fractal and fractional

Article

Existence and U-H Stability Results for Nonlinear Coupled Fractional Differential Equations with Boundary Conditions Involving Riemann–Liouville and Erdélyi–Kober Integrals

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Abstract: The purpose of this article is to discuss the existence, uniqueness, and Ulam–Hyers stability of solutions to a coupled system of fractional differential equations with Erdélyi–Kober and Riemann–Liouville integral boundary conditions. The Banach fixed point theorem is used to prove the uniqueness of solutions, while the Leray–Schauder alternative is used to prove the existence of solutions. Furthermore, we conclude that the solution to the discussed problem is Hyers–Ulam stable. The results are illustrated with examples.

Keywords: coupled system; Erdélyi–Kober integrals; Riemann–Liouville integrals; existence; Ulam–Hyers stability; fixed point

MSC: 26A33; 34A08; 34B10

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Citation: Subramanian, M.; Duraisamy, P.; Kamaleshwari, C.; Unyong, B.; Vadivel, R. Existence and U-H Stability Results for Nonlinear Coupled Fractional Differential Equations with Boundary Conditions Involving Riemann–Liouville and Erdélyi–Kober Integrals. *Fractal Fract.* **2022**, *6*, 266. https://doi.org/10.3390/fractalfract6050266

Academic Editor: Agnieszka B. Malinowska

Received: 13 April 2022
Accepted: 9 May 2022
Published: 13 May 2022

1. Introduction

The mathematical modeling of systems and processes in the fields of astrophysics, chemistry, polymer rheology, chemical physics, aerodynamics, physics, engineering, and scientific disciplines requires differential equations of fractional order. Additionally, fractional differential Equations (FDEs) are an effective tool for describing the inherited properties of diverse materials and processes. As a result, FDEs are becoming increasingly important and popular. See [1–4] and the references therein for more information.

In the realm of differential equations, the study of boundary-value problems (BVPs) for both linear and nonlinear differential equations is a popular area of study with numerous applications in a wide range of fields in both the pure and applied sciences. Recent years have seen a surge in interest in BVPs of fractional order. Thus, the literature on the subject has a variety of results of varying importance, ranging from theoretical to applied aspects. See [5–12] and the references therein for some recent work on the topic.

A large part of the research on fractional-order boundary problems is concerned with integral boundary conditions of the classical, Riemann–Liouville, or Hadamard types. In addition to the aforementioned criteria, the Erdélyi–Kober fractional integral operator is used in another sort of integral boundary condition (introduced by Arthur Erdélyi and Hermann Kober [13] in 1940). These operators are critical in solving single, dual, and triple integral equations with kernels that contain special functions of mathematical physics. Furthermore, the applications of the Erdélyi–Kober fractional integrals have been discussed in [14–17].

Furthermore, the analysis of coupled systems of fractional-order differential equations is crucially significant since systems of this type exist in a wide variety of applications in numerous fields, particularly in the biosciences. We refer the reader to the works [18–27] for the sources referenced therein for more information and examples.

The study of coupled systems with fractional differential equations is particularly important because these systems are used to solve a wide range of real-world problems. Additionally, numerous research studies have investigated coupled systems of fractional differential equations.

Stability analysis is another field of research that has received great attention in the last few decades for fractional differential equations. Various kinds of stability have been investigated in the literature, including Mittag–Leffler, Lyapunov, and others. To our knowledge, the Ulam–Hyers stability of a coupled system of fractional differential equations has been studied very rarely. Ulam and Hyers discovered a novel type of stability called Ulam–Hyers stability [28,29].

This type of research can aid in understanding biochemical processes and fluid motion, as well as semiconductors, population dynamics, heat conduction, and elasticity. Researchers have recently started investigating the coupled fractional BVPs. The authors in [30] discussed the solvability of the following coupled FDEs with integral boundary conditions:

$$\begin{cases} {}^{c}\mathcal{D}^{q}x(t) = f(t, x(t), y(t)), \\ {}^{c}\mathcal{D}^{p}y(t) = h(t, x(t), y(t)), \\ x^{'}(0) = \alpha \int_0^{\xi} x^{'}(s)ds, \;\; x(1) = \beta \int_0^1 g(x^{'}(s))ds, \\ y^{'}(0) = \alpha_1 \int_0^{\theta} y^{'}(s)ds, \;\; y(1) = \beta_1 \int_0^1 g(y^{'}(s))ds, \\ \;\; t \in [0,1], \;\; 1 < q, p \leq 2, \;\; 0 \leq \xi, \theta \leq 1, \end{cases}$$

where ${}^{c}\mathcal{D}^{q}$ and ${}^{c}\mathcal{D}^{p}$ denote the Caputo fractional derivatives (CFDs) of order q, p; f, h: $[0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions; α, β, α_1, and β_1 are real constants. The FDEs with integral and ordinary-fractional flux boundary conditions

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}x(t) = f(t, x(t), y(t)), \\ {}^{c}\mathcal{D}^{\beta}y(t) = h(t, x(t), y(t)), \\ x(0) + x(1) = a \int_0^1 x(s)ds, \;\; x^{'}(0) = b^{c}\mathcal{D}^{\gamma}x(1), \\ y(0) + y(1) = a_1 \int_0^1 y(s)ds, \;\; y^{'}(0) = b_1{}^{c}\mathcal{D}^{\delta}y(1), \\ t \in [0,1], \;\; 1 < \alpha, \beta \leq 2, \;\; 0 < \gamma, \delta \leq 1, \end{cases}$$

was discussed in [31], where ${}^{c}\mathcal{D}^{\alpha}$,${}^{c}\mathcal{D}^{\beta}$, and ${}^{c}\mathcal{D}^{\gamma}$,${}^{c}\mathcal{D}^{\delta}$ denote the CFDs of order $\alpha, \beta, \gamma, \delta$; f, h: $[0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions; and a, b, a_1, and b_1 are real constants. In this article, we investigate a novel class of coupled Caputo FDE boundary value problem:

$$\begin{cases} {}^{c}\mathcal{D}^{\xi}p(\iota) = f(\iota, p(\iota), q(\iota)), & \iota \in [0,T], \quad 1 < \xi \leq 2 \\ {}^{c}\mathcal{D}^{\zeta}q(\iota) = g(\iota, p(\iota), q(\iota)), & \iota \in [0,T], \quad 1 < \zeta \leq 2, \end{cases} \tag{1}$$

supported by integral boundary conditions of the form:

$$\begin{cases} p(0) = 0, \quad \mathcal{I}^{\epsilon}p(\alpha) = \lambda \, \mathcal{J}_{\rho}^{\gamma,\vartheta}q(T) \\ q(0) = 0, \quad \mathcal{I}^{\delta}q(\beta) = \mu \, \mathcal{J}_{\sigma}^{\eta,\omega}p(T), \end{cases} \tag{2}$$

where ${}^{c}\mathcal{D}^{i}$ denotes the Caputo fractional derivatives (CFDs) of order i; $\mathcal{I}^{\epsilon}$; and $\mathcal{I}^{\delta}$ are the Riemann–Liouville fractional integral (RLFI) of order $\epsilon, \delta > 0$; $\mathcal{J}_{\rho}^{\gamma,\vartheta}$, $\mathcal{J}_{\sigma}^{\eta,\omega}$ is the Erdélyi–Kober fractional integral (EKFI) of order $\vartheta, \omega > 0$, $\rho, \sigma > 0$ and $\gamma, \eta \in \mathbb{R}$; $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions; and $\lambda, \mu, \alpha, \beta$ are real constants. The manuscript is structured as follows. Section 2 is dedicated to some elemental concepts of fractional calculus with primitive lemmas to the given problem. The existence, uniqueness, and Ulam–Hyers stability results are based on fixed point theory, and numerical examples are obtained in Section 3.

2. Preliminaries

To begin, let us recall some fundamental definitions and lemmas of fractional calculus.

Definition 1 ([2]). *The RLFI of order $\epsilon > 0$ for a function $g(\iota)$ is defined as*

$$\mathcal{I}^{\epsilon} g(\iota) = \frac{1}{\Gamma(\epsilon)} \int_0^{\iota} (\iota - s)^{\epsilon-1} g(s) ds, \quad \iota > 0,$$

provided that the RHS is point-wise defined on $[0, \infty)$.

