the formal solutions of system (26) can be expressed as

$$\begin{aligned} \mathscr{F}(\xi) = p_0 + p_1 Q_{\jmath}(\xi) + p_2 R_{\jmath}(\xi) + p_3 S_{\jmath}(\xi),\ \jmath = 1,2,\ldots,\ 12, \\ \mathscr{G}(\xi) = q_0 + q_1 Q_{\jmath}(\xi) + q_2 R_{\jmath}(\xi) + q_3 S_{\jmath}(\xi),\ \jmath = 1,2,\ldots,\ 12. \end{aligned} \tag{27}$$

where $p_{\hbar},\ q_{\hbar},\ \hbar = 0,1,2,3$ are constants to be determined. Let $\jmath = 1$. Then, the formal solutions (27) becomes

$$\begin{aligned} \mathscr{F}(\xi) = p_0 + p_1 \tfrac{1}{\rho+dn(\xi)} + p_2 \tfrac{sn(\xi)}{\rho+dn(\xi)} + p_3 \tfrac{cn(\xi)}{\rho+dn(\xi)}, \\ \mathscr{G}(\xi) = q_0 + q_1 \tfrac{1}{\rho+dn(\xi)} + q_2 \tfrac{sn(\xi)}{\rho+dn(\xi)} + q_3 \tfrac{cn(\xi)}{\rho+dn(\xi)}. \end{aligned} \tag{28}$$

Substitute the solutions from (28) into the system from (26), and separate the real and imaginary parts so that the denominators are canceled in both parts. Then, collect the coefficients of $sn^{d_1}(\eta)cn^{d_2}(\eta)dn^{d_3}(\eta)(d_1, d_2, d_3 = 0,\ 1,\ 2, 3)$, and set each coefficient to zero. Consequently, two sets of over-determined algebraic equations are constructed in terms of $\rho,\ k_1,\ k_2,\ r_1,\ r_2,\ p_{\ \jmath},\ q_{\ \jmath}, \jmath = 0,1,2,3$. The obtained sets of these algebraic equations are solved via the computer software of Mathematica, so that the resulting form of the imaginary part yields

$$k_1 = k_2 \text{ or } k_1 = -k_2. \tag{29}$$

and the resulting form of the real part yields the following solution families:

Family I: When $p_1 = q_1 = 0$, let $q_0,\ p_2,\ q_3,\ r_1$ and r_2 be arbitrary constants. Then, we get the following cases for and :

Case 1:

$$p_0 = \frac{-q_0}{q_0^2 - (r_1+r_2)}, \quad q_2 = \frac{p_2\left(q_0^2 - (r_1+r_2)\right)}{q_0^2 + (r_1+r_2)}, \quad p_3 = \frac{-2q_3(r_1+r_2)}{\left(q_0^2 - (r_1+r_2)\right)^2}. \tag{30}$$

Case 2:

$$p_0 = \frac{-q_0}{q_0^2 - (r_1+r_2)}, \quad q_2 = 2p_2\left(q_0^2 - (r_1+r_2)\right), \quad p_3 = \frac{q_3}{\left(q_0^2 - (r_1+r_2)\right)}. \tag{31}$$

Case 3:

$$p_0 = \frac{-q_0}{q_0^2 - (r_1+r_2)}, \quad q_2 = -p_2\left(q_0^2 - (r_1+r_2)\right), \quad p_3 = \frac{-2q_3}{\left(q_0^2 - (r_1+r_2)\right)}. \tag{32}$$

Case 4:

$$p_0 = \frac{-q_0}{q_0^2 - (r_1+r_2)}, \quad q_2 = \frac{p_2\left(q_0^2 - (r_1+r_2)\right)^2}{2(r_1+r_2)}, \quad p_3 = \frac{q_3\left(q_0^2 + (r_1+r_2)\right)}{\left(q_0^2 - (r_1+r_2)\right)^2}. \tag{33}$$

By substituting the results above into (28) and combining with (23), we can obtain four exact solutions $F_{\jmath}(x,t)$ and $G_{\jmath}(x,t),\ \jmath = 1,2,3,4$, for the space–time fractional MTM system (5) in the forms of Jacobi elliptic functions. For example, some graphical representations of these solutions are presented in the following figures. Figure 1 shows the 3D plots of $|F_1(x,t)|^2$ and $|G_1(x,t)|^2$ at some parameters that were chosen randomly,

$r_1 = r_2 = 1, k_1 = k_2 = 1,\ q_0 = 0.6,\ \ p_2 = -2,\ \ q_3 = 0.8,\ \rho = 1$ and $\tau = 0$, in the intervals $0 \le x \le 20$ and $0 \le t \le 10$ at different values of the fractional derivative such that $\alpha = 1$ and $\alpha = 0.75$. While Figure 2 presents 3D plots of the real and imaginary parts of the periodic wave solutions $F_2(x,t)$ and $G_2(x,t)$ in $(x,t) \in [0,20] \times [0,10]$ with $r_1 = r_2 = 1, k_1 = k_2 = 1, q_0 = 0.2,\ p_2 = -1,\ q_3 = 0.5$ and $\rho = \tau = 1$ for the fractional order $\alpha = 0.85$. From these figures, it is observed that the propagation of the periodic wave forms propagation along the space direction over time by maintaining its shape and amplitude. The fractional order affects only the velocity of propagation.

