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On a Partial Fractional Hybrid Version of Generalized Sturm–Liouville–Langevin Equation

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Abstract: As we know one of the most important equations which have many applications in various areas of physics, mathematics, and financial markets, is the Sturm–Liouville equation. In this paper, by using the α-ψ-contraction technique in fixed point theory and employing some functional inequalities, we study the existence of solutions of the partial fractional hybrid case of generalized Sturm–Liouville–Langevin equations under partial boundary value conditions. Towards the end, we present two examples with numerical and graphical simulation to illustrate our main results.

Keywords: Sturm–Liouville equation; Langevin equation; α-ψ-contraction; functional inequality; fixed point theory; Caputo derivative

1. Introduction and Preliminaries

As we know, fractional calculus have been the focus of many researchers in recent years due to their wide application in various fields of engineering, modeling of natural phenomena, optimal control, and biological mathematics ([1–11]). Given the wide application of this branch of mathematics in human life, it makes sense for researchers to spend more time identifying equations that can interpret many physical phenomena and come up with newer and more powerful solutions to them. For this reason, in the last decade, many articles have been published in the field of ordinary and partial differential equations (see, for example, [12–19]). On the other hand, the theory of inequalities has always led to the expansion of fractional calculus and the introduction of new operators due to its many applications ([20,21]). For this reason, many researchers today study the existence and uniqueness of ordinary and partial differential equations with the help of such inequalities ([22–24]).

The history of mathematics has always been full of the study of a famous problem in various modes. One of these popular interdisciplinary problems is the Sturm–Liouville equation. The classical Sturm–Liouville problem for a linear differential equation of second order is a boundary-value problem which reads as follows:

$$\begin{cases} -\frac{d}{ds}[m(s)\frac{du}{ds}] + n(s)u = \lambda p(s)u, & s \in [a,b], \\ a_1 u(a) + a_2 u'(a) = 0, \\ b_1 u(b) + b_2 u'(b) = 0. \end{cases}$$

which is one of the most widely used equations in mathematics and physics, for instance the equation $u'' + \lambda u = 0$, that is obtained by appling the method separating variables to the equation that studies the problem of heat conduction in a bar. Also, other differential equations can be transformed into Sturm–Liouville equations, for example Hermite, Jacobi, Bessel and Legendre equations [25]. In the last decade, researchers have tried to study the fractional version of this equation from different aspects. Some of them investigated the eigenvalues and eigenfunctions associated with these operators and also their properties ([26,27]). Al-Mdallal proposed a numerical method by using the Adomian decomposition method to solve this problem, and investigated very interesting application of the fractional Sturm–Liouville problems which is the fractional diffusion-wave equation [28]. Due to the special importance of this equation in quantum mechanics and classical mechanics ([29,30]), its various properties have always been studied by researchers (see, for example, [28,31–45]).

Another very important equation that plays a key role in physics and interest rate modeling in financial markets is the Langevin equation, which is itself a special case of the Sturm–Liouville equation. The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [46]. For more details about the contributions in the Langevin equation refer to ([47–53]).

In addition to the above, the study of fractional differential equations in hybrid mode has always been of particular interest. In recent years, quadratic perturbations of nonlinear differential equations have attracted much attention. These types of differential equations are called hybrid differential equations in literature. Dhage and Lakshmikantham established the existence and uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems [54]. Perturbation techniques are very useful in the subject of nonlinear analysis for studying the dynamical systems represented by nonlinear differential and integral equations in a nice way [55]. For more details about the theory of hybrid differential equations, we refer to ([54–59]). Although researchers in the fields of mathematics, physics, and engineering have each taken different approaches to the Sturm–Liouville equation according to their needs and perspectives, in this work, motivated by the above, we want to develop the theory of fractional hybrid differential equations involving Sturm–Liouville operators. Theorems of existence for fractional hybrid differential equations are proved under the fixed point technique. We will consider some functional inequalities which are relevant for a further study of the existence and properties of solutions of our hybrid boundary value problems.

In 2011 [59], Zhao et al. investigated the following hybrid problem

$$\begin{cases} \mathcal{D}_c^\ell \left(\frac{h(x)}{k(x,h(x))} \right) = p(x, h(x)), \\ h(0) = 0, \end{cases}$$

such that $\mathcal{D}_c^\ell$ is the Caputo fractional derivative, $\ell \in (0,1)$ and $k \in \mathcal{C}(\mathfrak{I} \times \mathbb{R}, \mathbb{R} - \{0\})$, $p \in \mathcal{C}(\mathfrak{I} \times \mathbb{R}, \mathbb{R})$.

In 2016 [36], Kiataramkul et al. introduced an important class of boundary value problems by combining Sturm-Lioville and Langevin fractional differential equations as follows

$$\begin{cases} \mathcal{D}_c^{\ell_1} \left([h(t) \mathcal{D}_c^{\ell_2} + k(t)] \sigma(t) \right) = \zeta(t, \sigma(t), \quad 1 < t < \mathbf{T}^*, \\ \sigma(1) = -\sigma(\mathbf{T}^*), \\ \mathcal{D}_c^{\ell_1} \sigma(1) = -\mathcal{D}_c^{\ell_1} \sigma(\mathbf{T}^*), \end{cases}$$

where $\mathcal{D}_c^{\ell_1}$ and $\mathcal{D}_c^{\ell_2}$ denotes the fractional derivative of order $\ell_1, \ell_2 \in (0,1)$, $h \in \mathcal{C}([1, \mathbf{T}^*], \mathbb{R})$ where $|h(t)| \geq K > 0$, and $k \in \mathcal{C}([1, \mathbf{T}^*] \times \mathbb{R}, \mathbb{R})$.