Definition 2 ([2]). *The CFD of order $\zeta > 0$ of a function $g : [0, \infty) \to \mathbb{R}$ is defined as*

$${}^{c}\mathcal{D}^{\zeta} g(\iota) = \frac{1}{\Gamma(n-\zeta)} \int_0^{\iota} (\iota - s)^{n-\zeta-1} g^{(n)}(s) ds, \quad n-1 < \zeta < n,$$

where $n = \lceil \zeta \rceil + 1$ and $\lceil \zeta \rceil$ denotes the integral part of the real number.

Definition 3 ([2]). *The EKFI of order $\vartheta > 0$ with $\rho > 0$ and $\gamma \in \mathbb{R}$ of a continuous function $g : (0, \infty) \to \mathbb{R}$ is defined by*

$$\mathcal{J}_{\rho}^{\gamma,\vartheta} g(\iota) = \frac{\rho \iota^{-\rho(\vartheta+\gamma)}}{\Gamma(\vartheta)} \int_0^{\iota} \frac{s^{\rho\gamma+\rho-1}}{(\iota^{\rho} - s^{\rho})^{1-\vartheta}} g(s) ds,$$

provided the RHS is point-wise defined on $\mathbb{R}_+$.

Remark 1. *For $\rho = 1$, the above operator is reduced to the Kober operator*

$$\mathcal{J}_{1}^{\gamma,\vartheta} g(\iota) = \frac{\iota^{-(\vartheta+\gamma)}}{\Gamma(\vartheta)} \int_0^{\iota} \frac{s^{\gamma}}{(\iota - s)^{1-\vartheta}} g(s) ds, \quad \rho, \vartheta > 0,$$

which was introduced for the first time by Kober in [13]. For $\gamma = 0$, the Kober operator is reduced to the RLFI with a power weight:

$$\mathcal{J}_{\rho}^{0,\vartheta} g(\iota) = \frac{\iota^{-\vartheta}}{\Gamma(\vartheta)} \int_0^{\iota} \frac{1}{(\iota - s)^{1-\vartheta}} g(s) ds, \quad \vartheta > 0.$$

Lemma 1 ([13]). *Let $\rho, \vartheta > 0$ and $\gamma, \zeta \in \mathbb{R}$. Then, we have*

$$\mathcal{J}_{\rho}^{\gamma,\vartheta} \iota^{\zeta} = \frac{\iota^{\zeta}\, \Gamma\left(\gamma + \left(\frac{\zeta}{\rho}\right) + 1\right)}{\Gamma\left(\gamma + \left(\frac{\zeta}{\rho}\right) + \vartheta + 1\right)}. \tag{3}$$

Lemma 2 ([13]). *Let $\zeta, r > 0$, and $n = \lceil \zeta \rceil + 1$. Then,*

$$\mathcal{I}^{\zeta}\, \iota^{r-1}(\iota) = \frac{\Gamma(r)}{\Gamma(\zeta + r)}\, \iota^{r+\zeta+1},$$

$${}^{c}\mathcal{D}^{\zeta}\, \iota^{r-1}(\iota) = \frac{\Gamma(r)}{\Gamma(r-\zeta)}\, \iota^{r-\zeta-1}, \tag{4}$$

$$\text{and} \quad {}^{c}\mathcal{D}^{\zeta}\, \iota^{k} = 0, \quad k = 0, 1, \ldots, n-1. \tag{5}$$

Lemma 3 ([2])**.** *For $\zeta > 0$ and $x \in C([0,T],\mathbb{R})$. Then, the FDEs ${}^{c}\mathcal{D}^{\zeta}u(\iota) = 0$ has a unique solution $u(\iota) = d_0 + d_1\,\iota + \ldots + d_{n-1}\,\iota^{n-1}$, and then*

$$\mathcal{I}^{\zeta}\,{}^{c}\mathcal{D}^{\zeta}u(\iota) = u(\iota) + d_0 + d_1\,\iota + \ldots + d_{n-1}\iota^{n-1},$$

where $n-1 < \zeta < n$ and $d_i \in \mathbb{R}, i = 0,1,\ldots,n-1$.

Lemma 4. *Let $f_1,\ g_1 \in C([0,T],\mathbb{R})$. Then, the integral solution for the linear system of FDEs:*

$$\begin{aligned} {}^{c}\mathcal{D}^{\xi}p(\iota) &= f_1(\iota), \quad 1 < \xi \le 2, \\ {}^{c}\mathcal{D}^{\zeta}q(\iota) &= g_1(\iota), \quad 1 < \zeta \le 2, \end{aligned} \tag{6}$$

augmented by the boundary conditions (2) is given by

$$\begin{aligned} p(\iota) &= \mathcal{I}^{\xi}f_1(\iota) + \varpi(\iota)\Big[\mu\kappa_3\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}f_1(T) - \kappa_2\mathcal{I}^{\epsilon+\xi}f_1(\alpha) + \lambda\kappa_2\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}g_1(T) \\ &\quad -\kappa_3\mathcal{I}^{\delta+\zeta}g_1(\beta)\Big], \end{aligned} \tag{7}$$

and

$$\begin{aligned} q(\iota) &= \mathcal{I}^{\zeta}g_1(\iota) + \varpi(\iota)\Big[\lambda\kappa_4\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}g_1(T) - \kappa_1\mathcal{I}^{\delta+\zeta}g_1(\beta) + \mu\kappa_1\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}f_1(T) \\ &\quad -\kappa_4\mathcal{I}^{\epsilon+\xi}f_1(\alpha)\Big], \end{aligned} \tag{8}$$

where

$$\kappa_1 = \frac{\Gamma(3)}{\Gamma(\epsilon+3)}\,\alpha^{\epsilon+2}, \quad \kappa_2 = \frac{\Gamma(3)}{\Gamma(\delta+3)}\,\beta^{\delta+2}, \quad \kappa_3 = \frac{\lambda T\,\Gamma\Big(\gamma + \Big(\frac{1}{\rho}\Big) + 1\Big)}{\Gamma\Big(\gamma + \Big(\frac{1}{\rho}\Big) + \vartheta + 1\Big)} \tag{9}$$

$$\kappa_4 = \frac{\mu T\,\Gamma\Big(\eta + \Big(\frac{1}{\sigma}\Big) + 1\Big)}{\Gamma\Big(\eta + \Big(\frac{1}{\sigma}\Big) + \omega + 1\Big)}, \quad \varpi(\iota) = \frac{\iota}{\kappa_1\kappa_2 - \kappa_3\kappa_4}, \text{ where } \kappa_1\kappa_2 - \kappa_3\kappa_4 \neq 0. \tag{10}$$

Proof. The general solution of the FDEs in (6) is defined as

$$p(\iota) = \mathcal{I}^{\xi}f_1(\iota) + c_1 + c_2\,\iota, \tag{11}$$
$$q(\iota) = \mathcal{I}^{\xi}g_1(\iota) + d_1 + d_2\,\iota. \tag{12}$$

Using the boundary conditions (2) in (11) and (12), we deduce that $c_1 = 0,\ d_1 = 0$. Moreover, we have

$$c_2\,\kappa_1 - d_2\kappa_3 = \lambda\,\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}g_1(T) - \mathcal{I}^{\epsilon+\xi}f_1(\alpha), \tag{13}$$
$$d_2\,\kappa_2 - c_2\kappa_4 = \mu\,\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}f_1(T) - \mathcal{I}^{\delta+\zeta}g_1(\beta). \tag{14}$$