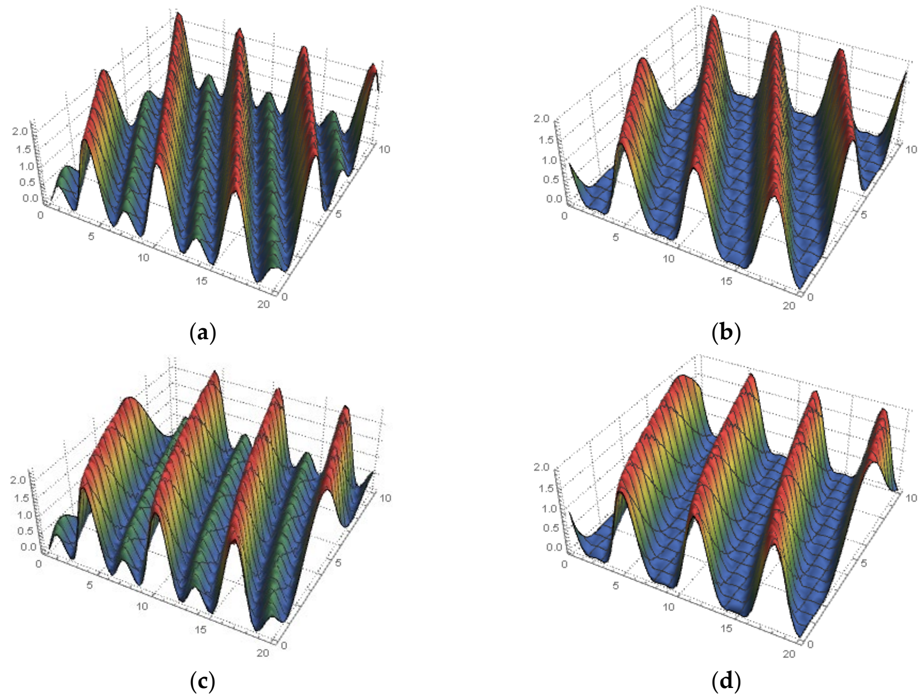


Figure 1. The 3D plots of $|F_1(x,t)|^2$ and $|G_1(x,t)|^2$ with the parameters $r_1 = r_2 = 1, k_1 = k_2 = 1$, $q_0 = 0.6,\ p_2 = -2,\ q_3 = 0.8, \rho = 1$ and $\tau = 0$ for various α values: (**a**) $|F_1|^2, \alpha = 1$, (**b**) $|G_1|^2, \alpha = 1$, (**c**) $|F_1|^2, \alpha = 0.75$ and (**d**) $|G_1|^2, \alpha = 0.75$.

Family II: When $p_2 = q_2 = 0$, let $q_0,\ p_3,\ q_1,\ r_1$ and r_2 be arbitrary constants. Then, we get the following cases for $p_0,\ p_1$ and q_3:

Case 1:

$$p_0 = \frac{-q_0}{q_0^2 - (r_1 + r_2)}, \quad p_1 = \frac{-q_1}{\left(q_0^2 - (r_1 + r_2)\right)}, \quad q_3 = 2p_3\left(q_0^2 - (r_1 + r_2)\right). \tag{34}$$

Case 2:

$$\begin{gathered} p_0 = \frac{-q_0}{q_0^2-(r_1+r_2)}, \quad p_1 = \frac{-2q_1\left(q_0^2+(r_1+r_2)\right)-\rho q_0\left(q_0^2+3(r_1+r_2)\right)}{\left(q_0^2-(r_1+r_2)\right)^2}, \\ q_3 = \frac{p_3\left(q_0^2-(r_1+r_2)\right)^2}{\left(q_0^2+(r_1+r_2)\right)}. \end{gathered} \tag{35}$$

Case 3:

