*fractal and fractional*

MDPI

*Article*

# New Results Involving Riemann Zeta Function Using Its Distributional Representation

**Asifa Tassaddiq [1,*] and Rekha Srivastava [2,*]**

[1] Department of Basic Sciences and Humanities, College of Computer and Information Sciences, Majmaah University, Al Majmaah 11952, Saudi Arabia
[2] Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8P 5C2, Canada
* Correspondence: a.tassaddiq@mu.edu.sa (A.T.); rekhas@uvic.ca (R.S.)

**Abstract:** The relation of special functions with fractional integral transforms has a great influence on modern science and research. For example, an old special function, namely, the Mittag–Leffler function, became the queen of fractional calculus because its image under the Laplace transform is known to a large audience only in this century. By taking motivation from these facts, we use distributional representation of the Riemann zeta function to compute its Laplace transform, which has played a fundamental role in applying the operators of generalized fractional calculus to this well-studied function. Hence, similar new images under various other popular fractional transforms can be obtained as special cases. A new fractional kinetic equation involving the Riemann zeta function is formulated and solved. Thereafter, a new relation involving the Laplace transform of the Riemann zeta function and the Fox–Wright function is explored, which proved to significantly simplify the results. Various new distributional properties are also derived.

**Keywords:** delta function; Riemann zeta-function; fractional transforms; Fox–Wright-function; generalized fractional kinetic equation



**Citation:** Tassaddiq, A.; Srivastava, R. New Results Involving Riemann Zeta Function Using Its Distributional Representation. *Fractal Fract.* **2022**, *6*, 254. https://doi.org/10.3390/fractalfract6050254

Academic Editor: Ricardo Almeida

Received: 10 March 2022
Accepted: 21 April 2022
Published: 6 May 2022



## 1. Introduction

In general, the Riemann zeta function and its generalizations have always been of fundamental importance [1–10] due to their widespread applications. For instance, the role of the Riemann zeta function is vital in fractal geometry for studying the complex dimensions of fractal strings [1]. More recently, new representations of special functions are discussed [10–21] in terms of the complex delta function [22,23]. In this article, we use a distributional representation [10], Equation (33), of the Riemann zeta function to obtain further new results. On the one hand, several fractional calculus images involving the Riemann zeta function are obtained under multiple E–K fractional operators, and on the other hand, a non-integer-order kinetic equation including the Riemann zeta function is formulated and solved. The Laplace transform of the Riemann zeta function is computed using its distributional representation, which played a fundamental role in accomplishing the goals of this research. Several new properties and results for this function are also discussed.

Calculation of the images of special functions using the fractional calculus operators has emerged as a popular subject in the data of various newly published papers [24–26]. This number is rising regularly, and such research has commented [24] further in mentioning Kiryakova's unified approach. Taking a cue from these facts, the author has followed the recommendation of [24] and obtained fractional calculus images involving the Riemann zeta function and its simpler cases using the unified approach [24–28]. The Marichev–Saigo–Maeda (M-S-M) operators and the Saigo, Erdélyi–Kober, and Riemann–Liouville (R–L) fractional operators for $m = 3,\ \ m = 2,\ \ m = 1$, respectively, are discussed as special cases of generalized fractional calculus operators (namely, multiple E–K operators of the multiplicity $m$). It is recommended in the conclusion section of [24] to examine whether the special function can be formulated as a general function, namely, the Fox–Wright function ${}_p\Psi_q$, then to use a general result such as [24] and Theorem 3 and 4 therein. It can be observed that it is not possible to apply these theorems for the Riemann zeta function using its classical representations, as already mentioned (see [24], p. 2). It is important to note that the results obtained in this research are completely verifiable with these general results. The corresponding fractional derivatives in the Riemann–Liouville and Caputo sense, as discussed in [24] (p. 9, Definition 6; and p. 17, Theorem 4), can now be used for the Laplace transform of the Riemann zeta function and also straightforwardly using its new representation.

The remaining paper is organized as follows: Necessary preliminaries related to the family of the Fox-H function and the generalized fractional integrals (multiple E–K operators) form part of Section 2. Section 3.1 contains fractional calculus images involving the Riemann zeta-function. The next Section 3.2, is devoted to the formulation and solution of a non-integer-order kinetic equation containing the Riemann zeta-function. Further new properties and results involving the Reimann zeta function are discussed in Section 3.3. The conclusion is given in Section 4. Related special cases of generalized fractional integrals (multiple E–K operators) are listed in Appendix A.

Hence, in order to achieve our purpose, let us first go through the basic definitions and preliminaries in the subsequent section.

## 2. Materials and Methods

Throughout this article, $\mathbb{C}$ and $\mathbb{R}$ represent the set of complex and real numbers. The real part of any complex number is denoted by $\Re$, $\mathbb{Z}_0^-$ denotes a set of negative integers containing *0*, and $\mathbb{R}^+$ symbolizes the set containing positive reals.

The Riemann zeta function is a classical function investigated by Riemann [2], defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}; (s = \sigma + i\tau\,; \Re(s) > 1). \tag{1}$$

With the exception of a simple pole at $s = 1$, the meromorphic continuation of this function extends it to the entire complex $s$-plane. As shown in [2] (p. 13, Equation (2.1.1)), this function satisfies the following result (also known as Riemann's Functional Equation):

$$\zeta(s) := 2^s \pi^{s-1} \Gamma(1-s)\zeta(1-s), \tag{2}$$

where $\Gamma(s)$ represents the gamma function [3,4] (a generalization of the factorial). The Riemann zeta function has simple zeros at negative even integers that are its trivial zeros. The remaining zeros of the zeta function are known as its nontrivial zeros, which are symmetrically placed on the line $\Re(s) = 1/2$. This unproved fact is also famously known as the "Riemann Hypothesis". Several authors have investigated and analyzed different generalizations of the zeta function. It has different integral representations, for example [2],

$$\begin{gathered} \zeta(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt; (\Re(s) > 1); \\ \zeta(s) := \frac{1}{\Gamma(s)} \int_0^\infty \left[\frac{1}{e^t - 1} - \frac{1}{t}\right] t^{s-1} dt; (0 < \Re(s) < 1); \\ \zeta(s) := \frac{1}{\Gamma(s)(1 - 2^{1-s})} \int_0^\infty \frac{t^{s-1}}{e^t + 1} dt; (\Re(s) > 0). \end{gathered} \tag{3}$$

For more details about the zeta function, the interested reader is referred to the references [4–9] and the cited bibliographies therein. More recently, the distributional representation of different special functions has been discussed in [10–20]. In this article, for $\Re(s) > 1$, the following representation [10], Equation (33),

$$\Gamma(s)\zeta(s) = 2\pi \sum_{n,l=0}^{\infty} \frac{(-(n+1))^l}{l!} \delta(s+l) \tag{4}$$

is the main focus to achieve the purpose of the current research. For similar studies, the interested reader is referred to [10–20]. For any suitable function $f$ and the number $\omega$, the delta function is a famous generalized function (distribution) defined by [22,23]:

$$\langle \delta(s-\omega), \wp(s) \rangle \ = \ \wp(\omega);\ \delta(-s) = \delta(s);\ \delta(\omega s) = \frac{\delta(s)}{|\omega|},\ \text{where } \omega \neq 0. \tag{5}$$

It has several interesting properties, such as the following (see [22,23]):

$$\delta(s+l) = \sum_{p=0}^{\infty} \frac{(l)^p}{p!}\, \delta^{(p)}(s); \tag{6}$$

$$\begin{gathered} \delta(z-c) * \vartheta(z) = \vartheta(z-c); \\ \delta^{(i)}(z-c) * \vartheta(z) = \vartheta^{(i)}(z-c); \\ \left(\sum_{i=0}^{\infty} \delta^{(i)}(z-v)\right) * \left(\sum_{i=0}^{\infty} \delta(z-v)\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} \delta^{(j)}(z-v); \\ \left(\sum_{i=0}^{\infty} \delta^{(i)}(z-v)\right) * \left(\sum_{i=0}^{\infty} \delta^{(i)}(z-v)\right) = \left(\sum_{j=0}^{\infty} (v+1)\delta^{(j)}(z-v)\right). \end{gathered} \tag{7}$$

Furthermore, the Laplace transform of an arbitrary function $\varepsilon(t)$ is defined by [23] (Chapter 8):

$$\varepsilon(s) = L[\varepsilon(t) : s] = \int_0^{\infty} e^{-st}(t)dt, \Re(s) > 0 \tag{8}$$

and we will also use [23] (p. 227):

$$L\left\{\delta^{(r)}(z); \xi\right\} = \xi^r. \tag{9}$$

The generalized fractional integrals, namely (multiple) E–K operators of multiplicity $m$, are defined by [24] (p. 8, Equation (18)):