Solving the system (13) and (14) for c_2 and d_2, we find that

$$\begin{aligned} c_2 &= \frac{1}{\kappa_1\kappa_2 - \kappa_3\kappa_4}\Big[\mu\kappa_3\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}f_1(T) - \kappa_2\mathcal{I}^{\epsilon+\xi}f_1(\alpha) + \lambda\kappa_2\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}g_1(T) - \kappa_3\mathcal{I}^{\delta+\zeta}g_1(\beta)\Big] \\ d_2 &= \frac{1}{\kappa_1\kappa_2 - \kappa_3\kappa_4}\Big[\lambda\kappa_4\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}g_1(T) - \kappa_1\mathcal{I}^{\delta+\zeta}g_1(\beta) + \mu\kappa_1\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}f_1(T) - \kappa_4\mathcal{I}^{\epsilon+\xi}f_1(\alpha)\Big], \end{aligned} \tag{15}$$

Substituting the values of c_1, c_2, d_1, d_2 in (11) and (12), we obtain the solution given by (7) and (8). □

3. Main Results

Let us define $\mathcal{P} = \{p(\iota) : p(\iota) \in C([0,T],\mathbb{R})\}$ and $\mathcal{Q} = \{q(\iota) : q(\iota) \in C([0,T],\mathbb{R})\}$ to denote the spaces equipped, respectively, with the norms $\|p\| = \sup_{\iota\in[0,T]} |p(\iota)|$ and $\|q\| = \sup_{\iota\in[0,T]} |q(\iota)|$ as Banach spaces. As a consequence, the product space $(\mathcal{P} \times \mathcal{Q}, \|(p,q)\|)$ is a Banach space endowed with the norm $\|(p,q)\| = \|p\| + \|q\|$ for $(p,q) \in \mathcal{P} \times \mathcal{Q}$. Using Lemma 4, we introduce an operator $\mathcal{H} : \mathcal{P} \times \mathcal{Q} \to \mathcal{P} \times \mathcal{Q}$ connected with (1) and (2) in the problem as follows:

$$\mathcal{H}(p,q)(\iota) = (\mathcal{H}_1(p,q)(\iota),\ \mathcal{H}_2(p,q)(\iota)), \tag{16}$$

where

$$\begin{aligned}\mathcal{H}_1(p,q)(\iota) &= \mathcal{I}^{\xi} f(s,p(s),q(s))(\iota) + \varpi(\iota)\Big[\mu\kappa_3 \mathcal{J}_{\sigma}^{\eta,\omega}\, \mathcal{I}^{\xi} f(s,p(s),q(s))(T) \\ &\quad -\kappa_2 \mathcal{I}^{\epsilon+\xi} f(s,p(s),q(s))(\alpha) + \lambda\kappa_2 \mathcal{J}_{\rho}^{\gamma,\vartheta}\, \mathcal{I}^{\zeta} g(s,p(s),q(s))(T) \\ &\quad -\kappa_3 \mathcal{I}^{\delta+\zeta} g(s,p(s),q(s))(\beta)\Big],\end{aligned} \tag{17}$$

and

$$\begin{aligned}\mathcal{H}_2(p,q)(\iota) &= \mathcal{I}^{\zeta} g(s,p(s),q(s))(\iota) + \varpi(\iota)\Big[\lambda\kappa_4 \mathcal{J}_{\rho}^{\gamma,\vartheta}\, \mathcal{I}^{\zeta} g(s,p(s),q(s))(T) \\ &\quad -\kappa_1 \mathcal{I}^{\delta+\zeta} g(s,p(s),q(s))(\beta) + \mu\kappa_1 \mathcal{J}_{\sigma}^{\eta,\omega}\, \mathcal{I}^{\xi} f(s,p(s),q(s))(T) \\ &\quad -\kappa_4 \mathcal{I}^{\epsilon+\xi} f(s,p(s),q(s))(\alpha)\Big].\end{aligned} \tag{18}$$

Theorem 1 (Leray–Schauder alternative [32])**.** *Let $\mathcal{H} : \mathcal{E} \to \mathcal{E}$ be a completely continuous operator. Let $\Phi(\mathcal{H}) = \{x \in \mathcal{E} : x = \kappa\, \mathcal{H}(x) \text{for some } 0 < \kappa < 1\}$. Then, either the set $\Phi(\mathcal{H})$ is unbounded or $\mathcal{H}$ has at least one fixed point.*

Theorem 2 (Arzela–Ascoli Theorem [32])**.** *A subset $\mathcal{G}$ in $\mathcal{E}([c,d],\mathbb{R})$ is relatively compact if it is uniformly bounded and equicontinuous on $[c,d]$.*

Theorem 3 (Banach Fixed Point Theorem [32])**.** *Let $(\mathcal{U},d)$ be a nonempty complete metric space, let $0 < \nu < 1$, and let $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ be the map such that, for every $u,v \in \mathcal{U}$, the relation $d(\mathcal{T}u,\mathcal{T}v) \le \nu\, d(u,v)$ holds. Then, the operator $\mathcal{T}$ has a unique fixed point u of $\mathcal{T}$ in $\mathcal{U}$.*

Next, we provide our result, which is concerned with the existence of a solution to the problem and is employing the Leray–Schauder alternative. For the sake of computing efficiency, we establish the following notations and hypothesis:

$$\mathcal{A}_1 = \frac{T^{\xi}}{\Gamma(\xi+1)} + \varpi\left[\frac{\mu\,\kappa_3\, T^{\xi}\Gamma\Big(\eta+\Big(\frac{\xi}{\sigma}\Big)+1\Big)}{\Gamma(\xi+1)\Gamma\Big(\eta+\Big(\frac{\xi}{\sigma}\Big)+\omega+1\Big)} + \frac{\kappa_2\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right], \tag{19}$$

$$\mathcal{A}_2 = \varpi\left[\frac{\mu\,\kappa_1\, T^{\xi}\Gamma\Big(\eta+\Big(\frac{\xi}{\sigma}\Big)+1\Big)}{\Gamma(\xi+1)\Gamma\Big(\eta+\Big(\frac{\xi}{\sigma}\Big)+\omega+1\Big)} + \frac{\kappa_4\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right], \tag{20}$$

$$\mathcal{B}_1 = \varpi\left[\frac{\lambda\,\kappa_2\,T^{\zeta}\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+1\right)}{\Gamma(\zeta+1)\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+\vartheta+1\right)}+\frac{\kappa_3\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right] \tag{21}$$

$$\mathcal{B}_2 = \frac{T^{\zeta}}{\Gamma(\zeta+1)}+\varpi\left[\frac{\lambda\,\kappa_4\,T^{\zeta}\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+1\right)}{\Gamma(\zeta+1)\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+\vartheta+1\right)}+\frac{\kappa_1\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right] \tag{22}$$

$$\Lambda = \min\left\{1-(\mathcal{A}_1+\mathcal{A}_2)\varrho_1-(\mathcal{B}_1+\mathcal{B}_2)\hat{\varrho}_1,\ 1-(\mathcal{A}_1+\mathcal{A}_2)\varrho_2-(\mathcal{B}_1+\mathcal{B}_2)\hat{\varrho}_2\right\} \tag{23}$$

Let $f, g : [0,T]\times\mathbb{R}^2\to\mathbb{R}$ be continuous functions.

(F1) $\exists$ non-negative constants $\varrho_i, \hat{\varrho}_i (i=0,1,2)$ such that, for all $p_i\in\mathbb{R}$ $(i=1,2)$, we have

$$\begin{aligned}|f(\iota,p_1,p_2)| &\le \varrho_0+\varrho_1|p_1|+\varrho_2|p_2|,\\ |g(\iota,p_1,p_2)| &\le \hat{\varrho}_0+\hat{\varrho}_1|p_1|+\hat{\varrho}_2|p_2|.\end{aligned}$$

(F2) $\exists$ non-negative constants $K_i>0, L_i>0$ $(i=1,2)$ such that for all $\iota\in[0,T]$ and $p_i, q_i\in\mathbb{R}$ $(i=1,2)$. We have

$$\begin{aligned}|f(\iota,p_1,q_1)-f(\iota,p_2,q_2)| &\le K_1|p_1-q_1|+K_2|p_2-q_2|,\\ |g(\iota,p_1,q_1)-g(\iota,p_2,q_2)| &\le L_1|p_1-q_1|+L_2|p_2-q_2|.\end{aligned} \tag{24}$$

In the following result, we establish the existence of solutions for problems (1) and (2) using the Leray–Schauder alternative [32].