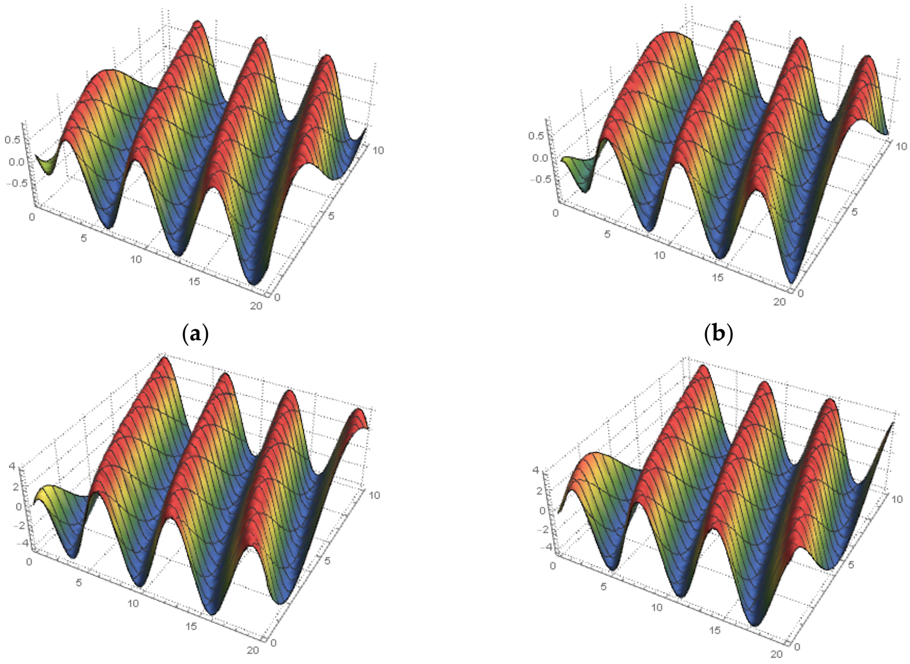
$$\begin{gathered} p_0 = \frac{-q_0}{q_0^2-(r_1+r_2)}, \quad p_1 = \frac{\rho^2 q_0\left(q_0^2+3(r_1+r_2)\right)+2q_0q_1^2+\rho q_1\left(3q_0^2+2(r_1+r_2)\right)}{\left(q_0^2-(r_1+r_2)\right)\left(q_0q_1+\rho\left(q_0^2+(r_1+r_2)\right)\right)}, \\ q_3 = \frac{-p_3\left(q_0^2-(r_1+r_2)\right)\left(q_0q_1+\rho\left(q_0^2+(r_1+r_2)\right)\right)}{q_0q_1+\rho(r_1+r_2)}. \end{gathered} \tag{36}$$

Case 4:

$$\begin{aligned} p_0 &= \frac{-q_0}{q_0^2-(r_1+r_2)}, \\ p_1 &= \frac{-q_1\left(q_0q_1^2+\rho q_1\left(q_0^2+(r_1+r_2)\right)+2\rho^2 q_0(r_1+r_2)\right)}{\left(q_0^2-(r_1+r_2)\right)\left(q_0q_1^2+2\rho q_1\left(q_0^2-(r_1+r_2)\right)+\rho^2 q_0\left(q_0^2-(r_1+r_2)\right)\right)}, \\ q_3 &= \frac{p_3\left(q_0^2-(r_1+r_2)\right)\left(\mathrm{q0q1}^2+2\mathrm{q1}\rho\left(\mathrm{q0}^2-(\mathrm{r1}+\mathrm{r2})\right)\right)+\rho^2 q_0\left(q_0^2-(r_1+r_2)\right)}{2q_1\left(q_0q_1+\rho\left(q_0^2+(r_1+r_2)\right)\right)+\rho^2 q_0\left(q_0^2+(r_1+r_2)\right)}. \end{aligned} \tag{37}$$

Case 5:

$$p_0 = \frac{-q_0}{q_0^2-(r_1+r_2)}, \quad p_1 = \frac{2q_1(r_1+r_2)}{\left(q_0^2-(r_1+r_2)\right)^2}, \quad q_3 = \frac{p_3\left(q_0^2-(r_1+r_2)\right)}{\left(q_0^2+(r_1+r_2)\right)}. \tag{38}$$



(**a**) (**b**) (**c**) (**d**)

Figure 2. The 3D plots of the real and imaginary parts of $F_2(x,t)$ and with the parameters $r_1 = r_2 = 1$, $k_1 = k_2 = 1, q_0 = 0.2$, $p_2 = -1$, $q_3 = 0.5$ and $\rho = \tau = 1$ for the fractional order $\alpha = 0.85$: (**a**) $Re[F_2(x,t)]$, (**b**) $Im[F_2(x,t)]$, (**c**) $Re[G_2(x,t)]$ and (**d**) $Im[G_2(x,t)]$.