In 2019, El-Sayed et al. [60], studied the following fractional Sturm–Liouville

$$
\begin{cases}
\mathcal{D}_c^\ell(k(t)h'(t)) + g(t)h(t) = m(t)p(h(t)), \\
h'(t) = 0, \\
\sum_{i=1}^r \lambda_i h(a_i) = \mu \sum_{j=1}^s \alpha_j h(b_j),
\end{cases}
$$

which $\mathcal{D}_c^\ell$ is the Caputo fractional derivative, $\ell \in (0,1]$ and $k \in \mathcal{C}^1(\mathfrak{I}, \mathbb{R})$, $g(t)$ and $m(t)$ are absolutely continuous functions on $\mathfrak{I} = [0,T]$, $T < \infty$ with $\forall t \in \mathfrak{I}$, $k(t) \neq 0$, $p : \mathbb{R} \to \mathbb{R}$ is defined and differentiable on the interval $\mathfrak{I}$, $0 \le a_1 < a_2 < \cdots < a_r < c$, $d \le b_1 < b_2 < \cdots < b_s < T$, $c < d$ and $\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s$ and μ are real numbers.

Let $\Lambda = (\lambda_1, \lambda_2) \in (0, \infty) \times (0, \infty)$, also $\mathcal{I}_{\mathbf{T}_1} = [0, \mathbf{T}_1]$, and $\mathcal{I}_{\mathbf{T}_2} = [0, \mathbf{T}_2]$, where $\mathbf{T}_1, \mathbf{T}_2 > 0$. For $\sigma \in \mathcal{L}^1(\mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}, \mathbb{R}) = \mathcal{L}^1(\mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2})$, the partial left-sided mixed Riemann-Liouville integral (of order Λ) can be introduced by the following framework [61].

$$
\mathcal{I}^\Lambda \sigma(w, u) = \int_0^w \int_0^u \frac{(w-s)^{\lambda_1-1}(u-t)^{\lambda_2-1}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \sigma(s,t) \, dt \, ds.
$$

In addition, the partial derivative in the sense of Caputo (of order Λ) is defined in the following manner

$$
\mathcal{D}_c^\Lambda \sigma(w, u) = \mathcal{I}_0^{1-\Lambda}\left(\frac{\partial^2}{\partial w \partial u}\sigma(w,u)\right) = \int_0^w \int_0^u \frac{(w-s)^{\lambda_1-1}(u-t)^{\lambda_2-1}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \frac{\partial^2}{\partial s \partial t}\sigma(s,t)\, dt\, ds.
$$

In this paper, motivated by the above, we want to examine the partial fractional hybrid case of generalized Sturm–Liouville–Langevin equation which reads as follows:

$$
\begin{cases}
\mathcal{D}_c^{\Lambda_1}\left[\breve{\mathbf{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))}\right) + \breve{\mathbf{r}}(w,u)\left(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u))\right)\right] = \zeta(w,u,\sigma(w,u)), \\
\left[\breve{\mathbf{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))}\right) + \breve{\mathbf{r}}(w,u)\left(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u))\right)\right]_{w=0} = \wp_2(u), \\
\left[\breve{\mathbf{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))}\right) + \breve{\mathbf{r}}(w,u)\left(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u))\right)\right]_{u=0} = \wp_1(w), \\
\left[\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))}\right]_{w=0} = \Im_2(u), \\
\left[\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))}\right]_{u=0} = \Im_1(w),
\end{cases} \tag{1}
$$

where $\Lambda = (\lambda_1, \lambda_2) \in (0, \infty) \times (0, \infty)$, $\mathcal{I}_{\mathbf{T}_1} = [0, \mathbf{T}_1]$, and $\mathcal{I}_{\mathbf{T}_2} = [0, \mathbf{T}_2]$, such that $\mathbf{T}_1, \mathbf{T}_2 > 0$. Moreover $(w,u) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}$, $\Lambda_1, \Lambda_2 \in (0,1] \times (0,1]$, $\mathcal{D}_c^{\Lambda_1}$ and $\mathcal{D}_c^{\Lambda_2}$ denote Caputo partial fractional derivatives, $\breve{\mathbf{p}} \in \mathcal{C}(\mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2})$ with $\breve{\mathbf{p}}(w,u) \neq 0$ for all $(w,u) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}$, $\vartheta_2, \zeta : \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2} \times \mathbb{R} \to \mathbb{R}$ and $\vartheta_1 : \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2} \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$ are given function with appropriate properties, the functions $\wp_1 : \mathcal{I}_{\mathbf{T}_1} \to \mathbb{R}$, $\wp_2 : \mathcal{I}_{\mathbf{T}_2} \to \mathbb{R}$, $\Im_1 : \mathcal{I}_{\mathbf{T}_1} \to \mathbb{R}$ and $\Im_2 : \mathcal{I}_{\mathbf{T}_1} \to \mathbb{R}$ are absolutely continuous with $\wp_1(0) = \wp_2(0)$ and $\Im_1(0) = \Im_2(0)$.

If in (1) for all $(w,u) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}$, and for all $(w,u,r) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2} \times \mathbb{R}$, we take $\breve{\mathbf{p}}(w,u) = 1$, $\vartheta_1(w,u,r) = 1$ and $\vartheta_2(w,u,r) = r\lambda$, which $\lambda \in \mathbb{R}$, then we obtain the following equation

$$
\begin{cases}
\mathcal{D}_c^{\Lambda_2}(\mathcal{D}_c^{\Lambda_1} + \lambda)\sigma(w,u) = \zeta(w,u,\sigma(w,u)), \\
\left[(\mathcal{D}_c^{\Lambda_1} + \lambda)\sigma(w,u)\right]_{w=0} = \wp_2(u), \\
\left[(\mathcal{D}_c^{\Lambda_1} + \lambda)\sigma(w,u)\right]_{u=0} = \wp_1(w), \\
\sigma(0,u) = \Im_2(u) \quad \text{and} \quad \sigma(w,0) = \Im_1(w),
\end{cases} \tag{2}
$$

which is a partial fractional Lagevin equation.