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) = \begin{cases} \int_0^1 f(z\sigma)\ H_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \end{matrix} \right.\right] d\sigma; \sum_k \delta_k > 0 \\ = z^{-1} \int_0^z f(\xi)\ H_{m,m}^{m,0}\left[\frac{\xi}{z} \left| \begin{matrix} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \end{matrix} \right.\right] d\xi; \sum_k \delta_k > 0 \\ f(z); \delta_k = 1 \end{cases} \tag{10}$$

where $\delta_k{}'s$ are concerned with the order of integration, $\gamma_k{}'s$ are weights, and $\beta_k{}'s$ are additional parameters. $H_{p,q}^{m,n}$ is the $H$-function defined in the subsequent paragraph. The limits of integration $(0,\ 1)$ and $(0,\ z)$ in the above equation can be changed to $(0, \infty)$ using the fact that $H_{m,m}^{m,0}$ vanishes for $|\sigma| \ > \ 1$ (To avoid prolonging this section, the special cases of (10) in relation to the results of this article are given in Appendix A). However, the corresponding multiple (m-tuple) Erdélyi–Kober fractional derivative of the R–L type of multi-order $\delta = (\delta_1 \geq\ 0,\ \ldots\ ,\ \delta_m \geq\ 0)$ is defined by [24] (p. 9):

$$D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}\ (f(z)) := D_\eta\ I_{(\beta k\ ),m}^{(\gamma k+\delta k\ ),(\eta k-\delta k\ )}\ \ f(z) D_\eta \int_0^1 f(z\sigma) H_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} \left(\gamma_k+\eta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \end{matrix} \right.\right] d\sigma \tag{11}$$

where $D_\eta$, is a polynomial of variable $z\left(\frac{d}{dz}\right)$ of degree $\eta_1 + \ldots + \eta_m$, given by

$$D_\eta = \prod\nolimits_{r=1}^{m} \prod\nolimits_{j=1}^{\eta_r} \left(\frac{1}{\beta_r} z \frac{d}{dz} + \gamma_r + j\right);\ \eta_k = \left\{ \begin{array}{c} [\delta_k] + 1; \delta_k \notin \mathbb{Z} \\ \delta_k; \delta_k \in \mathbb{Z} \end{array} \right. \tag{12}$$

and the corresponding multiple (m-tuple) Erdélyi–Kober fractional derivative of the Caputo type is given as (see [24] (p. 9) and references therein):

$$ {}^{*}D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)} f(z) = I_{(\beta k\ ),m}^{(\gamma k+\delta k\ ),(\eta k-\delta k\ )} D_\eta f(z). \quad (13) $$

The action of the E–K operators on the power function yields [24] (p. 9; Equation (27)):

$$ I_{(\beta k),m}^{(\gamma k),(\delta k)}\{z^p\} = \prod_{i=1}^{m} \frac{\Gamma\left(\gamma_i + 1 + \frac{p}{\beta_i}\right)}{\Gamma\left(\gamma_i + \delta_i + 1 + \frac{p}{\beta_i}\right)} z^p; \ \ [-\beta_k(1+\gamma_k)] < p; \delta_k \geq 0; k = 1, \ldots, m. \quad (14) $$

The integrand of (10) involves the Fox $H$-function defined by [24] (p. 3; see also [25,29]), which is given here in its integral and series form as follows:

$$ \begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i, A_i)\\(b_j,\ B_j)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1, A_1), \ldots, & (a_i, A_i)\\(b_1,\ B_1), \ldots, & (b_j,\ B_j)\end{matrix}\right.\right] \\ &= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma\left(b_j + B_j\mathfrak{s}\right)\prod_{i=1}^{n}\Gamma\left(1-a_j-A_j\mathfrak{s}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-B_j\mathfrak{s}\right)\prod_{i=n+1}^{p}\Gamma\left(a_j+A_j\mathfrak{s}\right)} z^{-\mathfrak{s}}\, d\mathfrak{s}, \end{aligned} \quad (15) $$

where $m,\ n, p,$ and $q$ are related as $1 \leqq m \leqq q; 0 \leqq n \leqq p,\ A_i > 0\ (\ i = 1, \cdots, p)$; $B_j > 0\ (\ j = 1, \cdots, q),\ a_i \in \mathbb{C}\ (\ i = 1, \cdots, p); b_j \in \mathbb{C}\ (\ j = 1, \cdots, q)$; and $\mathcal{L}$ is an appropriate Mellin–Barnes type of contour that separates the singularities of $\left\{\Gamma(\ b_j\ + B_j\mathfrak{s})\right\}_{j=1}^{m}$ from the singularities of $\left\{\Gamma(\ 1 - a_j\ \ - A_j\mathfrak{s})\right\}_{j=1}^{n}$. Here, $\Gamma(z)$ denotes the familiar gamma function [4], and when all $A_p = B_q = 1$, then the $H$-function becomes the Meijer $G$-function [24] (p.4; see also [25,29]):

$$ \begin{aligned} H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1, A_1), \ldots, & (a_i, A_i)\\(b_1,\ B_1), \ldots, & (b_j,\ B_j)\end{matrix}\right.\right] &= \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}\Gamma\left(\ b_j\ + B_j m\right)\prod_{i=1}^{n}\Gamma\left(\ 1 - a_i\ \ - A_i m\right)}{\prod_{j=m+1}^{q}\Gamma\left(\ 1 - b_j\ - B_j m\right)\prod_{i=n+1}^{p}\Gamma\left(\ a_i\ + i m\right)} \frac{z^m}{m!} \\ H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1, 1), \ldots, & (a_i, 1)\\(b_1, 1), \ldots, & (b_j, 1)\end{matrix}\right.\right] &= G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1 \ldots, & a_i\\b_1, \ldots, & b_j\end{matrix}\right.\right]. \end{aligned} \quad (16) $$

The basic Fox–Wright function denoted by ${}_p\Psi_q$ is defined and related to the $H$-function:

$$ \begin{aligned} {}_p\Psi_q\left[\begin{matrix}(a_i,\ A_i)\\(b_j, B_j)\end{matrix}; z\right] &= \sum_{m=0}^{\infty}\frac{\prod_{l=1}^{p}\Gamma(a_i + A_i m)}{\prod_{l=1}^{q}\Gamma((b_j+B_j m)}\frac{z^m}{m!} = H_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-a_1, A_1), \ldots, & (1-a_i,\ A_i)\\(0,1), (1-b_1,\ B_1), \ldots, & ((1-b_j, B_j)\end{matrix}\right.\right] \\ &\left(a_i \epsilon \mathbb{R}^+ (i = 1, \ldots, p); B_j \epsilon \mathbb{R}^+ (j = 1, \ldots, q); 1 + \sum_{i=1}^{q} B_i - \sum_{j=1}^{p} A_j > 0\right) \end{aligned} \quad (17) $$

and contains the hypergeometric and other important functions as [24] (p. 4; see also [25,29]):

$$ \begin{aligned} {}_p\Psi_q\left[\begin{matrix}(a_i,\ 1)\\(b_j, 1)\end{matrix}; z\right] &= G_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-a_1, 1), \ldots, & (1-a_i, 1)\\0, (1-b_1, 1), \ldots, & (1-b_j, 1)\end{matrix}\right.\right] \\ &= {}_pF_q\left[\begin{matrix}a_i\\b_j\end{matrix}; z\right] \cdot \frac{\Gamma(a_1)\ldots\Gamma(a_i)}{\Gamma(b_1)\ldots\Gamma(b_j)}. \qquad (a_j > 0; b_j \notin \mathbb{Z}_0^-). \end{aligned} \quad (18) $$

Furthermore, many other special functions studied in the literature are connected with this class of special functions. For example, the Mittag–Leffler function [30] of parameters 1, 2, and 3 is related with the abovementioned special functions as follows:

$$ \begin{aligned} E_{\alpha,\beta}^{\gamma}(z) &= \sum_{r=0}^{\infty}\frac{(\gamma)_r z^r}{\Gamma(\alpha r+\beta)} = \frac{1}{\Gamma(\gamma)}{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta, \alpha)\end{matrix}; z\right] = H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(1-\gamma, 1)\\(0,1), (1-\beta, \alpha)\end{matrix}\right.\right]; \\ E_{\alpha,\beta}^{1}(z) = E_{\alpha,\beta}(z) &= \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+\beta)} = {}_1\Psi_1\left[\begin{matrix}(1,\ 1)\\(\beta, \alpha)\end{matrix}; z\right] = H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(0,1)\\(0,1), (1-\beta, \alpha)\end{matrix}\right.\right]; \\ E_{\alpha,1}^{1}(z) = E_{\alpha}(z) &= \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+1)} = {}_1\Psi_1\left[\begin{matrix}(1,\ 1)\\(1, \alpha)\end{matrix}; z\right] = H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(0,1)\\(0,1), (0, \alpha)\end{matrix}\right.\right]. \end{aligned} \quad (19) $$

Furthermore, $(s)_k$ is the Pochhammer symbols defined in terms of the gamma function as follows:

$$(s)_\rho = \frac{\Gamma(s+\rho)}{\Gamma(s)} = \begin{cases} 1 \ (\rho = 0, s \in C\backslash\{0\}) \\ s(s+1)\ldots(s+k-1) \ (\rho = k \in \mathbb{N}; s \in \mathbb{C}). \end{cases} \tag{20}$$