By substituting the results above into (28) and combining with (23), we can obtain five exact solutions $F_{\mathcal{J}}(x,t)$ and $G_{\mathcal{J}}(x,t)$, $\mathcal{J} = 5,6,7,8,9$, for the space–time fractional MTM system (5) in the forms of Jacobi elliptic functions. For physical illustration, some graphical representations of these solutions are drawn and introduced in the following figures. Figure 3 reveals the 3D plots of $|F_5(x,t)|^2$ and $|G_5(x,t)|^2$ with some selected parameters $r_1 = r_2 = -1$, $k_1 = k_2 = 1$, $q_0 = 0.05$, $p_3 = -0.3$, $q_1 = 2$, $\rho = 1$, and $\tau = 0$ over the intervals $0 \le x \le 20$ and $0 \le t \le 10$ for various values of $\alpha \in \{0.75, 1\}$.

Figures 4 and 5 show 3D plots of the real and imaginary parts of the periodic wave solutions $F_6(x,t), F_7(x,t), G_6(x,t)$, and $G_7(x,t)$ in $(x,t) \in [0,20] \times [0,10]$ with some selected physical free parameters and different fractional orders. The regularity, harmony and compatibility of the periodic wave solutions can be observed for different α values in all cases.

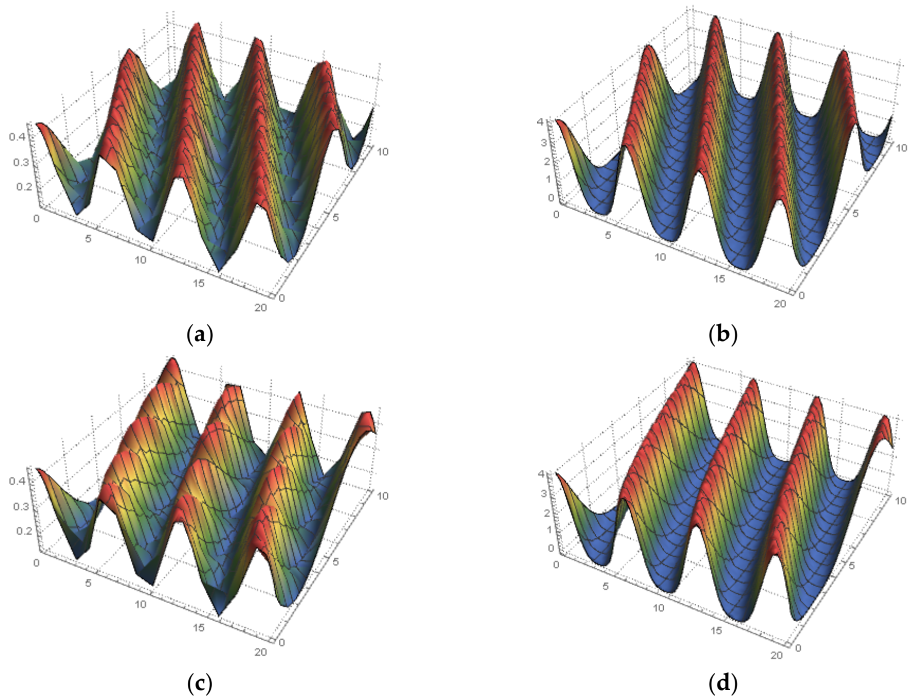


Figure 3. The 3D plots of $|F_5(x,t)|^2$ and $|G_5(x,t)|^2$ with the parameters $r_1 = r_2 = -1, k_1 = k_2 = 1$, $q_0 = 0.05,\ p_3 = -0.3,\ q_1 = 2, \rho = 1$ and $\tau = 0$ for various α values: (**a**) $|F_5|^2, \alpha = 1$, (**b**) $|G_5|^2, \alpha = 1$, (**c**) $|F_5|^2, \alpha = 0.75$ and (**d**) $|G_5|^2, \alpha = 0.75$.