If in (1) we take $\vartheta_1(w, u, r) - 1 = \vartheta_2(w, u, r) = 0$ for all $(w, u, r) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2} \times \mathbb{R}$ then we obtain the following equation

$$\begin{cases} \mathcal{D}_c^{\Lambda_2}(\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_1}\sigma(w, u)) = \zeta(w, u, \sigma(w, u)), \\[2mm] \left[\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_1}\sigma(w, u)\right]_{w=0} = \wp_2(u) \\[2mm] \left[\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_1}\sigma(w, u)\right]_{u=0} = \wp_1(w) \\[2mm] \sigma(0, u) = \Im_2(u) \quad \text{and} \quad \sigma(w, 0) = \Im_1(w), \end{cases}$$

which is a partial fractional Sturm-Lioville equation.

Next, we study partial fractional a hybrid version of the generalized Sturm–Liouville–Langevin equation which reads as

$$\begin{cases} \mathcal{D}_c^{\Lambda_1}\left(\dfrac{\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\sigma(w, u) + \check{\mathbf{r}}(w, u)\sigma(w, u)}{\Theta\left(w, u, \sigma(w, u), \mathcal{I}_c^{Y_1}\sigma(w, u), \mathcal{I}_c^{Y_2}\sigma(w, u), ..., \mathcal{I}_c^{Y_\kappa}\sigma(w, u)\right)}\right) = \ell(w, u, \sigma(w, u)), \\[4mm] \left[\dfrac{\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\sigma(w, u) + \check{\mathbf{r}}(w, u)\sigma(w, u)}{\Theta\left(w, u, \sigma(w, u), \mathcal{I}_c^{Y_1}\sigma(w, u), \mathcal{I}_c^{Y_2}\sigma(w, u), ..., \mathcal{I}_c^{Y_\kappa}\sigma(w, u)\right)}\right]_{w=0} = \Re_2(u), \\[4mm] \left[\dfrac{\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\sigma(w, u) + \check{\mathbf{r}}(w, u)\sigma(w, u)}{\Theta\left(w, u, \sigma(w, u), \mathcal{I}_c^{Y_1}\sigma(w, u), \mathcal{I}_c^{Y_2}\sigma(w, u), \sigma(w, u), ..., \mathcal{I}_c^{Y_\kappa}\sigma(w, u)\right)}\right]_{u=0} = \Re_1(w), \\[4mm] \sigma(0, u) = \hbar_2(u) \quad \text{and} \quad \sigma(w, 0) = \hbar_1(w), \end{cases} \qquad (3)$$

where $\Lambda_1 = (\lambda_{11}, \lambda_{12})$, $\Lambda_2 = (\lambda_{21}, \lambda_{22})$, $Y_1 = (\epsilon_{11}, \epsilon_{12})$, $Y_2 = (\epsilon_{21}, \epsilon_{22})$, ..., $Y_\kappa = (\epsilon_{\kappa1}, \epsilon_{\kappa2}) \in (0, 1] \times (0, 1]$ ($\kappa \in \mathbb{N}$), $\mathcal{D}_c^{\Lambda_1}$ and $\mathcal{D}_c^{\Lambda_2}$ denote Caputo partial fractional derivatives, $\check{\mathbf{p}} \in \mathcal{C}(\mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2})$ with $\check{\mathbf{p}}(w, u) \neq 0$ for all $(w, u) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}$, $\Theta : \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2} \times \mathbb{R}^{\kappa+1} \to \mathbb{R} - \{0\}$, the functions $\hbar_1 : \mathcal{I}_{\mathbf{T}_1} \to \mathbb{R}$, $\hbar_2 : \mathcal{I}_{\mathbf{T}_2} \to \mathbb{R}$, $\Re_1 : \mathcal{I}_{\mathbf{T}_2} \to \mathbb{R}$ and $\Re_2 : \mathcal{I}_{\mathbf{T}_2} \to \mathbb{R}$ are absolutely continuous with $\hbar_1(0) = \hbar_2(0)$ and $\Re_1(0) = \Re_2(0)$.

The existence of fixed points of different maps on ordered spaces has an old history. Some researchers investigated different versions of the Tarski's fixed point theorem (see for example, [62,63]). In 2012 [64], Samet et al. introduced the following concepts in fixed point theory, which will play a key role in this paper. Throughout this work, the symbol Ψ will be shorthand for the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$, such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ. Let $T : \mathcal{Y} \to \mathcal{Y}$ be a selfmap and $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$ be a function. We say that T is α-admissible whenever $\alpha(x, y) \geq 1$ yield that $\alpha(Tx, Ty) \geq 1$. Also suppose that $\psi \in \Psi$, then a self-map $T : \mathcal{Y} \to \mathcal{Y}$ is called an α-ψ-contraction whenever $\forall x, y \in \mathcal{Y}$, we have $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$. The notion of α-ψ-contractions generalized many old techniques in the literature. It let us to consider ordered spaces or graphs in our works. Also, it able us to work with some new notions such approximate fixed points (see for example, [65–67]). To continue, we need the following lemma.

Lemma 1 ([64]). *Assume that $(\mathcal{Y}, d)$ be a complete metric space and $T : \mathcal{Y} \to \mathcal{Y}$ is α-admissible and α-ψ-contraction, also let $\exists y_0 \in \mathcal{Y}$ which satisfies $\alpha(y_0, Ty_0) \geq 1$. Moreover suppose that every sequence $\{y_n\}$ in $\mathcal{Y}$ such that $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \geq 1$ and for which there exists $y \in \mathcal{Y}$ such that $y_n \to y$, it follows that $\alpha(y_n, y) \geq 1$ for all $n \geq 1$. Then T has a fixed point.*

2. Main Results

Consider the Banach space $\mathcal{Y} = \{\sigma \mid \sigma \in \mathcal{C}(\mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2})\}$ equipped with the supremum norm $\|\sigma\| = \sup_{(w, u) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}} |\sigma(w, u)|$, where $\sigma \in \mathcal{Y}$.