Furthermore, it is important to mention that if any function can be expressed in the form of the Fox–Wright function, then the generalized (multiple E–K) fractional integrals and derivatives involving this function can be obtained directly using the general results of [24], Theorem 3:

$$I^{(\gamma k)_1^m,(\delta k)}_{(\beta k),m}\left\{z^c{}_p\Psi_q\left[\begin{matrix}(a_k,\ \alpha_k)_1^p \\ (b_k,\beta_k)_1^q\end{matrix};\lambda z^\mu\right]\right\} = z^c\left\{{}_{p+m}\Psi_{q+m}\left[\begin{matrix}(a_k,\ \alpha_k)_1^p, \left(\gamma_k+1+\frac{c}{\beta_k},\frac{\mu}{\beta_k}\right)_1^m \\ (b_k,\beta_k)_1^q, \left(\gamma_k+\delta_k+1+\frac{c}{\beta_k},\frac{\mu}{\beta_k}\right)_1^m\end{matrix};\lambda z^\mu\right]\right\} \tag{21}$$

$$(\delta k \geq 0,\ \gamma_k > -1,\ \beta_k > 0,\ k = 1,\ \ldots,\ m\ \wedge\ \mu > 0,\ \lambda \neq 0)$$

and [24], Theorem 4:

$$D^{(\gamma k)_1^m,(\delta k)}_{(\beta k),m}\left\{z^c{}_p\Psi_q\left[\begin{matrix}(a_i,\ \alpha_i)_1^p \\ (b_j,\beta_j)_1^q\end{matrix};\lambda z^\mu\right]\right\} = z^c\left\{{}_{p+m}\Psi_{q+m}\left[\begin{matrix}(a_i,\ \alpha_i)_1^p, \left(\gamma_k+\delta_k+1+\frac{c}{\beta_k},\frac{\mu}{\beta_k}\right)_1^m \\ (b_j,\beta_j)_1^q, \left(\gamma_k+1+\frac{c}{\beta_k},\frac{\mu}{\beta_k}\right)_1^m\end{matrix};\lambda z^\mu\right]\right\}. \tag{22}$$

Unless otherwise stated, the conditions of parameters will remain similar to this Section 2 and references therein.

## 3. Results

### *3.1. Fractional Integrals and Derivatives Formulae Involving the Riemann Zeta-Function*

The following lemma has significant importance for the application of Equations (21) and (22).

**Lemma 1.** *Prove that the following result involving the Fox–Wright function holds true:*

$$2\pi\sum_{n=0}^{\infty}{}_0\Psi_0\left[\begin{matrix}- \\ -\end{matrix}\middle| -(n+1)e^\omega\right] = \sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}{}_0\Psi_0\left[\begin{matrix}- \\ -\end{matrix}\middle| l\omega\right]. \tag{23}$$

**Proof .** First of all, let us use (6) in (4) to get the following form:

$$\Gamma(s)\zeta(s) = 2\pi\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}\delta^{(p)}(s). \tag{24}$$

Then, by applying the Laplace transform to (24), and by making use of (9), we are led to the following:

$$L(\Gamma(s)\zeta(s);\omega) = 2\pi\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}\omega^p = 2\pi\sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}{}_0\Psi_0\left[\begin{matrix}- \\ -\end{matrix}\middle| l\omega\right]. \tag{25}$$

From (25), it can be further noticed that

$$L(\Gamma(s)\zeta(s);\omega) = \frac{2\pi}{\exp(e^\omega)-1} = 2\pi\sum_{n=0}^{\infty}\exp(-(r+1)e^\omega) = 2\pi\sum_{n=0}^{\infty}{}_0\Psi_0\left[\begin{matrix}- \\ -\end{matrix}\middle| -(n+1)e^\omega\right]. \tag{26}$$

From (25) and (26), the required result follows. □

**Theorem 1.** *The multiple E–K fractional transform of the Riemann zeta function is given by:*

$$I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}\right)=2\pi x^{\chi-1}\sum_{r=0}^{\infty}{}_m\Psi_m\left[\begin{array}{c}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{array}\middle|-(r+1)e^{\omega}\right] \tag{27}$$
$$[-\beta_k(1+\gamma_k)]<p;\delta_k\geq 0;k=1,\ldots,m.$$

**Proof.** Let us first consider multiple E–K's fractional transform using (25):

$$I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}\right)=I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}2\pi\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(m)^p}{l!p!}\,\omega^p\right), \tag{28}$$

exchanging the summation and integration,

$$I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}\right)=2\pi\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(m)^p}{l!p!}I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}\omega^p\right), \tag{29}$$

and then using (14) yields

$$\begin{aligned}I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}\right)&=2\pi\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}\prod_{i=1}^{m}\frac{\Gamma\left(\gamma_i+1+\frac{\chi+p-1}{\beta_i}\right)}{\Gamma\left(\gamma i+\delta i+1+\frac{\chi+p-1}{\beta i}\right)}\omega^{p+\chi-1}\\&=2\pi\omega^{\chi-1}\sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}{}_m\Psi_m\left[\begin{array}{c}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{array}\middle|l\omega\right],\end{aligned} \tag{30}$$

which, after using Lemma 1, leads to the required result. □

**Remark 1.** *Hence the result (30) is completely verifiable with ([24], Theorem 3) in view of (25). Similarly, the generalized fractional derivatives involving the Riemann zeta function can be obtained using the methodology of theorem 1 or by using directly (22) and (25) as follows:*

$$\begin{aligned}D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}\{z^cL(\Gamma(s)\zeta(s);z)\}&=2\pi z^c\sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}{}_m\Psi_m\left[\begin{array}{c}\left(\gamma_k+\delta_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{array}\middle|l\omega\right]\\&=2\pi z^c\sum_{n=0}^{\infty}{}_m\Psi_m\left[\begin{array}{c}\left(\gamma_k+\delta_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{array}\middle|-(n+1)e^{\omega}\right].\end{aligned} \tag{31}$$

Continuing in this way, we obtain the following Table 1 of fractional integrals and derivatives formulae involving the Riemann zeta function by following the methodology of Theorem 1 and using Equations (27), (30), and (31), respectively. (As already mentioned in Section 2, the definitions of the Marichev–Saigo–Maeda, Saigo, Erdélyi–Kober, and Riemann–Liouville (R–L) fractional operators and their relation to (10) for $m=3$, m = 2, $m=1$, respectively, are given in Appendix A; see also [31–34]).

**Table 1.** Fractional integrals and derivatives formulae involving Riemann zeta-function.

| **m = 3** | **Marichev–Saigo–Maeda fractional integrals and derivatives [31–34]** |
|---|---|
| | $I_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}\left(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}\right)=$ $2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\sum_{n=0}^{\infty}{}_3\Psi_3\left[\begin{array}{ccc}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\ (\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & \chi+\delta-\gamma_1'-\gamma_2\end{array}\middle|-(n+1)e^{\omega}\right]$ |