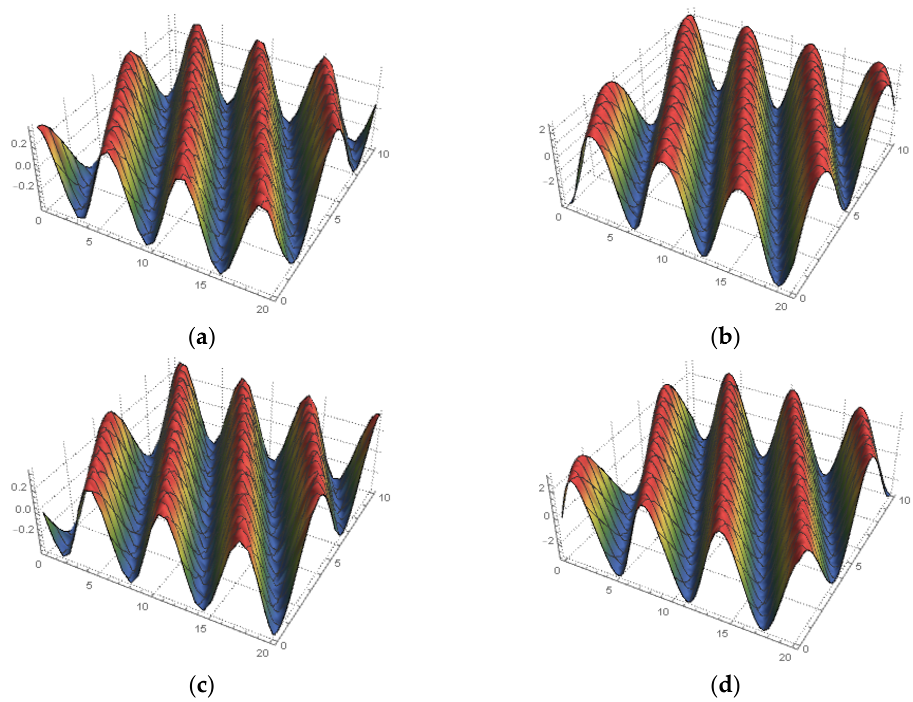
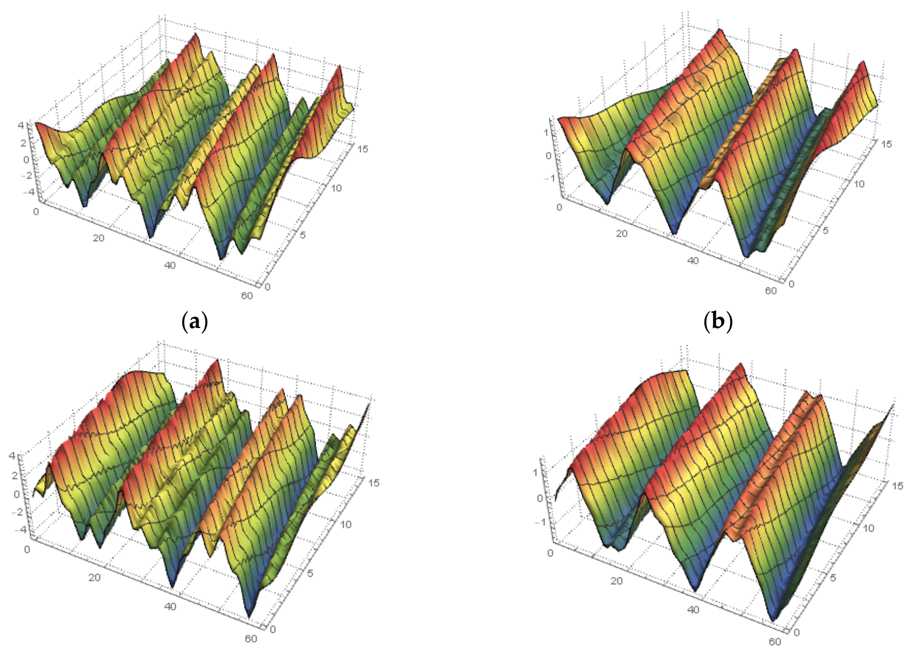


Figure 4. The 3D plots of the real and imaginary parts of $F_6(x,t)$ and $G_6(x,t)$ with the parameters $r_1 = r_2 = -1, k_1 = k_2 = 1, q_0 = -2.5,\ p_3 = 0,\ q_1 = -1, \rho = 1$ and $\tau = 0$ for the fractional order $\alpha = 1$: (**a**) $Re[F_6(x,t)]$, (**b**) $Re[G_6(x,t)]$, (**c**) $Im[F_6(x,t)]$ and (**d**) $Im[G_6(x,t)]$.



(**a**) (**b**) (**c**) (**d**)

Figure 5. The 3D plots of the real and imaginary parts of $F_7(x,t)$ and $G_7(x,t)$ with the parameters $r_1 = r_2 = 0.3, k_1 = k_2 = 1, q_0 = 0.9,\ p_3 = 4,\ q_1 = 5/7, \rho = 1$ and $\tau = 0$ for the fractional order $\alpha = 0.65$: (**a**) $Re[F_7(x,t)]$, (**b**) $Re[G_7(x,t)]$, (**c**) $Im[F_7(x,t)]$ and (**d**) $Im[G_7(x,t)]$.