Lemma 2. *Let $\ell \in \mathcal{L}^1(\mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2})$, $\Lambda_1 = (\lambda_{11}, \lambda_{12}) \in (0, 1] \times (0, 1]$ and also $\Lambda_2 = (\lambda_{21}, \lambda_{22}) \in (0, 1] \times (0, 1]$. Consider the equation*

$$\mathcal{D}_c^{\Lambda_1}\left[\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w, u)}{\vartheta_1(w, u, \sigma(w, u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right] = \ell(w, u) \quad (4)$$

with boundary conditions

$$\begin{cases} \left[\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right]_{w=0} = \wp_2(u), \\[2mm] \left[\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right]_{u=0} = \wp_1(w), \\[2mm] \left[\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right]_{w=0} = \Im_2(u), \\[2mm] \left[\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right]_{u=0} = \Im_1(w). \end{cases} \quad (5)$$

Then, the function $\sigma \in \mathcal{C}(\mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2})$ is a solution of the problem (4) *and* (5) *whenever*

$$\sigma(w, u) = \vartheta_1\big(w, u, \sigma(w, u)\big)\left[\int_0^w \int_0^u \int_0^{\varrho_1} \int_0^{\varrho_2} \right.$$

$$\frac{(\varrho_1 - \varsigma_1)^{\lambda_{11}-1}(\varrho_2 - \varsigma_2)^{\lambda_{12}-1}(w - \varrho_1)^{\lambda_{21}-1}(u - \varrho_2)^{\lambda_{22}-1}\sigma(\varsigma_1, \varsigma_2)}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})\check{\mathbf{p}}(\varrho_1, \varrho_2)} d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1$$

$$- \int_0^w \int_0^u \frac{(w - \jmath)^{\lambda_{21}-1}(u - \imath)^{\lambda_{22}-1}\check{\mathbf{r}}(\imath, \jmath)\sigma(\imath, \jmath)}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\check{\mathbf{p}}(\imath, \jmath)} d\imath d\jmath$$

$$+ \int_0^w \int_0^u \frac{(w - \jmath)^{\lambda_{21}-1}(u - \imath)^{\lambda_{22}-1}\check{\mathbf{r}}(\imath, \jmath)\vartheta_2(\imath, \jmath, \sigma(\imath, \jmath))}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\check{\mathbf{p}}(\imath, \jmath)} d\imath d\jmath$$

$$+ \left. \int_0^w \int_0^u \frac{(w - \jmath)^{\lambda_{21}-1}(u - \imath)^{\lambda_{22}-1}\check{\mathbf{r}}(\imath, \jmath)\triangle_1(\imath, \jmath)}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\check{\mathbf{p}}(\imath, \jmath)} d\imath d\jmath + \triangle_2(w, u) \right],$$

where $\triangle_1(w, u) = \wp_1(w) + \wp_2(u) - \wp_1(0)$ and $\triangle_2(w, u) = \Im_1(w) + \Im_2(u) - \Im_1(0)$.

Proof. We know that, the Equation (4) can be written as

$$\mathcal{I}^{1-\Lambda_1}\left[\frac{\partial^2}{\partial w \partial u}\left(\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w, u)}{\vartheta_1(w, u, \sigma(w, u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right) \right] = \ell(w, u)$$

and by applying the operator $\mathcal{I}^{\Lambda_1}$ on both sides, we get

$$\mathcal{I}^1\left[\frac{\partial^2}{\partial w \partial u}\left(\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w, u)}{\vartheta_1(w, u, \sigma(w, u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right) \right] = \mathcal{I}^{\Lambda_1}\ell(w, u).$$

Since

$$\mathcal{I}^1\left[\frac{\partial^2}{\partial w \partial u}\left(\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w, u)}{\vartheta_1(w, u, \sigma(w, u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right) \right]$$

$$= \left(\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w, u)}{\vartheta_1(w, u, \sigma(w, u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right)$$

$$- \left[\left(\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w, u)}{\vartheta_1(w, u, \sigma(w, u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right) \right]_{u=0}$$

$$- \left[\left(\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w, u)}{\vartheta_1(w, u, \sigma(w, u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right) \right]_{w=0}$$

$$+ \left[\left(\check{\mathbf{p}}(w, u)\mathcal{D}_c^{\Lambda_2}\left(\frac{\sigma(w, u)}{\vartheta_1(w, u, \sigma(w, u))} \right) + \check{\mathbf{r}}(w, u)\Big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big) \Big) \right) \right]_{(w,u)=(0,0)}$$

$$= \left(\check{\mathbf{p}}(w,u) \mathcal{D}_c^{\Lambda_2} \left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right) + \check{\mathbf{r}}(w,u) \Big(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u)) \Big) \right)$$

$$- \wp_2(u) - \wp_1(w) + \wp_1(0),$$

we get

$$\check{\mathbf{p}}(w,u) \mathcal{D}_c^{\Lambda_2} \left(\frac{\sigma(w,u) - \vartheta_1(w,u,\sigma(w,u))}{\vartheta_1(w,u,\sigma(w,u))} \right) + \check{\mathbf{r}}(w,u)(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u)))$$

$$= \mathcal{I}^{\Lambda_1} \ell(w,u) + \wp_2(u) + \wp_1(w) - \wp_1(0).$$

Hence,

$$\mathcal{D}_c^{\Lambda_2} \left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right) = \frac{1}{\check{\mathbf{p}}(w,u)} \mathcal{I}^{\Lambda_1} \ell(w,u) + \frac{\wp_2(u) + \wp_1(w) - \wp_1(0)}{\check{\mathbf{p}}(w,u)}$$

$$- \frac{\check{\mathbf{r}}(w,u) \Big(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u)) \Big)}{\check{\mathbf{p}}(w,u)}.$$