**Table 1.** *Cont.*

| | |
|---|---|
| ***m* = 3** | **Marichev–Saigo–Maeda fractional integrals and derivatives [31–34]** |
| | $I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}) = 2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\sum_{n=0}^{\infty}{}_3\Psi_3\left[\begin{array}{ccc}(1-\chi-\delta+\gamma_1+\gamma_1',-1) & (1-\chi+\gamma_1+\gamma_2'-\delta,-1) & (1-\chi-\gamma_1,-1)\\ (1-\chi,-1) & (1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,-1) & (1-\chi+\gamma_1-\gamma_2,-1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $D_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}(\omega^{\chi-1}L(\Gamma(s)\zeta(s);\omega)) = 2\pi\omega^{\chi-1}\sum_{n=0}^{\infty}{}_3\Psi_3\left[\begin{array}{ccc}(\chi,1) & (\chi-\gamma_2+\gamma_1,1) & (\chi+\gamma_1+\gamma_1'+\gamma_2'-\delta,1)\\ (\chi-\gamma_2,1) & (\chi-\delta+\gamma_1++\gamma_2',1) & (\chi-\delta+\gamma_1'+\gamma_1,1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $D_{-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}(\omega^{\chi-1}L(\Gamma(s)\zeta(s);\omega)) = 2\pi\omega^{\chi-1}\sum_{n=0}^{\infty}{}_3\Psi_3\left[\begin{array}{ccc}(1-\chi+\gamma_2',1) & (1+\gamma_2'-\chi-\gamma_2+\gamma_1,1) & (1-\chi-\gamma_1-\gamma_1'+\delta,-1)\\ (1-\chi,1) & (1-\chi-\gamma_1'+\gamma_2',1) & (1-\chi+\delta-\gamma_1'-\gamma_1-\gamma_2,-1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| ***m* = 2** | **Saigo fractional integrals and derivatives [31–34]** |
| | $I_{0+}^{\gamma_1,\gamma_2,\delta}(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}) = 2\pi\omega^{\chi-\gamma_1-1}\sum_{n=0}^{\infty}{}_2\Psi_2\left[\begin{array}{cc}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\ (\chi-\gamma_2,1) & (\chi+\delta+\gamma_2)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $I_{-}^{\gamma_1,\gamma_2,\delta}\omega^{\chi-1}(L\{\Gamma(s)\zeta(s);\omega\}) = 2\pi\omega^{\chi-\gamma_1-1}\sum_{n=0}^{\infty}{}_2\Psi_2\left[\begin{array}{cc}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,-1)\\ (1-\chi,1) & ((\gamma_1+\gamma_2+\delta-\chi+1,-1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $D_{0+}^{\gamma_1,\gamma_2,\delta}(t^{\chi-1}L(\Gamma(s)\zeta(s);\omega)) = 2\pi\sum_{n=0}^{\infty}{}_2\Psi_2\left[\begin{array}{cc}(\chi,1) & (\chi+\delta+\gamma_2+\gamma_1,1)\\ (\chi+\gamma_2,1) & (\chi+\delta,1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $D_{-}^{\gamma_1,\gamma_2,\delta}(t^{\chi-1}L(\Gamma(s)\zeta(s);\omega)) = 2\pi\sum_{n=0}^{\infty}{}_2\Psi_2\left[\begin{array}{cc}(1-\chi-\gamma_2,1) & (1-\chi+\delta+\gamma_1,-1)\\ (1-\chi+\delta-\gamma_2,1) & (1-\chi,-1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| ***m* = 1** | **Erdélyi–Kober, Riemann–Liouville (R–L) fractional integrals and derivatives [31–34]** |
| | $I_{0+}^{\gamma,\delta}(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}) = 2\pi\omega^{\chi-1}\sum_{n=0}^{\infty}{}_1\Psi_1\left[\begin{array}{c}(\chi+\gamma,1)\\ (\chi+\gamma+\delta,1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $I_{0-}^{\gamma,\delta}(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}) = 2\pi\omega^{\chi+\delta-1}\sum_{n=0}^{\infty}{}_1\Psi_1\left[\begin{array}{c}(\gamma-\chi+1,-1)\\ (\gamma+\delta-\chi+1,-1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $D_{0+}^{\gamma,\delta}\{\omega^{\chi-1}L(\Gamma(s)\zeta(s);\omega)\} = 2\pi\omega^{\chi-1}\sum_{n=0}^{\infty}{}_1\Psi_1\left[\begin{array}{c}(\gamma+\delta+\chi,1)\\ (\gamma+\chi,1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $D_{-}^{\gamma,\delta}\{\omega^{\chi-1}L(\Gamma(s)\zeta(s);\omega)\} = 2\pi\omega^{\chi-1}\sum_{n=0}^{\infty}{}_1\Psi_1\left[\begin{array}{c}(1-\chi+\gamma+\delta,-1)\\ (1-\chi+\gamma,-1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $I_{+}^{\delta}(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}) = 2\pi\omega^{\chi+\delta-1}\sum_{n=0}^{\infty}{}_1\Psi_1\left[\begin{array}{c}(\chi,1)\\ (\delta+\chi,1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $I_{-}^{\delta}(\omega^{\chi-1}L\{\Gamma(s)\zeta(s);\omega\}) = 2\pi\omega^{\chi+\delta-1}\sum_{n=0}^{\infty}{}_1\Psi_1\left[\begin{array}{c}(1-\delta-\chi,-1)\\ (1-\chi,-1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $D_{0+}^{\delta}\{\omega^{\chi-1}L(\Gamma(s)\zeta(s);\omega)\} = 2\pi\omega^{\chi-1-\delta}\sum_{n=0}^{\infty}{}_1\Psi_1\left[\begin{array}{c}(\chi,1)\\ (\chi-\delta,1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |
| | $D_{-}^{\delta}\{\omega^{\chi-1}L(\Gamma(s)\zeta(s);\omega)\} = 2\pi\omega^{\chi-1-\delta}\sum_{n=0}^{\infty}{}_1\Psi_1\left[\begin{array}{c}(\delta-\chi+1,-1)\\ (1-\chi,-1)\end{array}\Big\vert -(n+1)e^{\omega}\right]$ |

**Remark 2.** *It is mentionable that the succeeding result involving the products of a large class of special functions is because of (26) and (27):*

$$\int_0^1 \frac{\omega^{\rho-1}}{\exp(e^{\omega})-1}\, H_{m,m}^{m,0}\left[\omega\,\middle|\,\begin{array}{c}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{array}\right]d\omega = \omega^{\rho-1}\sum_{n=0}^{\infty}{}_m\Psi_m\left[\begin{array}{c}\left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{array}\,\middle|\,-(n+1)e^{\omega}\right]. \tag{32}$$

**Remark 3.** *Using the principle of mathematical induction for (23), it can be proved that*

$$\sum_{n=0}^{\infty}{}_p\Psi_q\left[\begin{array}{c}(a_i,\ A_i)\\ (b_j,B_j)\end{array};-(n+1)e^{\omega}\right] = \sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}\,{}_p\Psi_q\left[\begin{array}{c}(a_i,\ A_i)\\ (b_j,B_j)\end{array};l\omega\right]. \tag{33}$$

*3.2. Formulation of Fractional Kinetic Equation Involving Riemann Zeta-Function*

The use of non-integer operators has emerged recently in the different disciplines of engineering and the physical sciences [35–40]. For instance, the fractional kinetic equation is important to investigate the theory of gases, aerodynamics, and astrophysics [41–48]. By reviewing the literature, it is found that the fractional kinetic equation comprising the Riemann zeta function is not formulated. The main purpose of this section is to formulate and solve this problem.

The change in the rates of production using subsequent kinetic equations to analyze the reaction and destruction is described in [41]:

$$\frac{d\varepsilon}{dt} = -d(\varepsilon_t) + p(\varepsilon_t), \tag{34}$$

where $\varepsilon_t$ is given by $\varepsilon_t(t^*) = \varepsilon(t - t^*), t^* > 0$. Further to this $\varepsilon = \varepsilon(t)$ = change in reaction, $d = d(\varepsilon)$ = change in destruction, and $p = p(\varepsilon)$ = change in production. The following is obtainable by ignoring the spatial fluctuation and inhomogeneity of $\varepsilon(t)$ with the concentration of species, $\varepsilon_j(t = 0) = \varepsilon_0$:

$$\frac{d\varepsilon_j}{dt} = -c_j\varepsilon_j(t). \tag{35}$$

Next, ignoring subscript $j$ and integrating (35) yields

$$\varepsilon(t) - \varepsilon_0 = -c\ I_{0+}^{-1}\varepsilon(t).$$

The non-integer-order kinetic equation is due to [41]:

$$\varepsilon(t) - \varepsilon_0 = -c^{\delta} I_{0+}^{\delta}\varepsilon(t), \tag{36}$$

where $I_{0+}^{\delta}, \delta > 0$ is the Riemann–Liouville fractional integral, $c$ is a constant, and its Laplace transform is given by

$$L\left\{\ I_{0+}^{\delta}\varepsilon(t);\omega\right\} = \omega^{-\delta}\varepsilon(\omega). \tag{37}$$

Following to Haubold and Mathai [41], we next formulate and solve the fractional kinetic equation so that for any integrable function *f(t)*, we have

$$\varepsilon(t) - f(t)\varepsilon_0 = -d^{\delta}\ I_{0+}^{\delta}\varepsilon(t). \tag{38}$$

In light of this discussion, the fractional kinetic equation involving the Riemann zeta function is formulated and solved in Theorem 2. This becomes possible only due to the Reimann zeta-function's new representation involving the delta function; otherwise, the Laplace transform is not found before the w.r.t variable $s$ (see [49]).

**Theorem 2.** *For $\delta > 0$ , the solution of a given fractional kinetic equation containing the Riemann zeta function is*

$$\varepsilon(t) - \varepsilon_0\Gamma(t)\zeta(t) = -d^{\delta}\ I_{0+}^{\delta}\varepsilon(t) \tag{39}$$

$$\varepsilon(t) = \frac{2\pi\varepsilon_0}{t}\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l\left(\frac{l}{t}\right)^p}{l!p!}E_{\delta,-p}\left(-d^{\delta}t^{\delta}\right). \tag{40}$$

**Proof.** Applying Laplace's transformation to (39) and making use of (25) as well as (37) gives

$$\varepsilon(\omega) = 2\pi\varepsilon_0\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}\ \omega^p - \left(\frac{\omega}{d}\right)^{-\delta}\varepsilon(\omega). \tag{41}$$

Therefore, we have

$$\varepsilon(\omega)\left[1+\left(\frac{\omega}{d}\right)^{-\delta}\right]=2\pi\varepsilon_0\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}\,\omega^p, \tag{42}$$

and

$$\varepsilon(\omega)=2\pi\varepsilon_0\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}\omega^p\sum_{m=0}^{\infty}\left[-\left(\frac{\omega}{d}\right)^{-\delta}\right]^m. \tag{43}$$

By considering $\delta m-p>0;\delta>0$ and using $L^{-1}\{\omega^{-\delta};t\}=\frac{t^{\delta-1}}{\Gamma(\delta)}$, the inverse Laplace transform of (43) is given by

$$\varepsilon(t)=2\pi\varepsilon_0\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}t^{-p-1}\times\sum_{m=0}^{\infty}\frac{\left(-d^\delta t^\delta\right)^m}{\Gamma(\delta m-p)}. \tag{44}$$