Family III: When $p_3 = q_3 = 0$, let $q_0,\ p_2,\ q_1,\ r_1$ and r_2 be arbitrary constants. Then, we get the following cases for p_0, p_1 and q_2:

Case 1:

$$\begin{aligned} p_0 &= \frac{-q_0}{q_0^2-(r_1+r_2)}, \\ p_1 &= \frac{2q_0q_1^2+\rho q_1\left(3q_0^2+2(r_1+r_2)\right)+\rho^2 q_0\left(q_0^2+3(r_1+r_2)\right)}{\left(q_0^2-(r_1+r_2)\right)\left(q_0q_1+\rho\left(q_0^2+(r_1+r_2)\right)\right)}, \\ q_2 &= \frac{-p_2\left(q_0^2-(r_1+r_2)\right)\left(q_0q_1+\rho\left(q_0^2+(r_1+r_2)\right)\right)}{q_0q_1+\rho(r1+r2)}. \end{aligned} \tag{39}$$

Case 2:

$$\begin{aligned} p_0 &= \frac{-q_0}{q_0^2-(r_1+r_2)}, \\ p_1 &= \frac{-q_1\left(q_0q_1^2+\rho q_1\left(q_0^2+(r_1+r_2)\right)+2\rho^2 q_0(r_1+r_2)\right)}{\left(q_0^2-(r_1+r_2)\right)\left(q_0q_1^2+2\rho q_1\left(q_0^2-(r_1+r_2)\right)+\rho^2 q_0\left(q_0^2-(r_1+r_2)\right)\right)}, \\ q_2 &= \frac{p_2\left(q_0^2-(r_1+r_2)\right)\left(q_0q_1^2+2\rho q_1\left(q_0^2-(r_1+r_2)\right)+\rho^2 q_0\left(q_0^2-(r_1+r_2)\right)\right)}{2q_0q_1^2+2\rho q_1\left(q_0^2+(r_1+r_2)\right)+\rho^2 q_0\left(q_0^2+(r_1+r_2)\right)} \end{aligned} \tag{40}$$

Case 3:

$$\begin{aligned} p_0 &= \frac{-q_0}{q_0^2-(r_1+r_2)}, \\ p_1 &= \frac{\rho\left(3q_0q_1^2+2\rho q_1\left(2q_0^2+(r_1+r_2)\right)+\rho^2 q_0\left(3q_0^2+(r_1+r_2)\right)\right)}{\left(q_0^2-(r_1+r_2)\right)\left(3q_1^2+6\rho q_0q_1+\rho^2\left(3q_0^2+(r_1+r_2)\right)\right)}, \\ q_2 &= \frac{p_2(3q_1^2+6q_0q_1\rho+\rho^2\left(3q_0^2+(r_1+r_2)\right)}{\rho^2}. \end{aligned} \tag{41}$$

Case 4:

$$p_0 = \frac{-q_0}{q_0^2-(r_1+r_2)}, \quad p_1 = \frac{-q_1}{\left(q_0^2-(r_1+r_2)\right)}, \quad q_2 = 2p_2\left(q_0^2-(r_1+r_2)\right). \tag{42}$$

Case 5:

$$p_0 = \frac{-q_0}{q_0^2-(r_1+r_2)}, \quad p_1 = \frac{-2q_1\left(q_0^2-(r_1+r_2)\right)-\rho q_0\left(q_0^2+3(r_1+r_2)\right)}{\left(q_0^2-(r_1+r_2)\right)^2}, \quad q_2 = \frac{q_1\left(q_0^2-(r_1+r_2)\right)^2}{\left(q_0^2+(r_1+r_2)\right)}. \tag{43}$$

By substituting the results above into (28) and combining with (23), we can obtain five exact solutions $F_{\jmath}(x,t)$ and $G_{\jmath}(x,t)$, $\jmath = 10,\ldots,14$, of the fractional MTM system (5) in the forms of Jacobi elliptic functions. The illustrations of these acquired solutions, for various values of α, are depicted in Figures 6–9.