Since $\mathcal{D}_c^{\Lambda_2} \left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right) = \mathcal{I}^{1-\Lambda_2} \left[\frac{\partial^2}{\partial w \partial u} \left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right) \right]$, we have

$$\mathcal{I}^{1-\Lambda_2} \left[\frac{\partial^2}{\partial w \partial u} \left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right) \right] = \frac{1}{\check{\mathbf{p}}(w,u)} \mathcal{I}^{\Lambda_1} \ell(w,u) + \frac{\wp_2(u) + \wp_1(w) - \wp_1(0)}{\check{\mathbf{p}}(w,u)}$$

$$- \frac{\check{\mathbf{r}}(w,u) \Big(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u)) \Big)}{\check{\mathbf{p}}(w,u)}.$$

By applying the operator $\mathcal{I}^{\Lambda_2}$ on both sides, we get

$$\mathcal{I}^1 \left[\frac{\partial^2}{\partial w \partial u} \left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right) \right] = \mathcal{I}^{\Lambda_2} \Big(\frac{1}{\check{\mathbf{p}}(w,u)} \mathcal{I}^{\Lambda_1} \ell(w,u) \Big) + \mathcal{I}^{\Lambda_2} \Big(\frac{\wp_2(u) + \wp_1(w) - \wp_1(0)}{\check{\mathbf{p}}(w,u)} \Big)$$

$$- \mathcal{I}^{\Lambda_2} \left[\frac{\check{\mathbf{r}}(w,u) \Big(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u)) \Big)}{\check{\mathbf{p}}(w,u)} \right].$$

On the other hand, we have

$$\mathcal{I}^1 \left[\frac{\partial^2}{\partial w \partial u} \left(\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} \right) \right] = \frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} - \Im_1(w) - \Im_2(u) + \Im_1(0).$$

Thus, we find

$$\frac{\sigma(w,u)}{\vartheta_1(w,u,\sigma(w,u))} = \mathcal{I}^{\Lambda_2} \Big(\frac{1}{\check{\mathbf{p}}(w,u)} \mathcal{I}^{\Lambda_1} \ell(w,u) \Big) + \mathcal{I}^{\Lambda_2} \Big(\frac{\wp_2(u) + \wp_1(w) - \wp_1(0)}{\check{\mathbf{p}}(w,u)} \Big)$$

$$- \mathcal{I}^{\Lambda_2} \left[\frac{\check{\mathbf{r}}(w,u) \Big(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u)) \Big)}{\check{\mathbf{p}}(w,u)} \right] + \Im_2(u) + \Im_1(w) - \Im(0)$$

$$= \mathcal{I}^{\Lambda_2} \Big(\frac{1}{\check{\mathbf{p}}(w,u)} \mathcal{I}^{\Lambda_1} \ell(w,u) \Big) - \mathcal{I}^{\Lambda_2} \left[\frac{\check{\mathbf{r}}(w,u) \Big(\sigma(w,u) - \vartheta_2(w,u,\sigma(w,u)) \Big)}{\check{\mathbf{p}}(w,u)} \right]$$

$$+ \mathcal{I}^{\Lambda_2} \Big(\frac{\triangle_1(w,u)}{\check{\mathbf{p}}(w,u)} \Big) + \triangle_2(w,u).$$

Hence,

$$\sigma(w, u) = \vartheta_1\big(w, u, \sigma(w, u)\big)\left[\mathcal{I}^{\Lambda_2}\big(\frac{1}{\breve{\mathbf{p}}(w, u)}\mathcal{I}^{\Lambda_1}\ell(w, u)\big)\right.$$

$$-\mathcal{I}^{\Lambda_2}\left[\frac{\breve{\mathbf{r}}(w, u)\big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big)\big)}{\breve{\mathbf{p}}(w, u)}\right]$$

$$\left. +\mathcal{I}^{\Lambda_2}\big(\frac{\triangle_1(w, u)}{\breve{\mathbf{p}}(w, u)}\big) + \triangle_2(w, u)\right].$$

Note that,

$$\mathcal{I}^{\Lambda_2}\big(\frac{1}{\breve{\mathbf{p}}(w, u)}\mathcal{I}^{\Lambda_1}\ell(w, u)\big) = \int_0^w\int_0^u \frac{(w - \varrho_1)^{\lambda_{21}-1}(u - \varrho_2)^{\lambda_{22}-1}}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\breve{\mathbf{p}}(\varrho_1, \varrho_2)}\mathcal{I}^{\Lambda_1}\ell(\varrho_1, \varrho_2)d\varrho_2 d\varrho_1$$

$$= \int_0^w\int_0^u\int_0^{\varrho_1}\int_0^{\varrho_2} \frac{(\varrho_1 - \varsigma_1)^{\lambda_{11}-1}(\varrho_2 - \varsigma_2)^{\lambda_{12}-1}(w - \varrho_1)^{\lambda_{21}-1}(u - \varrho_2)^{\lambda_{22}-1}\ell(\varsigma_1, \varsigma_2)}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})\breve{\mathbf{p}}(\varrho_1, \varrho_2)}d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1$$

and

$$\mathcal{I}^{\Lambda_2}\left[\frac{\breve{\mathbf{r}}(w, u)\big(\sigma(w, u) - \vartheta_2\big(w, u, \sigma(w, u)\big)\big)}{\breve{\mathbf{p}}(w, u)}\right] = \int_0^w\int_0^u \frac{(w - \jmath)^{\lambda_{21}-1}(u - \imath)^{\lambda_{22}-1}\breve{\mathbf{r}}(\imath, \jmath)\sigma(\imath, \jmath)}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\breve{\mathbf{p}}(\imath, \jmath)}d\imath d\jmath$$

$$-\int_0^w\int_0^u \frac{(w - \jmath)^{\lambda_{21}-1}(u - \imath)^{\lambda_{22}-1}\breve{\mathbf{r}}(\imath, \jmath)\vartheta_2(\imath, \jmath, \sigma(\imath, \jmath))}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\breve{\mathbf{p}}(\imath, \jmath)}d\imath d\jmath.$$

This completes the proof. $\square$

Next we establish and prove our first main theorem.