Lastly, making use of (19) in the above equation (44) provides the solution as stated in (39) and (40). □

**Remark 4.** *It can be noted that the solution methodology of Theorem 2 is in line with the existing methods [41–48], and, as expected, the reaction rate $\varepsilon(t)$ contains the Mittag–Leffler function governed by the non-integer parameter $\delta$. Furthermore, the sum over the coefficients in (40) is well-defined and can be computed as follows:*

$$C(t)=\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l\left(\frac{l}{t}\right)^p}{l!p!}=\frac{1}{exp\left(e^{\frac{1}{t}}\right)-1}. \tag{45}$$

*Likewise,* $\lim_{t\to\infty}C(t)=\frac{1}{\exp(1)-1}$ *and* $\lim_{t\to 0}C(t)=0.$

### *3.3. Further New Properties of the Riemann Zeta function as a Distribution*

The Dirac delta function is a linear functional, which transforms each function to its value at zero. Hence, using (4), we have

$$\int_{s\epsilon\mathbb{C}}\wp(s)\Gamma(s)\zeta(s)ds=2\pi\sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}\,\delta(s+l),\wp(s)=2\pi\sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}\wp(-l), \tag{46}$$

or, for a real $t$, using (24), we have

$$\int_{t\epsilon\mathbb{R}}\wp(t)\Gamma(t)\zeta(t)dt=2\pi\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}\,\delta^{(p)}(t),\wp(t)=2\pi\sum_{n,l,p=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}\,(-1)^p\wp^{(p)}(0) \tag{47}$$

and from the above equations it follows that the most properties that hold for the delta function will also hold for the Riemann zeta-function. It can be noted that the sum over the co-efficient in (46) and (47) is finite and well-defined, as well as rapidly decreasing. This sum also defines a new transform named the zeta transform, and the following formulae given in Table 2 and many others can be obtained using it.

**Table 2.** Zeta Transform.

| **Function** | **Zeta Transform** |
|---|---|
| $e^{at}$ | $\frac{2\pi}{exp(e^{-a})-1}$ |
| $sinat$ | $IMG\left(\frac{2\pi}{exp(e^{-ia})-1}\right)$ |

**Table 2.** *Cont.*

| Function | Zeta Transform |
|---|---|
| $cosat$ | $Re\left(\frac{2\pi}{exp(e^{-ia})-1}\right)$ |
| $E_\alpha(s)$ | $2\pi\sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}E_\alpha(-l)$ |
| $K_v\ (s)$ [McDonald function [4]] | $2\pi\sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}K_v(-l)$ |
| $H_{r,r}^{r,0}\left[\xi\left\vert\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^r\\\left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^r\end{matrix}\right.\right]$ | $2\pi\sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}H_{r,r}^{r,0}\left[-l\left\vert\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^r\\\left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^r\end{matrix}\right.\right]$ |

The purpose of the remaining section is to enlist the new properties of the Riemann zeta function as a distribution by following the concepts and methodology of [23] (Chapter 7, pp. 199–207), which is achieved due to the zeta function's new representations (4) and (24) in terms of the delta function. First note that the frequently used test functions [22,23] are of either compact support, or they are rapidly decreasing as well as infinitely differentiable. These domains of test functions are commonly denoted by $\mathcal{D}$ and $\mathcal{S}$, respectively, and their codomains are the spaces $\mathcal{D}'$ and $\mathcal{S}'$ (also known as their dual spaces). Actually, $\mathcal{D}$ and $\mathcal{D}\prime$ are not closed w.r.t Fourier transforms, but $\mathcal{S}$ and $\mathcal{S}'$ are closed. Another space of test functions is denoted by $\mathcal{Z}$, which is the space of the analytic and its entire functions. Hence, Fourier transforms of the elements of $\mathcal{D}'$ to belong to $\mathcal{Z}'$, which is dual to $\mathcal{Z}$, and Fourier transforms of the elements of $-\mathcal{Z}$ into $\mathcal{D}$ [22,23]. Therefore, the Fourier transform as well as its inverse are continuous linear functionals from $\mathcal{D}\prime$ to $\mathcal{Z}'$ ([23], p. 203). Since the complex delta function is an element of $\mathcal{Z}'$, from (4), it is therefore obvious that $\Gamma(s)\zeta(s)$ is also an element of $\mathcal{Z}'$. In light of this discussion, the following theorem follows.

**Theorem 3.** *Suppose $f$ is a distribution of bounded support; then,*

$$\mathcal{F}\left[f(y)*\frac{\sqrt{2\pi}e^{\sigma y}}{exp(e^{y}\ )-1};s\right]=\mathcal{F}[f(s)]\Gamma(s)\zeta(s). \tag{48}$$

**Proof.** Because $\Gamma(s)\zeta(s)\in\mathcal{Z}'$ and $\mathcal{F};\mathcal{F}^{-1}$ are continuous linear functionals from $\mathcal{D}'$ to $\mathcal{Z}'$. Further, we have [14], Equation (42):

$$\mathcal{F}\left[\frac{\sqrt{2\pi}e^{\sigma y}}{exp(e^{y})-1};\tau\right]=\Gamma(s)\zeta(s);(s=\sigma+i\tau). \tag{49}$$

Therefore, $\frac{\sqrt{2\pi}e^{\sigma y}}{exp(e^{y}\ )-1}$ is an element of $\mathcal{D}'$ being a Fourier transform of an element of space $\mathcal{Z}'$. Hence, the proof of the result (48) is complete using ([23], p. 206, Theorem 7.9.1). □

**Example 1.** *Consider a function$f$ with bounded support defined by*

$$f(y)=\begin{cases}1 & |y|<1\\ 0 & |y|\geq 1\end{cases} \tag{50}$$

*Then, according to Theorem 3,*

$$\mathcal{F}\left[f(y)*\frac{\sqrt{2\pi}e^{\sigma y}}{exp(e^{y}\ )-1}\right]=\mathcal{F}[f(y);s]\mathcal{F}\left[\frac{\sqrt{2\pi}e^{\sigma y}}{exp(e^{y}\ )-1};s\right]=\frac{sins}{s}\Gamma(s)\zeta(s), \tag{51}$$

*yields a valuable consequence of distributional representation.*

Continuing in this way, we can apply the elements of distributions to the Riemann zeta function using its distributional representation (4) and (24). Some of these are listed below in Table 3, and it is mentionable that the proof of all these properties simply follows from the properties of the delta function and are therefore omitted. Here, we restrict these over the space of complex analytic functions $\wp(s)\epsilon\mathcal{Z}$, as defined in [22,23], but these properties may hold for a large space of test functions, and it is supposed that $c_1$, $\gamma$, and $c_2$ are constants.

**Table 3.** Properties of Riemann Zeta function as a distribution.

| | |
|---|---|
| addition with an arbitrary distribution $f$ | $\langle\Gamma(s)\zeta(s)+f,\ \wp(s)\rangle = \langle\Gamma(s)\zeta(s),\wp(s)+f,\ \wp(s)\rangle$ |
| multiplication with an arbitrary constant $c_1$ | $\langle c_1\Gamma(s)\zeta(s),\wp(s)\rangle = \langle\Gamma(s)\zeta(s),c_1\wp(s)\rangle$ |
| shifting by an arbitrary complex constant $\gamma$ | $\langle\Gamma(s-\gamma)\zeta(s-\gamma),\wp(s)\rangle = \langle\Gamma(s)\zeta(s),\wp(s+\gamma)\rangle$ |
| transposition | $\langle\Gamma(-s)\zeta(-s),\wp(s)\rangle = \langle\Gamma(s)\zeta(s),\wp(-s)\rangle$ |
| multiplication of the independent variable with a positive constant $c_1$ | $\langle\Gamma(c_1 s)\zeta(c_1 s),\wp(s)\rangle = \langle\Gamma(s)\zeta(s),\frac{1}{c_1}\wp\left(\frac{s}{c_1}\right)\rangle$ |
| distributional differentiation | $\langle\frac{d^k}{ds^k}(\Gamma(s)\zeta(s)),\wp(s)\rangle = \sum_{n,l=0}^{\infty}\frac{(-(n+1))^l}{l!}(-1)^k\wp^k(-l)$ |
| distributional Fourier transform | $\langle\mathcal{F}[\Gamma(s)\zeta(s)],\wp(s)\rangle = \langle\Gamma(s)\zeta(s),\mathcal{F}[\wp](s)\rangle$ |
| duality property of Fourier transform | $\langle\mathcal{F}[\Gamma(s)\zeta(s)],\mathcal{F}[\wp(s)]\rangle = \langle 2\pi\Gamma(s)\zeta(s),\wp(-s)\rangle$ |
| Parseval's identity of Fourier transform | $\langle\mathcal{F}[\Gamma(s)\zeta(s)],\overline{\mathcal{F}[\wp(s)]}\rangle = \langle\overline{\mathcal{F}[\Gamma(s)\zeta(s)]},\mathcal{F}[\wp(s)]\rangle =$ $2\pi\langle[\Gamma(\sigma)\zeta(\sigma)],\overline{[\wp(\sigma)]}\rangle;\sigma=\Re(s)$ |
| differentiation property of Fourier transform | $\langle\mathcal{F}\left[\frac{d^k}{ds^k}(\Gamma(s)\zeta(s))\right],\wp(s)\rangle = \langle(-it)^m\Gamma(s)\zeta(s),\mathcal{F}[\wp](s)\rangle$ |
| Taylor series | $\langle\Gamma(s+c_1)\zeta(s+c_1),\wp(s)\rangle = \langle\sum_{n=0}^{\infty}\frac{(c_1)^n}{n!}\frac{d^n}{ds^n}(\Gamma(s)\zeta(s)),\wp(s)\rangle$ |
| Convolution property | $\Gamma(t)\zeta(t)*f(t) = 2\pi\sum_{n,l=0}^{\infty}\frac{(-(n+1))^l(l)^p}{l!p!}\frac{d^p}{dt^p}(f(t))$ |
| $\Gamma(s)\zeta(s)*exp(as)$ | $\frac{2\pi e^{as}}{exp(e^a)-1}$ |