Theorem 4. *Assume that (F1) holds. In addition, let us assume that* $(\mathcal{A}_1+\mathcal{A}_2)\varrho_1-(\mathcal{B}_1+\mathcal{B}_2)\hat{\varrho}_1<1$ *and* $(\mathcal{A}_1+\mathcal{A}_2)\varrho_2-(\mathcal{B}_1+\mathcal{B}_2)\hat{\varrho}_2<1$*. Where* $\mathcal{A}_1,\mathcal{A}_2,\mathcal{B}_1,\mathcal{B}_2$ *are given by (19)–(22), respectively. Then, there exists at least one solution for the BVP (1) and (2) on* $[0,T]$*.*

Proof. In the first step, we will show that the operator $\mathcal{H}:\mathcal{P}\times\mathcal{Q}\to\mathcal{P}\times\mathcal{Q}$ is completely continuous. By continuity of functions f, g, it follows that the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are continuous. As a consequence, the operator $\mathcal{H}$ is continuous. Next, we show that the operator $\mathcal{H}$ is uniformly bounded. Let $\Theta\subset\mathcal{P}\times\mathcal{Q}$ be bounded. Then, there exist positive constants L_f and L_g such that

$$|f(\iota,p(\iota),q(\iota))|\le L_f,\quad |g(\iota,p(\iota),q(\iota))|\le L_g,\quad \forall(p,q)\in\Theta.$$

Then, for any $(p,q)\in\Theta$, we find that

$$\begin{aligned}|\mathcal{H}_1(p,q)(\iota)| &\le \mathcal{I}^{\xi}|f(s,p(s),q(s))|(T)+\varpi(\iota)\Big[\mu\kappa_3\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}|f(s,p(s),q(s))|(T)\\ &\quad+\kappa_2\mathcal{I}^{\epsilon+\xi}|f(s,p(s),q(s))|(\alpha)+\lambda\kappa_2\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}|g(s,p(s),q(s))|(T)\\ &\quad+\kappa_3\mathcal{I}^{\delta+\zeta}|g(s,p(s),q(s))|(\beta)\Big]\\ &\le L_f\left[\frac{T^{\xi}}{\Gamma(\xi+1)}+\varpi\left(\frac{\mu\,\kappa_3\,T^{\xi}\Gamma\left(\eta+\left(\frac{\xi}{\sigma}\right)+1\right)}{\Gamma(\xi+1)\Gamma\left(\eta+\left(\frac{\xi}{\sigma}\right)+\omega+1\right)}+\frac{\kappa_2\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right)\right]\\ &\quad+L_g\left[\varpi\left(\frac{\lambda\,\kappa_2\,T^{\zeta}\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+1\right)}{\Gamma(\zeta+1)\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+\vartheta+1\right)}+\frac{\kappa_3\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right)\right]\\ &\le L_f\,\mathcal{A}_1+L_g\,\mathcal{B}_1.\end{aligned} \tag{25}$$

In this same way, we obtain that

$$
\begin{aligned}
|\mathcal{H}_2(p,q)(\iota)| &\leq L_g\left[\frac{T^{\zeta}}{\Gamma(\zeta+1)}+\varpi\left(\frac{\lambda\,\kappa_4\,T^{\zeta}\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+1\right)}{\Gamma(\zeta+1)\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+\vartheta+1\right)}+\frac{\kappa_1\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right)\right] \\
&\quad +L_f\left[\varpi\left(\frac{\mu\,\kappa_1\,T^{\xi}\Gamma\left(\eta+\left(\frac{\xi}{\sigma}\right)+1\right)}{\Gamma(\xi+1)\Gamma\left(\eta+\left(\frac{\xi}{\sigma}\right)+\omega+1\right)}+\frac{\kappa_4\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right)\right] \\
&\leq L_g\,\mathcal{B}_2+L_f\,\mathcal{A}_2.
\end{aligned}
\tag{26}
$$

We derive from the inequalities (26) and (27) that the variables $\mathcal{H}_1$ and $\mathcal{H}_2$ are uniformly bounded, and thus the operator $\mathcal{H}$ is uniformly bounded as well. Following that, let us demonstrate that $\mathcal{H}$ is equicontinuous. Consider $\iota_1,\iota_2\in[0,T]$ with $\iota_1<\iota_2$. Then, we have

$$
\begin{aligned}
&|\mathcal{H}_1(p,q)(\iota_2)-\mathcal{H}_1(p,q)(\iota_1)| \\
&\leq \frac{1}{\Gamma(\xi)}\int_0^{\iota_1}\left|(\iota_2-\epsilon)^{\xi-1}-(\iota_1-\epsilon)^{\xi-1}\right||f(s,p(s),q(s))|ds+\frac{1}{\Gamma(\xi)}\int_{\iota_1}^{\iota_2}\left|(\iota_2-\iota_1)^{\xi-1}\right| \\
&\quad\times|f(s,p(s),q(s))|ds+|\varpi(\iota_2)-\varpi(\iota_1)|\Big[\mu\kappa_3\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}|f(s,p(s),q(s))|(T) \\
&\quad+\kappa_2\mathcal{I}^{\epsilon+\xi}|f(s,p(s),q(s))|(\alpha)+\lambda\kappa_2\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}|g(s,p(s),q(s))|(T) \\
&\quad+\kappa_3\mathcal{I}^{\delta+\zeta}|g(s,p(s),q(s))|(\beta)\Big], \\
&\leq \frac{L_f}{\Gamma(\xi+1)}\Big[(\iota_2-\iota_1)^{\xi}+(\iota_2^{\xi}-\iota_1^{\xi})\Big]+|\varpi(\iota_2)-\varpi(\iota_1)|\Big[\mu\kappa_3\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}|f(s,p(s),q(s))|(T) \\
&\quad+\kappa_2\mathcal{I}^{\epsilon+\xi}|f(s,p(s),q(s))|(\alpha)+\lambda\kappa_2\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}|g(s,p(s),q(s))|(T) \\
&\quad+\kappa_3\mathcal{I}^{\delta+\zeta}|g(s,p(s),q(s))|(\beta)\Big].
\end{aligned}
$$

Evidently, $|\mathcal{H}_1(p,q)(\iota_2)-\mathcal{H}_1(p,q)(\iota_1)|\to 0$ independent of p,q as $\iota_2\to\iota_1$. Similarly, we can obtain that

$$
\begin{aligned}
&|\mathcal{H}_2(p,q)(\iota_2)-\mathcal{H}_2(p,q)(\iota_1)| \\
&\leq \frac{L_g}{\Gamma(\zeta+1)}\Big[(\iota_2-\iota_1)^{\zeta}+(\iota_2^{\zeta}-\iota_1^{\zeta})\Big]+|\varpi(\iota_2)-\varpi(\iota_1)|\Big[\lambda\kappa_4\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}|g(s,p(s),q(s))|(T) \\
&\quad+\kappa_1\mathcal{I}^{\delta+\zeta}|g(s,p(s),q(s))|(\beta)+\mu\kappa_1\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}|f(s,p(s),q(s))|(T) \\
&\quad+\kappa_4\mathcal{I}^{\epsilon+\xi}|f(s,p(s),q(s))|(\alpha)\Big],
\end{aligned}
$$

which implies that $|\mathcal{H}_2(p,q)(\iota_2)-\mathcal{H}_2(p,q)(\iota_1)|\to 0$ independent of p,q as $\iota_2\to\iota_1$. As a result of the equicontinuity of $\mathcal{H}_1$ and $\mathcal{H}_2$, the operator $\mathcal{H}$ is equicontinuous as well. As a result of the Arzela–Ascoli theorem [32], we can conclude that the operator $\mathcal{H}$ is completely continuous.