Theorem 1. *Assume that the following conditions hold*

$(\mathcal{Q}_1)$*There exist* $\chi, \varpi_1, \varpi_2 \in \mathcal{C}(\mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}, \mathbb{R}^+)$ *such that the following functional inequalities are satisfied*

$$\begin{cases} |\zeta(w, u, \mathbf{r}_1) - \zeta(w, u, \mathbf{r}_2)| \leq \chi(w, u)|\mathbf{r}_1 - \mathbf{r}_1|, \\ |\vartheta_1(w, u, \mathbf{r}_1) - \vartheta_1(w, u, \mathbf{r}_2)| \leq \varpi_1(w, u)|\mathbf{r}_1 - \mathbf{r}_1|, \\ |\vartheta_2(w, u, \mathbf{r}_1) - \vartheta_2(w, u, \mathbf{r}_2)| \leq \varpi_2(w, u)|\mathbf{r}_1 - \mathbf{r}_1|. \end{cases}$$

for all $(w, u, \mathbf{r}_1, \mathbf{r}_2) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2} \times \mathbb{R} \times \mathbb{R}.$

$(\mathcal{Q}_2)$*There exists* $\eta > 0$ *such that*

$$\eta \geq (\varpi_1^\star\eta + \vartheta_1^\star)(\mathbf{U}_1\eta + \mathbf{U}_2 + \triangle_2^\star) \quad and \quad (\mathbf{U}_1\eta + \mathbf{U}_2 + \triangle_2^\star)\varpi_1^\star + (\varpi_1^\star\eta + \vartheta_1^\star)\mathbf{U}_1 < 1,$$

where

$$\begin{cases} \mathbf{U}_1 = \frac{\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star}\left[\frac{\mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}\chi^\star}{\Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)} + \frac{\breve{\mathbf{r}}^\star(1+\varpi_2^\star)}{\Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}\right], \\ \mathbf{U}_2 = \frac{\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star}\left[\frac{\mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}\zeta^\star}{\Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)} + \frac{\breve{\mathbf{r}}^\star(\triangle_1^\star+\vartheta_2^\star)}{\Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}\right], \end{cases}$$

and $\zeta^\star, \varpi_1^\star, \varpi_2^\star, \breve{\mathbf{r}}^\star, \triangle_1^\star, \triangle_2^\star, \chi^\star, \breve{\mathbf{p}}^\star$ *indicate the supremum of* $\zeta(w, u, 0), \varpi_1(w, u, 0), \varpi_2(w, u, 0),$ $\breve{\mathbf{r}}(w, u), \triangle_1(w, u), \triangle_2(w, u), \chi(w, u)$ *and* $\breve{\mathbf{p}}(w, u)$, *respectively. Then the problem formulated in* (3) *has a solution.*

Proof. Define the map $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, \infty)$ by

$$
\alpha(\sigma, \tilde{\sigma}) =
\begin{cases}
1 & \text{if} \quad \sigma(w, u) \leq \eta, \quad \forall (w, u) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}, \\
0 & \text{otherwise.}
\end{cases}
$$

Addotionally, define the operator $\mathbb{H} : \mathcal{Y} \to \mathcal{Y}$ by $\mathbb{H}\sigma(w, u) = \vartheta_1(w, u, \sigma(w, u))\mathbb{K}\sigma(w, u)$, where

$$
\mathbb{K}\sigma(w, u) = \int_0^w \int_0^u \int_0^{\varrho_1} \int_0^{\varrho_2}
$$

$$
\frac{(\varrho_1 - \varsigma_1)^{\lambda_{11}-1}(\varrho_2 - \varsigma_2)^{\lambda_{12}-1}(w - \varrho_1)^{\lambda_{21}-1}(u - \varrho_2)^{\lambda_{22}-1}\zeta(\varsigma_1, \varsigma_2, \sigma(\varsigma_1, \varsigma_2))}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})\breve{\mathbf{p}}(\varrho_1, \varrho_2)} d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1
$$

$$
- \int_0^w \int_0^u \frac{(w - \jmath)^{\lambda_{21}-1}(u - \imath)^{\lambda_{22}-1}\breve{\mathbf{r}}(\imath, \jmath)\sigma(\imath, \jmath)}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\breve{\mathbf{p}}(\imath, \jmath)} d\imath d\jmath
$$

$$
+ \int_0^w \int_0^u \frac{(w - \jmath)^{\lambda_{21}-1}(u - \imath)^{\lambda_{22}-1}\breve{\mathbf{r}}(\imath, \jmath)\vartheta_2(\imath, \jmath, \sigma(\imath, \jmath))}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\breve{\mathbf{p}}(\imath, \jmath)} d\imath d\jmath
$$

$$
+ \int_0^w \int_0^u \frac{(w - \jmath)^{\lambda_{21}-1}(u - \imath)^{\lambda_{22}-1}\breve{\mathbf{r}}(\imath, \jmath)\triangle_1(\imath, \jmath)}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\breve{\mathbf{p}}(\imath, \jmath)} d\imath d\jmath + \triangle_2(w, u).
$$