## 4. Conclusions

The calculation of the images of special functions using the fractional calculus operators has emerged as a popular subject. In this research, we have obtained fractional calculus images involving Riemann zeta-functions and their simpler cases. Specifications of these results were discussed for $m=3$, $m=2$, and $m=1$. It is reasonable to verify, in view of (25), that Theorems 3 and 4 of [24] are applicable, and the main result (27) and its several special cases are completely verifiable with these theorems. A new fractional kinetic equation involving the Riemann zeta function was formulated and solved. A newly obtained representation of the Riemann zeta function and its Laplace transform has played a crucial role in accomplishing the goals of this research. Certain distributional properties of the Riemann zeta function and examples were also discussed. We hope that this confluence of distribution theory and the function of analytic number theory will have far-reaching applications in the future.

**Author Contributions:** Conceptualization, A.T.; methodology, A.T.; validation, A.T. and R.S.; formal analysis, A.T. and R.S.; investigation, A.T.; resources, A.T. and R.S.; data curation, R.S.; writing—original draft preparation, A.T.; writing—review and editing, R.S.; visualization, R.S.; supervision, R.S.; project administration, R.S.; funding acquisition, A.T. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** Asifa Tassaddiq would like to thank the Deanship of Scientific Research at Majmaah University for supporting this work under Project No. R-2022-133. The authors are also thankful to the worthy reviewers and editors for their useful and valuable suggestions for the improvement of this paper which led to a better presentation.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A

Related Special Cases to (10)

*Case 1: Marichev–Saigo–Maeda fractional integral operator*

First of all, let us consider the case $m = 3$ and further take $\beta_1 = \beta_2 = \beta_3 = \beta = 1$ in (10). Then, the kernel of (10) will reduce to a special case of the $H$-function $H^{3,0}_{3,3}$ that has the following relation with the Meijer $G$-function $G^{3,0}_{3,3}$–function and the Appel function (Horn function) $F_3$ ([2], Vol. 1):

$$H^{3,0}_{3,3}\left(\frac{t}{x}\right) = G^{3,0}_{3,3}\left[\frac{t}{x}\middle| \begin{matrix} \gamma_1{}' + \gamma_2{}', & \delta - \gamma_1, \delta - \gamma_2 \\ \gamma_1{}', \gamma_2{}', & \delta - \gamma_1 - \gamma_2 \end{matrix}\right] = \frac{x^{-\gamma_1}}{\Gamma(\delta)}(x-t)^{\delta-1} t^{-\gamma_1{}'} F_3\left(\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}', \delta; 1 - \frac{t}{x}; 1 - \frac{x}{t}\right) \quad \text{(A1)}$$

where

$$F_3(\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}', \delta; u; v) = \sum_{k,l=0}^{\infty} \frac{(\gamma_1)_k (\gamma_1{}')_l (\gamma_2)_k (\gamma_2{}')_l}{(\delta)_{l+m}} \frac{u^k}{k!} \frac{v^l}{l!}, max(|u|, |v|) < 1. \quad \text{(A2)}$$

Hence, due to (A1), for the complex parameters $\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}'$, $\Re\,(\delta) > 0$, the Marichev–Saigo–Maeda fractional integral operator of integration (see ([2], Vol. 1) is also [31–34]) defined as

$$\left(I^{\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}', \delta}_{0+} f\right)(x) = \frac{x^{-\gamma_1}}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} t^{-\gamma_1{}'} F_3\left(\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}', \delta; 1 - \frac{t}{x}; 1 - \frac{x}{t}\right) f(t) \mathrm{d}t \quad \text{(A3)}$$

and

$$\left(I^{\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}', \delta}_{0-} f\right)(x) = \frac{t^{-\gamma_1{}'}}{\Gamma(\delta)} \int_x^{\infty} (x-t)^{\delta-1} t^{-\gamma_1} F_3\left(\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}', \delta; 1 - \frac{x}{t}; 1 - \frac{t}{x}\right) f(t) \mathrm{d}t. \quad \text{(A4)}$$

Both of the above forms have significant importance. Furthermore, it is now obvious from (20)–(23) that the Marichev–Saigo–Maeda fractional integral operator is related to the multiple E–K fractional integral operators as given in (10) for $m = 3$. The Marichev–Saigo–Maeda fractional integral operator can also be expressed as a composition of three commutable classical E–K integrals (see Kiryakova [24,25]) as follows:

$$I^{\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}', \delta}_{0+} f(x) = I^{(0, \delta - \gamma_1 - \gamma_1{}' - \gamma_2, \gamma_2{}' - \gamma_1{}'), (\gamma_2{}', \gamma_2, \delta - \gamma_2 - \gamma_2{}')}_{(1,1,1),3} f(x) = I^{(0, \gamma_2{}')}_1 I^{(\delta - \gamma_1 - \gamma_1{}' - \gamma_2, \gamma_2)}_1 I^{(\gamma_2{}' - \gamma_1{}', \delta - \gamma_2 - \gamma_2{}')}_1 f(x) \quad \text{(A5)}$$

Many such representations are found (see Kiryakova [24,25] and cited references) because of the symmetry of variables $\gamma_1, \gamma_1{}'$ and $\gamma_2, \gamma_2{}'$ in $F_3$, as well as the symmetry in the upper and lower rows of the $G$-function in (A-1). Hence, the following result holds true in view of (A1)–(A4) and (10) (see also [31–34]).

Let $\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}' \in \mathbb{C}$, $\omega > 0 \wedge \Re(\chi) > \max\{0, \Re(\gamma_1 + \gamma_1{}' + \gamma_2 - \delta), \Re(\gamma_1{}' - \gamma_2{}')\}$, $\Re(\delta) > 0$, then

$$I^{\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}', \delta}_{0+}\left(\omega^{\chi-1}\right) = \frac{\Gamma(\chi)\Gamma(\chi + \delta - \gamma_1 - \gamma_1{}' - \gamma_2)\Gamma(\chi + \gamma_2{}' - \gamma_1{}')}{\Gamma(\chi + \gamma_2{}')\Gamma(\chi + \delta - \gamma_1 - \gamma_1{}')\Gamma(\chi + \delta - \gamma_1{}' - \gamma_2)} \omega^{\delta + \chi - \gamma_1 - \gamma_1{}' - 1} \quad \text{(A6)}$$

Similarly, let $\gamma_1, \gamma_1', \gamma_2, \gamma_2' \in \mathbb{C}$, $\omega > 0$, and if $\Re(\delta) > 0$, $\Re(\chi) < 1 + \min\{\Re(-\gamma_2), \Re(\gamma_1 + \gamma_1' - \delta), \Re(\gamma_1 + \gamma_2' - \delta)\}$; then, the following image formula holds true in view of (A1)–(A4) and (10) (see also [31–34]):

$$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}\left(\omega^{\chi-1}\right) = \frac{\Gamma(1-\chi-\delta+\gamma_1+\gamma_1')\Gamma(1-\chi+\gamma_1+\gamma_2'-\delta)\Gamma(1-\chi-\gamma_1)}{\Gamma(1-\chi)\Gamma(1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta)\Gamma(1-\chi+\gamma_1-\gamma_2)}\omega^{\delta+\chi-\gamma_1-\gamma_1'-1} \quad \text{(A7)}$$

*Case 2: Saigo fractional operator*

Next, let us consider the case m = 2 with $\beta_1 = \beta_2 = \beta > 0$.; then, the kernel-functions of (10) reduce to the Gauss function [24]:

$$\begin{aligned} H_{2,2}^{2,0}\left[\sigma\middle| \begin{matrix} \left(\gamma_1+\delta_1+1-\frac{1}{\beta},\frac{1}{\beta}\right),\left(\gamma_2+\delta_2+1-\frac{1}{\beta},\frac{1}{\beta}\right) \\ \left(\gamma_1+1-\frac{1}{\beta},\frac{1}{\beta}\right),\left(\gamma_2+1-\frac{1}{\beta},\frac{1}{\beta}\right) \end{matrix}\right] &= G_{2,2}^{2,0}\left[\sigma^{\beta}\middle| \begin{matrix} \gamma_1+\delta_2,\gamma_2+\delta_2 \\ \gamma_1\gamma_2 \end{matrix}\right] \\ &= \beta\,\frac{\sigma^{\beta\gamma_2}\left(1-\sigma^{\beta}\right)^{\delta_1+\delta_2-1}}{\Gamma(\delta_1+\delta_2)}\,{}_2F_1\left(\gamma_2+\delta_2-\gamma_1,\delta_1;\delta_1+\delta_2;1-\sigma^{\beta}\right) \end{aligned} \quad \text{(A8)}$$