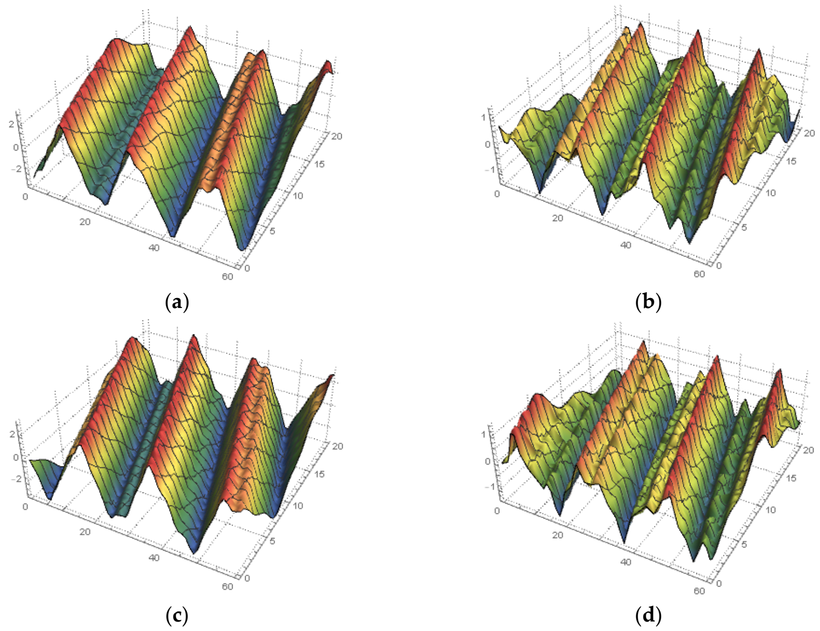


Figure 6. The 3D plots of the real and imaginary parts of $F_{10}(x,t)$ and $G_{10}(x,t)$ with the parameters $r_1 = r_2 = 0.3, k_1 = k_2 = 1, q_0 = 0.9,\ p_2 = 1,\ q_1 = -0.2, \rho = 0.5$ and $\tau = 0$ for the fractional order $\alpha = 0.9$: (**a**) $Re[F_{10}(x,t)]$, (**b**) $Re[G_{10}(x,t)]$, (**c**) $Im[F_{10}(x,t)]$ and (**d**) $Im[G_{10}(x,t)]$.

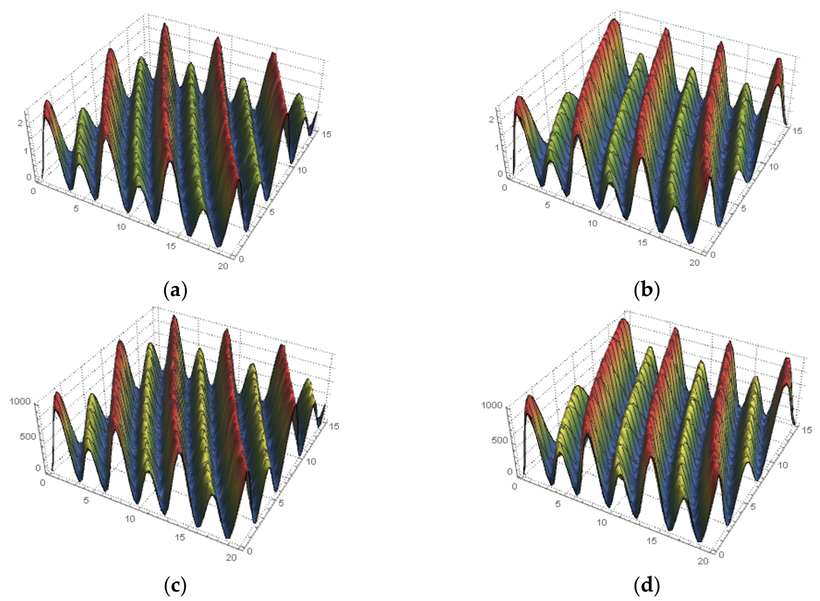


Figure 7. The 3D plots of $|F_{12}(x,t)|^2$ and $|G_{12}(x,t)|^2$ with the parameters $r_1 = r_2 = -3$, $k_1 = k_2 = 1, q_0 = 7,\ p_2 = 2,\ q_1 = -5, \rho = 0.5$ and $\tau = 0$ for various α values: (**a**) $|F_{12}|^2, \alpha = 0.95$, (**b**) $|G_{12}|^2, \alpha = 0.95$, (**c**) $|F_{12}|^2, \alpha = 0.75$ and (**d**) $|G_{12}|^2, \alpha = 0.75$.