Next, it will be proven that the set $\mathcal{F}=\{(p,q)\in\mathcal{P}\times\mathcal{Q}\,|(p,q)=\iota\,\mathcal{H}(p,q),\ 0<\iota<1\}$ is bounded. Let us define $(p,q)\in\mathcal{F}$, then $(p,q)=\iota\mathcal{H}(p,q)$. For any $\iota\in[0,T]$, we have

$$
p(\iota)=\iota\mathcal{H}_1(p,q)(\iota),\quad q(\iota)=\iota\mathcal{H}_2(p,q)(\iota).
$$

By using our assumption, we find that

$$
\begin{aligned}
|p(\iota)| \;&\leq\; \mathcal{I}^{\xi}|f(s,p(s),q(s))|(T)+\varpi(\iota)\Big[\mu\kappa_3\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}|f(s,p(s),q(s))|(T)\\
&\quad+\kappa_2\mathcal{I}^{\epsilon+\xi}|f(s,p(s),q(s))|(\alpha)+\lambda\kappa_2\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}|g(s,p(s),q(s))|(T)\\
&\quad+\kappa_3\mathcal{I}^{\delta+\zeta}|g(s,p(s),q(s))|(\beta)\Big],\\
&\leq\; (\varrho_0+\varrho_1\|p\|+\varrho_2\|q\|)\times\left[\frac{T^{\xi}}{\Gamma(\xi+1)}+\varpi\left(\frac{\mu\,\kappa_3\,T^{\xi}\Gamma\Big(\eta+\Big(\frac{\xi}{\sigma}\Big)+1\Big)}{\Gamma(\xi+1)\Gamma\Big(\eta+\Big(\frac{\xi}{\sigma}\Big)+\omega+1\Big)}\right.\right.\\
&\quad\left.\left.+\frac{\kappa_2\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right)\right]+(\hat{\varrho}_0+\hat{\varrho}_1\|p\|+\hat{\varrho}_2\|q\|)\\
&\quad\times\left[\varpi\left(\frac{\lambda\,\kappa_2\,T^{\zeta}\Gamma\Big(\gamma+\Big(\frac{\zeta}{\rho}\Big)+1\Big)}{\Gamma(\zeta+1)\Gamma\Big(\gamma+\Big(\frac{\zeta}{\rho}\Big)+\vartheta+1\Big)}+\frac{\kappa_3\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right)\right]\\
&\leq\; (\varrho_0+\varrho_1\|p\|+\varrho_2\|q\|)\,\mathcal{A}_1+(\hat{\varrho}_0+\hat{\varrho}_1\|p\|+\hat{\varrho}_2\|q\|)\,\mathcal{B}_1.
\end{aligned}
\tag{27}
$$

In similar way, we have

$$
\begin{aligned}
|q(\iota)| \;&\leq\; (\hat{\varrho}_0+\hat{\varrho}_1\|p\|+\hat{\varrho}_2\|q\|)\times\left[\frac{T^{\zeta}}{\Gamma(\zeta+1)}+\varpi\left(\frac{\lambda\,\kappa_4\,T^{\zeta}\Gamma\Big(\gamma+\Big(\frac{\zeta}{\rho}\Big)+1\Big)}{\Gamma(\zeta+1)\Gamma\Big(\gamma+\Big(\frac{\zeta}{\rho}\Big)+\vartheta+1\Big)}\right.\right.\\
&\quad\left.\left.+\frac{\kappa_1\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right)\right]+(\varrho_0+\varrho_1\|p\|+\varrho_2\|q\|)\\
&\quad\times\left[\varpi\left(\frac{\mu\,\kappa_1\,T^{\xi}\Gamma\Big(\eta+\Big(\frac{\xi}{\sigma}\Big)+1\Big)}{\Gamma(\xi+1)\Gamma\Big(\eta+\Big(\frac{\xi}{\sigma}\Big)+\omega+1\Big)}+\frac{\kappa_4\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right)\right]\\
&\leq\; (\hat{\varrho}_0+\hat{\varrho}_1\|p\|+\hat{\varrho}_2\|q\|)\,\mathcal{B}_2+(\varrho_0+\varrho_1\|p\|+\varrho_2\|q\|)\,\mathcal{A}_2.
\end{aligned}
\tag{28}
$$

From (27) and (29), together with the notation (19)–(23), we deduce that

$$
\begin{aligned}
\|p\|+\|q\| \;&\leq\; (\mathcal{A}_1+\mathcal{A}_2)\varrho_0+(\mathcal{B}_1+\mathcal{B}_2)\hat{\varrho}_0+\Big[(\mathcal{A}_1+\mathcal{A}_2)\varrho_1+(\mathcal{B}_1+\mathcal{B}_2)\hat{\varrho}_1\Big]\|p\|\\
&\quad+\Big[(\mathcal{A}_1+\mathcal{A}_2)\varrho_2+(\mathcal{B}_1+\mathcal{B}_2)\hat{\varrho}_2\Big]\|q\|,
\end{aligned}
$$

which yields $\|(p,q)\|\leq\dfrac{(\mathcal{A}_1+\mathcal{A}_2)\varrho_0+(\mathcal{B}_1+\mathcal{B}_2)\hat{\varrho}_0}{\Lambda}$. This shows that the set $\mathcal{F}$ is bounded. Thus, the operator $\mathcal{H}$ has at least one fixed point with the Leray–Schauder alternative [32]. Hence, the BVP (1) and (2) has at least one solution on $[0,T]$. □

Example 1. *Consider the following coupled system of fractional differential equations*

$$
\begin{aligned}
\mathcal{D}^{\frac{11}{6}}p(\iota) &= \frac{3}{\iota+1}+\frac{15}{200}\cos|p(\iota)|+\frac{|q(\iota)|}{25\,(1+|q(\iota)|)},\\
\mathcal{D}^{\frac{13}{8}}q(\iota) &= \frac{3t}{\iota+6}+\frac{4}{97}\frac{|p(\iota)|}{(1+|p(\iota)|)}+\frac{5}{60}\tan^{-1}|q(\iota)|,
\end{aligned}
\tag{29}
$$

augmented by boundary conditions:

$$\begin{aligned} p(0) = 0, \quad \mathcal{I}^{\frac{3}{5}}\, p\left(\frac{18}{15}\right) &= \mathcal{J}^{\frac{2}{7},\,\frac{11}{13}}_{\frac{\sqrt{3}}{4}}\, q(1), \\ q(0) = 0, \quad \mathcal{I}^{\frac{9}{18}}\, q\left(\frac{10}{7}\right) &= \mathcal{J}^{\frac{\sqrt{2}}{5},\,\frac{\sqrt{3}}{7}}_{\frac{\sqrt{5}}{9}}\, p(1). \end{aligned} \tag{30}$$

Here, $\xi = \frac{11}{6}$, $\zeta = \frac{13}{8}$, $\epsilon = \frac{3}{5}$, $\delta = \frac{9}{18}$, $\alpha = \frac{18}{15}$, $\beta = \frac{10}{7}$, $\lambda = 1$, $\mu = 1$, $\gamma = \frac{2}{7}$, $\vartheta = \frac{11}{13}$, $\rho = \frac{\sqrt{3}}{4}$, $\eta = \frac{\sqrt{2}}{5}$, $\omega = \frac{\sqrt{3}}{7}$, $\sigma = \frac{\sqrt{5}}{9}$, *and it is clear that*

$$\begin{aligned} |f(\iota, p(\iota), q(\iota))| &= \frac{3}{\iota+1} + \frac{15}{200}\cos|p(\iota)| + \frac{|q(\iota)|}{25\,(1+|q(\iota)|)}, \\ |g(\iota, p(\iota), q(\iota))| &= \frac{3t}{\iota+6} + \frac{4}{97}\,\frac{|p(\iota)|}{(1+|p(\iota)|)} + \frac{5}{60}\tan^{-1}|q(\iota)|. \end{aligned}$$

The functions f and g satisfy the condition with $\varrho_0 = \frac{3}{2}$, $\varrho_1 = \frac{15}{200}$, $\varrho_1 = \frac{1}{25}$, $\hat{\varrho}_0 = \frac{3}{7}$, $\hat{\varrho}_1 = \frac{4}{97}$, $\hat{\varrho}_1 = \frac{5}{60}$, $\varpi = 1.04118$, $\mathcal{A}_1 = 1.48920$, $\mathcal{A}_2 = 0.71859$, $\mathcal{B}_1 = 0.60859$, *and* $\mathcal{B}_2 = 1.66609$. *We find that* $\Lambda = \min\{1 - (\mathcal{A}_1 + \mathcal{A}_2)\varrho_1 - (\mathcal{B}_1 + \mathcal{B}_2)\hat{\varrho}_1,\ 1 - (\mathcal{A}_1 + \mathcal{A}_2)\varrho_2 - (\mathcal{B}_1 + \mathcal{B}_2)\hat{\varrho}_2\} \cong 0.72971 < 1$. *Clearly, all the conditions of Theorem 4 are satisfied, and the BVP (29) and (30) has a solution on* $[0,1]$.