For the purpose of this investigation, let us focus on two fractional integral operators that are defined for $\gamma_1, \gamma_2, \delta \in \mathbb{C}$ with $x; \Re(\delta) > 0$ by Saigo [33], which can also be obtained by taking $\beta = 1$; $\sigma = \frac{t}{x}$ and then $\sigma = \frac{x}{t}$, also appropriately specifying the other parameter values $\delta_1 + \delta_2 = \delta; \delta_1 = -\gamma_1$ in (A1) and (A8) (see also [31,32]).

$$I_{0+}^{\gamma_1,\gamma_2,\delta} = \frac{x^{-\delta-\gamma_1}}{\Gamma(\delta)}\int_0^x (x-t)^{\delta-1}{}_2F_1\left(\delta+\gamma_2\,,-\gamma_1;\delta;1-\frac{t}{x}\right)f(t)\mathrm{d}t \quad \text{(A9)}$$

and

$$I_{-}^{\gamma_1,\gamma_2,\delta}\,(f(x)) = \frac{1}{\Gamma(\delta)}\int_x^{\infty}(t-x)^{\delta-1}t^{-\delta-\gamma_1}{}_2F_1\left(\delta+\gamma_2\,,-\gamma_1;\delta;1-\frac{x}{t}\right)f(t)\mathrm{d}t \quad \text{(A10)}$$

where ${}_2F_1$ represents the Gauss hypergeometric function given by (see [34]):

$${}_2F_1(\gamma_1,\ \gamma_2,\gamma_3;u) = \sum_{k=0}^{\infty}\frac{(\gamma_1)_k(\gamma_2)_k}{(\gamma_3)_k}\frac{u^k}{k!}, |u|< 1; |u| = 1(u \neq 1), \Re(\gamma_3-\gamma_1-\gamma_2) >0. \quad \text{(A11)}$$

The Appell function $F_3$ diminishes to ${}_2F_1$ (Gauss hypergeomatric function) and also contends the following relationships (see [34], p. 301, Equation 9.4):

$$\begin{gathered} F_3(\gamma_1,\delta-\gamma_1,\gamma_2,\delta-\gamma_2;\delta;u;v) = {}_2F_1(\gamma_1,\ \gamma_2;\ \delta;u+v-\mathrm{uv}) \\ \text{and} \\ F_3(0,\gamma_1',\gamma_2,\gamma_2',\delta) = {}_2F_1(\gamma_1,\ \gamma_2;\ \ \delta;x); F_3(\gamma_1,0,\gamma_2,\gamma_2',\delta) = {}_2F_1(\gamma_1',\gamma_2',\delta;y) \end{gathered} \quad \text{(A12)}$$

Hence, the relation of the Marichev–Saigo–Maeda (A3) and (A4) and the Saigo fractional integral operators (A9) and (A10) is obvious using (31) for $\gamma_1 = 0 \vee \gamma_1^{'} = 0$, where both equations also interrelated with (10) in view of (A1) and (A8). Hence using these facts for (10) and (A9), we have (see also [31–34]):

$$I_{0+}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}\right) = \frac{\Gamma(\chi)\Gamma(\chi+\gamma_2-\gamma_1)}{\Gamma(\chi-\gamma_2)\Gamma(\chi+\delta+\gamma_2)}\omega^{\chi-\gamma_1-1},(\gamma_1,\ \gamma_2,\ \delta\ \in \mathbb{C}\,; \Re(\delta,) > 0, \Re(\chi) > \max[0, \Re(\gamma_1 -\ \gamma_2)]). \quad \text{(A13)}$$

Similar to (A13), we have the following right-handed formula (see also [31–34]):

$$\begin{gathered} I_{-}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}\right) = \frac{\Gamma(\gamma_1-\chi+1)\Gamma(\gamma_2-\chi+1)}{\Gamma(1-\chi)\Gamma(\gamma_1+\gamma_2+\delta-\chi+1)}\omega^{\chi-\gamma_1-1}; \\ \gamma_1, \gamma_2,\ \delta \in \mathbb{C} \wedge\ \Re(\delta) > 0 \wedge \Re(\chi) < 1 + \min[\Re(\gamma_1), \Re(\gamma_2)]. \end{gathered} \quad \text{(A14)}$$

*Case 3: Erdélyi–Kober (E–K) and the Riemann–Liouville (R–L) fractional operator*

Let us consider $m = 1$ in (10); then, the kernel function of (10) becomes

$$H_{1,0}^{1,1}\left[\sigma\left|\begin{array}{c}\left(\gamma+\delta,\frac{1}{\beta}\right)\\ \left(\gamma,\frac{1}{\beta}\right)\end{array}\right.\right]=\beta\sigma^{\beta-1}G_{1,0}^{1,1}\left[\sigma^{\beta}\left|\begin{array}{c}\gamma+\delta\\ \gamma\end{array}\right.\right]=\beta\,\frac{\sigma^{\beta\gamma+\beta-1}\left(1-\sigma^{\beta}\right)^{\delta-1}}{\Gamma(\delta)} \quad \text{(A15)}$$

and one can obtain the classical fractional operators, namely, the Erdélyi–Kober (E–K) operators:

$$I_{\beta}^{\gamma,\delta}\, f(z)=\frac{1}{\Gamma(\delta)}\int_0^1 \sigma^{\gamma}\,(1-\sigma)^{\delta-1}f\left(z\sigma^{\frac{1}{\beta}}\right)d\sigma,\delta\geq 0,\ \beta>0,\ \gamma\in\mathbb{R}. \quad \text{(A16)}$$

Further, for $\gamma_1 = 0, \gamma_2 = \gamma$, the Saigo operators (A9) and (A10) reduce the other fractional operators, namely, the Erdélyi–Kober integrals defined for complex $\gamma, \delta \in \mathbb{C}$, $\Re(\delta) > 0$, (see also [31–34]):

$$I_{0+}^{0,\gamma,\delta}(\,f(x))=\left(I_{0+}^{\gamma,\delta}\,f\right)(x)=\frac{x^{-\delta-\gamma}}{\Gamma(\chi)}\int_0^x (x-t)^{\delta-1}t^{\gamma}f(t)\mathrm{d}t\;(x>0) \quad \text{(A17)}$$

$$I_{0-}^{0,\gamma,\delta}(\,f(x))=\left(I_{0-}^{\gamma,\delta}\,f\right)(x)=\frac{x^{\gamma}}{\Gamma(\chi)}\int_x^{\infty}(t-x)^{\delta-1}t^{-\delta-\gamma}f(t)\mathrm{d}t\;(x>0) \quad \text{(A18)}$$

It is obvious that (A17) and (A18) are also obtainable from (A16) for specific values of $\beta = 1$; $\sigma = \frac{t}{x}$ and then $\sigma = \frac{x}{t}$. Similarly, the Saigo operators are also related with the E–K and the Riemann–Liouville (R–L) operators:

$$I_{0+}^{0,\gamma,\delta}(\,f(x))=I_{0+}^{\gamma_1,0,\delta}(\,f(x));\,I_{0+}^{\gamma,\delta}(\,f(x))=I_{0-}^{\gamma,\delta}\,(\,f(x));\Re(\delta)>0 \quad \text{(A19)}$$

Continuing in this way, if $\gamma_1 = -\delta$, the Saigo operators (A9) and (A10) reduce to the Riemann–Liouville (R–L) operators (see also [31–34]). The classical left-hand-sided Riemann–Liouville fractional integrals $I_{0+}^{\delta}$ and right-hand-sided Riemann–Liouville fractional integrals $I_{-}^{\delta}$ of order $\delta \in \mathbb{C}$, $\Re(\delta) > 0$ are defined by [7–9]:

$$I_{0+}^{\delta}(f(x))=\frac{1}{\Gamma(\delta)}\int_0^x (x-t)^{\delta-1}f(t)dt \qquad (x>0) \quad \text{(A20)}$$

and

$$I_{-}^{\delta}(f(x))=\frac{1}{\Gamma(\delta)}\int_x^{\infty}(x-t)^{\delta-1}f(t)dt \qquad (x>0) \quad \text{(A21)}$$

respectively. These are also related to the Weyl transform [7–9]. It can be noted that (A20) and (A21) are also obtainable from (A16) for specific values of the involved parameters.