According to Lemma 2, $\mathbb{H}\sigma$ is equivalent to problem (1). By using Lemma 1, σ_0, solves problem (1), if and only if σ_0 is a fixed point of $\mathbb{H}$. Now we are going to prove that all conditions of Lemma 1 hold for the operator $\mathbb{H}$. At first step, we find

$$
|\zeta(w, u, \sigma(w, u))| = |\zeta(w, u, \sigma(w, u)) - \zeta(w, u, 0) + \zeta(w, u, 0)|
$$

$$
\leq |\zeta(w, u, \sigma(w, u)) - \zeta(w, u, 0)| + |\zeta(w, u, 0)|
$$

$$
\leq \chi(w, u)|\sigma(w, u)| + \zeta^\star \leq \chi^\star \eta + \zeta^\star.
$$

Thus,

$$
\int_0^w \int_0^u \int_0^{\varrho_1} \int_0^{\varrho_2} \frac{(\varrho_1 - \varsigma_1)^{\lambda_{11}-1}(\varrho_2 - \varsigma_2)^{\lambda_{12}-1}(w - \varrho_1)^{\lambda_{21}-1}(u - \varrho_2)^{\lambda_{22}-1}|\zeta(\varsigma_1, \varsigma_2, \sigma(\varsigma_1, \varsigma_2))|}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{\mathbf{p}}(\varrho_1, \varrho_2)|} d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1
$$

$$
\leq \frac{\chi^\star \eta + \zeta^\star}{\breve{\mathbf{p}}^\star}
$$

$$
\times \int_0^w \int_0^u \int_0^{\varrho_1} \int_0^{\varrho_2} \frac{(\varrho_1 - \varsigma_1)^{\lambda_{11}-1}(\varrho_2 - \varsigma_2)^{\lambda_{12}-1}(w - \varrho_1)^{\lambda_{21}-1}(u - \varrho_2)^{\lambda_{22}-1}}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})} d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1
$$

$$
= \frac{\chi^\star \eta + \zeta^\star}{\breve{\mathbf{p}}^\star} \int_0^w \int_0^u \frac{\varrho_1^{\lambda_{11}} \varrho_2^{\lambda_{12}}(w - \varrho_1)^{\lambda_{21}-1}(u - \varrho_2)^{\lambda_{22}-1}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)\Gamma(\lambda_{21})\Gamma(\lambda_{22})} d\varrho_2 d\varrho_1
$$

$$
\leq \frac{\chi^\star \eta + \zeta^\star}{\breve{\mathbf{p}}^\star} \int_0^{\mathbf{T}_1} \int_0^{\mathbf{T}_2} \frac{\varrho_1^{\lambda_{11}} \varrho_2^{\lambda_{12}}(\mathbf{T}_1 - \varrho_1)^{\lambda_{21}-1}(\mathbf{T}_2 - \varrho_2)^{\lambda_{22}-1}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)\Gamma(\lambda_{21})\Gamma(\lambda_{22})} d\varrho_2 d\varrho_1
$$

$$
= \frac{\chi^\star \eta + \zeta^\star}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)\Gamma(\lambda_{21})\Gamma(\lambda_{22})} \int_0^{\mathbf{T}_1} \varrho_1^{\lambda_{11}}(\mathbf{T}_1 - \varrho_1)^{\lambda_{21}-1} d\varrho_1
$$

$$
\times \int_0^{\mathbf{T}_2} \varrho_2^{\lambda_{12}}(\mathbf{T}_2 - \varrho_2)^{\lambda_{22}-1} d\varrho_2.
$$

Since

$$
\int_0^{\mathbf{T}_1} \varrho_1^{\lambda_{11}}(\mathbf{T}_1 - \varrho_1)^{\lambda_{21}-1} d\varrho_1 \times \int_0^{\mathbf{T}_2} \varrho_2^{\lambda_{12}}(\mathbf{T}_2 - \varrho_2)^{\lambda_{22}-1} d\varrho_2
$$

$$
= \mathbf{T}_1^{\lambda_{11}+\lambda_{21}} \mathbf{T}_2^{\lambda_{12}+\lambda_{22}} \int_0^1 \left(\frac{\varrho_1}{\mathbf{T}_1}\right)^{\lambda_{11}} \left(1 - \frac{\varrho_1}{\mathbf{T}_1}\right)^{\lambda_{21}-1} d\left(\frac{\varrho_1}{\mathbf{T}_1}\right) \times \int_0^1 \left(\frac{\varrho_2}{\mathbf{T}_2}\right)^{\lambda_{12}} \left(1 - \frac{\varrho_2}{\mathbf{T}_2}\right)^{\lambda_{22}-1} d\left(\frac{\varrho_2}{\mathbf{T}_2}\right)
$$

$$= \mathbf{T}_1^{\lambda_{11}+\lambda_{21}} \mathbf{T}_2^{\lambda_{12}+\lambda_{22}} \mathbf{B}(\lambda_{11}+1, \lambda_{21}) \mathbf{B}(\lambda_{12}+1, \lambda_{22})$$

$$= \frac{\mathbf{T}_1^{\lambda_{11}+\lambda_{21}} \mathbf{T}_2^{\lambda_{12}+\lambda_{22}} \Gamma(\lambda_{11}+1)\Gamma(\lambda_{21})\Gamma(\lambda_{12}+1)\Gamma(\lambda_{22})}{\Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)},$$

where **B** denotes Euler's beta function. Hence, we obtain

$$\int_0^w \int_0^u \int_0^{\varrho_1} \int_0^{\varrho_2} \frac{(\varrho_1-\varsigma_1)^{\lambda_{11}-1}(\varrho_2-\varsigma_2)^{\lambda_{12}-1}(w-\varrho_1)^{\lambda_{21}-1}(u-\varrho_2)^{\lambda_{22}-1}|\zeta(\varsigma_1,\varsigma_2,\sigma(\varsigma_1,\varsigma_2))|}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{\mathbf{p}}(\varrho_1,\varrho_2)|} d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1$$