In the following result, we establish the uniqueness of solutions for problems (1) and (2) using the Banach Fixed Point Theorem [32].

In the sequel, we use the notations:

$$\psi_1 = \mathcal{N}_1(\mathcal{A}_1 + \mathcal{A}_2), \qquad \psi_2 = \mathcal{N}_2(\mathcal{B}_1 + \mathcal{B}_2), \tag{31}$$

$$\phi_1 = K_1\mathcal{A}_1 + K_2\mathcal{A}_1 + L_1\mathcal{B}_1 + L_2\mathcal{B}_1, \quad \phi_2 = K_1\mathcal{A}_2 + K_2\mathcal{A}_2 + L_1\mathcal{B}_2 + L_2\mathcal{B}_2. \tag{32}$$

Theorem 5. *Assume that (F2) holds. Further, we suppose that* $(\phi_1 + \phi_2) < 1$, *where* ϕ_1 *and* ϕ_2 *are given by (31). Then, there exists a unique solution for the BVP (1) and (2) on* $[0, T]$.

Proof. Let us define $\sup_{\iota\in[0,T]} |f(\iota, 0, 0| \le \mathcal{N}_1 < \infty$ and $\sup_{\iota\in[0,T]} |g(\iota, 0, 0| \le \mathcal{N}_2 < \infty$ such that $\hat{\rho} \ge (\psi_1 + \psi_2)\Big[1 - (\phi_1 + \phi_2)\Big]^{-1}$, where ψ_1 and ψ_2 are given by (31). Then, we will prove that $\mathcal{H}B_{\hat{\rho}} \subset B_{\hat{\rho}}$, where $B_{\hat{\rho}} = \{(p,q) \in \mathcal{P} \times \mathcal{Q} : \|(p,q)\| \le \hat{\rho}\}$, and the operator $\mathcal{H}$ is defined by (16). For $(p,q) \in B_{\hat{\rho}}$, we have

$$\begin{aligned}
|\mathcal{H}_1(p,q)(\iota)| &\le \mathcal{I}^{\xi}\big(|f(s,p(s),q(s))-f(s,0,0)|+|f(s,0,0)|\big)(T)+\varpi(\iota)\\
&\quad\times\Big[\mu\kappa_3\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}\big(|f(s,p(s),q(s))-f(s,0,0)|+|f(s,0,0)|\big)(T)\\
&\quad+\kappa_2\mathcal{I}^{\epsilon+\xi}\big(|f(s,p(s),q(s))-f(s,0,0)|+|f(s,0,0)|\big)(\alpha)\\
&\quad+\lambda\kappa_2\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}\big(|g(s,p(s),q(s))-g(s,0,0)|+|g(s,0,0)|\big)(T)\\
&\quad+\kappa_3\mathcal{I}^{\delta+\zeta}\big(|g(s,p(s),q(s))-g(s,0,0)|+|g(s,0,0)|\big)(\beta)\Big]\\
&\le (K_1\|p\|+K_2\|q\|+N_1)\times\left[\frac{T^{\xi}}{\Gamma(\xi+1)}+\varpi\left(\frac{\mu\,\kappa_3\,T^{\xi}\Gamma(\eta+(\frac{\xi}{\sigma})+1)}{\Gamma(\xi+1)\Gamma(\eta+(\frac{\xi}{\sigma})+\omega+1)}\right.\right.\\
&\quad\left.\left.+\frac{\kappa_2\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right)\right]+(L_1\|p\|+L_2\|q\|+N_2)\\
&\quad\times\left[\varpi\left(\frac{\lambda\,\kappa_2\,T^{\zeta}\Gamma(\gamma+(\frac{\zeta}{\rho})+1)}{\Gamma(\zeta+1)\Gamma(\gamma+(\frac{\zeta}{\rho})+\vartheta+1)}+\frac{\kappa_3\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right)\right]\\
&\le (K_1\|p\|+K_2\|q\|+N_1)\,\mathcal{A}_1+(L_1\|p\|+L_2\|q\|+N_2)\,\mathcal{B}_1. \qquad (33)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|\mathcal{H}_2(p,q)(\iota)| &\le (L_1\|p\|+L_2\|q\|+N_2)\times\left[\frac{T^{\zeta}}{\Gamma(\zeta+1)}+\varpi\left(\frac{\lambda\,\kappa_4\,T^{\zeta}\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+1\right)}{\Gamma(\zeta+1)\Gamma\left(\gamma+\left(\frac{\zeta}{\rho}\right)+\vartheta+1\right)}\right.\right.\\
&\quad\left.\left.+\frac{\kappa_1\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right)\right]+(K_1\|p\|+K_2\|q\|+N_1) \qquad (34)\\
&\quad\times\left[\varpi\left(\frac{\mu\,\kappa_1\,T^{\xi}\Gamma\left(\eta+\left(\frac{\xi}{\sigma}\right)+1\right)}{\Gamma(\xi+1)\Gamma\left(\eta+\left(\frac{\xi}{\sigma}\right)+\omega+1\right)}+\frac{\kappa_4\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right)\right]\\
&\le (L_1\|p\|+L_2\|q\|+N_2)\,\mathcal{B}_2+(K_1\|p\|+K_2\|q\|+N_1)\,\mathcal{A}_2.
\end{aligned}$$

Thus, it follows from (33) and (35) that $\|\mathcal{H}(p,q)\|\le\hat{\rho}$, which implies $\mathcal{H}B_{\hat{\rho}}\subset B_{\hat{\rho}}$.

Let us show that the operator $\mathcal{H}$ is a contraction. For $p_i,q_i\in B_{\hat{\rho}}$ and for any $\iota\in[0,T]$, by virtue of the condition $(F2)$, we obtain

$$
\begin{aligned}
&|\mathcal{H}_1(p_1,q_1)(\iota)-\mathcal{H}_1(p_2,q_2)(\iota)| \\
&\quad\leq \mathcal{I}^{\xi}|f(s,p_1(s),q_1(s))-f(s,p_2(s),q_2(s))|(T)+\varpi(\iota) \\
&\qquad\times\Big[\mu\kappa_3\mathcal{J}_{\sigma}^{\eta,\omega}\,\mathcal{I}^{\xi}|f(s,p_1(s),q_1(s))-f(s,p_2(s),q_2(s))|(T) \\
&\qquad+\kappa_2\mathcal{I}^{\epsilon+\xi}|f(s,p_1(s),q_1(s))-f(s,p_2(s),q_2(s))|(\alpha) \\
&\qquad+\lambda\kappa_2\mathcal{J}_{\rho}^{\gamma,\vartheta}\,\mathcal{I}^{\zeta}|g(s,p_1(s),q_1(s))-g(s,p_2(s),q_2(s))|(T) \\
&\qquad+\kappa_3\mathcal{I}^{\delta+\zeta}|g(s,p_1(s),q_1(s))-g(s,p_2(s),q_2(s))|(\beta)\Big], \qquad (35)\\
&\quad\leq (K_1\|p_1-p_2\|+K_2\|q_1-q_2\|)\times\left[\frac{T^{\xi}}{\Gamma(\xi+1)}+\varpi\left(\frac{\mu\,\kappa_3\,T^{\xi}\Gamma(\eta+(\frac{\xi}{\sigma})+1)}{\Gamma(\xi+1)\Gamma(\eta+(\frac{\xi}{\sigma})+\omega+1)}\right.\right. \\
&\qquad\left.\left.+\frac{\kappa_2\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right)\right]+(L_1\|p_1-p_2\|+L_2\|q_1-q_2\|) \\
&\qquad\times\left[\varpi\left(\frac{\lambda\,\kappa_2\,T^{\zeta}\Gamma(\gamma+(\frac{\zeta}{\rho})+1)}{\Gamma(\zeta+1)\Gamma(\gamma+(\frac{\zeta}{\rho})+\vartheta+1)}+\frac{\kappa_3\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right)\right] \\
&\quad\leq (K_1\|p_1-p_2\|+K_2\|q_1-q_2\|)\,\mathcal{A}_1+(L_1\|p_1-p_2\|+L_2\|q_1-q_2\|)\,\mathcal{B}_1.
\end{aligned}
$$