## References


1. Lapidus, M.L.; van Frankenhuijsen, M. *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*; Springer: Berlin/Heidelberg, Germany, 2012.
2. Titchmarsh, E.C. *The Theory of the Riemann Zeta Function*; Oxford University Press: Oxford, UK, 1951.
3. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F. *Higher Transcendental Functions*; McGraw-Hill Book Corp.: New York, NY, USA, 1953; Volume 1–2.
4. Chaudhry, M.A.; Zubair, S.M. *On a Class of Incomplete Gamma Functions with Applications*; Chapman and Hall (CRC Press Company): Boca Raton, FL, USA; London, UK; New York, NY, USA; Washington, DC, USA, 2001.
5. Srivastava, H.M.; Choi, J. *Series Associated with the Zeta and Related Functions*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 2001.
6. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.
7. Srivastava, H.M. An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. *J. Adv. Engrg. Comput.* **2021**, *5*, 135–166. [CrossRef]
8. Srivastava, H.M. Some general families of the Hurwitz-Lerch Zeta functions and their applications: Recent developments and directions for further researches. *Proc. Inst. Math. Mech. Nat. Acad. Sci. Azerbaijan* **2019**, *45*, 234–269. [CrossRef]

9. Srivastava, H.M. The Zeta and related functions: Recent developments. *J. Adv. Engrg. Comput.* **2019**, *3*, 329–354. [CrossRef]
10. Tassaddiq, A. A New Representation for Srivastava's λ-Generalized Hurwitz-Lerch Zeta Functions. *Symmetry* **2018**, *10*, 733. [CrossRef]
11. Tassaddiq, A. A New Representation of the generalized Krätzel function. *Mathematics* **2020**, *8*, 2009. [CrossRef]
12. Chaudhry, M.A.; Qadir, A. Fourier transform and distributional representation of Gamma function leading to some new identities. *Int. J. Math. Math. Sci.* **2004**, *37*, 2091–2096. [CrossRef]
13. Tassaddiq, A.; Qadir, A. Fourier transform and distributional representation of the generalized gamma function with some applications. *Appl. Math. Comput.* **2011**, *218*, 1084–1088. [CrossRef]
14. Tassaddiq, A.; Qadir, A. Fourier transform representation of the extended Fermi-Dirac and Bose-Einstein functions with applications to the family of the zeta and related functions. *Integral Transform. Spec. Funct.* **2018**, *22*, 453–466. [CrossRef]
15. Tassaddiq, A. Some Representations of the Extended Fermi-Dirac and Bose-Einstein Functions with Applications. Ph.D. Dissertation, National University of Sciences and Technology Islamabad, Islamabad, Pakistan, 2011.
16. Al-Lail, M.H.; Qadir, A. Fourier transform representation of the generalized hypergeometric functions with applications to the confluent and gauss hypergeometric functions. *Appl. Math. Comput.* **2015**, *263*, 392–397. [CrossRef]
17. Tassaddiq, A. A New Representation of the Extended Fermi-Dirac and Bose-Einstein Functions. *Int. J. Math. Appl.* **2017**, *5*, 435–446.
18. Tassaddiq, A.; Safdar, R.; Kanwal, T. A distributional representation of gamma function with generalized complex domain. *Adv. Pure Math.* **2017**, *7*, 441–449. [CrossRef]
19. Tassaddiq, A. A new representation of the k-gamma functions. *Mathematics* **2019**, *10*, 133. [CrossRef]
20. Tassaddiq, A. An application of theory of distributions to the family of λ-generalized gamma function. *AIMS Math.* **2020**, *5*, 5839–5858. [CrossRef]
21. Tassaddiq, A. A new representation of the extended k-gamma function with applications. *Math. Meth. Appl. Sci.* **2021**, *44*, 11174–11195. [CrossRef]
22. Gel'fand, I.M.; Shilov, G.E. *Generalized Functions: Properties and Operations*; Academic Press: New York, NY, USA, 1969; Volume 1.
23. Zamanian, A.H. *Distribution Theory and Transform Analysis*; Dover Publications: New York, NY, USA, 1987.
24. Kiryakova, V. Unified Approach to Fractional Calculus Images of Special Functions—A Survey. *Mathematics* **2020**, *8*, 2260. [CrossRef]
25. Kiryakova, V. A Guide to Special Functions in Fractional Calculus. *Mathematics* **2021**, *9*, 106. [CrossRef]
26. Agarwal, R.; Jain, S.; Agarwal, R.P.; Baleanu, D. A remark on the fractional integral operators and the image formulas of generalized Lommel-Wright function. *Front. Phys.* **2018**, *6*, 79. [CrossRef]
27. Kiryakova, V. Commentary: A remark on the fractional integral operators and the image formulas of generalized Lommel-Wright function. *Front. Phys.* **2019**, *7*, 145. [CrossRef]
28. Agarwal, R.; Jain, S.; Agarwal, R.P.; Baleanu, D. Response: Commentary: A remark on the fractional integral operators and the image formulas of generalized Lommel-Wright function. *Front. Phys.* **2020**, *8*, 72. [CrossRef]
29. Kilbas, A.A. *H-Transforms: Theory and Applications*, 1st ed.; CRC Press: Boca Raton, FL, USA, 2004. [CrossRef]
30. Mittag-Leffler, M.G. Sur la nouvelle fonction E(x). *Comptes Rendus Acad. Sci.* **1903**, *137*, 554–558.
31. Marichev, O.I. Volterra equation of Mellin convolutional type with a Horn function in the kernel. *Izv. AN BSSR Ser. Fiz. Mat. Nauk* **1974**, *1*, 128–129. (In Russian)
32. Saigo, M.; Maeda, N. More generalization of fractional calculus. In *Transformation Methods & Special Functions, Varna '96: Second International Workshop: Proceedings*; Rusev, P., Dimovski, I., Kiryakova, V., Eds.; Science Culture Technology Publishing: Singapore, 1998; pp. 386–400.
33. Saigo, M. A remark on integral operators involving the Gauss hypergeometric functions. *Math. Rep. Coll. Gen. Ed. Kyushu Univ.* **1978**, *11*, 135–143.
34. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series (Ellis Horwood Series in Mathematics and Its Applications)*; Halsted Press: Chichester, UK, 1985.
35. Abboubakar, H.; Kom Regonne, R.; Sooppy Nisar, K. Fractional Dynamics of Typhoid Fever Transmission Models with Mass Vaccination Perspectives. *Fractal Fract.* **2021**, *5*, 149. [CrossRef]
36. Yasmin, H. Numerical Analysis of Time-Fractional Whitham-Broer-Kaup Equations with Exponential-Decay Kernel. *Fractal Fract.* **2022**, *6*, 142. [CrossRef]
37. Delgado, B.B.; Macías-Díaz, J.E. On the General Solutions of Some Non-Homogeneous Div-Curl Systems with Riemann–Liouville and Caputo Fractional Derivatives. *Fractal Fract.* **2021**, *5*, 117. [CrossRef]
38. Feng, Z.; Ye, L.; Zhang, Y. On the Fractional Derivative of Dirac Delta Function and Its Application. *Adv. Math. Phys.* **2020**, *2020*, 1842945. [CrossRef]
39. Derbazi, C.; Baitiche, Z.; Abdo, M.S.; Shah, K.; Abdalla, B.; Abdeljawad, T. Extremal Solutions of Generalized Caputo-Type Fractional-Order Boundary Value Problems Using Monotone Iterative Method. *Fractal Fract.* **2022**, *6*, 146. [CrossRef]
40. Srivastava, H.M. Some families of Mittag-Leffler type functions and associated operators of fractional calculus. *TWMS J. Pure Appl. Math.* **2016**, *7*, 123–145.
41. Haubold, H.J.; Mathai, A.M. The fractional kinetic equation and thermonuclear functions. *Astrophys. Space Sci.* **2000**, *273*, 53–63. [CrossRef]

42. Saxena, R.K.; Mathai, A.M.; Haubold, H.J. On fractional kinetic equations. *Astrophys. Space Sci.* **2002**, *282*, 281–287. [CrossRef]
43. Saxena, R.K.; Mathai, A.M.; Haubold, H.J. On generalized fractional kinetic equations. *Phys. A* **2004**, *344*, 653–664. [CrossRef]
44. Saxena, R.K.; Mathai, A.M.; Haubold, H.J. Unified fractional kinetic equations and a fractional diffusion equation. *Astrophys. Space Sci.* **2004**, *290*, 299–310. [CrossRef]
45. Saxena, R.K.; Kalla, S.L. On the solutions of certain fractional kinetic equations. *Appl. Math. Comput.* **2008**, *199*, 504–511. [CrossRef]
46. Chaurasia, V.B.L.; Pandey, S.C. On the new computable solution of the generalized fractional kinetic equations involving the generalized function for the fractional calculus and related functions. *Astrophys. Space Sci.* **2008**, *317*, 213–219. [CrossRef]
47. Chaurasia, V.B.L.; Kumar, D. On the solutions of generalized fractional kinetic equations. *Adv. Stud. Theor. Phys.* **2010**, *4*, 773–780.
48. Chaurasia, V.B.L.; Singh, Y. A novel computable extension of fractional kinetic equations arising in astrophysics. *Int. J. Adv. Appl. Math. Mech.* **2015**, *3*, 1–9.
49. Srivastava, H.M. A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. *Symmetry* **2021**, *13*, 2294. [CrossRef]