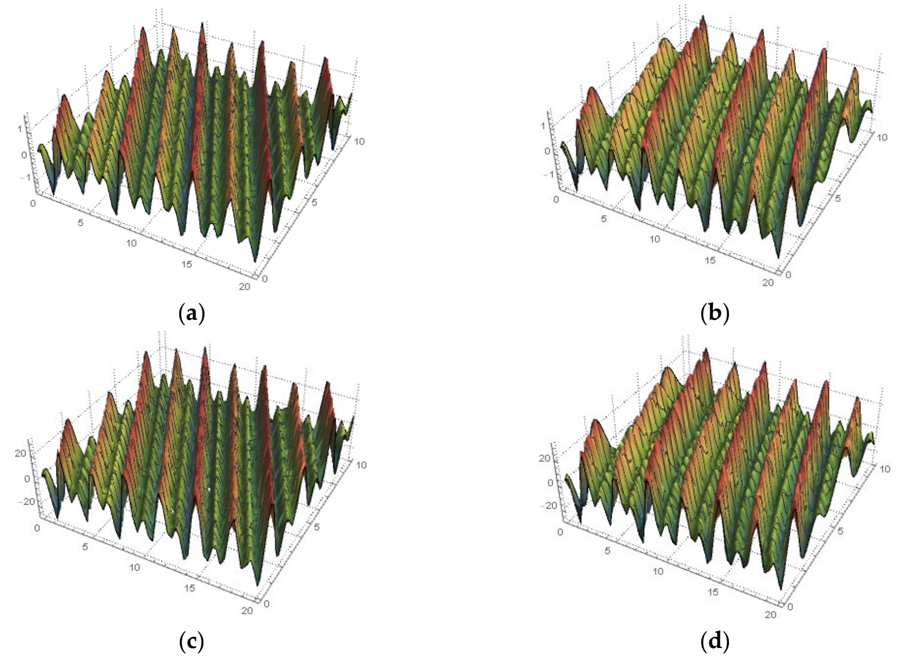


Figure 8. The 3D plots of the real and imaginary parts of $F_{13}(x,t)$ and $G_{13}(x,t)$ with the parameters $r_1 = r_2 = -3, k_1 = k_2 = 1, q_0 = 7,\ p_2 = 2,\ q_1 = -5, \rho = 0.5$, and $\tau = 0.1$ for the fractional order $\alpha = 0.8$: (**a**) $Re[F_{13}(x,t)]$, (**b**) $Re[G_{13}(x,t)]$, (**c**) $Im[F_{13}(x,t)]$, (**d**) $Im[G_{13}(x,t)]$.

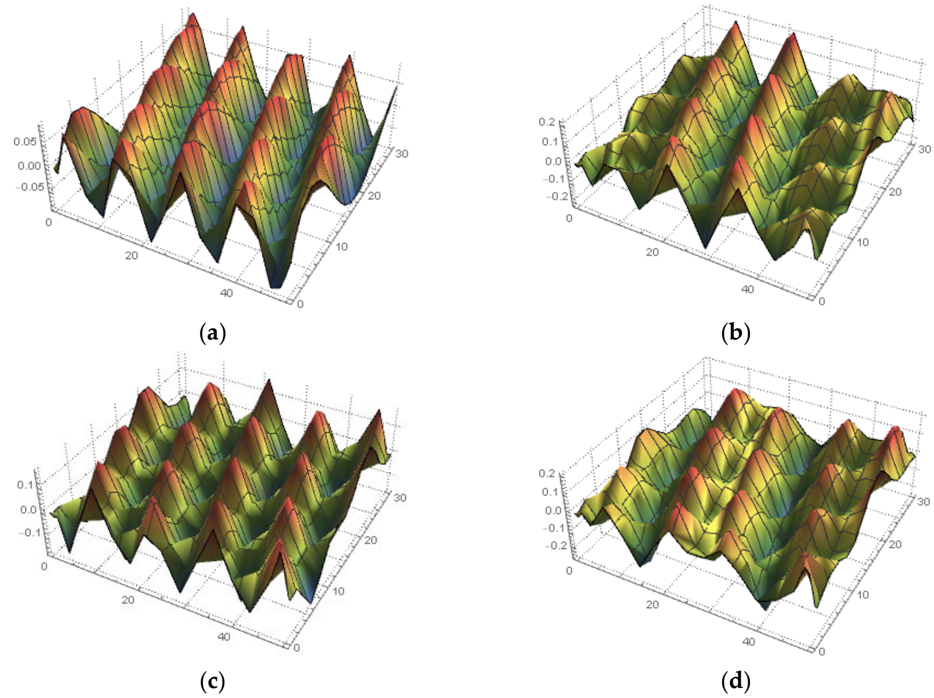


Figure 9. The 3D plots of the real and imaginary parts of $F_{14}(x,t)$ and $G_{14}(x,t)$ with the parameters $r_1 = r_2 = 0.5, k_1 = k_2 = 1, q_0 = 10,\ p_2 = 0.2,\ q_1 = 1, \rho = 1$ and $\tau = 0.1$ for the fractional order $\alpha = 0.75$: (**a**) $Re[F_{14}(x,t)]$, (**b**) $Re[G_{14}(x,t)]$, (**c**) $Im[F_{14}(x,t)]$ and (**d**) $Im[G_{14}(x,t)]$.

5. Conclusions

In this paper, the fractional massive Thirring model has been considered in the sense of the modified Riemann–Liouville fractional derivative. Based on the nonlinear fractional complex transformation, a series of exact traveling wave solutions for this model has been successfully obtained in terms of Jacobi elliptic functions. With the aid of the Mathematica wolfram computation package, the resulting algebraic system of free parameters was solved and graphical representations of some acquired solutions were performed in 3D plots. The proposed method provides a powerful and systematic tool for obtaining novel exact solutions and can be applied to deal with other governing nonlinear fractional evolution equations emerging in mathematical physics.

Author Contributions: Conceptualization, M.S.; methodology, M.A.-S.; software M.A.; validation, M.S., M.A., S.A.-O. and M.A.-S.; investigation, M.S.; writing—original draft preparation, M.A.; writing—review and editing, M.A. and S.A.-O.; visualization, M.A.-S.; supervision, S.A.-O.; project administration, M.A.; funding acquisition, M.A.-S. All authors have read and agreed to the published version of the manuscript.

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