In similar way, we can find that

$$
\begin{aligned}
&|\mathcal{H}_2(p_1,q_1)(\iota)-\mathcal{H}_2(p_2,q_2)(\iota)| \\
&\quad\leq (L_1\|p_1-p_2\|+L_2\|q_1-q_2\|)\mathcal{B}_2+(K_1\|p_1-p_2\|+K_2\|q_1-q_2\|)\mathcal{A}_2. \qquad (36)
\end{aligned}
$$

Consequently, it follows from (36) and (36) that

$$\|\mathcal{H}(p_1,q_1)(\iota)-\mathcal{H}(p_2,q_2)(\iota)\| \leq (\phi_1+\phi_2)(\|p_1-p_2\|+\|q_1-q_2\|).$$

By the assumption $(\phi_1+\phi_2)<1$, it follows that the operator $\mathcal{H}$ is a contraction. Hence, by the Banach fixed point theorem [32], the operator $\mathcal{H}$ has a unique fixed point, which corresponds to a unique solution of problems (1) and (2) on $[0,T]$. □

Example 2. *Consider the following coupled system of fractional differential equations*

$$
\begin{aligned}
\mathcal{D}^{\frac{15}{10}}p(\iota) &= \frac{3}{2}+\frac{2\,|p(\iota)|}{45\,(1+|p(\iota)|)}+\frac{1}{30}\cos|q(\iota)|, \\
\mathcal{D}^{\frac{12}{7}}q(\iota) &= \frac{1}{5}+\frac{5}{60}\sin|p(\iota)|+\frac{3}{75}\cos|q(\iota)|, \qquad (37)
\end{aligned}
$$

equipped with the integral boundary conditions:

$$
\begin{aligned}
p(0)=0,\quad &\mathcal{I}^{\frac{2}{5}}\,p\left(\frac{28}{25}\right)=2\,\mathcal{J}_{\frac{2}{3}}^{\sqrt{2},\,\frac{5}{3}}\,q(\pi), \\
q(0)=0,\quad &\mathcal{I}^{\frac{5}{8}}\,q(1)=\frac{1}{5}\,\mathcal{J}_{\frac{4}{3}}^{\frac{7}{3},\,\frac{4}{5}}\,p(\pi). \qquad (38)
\end{aligned}
$$

Here, $\xi=\frac{15}{10}$, $\zeta=\frac{12}{7}$, $\epsilon=\frac{2}{5}$, $\delta=\frac{5}{8}$, $\alpha=\frac{28}{25}$, $\beta=1$, $\lambda=2$, $\mu=\frac{1}{5}$, $\gamma=\sqrt{2}$, $\vartheta=\frac{5}{3}$, $\rho=\frac{2}{3}$, $\eta=\frac{7}{3}$, $\omega=\frac{4}{5}$, *and* $\sigma=\frac{4}{3}$. *Clearly,*

$$|f(\iota, p(\iota), q(\iota))| = \frac{3}{2} + \frac{2\,|p(\iota)|}{45\,(1+|p(\iota)|)} + \frac{1}{30}\cos|q(\iota)|,$$
$$|g(\iota, p(\iota), q(\iota))| = \frac{1}{5} + \frac{5}{60}\,\sin|p(\iota)| + \frac{3}{75}\,\cos|q(\iota)|.$$

The functions f and g satisfy the condition with $K_1 = \frac{2}{45}$, $K_2 = \frac{1}{30}$, $Ł_1 = \frac{12}{180}$, *and* $Ł_2 = \frac{3}{75}$. *Using the given data, we find that* $\kappa_1 = 0.63148$, $\kappa_2 = 0.53807$, $\kappa_3 = 0.56348$, $\kappa_4 = 0.20711$, $\varpi = 0.22307$, $\mathcal{A}_1 = 4.29968$, $\mathcal{A}_2 = 0.06635$, $\mathcal{B}_1 = 0.11426$, $\mathcal{B}_2 = 4.61069$, *and* $(\phi_1 + \phi_2) \cong 0.82499 < 1$. *Thus, all the conditions of Theorem 5 are satisfied, and there exists a unique solution of BVP (37) and (38) on* $[0, \pi]$.

In the following result, we demonstrate the stability of BVP solutions for Ulam–Hyers (1) and (2) by its integral solution with the provision that

$$p(\iota) = \mathcal{H}_1(p, q)(\iota), \qquad q(\iota) = \mathcal{H}_2(p, q)(\iota). \tag{39}$$

Define the following operators $\mathcal{S}_1, \mathcal{S}_2 \in C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$;

$$\begin{aligned} \mathcal{D}^{\xi} p(\iota) - f(\iota, p(\iota), q(\iota)) &= \mathcal{S}_1(p, q)(\iota), \quad \iota \in [0, T], \\ \mathcal{D}^{\zeta} q(\iota) - g(\iota, p(\iota), q(\iota)) &= \mathcal{S}_2(p, q)(\iota), \quad \iota \in [0, T]. \end{aligned}$$

For some $\mu_1, \mu_2 > 0$, the following inequalities are examined:

$$\|\mathcal{S}_1(p, q)\| \le \iota_1, \qquad \|\mathcal{S}_2(p, q)\| \le \iota_2. \tag{40}$$

Definition 4. *The BVP (1) and (2) is Ulam–Hyers stable if there exist real numbers* $\mathcal{R}_i > 0$ $(i = 1, 2)$ *such that, for each* $\iota_i > 0$ $(i = 1, 2)$ *and for each solution* $(p^*, q^*) \in C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R})$ *of inequalities, there exists a solution* $(p, q) \in C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R})$ *of (1) and (2) with* $\|(p, q) - (p^*, q^*)\| \le \mathcal{R}_1 \iota_1 + \mathcal{R}_2 \iota_2$.

Theorem 6. *Assume that* $(F2)$ *holds. Then, the BVP (1) and (2) is stable.*

Proof. Let us define (p, q) as the solution satisfying (17) and (18). Let (p^*, q^*) be any solution satisfying (40).

$$\begin{aligned} \mathcal{D}^{\xi} p(\iota) &= f(\iota, p(\iota), q(\iota)) + \mathcal{S}_1(p, q)(\iota), \quad \iota \in [0, T], \\ \mathcal{D}^{\zeta} q(\iota) &= g(\iota, p(\iota), q(\iota)) + \mathcal{S}_2(p, q)(\iota), \quad \iota \in [0, T]. \end{aligned}$$

Thus,

$$\begin{aligned} p^*(\iota) = {} & \mathcal{H}_1(p^*, q^*)(\iota) + \mathcal{I}^{\xi}\mathcal{S}_1(p^*, q^*)(\iota) + \varpi(\iota)\Big[\mu\kappa_3 \mathcal{J}_{\sigma}^{\eta,\omega}\, \mathcal{I}^{\xi}\mathcal{S}_1(p^*, q^*)(T) \\ & -\kappa_2 \mathcal{I}^{\epsilon+\xi}\mathcal{S}_1(p^*, q^*)(\alpha) + \lambda\kappa_2 \mathcal{J}_{\rho}^{\gamma,\vartheta}\, \mathcal{I}^{\zeta}\mathcal{S}_2(p^*, q^*)(T) - \kappa_3 \mathcal{I}^{\delta+\zeta}\mathcal{S}_2(p^*, q^*)(\beta)\Big] \end{aligned}$$

As a consequence,

$$\begin{aligned} |\mathcal{H}_1(p^*, q^*)(\iota) - p^*(\iota)| \le {} & \iota_1\left[\frac{T^{\xi}}{\Gamma(\xi+1)} + \varpi\left(\frac{\mu\,\kappa_3\,T^{\xi}\Gamma\left(\eta + \left(\frac{\xi}{\sigma}\right) + 1\right)}{\Gamma(\xi+1)\Gamma\left(\eta + \left(\frac{\xi}{\sigma}\right) + \omega + 1\right)} + \frac{\kappa_2\,\alpha^{\epsilon+\xi}}{\Gamma(\epsilon+\xi+1)}\right)\right] \\ & + \iota_2\left[\varpi\left(\frac{\lambda\,\kappa_2\,T^{\zeta}\Gamma\left(\gamma + \left(\frac{\zeta}{\rho}\right) + 1\right)}{\Gamma(\zeta+1)\Gamma\left(\gamma + \left(\frac{\zeta}{\rho}\right) + \vartheta + 1\right)} + \frac{\kappa_3\,\beta^{\delta+\zeta}}{\Gamma(\delta+\zeta+1)}\right)\right] \\ \le {} & \iota_1\,\mathcal{A}_1 + \iota_2\,\mathcal{B}_1. \end{aligned}$$

In a similar way, we can deduce that

$$|\mathcal{H}_2(p^*,q^*)(\iota) - q^*(\iota)| \leq \iota_2\,\mathcal{B}_2 + \iota_1\,\mathcal{A}_2.$$

By using the fixed-point property, we obtain

$$\begin{aligned} |p(\iota) - p^*(\iota)| &\leq |p(\iota) - \mathcal{H}_1(p^*,q^*)(\iota)| + \mathcal{H}_1|(p^*,q^*)(\iota) - p^*(\iota)| \\ &\leq K_1\mathcal{A}_1 + K_2\mathcal{A}_1 + L_1\mathcal{B}_1 + L_2\mathcal{B}_1 + \iota_1\,\mathcal{A}_1 + \iota_2\,\mathcal{B}_1. \end{aligned} \tag{41}$$

Analogously, we can obtain

$$|q(\iota) - q^*(\iota)| \leq K_1\mathcal{A}_2 + K_2\mathcal{A}_2 + L_1\mathcal{B}_2 + L_2\mathcal{B}_2 + \iota_2\,\mathcal{B}_2 + \iota_1\,\mathcal{A}_2. \tag{42}$$

Consequently, it follows from (41) and (42) that

$$\begin{aligned} \|(p,q) - (p^*,q^*)\| &\leq \iota_1(\mathcal{A}_1 + \mathcal{A}_2) + \iota_2(\mathcal{B}_1 + \mathcal{B}_2) + (\phi_1 + \phi_2)\|(p,q) - (p^*,q^*)\| \\ &\leq \mathcal{R}_1\iota_1 + \mathcal{R}_2\iota_2. \end{aligned}$$

where, $\mathcal{R}_1 = \dfrac{\mathcal{A}_1 + \mathcal{A}_2}{1 - (\phi_1 + \phi_2)}$ and $\mathcal{R}_2 = \frac{\mathcal{B}_1 + \mathcal{B}_2}{1 - (\phi_1 + \phi_2)}$.

Hence, the boundary value problem (1) and (2) is stable for Ulam–Hyers. □

Example 3. *Consider the following coupled system of fractional differential equations*

$$\begin{aligned} \mathcal{D}^{\frac{6}{5}}p(\iota) &= \frac{1}{2\,(\iota+1)^2} + \frac{1}{65}\,\tan^{-1}|p(\iota)| + \frac{5\,|q(\iota)|}{85\,(1+|q(\iota)|)}, \\ \mathcal{D}^{\frac{11}{8}}q(\iota) &= \frac{\sqrt{\iota}}{2\,(\iota+2)} + \frac{2}{55}\,\frac{|p(\iota)|}{(1+|p(\iota)|)} + \frac{1}{85}\,\sin|q(\iota)|, \end{aligned} \tag{43}$$

subject to the coupled integral boundary conditions:

$$\begin{aligned} p(0) = 0, \quad & \mathcal{I}^{\frac{3}{8}}\,p\Big(\frac{12}{15}\Big) = \frac{5}{4}\,\mathcal{J}_{\frac{1}{6}}^{\frac{3}{4},\,\frac{\sqrt{7}}{5}}\,q\big(\frac{5}{2}\big), \\ q(0) = 0, \quad & \mathcal{I}^{\frac{4}{6}}\,q\Big(\frac{7}{9}\Big) = 2\,\mathcal{J}_{\frac{11}{4}}^{\frac{\sqrt{2}}{3},\,\frac{1}{\sqrt{3}}}\,p\big(\frac{5}{2}\big). \end{aligned} \tag{44}$$

Here, $\xi = \frac{6}{5}$, $\zeta = \frac{11}{8}$, $\epsilon = \frac{3}{8}$, $\delta = \frac{4}{6}$, $\alpha = \frac{12}{15}$, $\beta = \frac{7}{9}$, $\lambda = \frac{5}{4}$, $\mu = 2$, $\gamma = \frac{3}{4}$, $\vartheta = \frac{\sqrt{7}}{5}$, $\rho = \frac{1}{6}$, $\eta = \frac{\sqrt{2}}{3}$, $\omega = \frac{1}{\sqrt{3}}$, $\sigma = \frac{11}{4}$, *and it is clear that*

$$\begin{aligned} |f(\iota, p(\iota), q(\iota))| &= \frac{1}{2\,(\iota+1)^2} + \frac{1}{65}\,\tan^{-1}|p(\iota)| + \frac{5\,|q(\iota)|}{85\,(1+|q(\iota)|)}, \\ |g(\iota, p(\iota), q(\iota))| &= \frac{\sqrt{\iota}}{2\,(\iota+2)} + \frac{2}{55}\,\frac{|p(\iota)|}{(1+|p(\iota)|)} + \frac{1}{85}\,\sin|q(\iota)|. \end{aligned}$$

The functions f and g satisfy the condition with $K_1 = \frac{1}{65}$, $K_2 = \frac{5}{85}$, $Ł_1 = \frac{2}{55}$, *and* $Ł_2 = \frac{1}{85}$. *Using the given data, we find that* $\kappa_1 = 0.39269$, $\kappa_2 = 0.18306$, $\kappa_3 = 1.27429$, $\kappa_4 = 3.61746$, $\varpi = 0.55093$, $\mathcal{A}_1 = 5.54328$, $\mathcal{A}_2 = 1.82866$, $\mathcal{B}_1 = 0.28159$, $\mathcal{B}_2 = 5.52986$, *and* $(\phi_1 + \phi_2) \cong 0.80661 < 1$. *Thus, all the conditions of Theorem 6 are satisfied, and there exists a unique solution of BVP (43) and (44) on* $\left[0, \frac{5}{2}\right]$, *that is stable.*

4. Conclusions

We established the existence, uniqueness, and Ulam–Hyers stability of some nonlinear Caputo type FDEs with Erdélyi–Kober and Riemann–Liouville integral boundary conditions in this study by employing some classic fixed point theorems and a nonlinear Leray–Schauder type alternative. Additionally, several examples are provided to illustrate the present work. The results of this paper are limited to a few intriguing instances with adequate values for the problem's parameters. For example, if we keep $\epsilon = 1 = \delta$ constant, our results correspond to those for the

$$\begin{cases} p(0) = 0, \quad \int_0^{\alpha} p(s)ds = \lambda\, \mathcal{J}_{\rho}^{\gamma,\vartheta} q(T) \\ q(0) = 0, \quad \int_0^{\beta} p(s)ds = \mu\, \mathcal{J}_{\sigma}^{\eta,\omega} p(T), \end{cases}$$

coupled Erdélyi–Kober and classical integral boundary conditions (2). We emphasize that all of the results that emerge as cases in our work are unique.

Author Contributions: Conceptualization, M.S. and P.D.; methodology and validation, C.K., R.V. and M.S.; investigation and formal analysis, R.V., P.D. resources, C.K.; data curation, P.D., C.K.; writing—original draft preparation, B.U.; writing—review and editing, M.S., B.U.; funding acquisition, B.U. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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