$$\leq \frac{\chi^\star \eta + \zeta^\star}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)\Gamma(\lambda_{21})\Gamma(\lambda_{22})} \times \frac{\mathbf{T}_1^{\lambda_{11}+\lambda_{21}} \mathbf{T}_2^{\lambda_{12}+\lambda_{22}} \Gamma(\lambda_{11}+1)\Gamma(\lambda_{21})\Gamma(\lambda_{12}+1)\Gamma(\lambda_{22})}{\Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)}$$

$$= \frac{\mathbf{T}_1^{\lambda_{11}+\lambda_{21}} \mathbf{T}_2^{\lambda_{12}+\lambda_{22}} (\chi^\star \eta + \zeta^\star)}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)}$$

and so

$$\int_0^w \int_0^u \int_0^{\varrho_1} \int_0^{\varrho_2} \frac{(\varrho_1-\varsigma_1)^{\lambda_{11}-1}(\varrho_2-\varsigma_2)^{\lambda_{12}-1}(w-\varrho_1)^{\lambda_{21}-1}(u-\varrho_2)^{\lambda_{22}-1}|\zeta(\varsigma_1,\varsigma_2,\sigma(\varsigma_1,\varsigma_2))|}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{\mathbf{p}}(\varrho_1,\varrho_2)|} d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1$$

$$\leq \frac{\mathbf{T}_1^{\lambda_{11}+\lambda_{21}} \mathbf{T}_2^{\lambda_{12}+\lambda_{22}} \chi^\star}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)} \eta + \frac{\mathbf{T}_1^{\lambda_{11}+\lambda_{21}} \mathbf{T}_2^{\lambda_{12}+\lambda_{22}} \zeta^\star}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)}. \tag{6}$$

On the other hand, we have

$$\int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\breve{\mathbf{r}}(\imath,\jmath)||\sigma(\imath,\jmath)|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{\mathbf{p}}(\imath,\jmath)|} d\imath d\jmath \leq \frac{\breve{\mathbf{r}}^\star \eta}{\breve{\mathbf{p}}^\star} \int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})} d\imath d\jmath$$

$$\leq \frac{\breve{\mathbf{r}}^\star \mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \eta \tag{7}$$

Thus, we see

$$|\vartheta_2(w,u,\sigma(w,u))| = |\vartheta_2(w,u,\sigma(w,u)) - \vartheta_2(w,u,0) + \vartheta_2(w,u,0)|$$
$$\leq |\vartheta_2(w,u,\sigma(w,u)) - \vartheta_2(w,u,0)| + |\vartheta_2(w,u,0)|$$
$$\leq \varpi_2(w,u)|\sigma(w,u)| + \vartheta_2^\star \leq \varpi_2^\star \eta + \vartheta_2^\star$$

and so

$$\int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\breve{\mathbf{r}}(\imath,\jmath)||\vartheta_2(\imath,\jmath,\sigma(\imath,\jmath))|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{\mathbf{p}}(\imath,\jmath)|} d\imath d\jmath$$
$$\leq \frac{\breve{\mathbf{r}}^\star \mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}} (\varpi_2^\star \eta + \vartheta_2^\star)}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} = \frac{\breve{\mathbf{r}}^\star \varpi_2^\star \mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \eta + \frac{\breve{\mathbf{r}}^\star \vartheta_2^\star \mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}. \tag{8}$$

Similarly, we obtain

$$\int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\breve{\mathbf{r}}(\imath,\jmath)||\triangle_1(\imath,\jmath)|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{\mathbf{p}}(\imath,\jmath)|} d\imath d\jmath + |\triangle_2(w,u)| \leq \frac{\breve{\mathbf{r}}^\star \triangle_1^\star \mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} + \triangle_2^\star. \tag{9}$$

Since

$$|\mathbb{K}\sigma(w,u)| \leq \int_0^w \int_0^u \int_0^{\varrho_1} \int_0^{\varrho_2}$$

$$\frac{(\varrho_1 - \varsigma_1)^{\lambda_{11}-1}(\varrho_2 - \varsigma_2)^{\lambda_{12}-1}(w - \varrho_1)^{\lambda_{21}-1}(u - \varrho_2)^{\lambda_{22}-1}|\zeta(\varsigma_1,\varsigma_2,\sigma(\varsigma_1,\varsigma_2))|}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{p}(\varrho_1,\varrho_2)|} d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1$$

$$+ \int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\breve{r}(\imath,\jmath)||\sigma(\imath,\jmath)|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{p}(\imath,\jmath)|} d\imath d\jmath$$

$$+ \int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\breve{r}(\imath,\jmath)||\vartheta_2(\imath,\jmath,\sigma(\imath,\jmath))|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{p}(\imath,\jmath)|} d\imath d\jmath$$

$$+ \int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\breve{r}(\imath,\jmath)||\triangle_1(\imath,\jmath)|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{p}(\imath,\jmath)|} d\imath d\jmath + |\triangle_2(w,u)|,$$

by using (6)–(9), we conclude that

$$|\mathbb{K}\sigma(w,u)| \leq \frac{\mathbf{T}_1^{\lambda_{11}+\lambda_{21}}\mathbf{T}_2^{\lambda_{12}+\lambda_{22}}\chi^\star}{\breve{p}^\star\Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)}\eta + \frac{\mathbf{T}_1^{\lambda_{11}+\lambda_{21}}\mathbf{T}_2^{\lambda_{12}+\lambda_{22}}\zeta^\star}{\breve{p}^\star\Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)}$$

$$+ \frac{\breve{r}^\star\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\breve{p}^\star\Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}\eta + \frac{\breve{r}^\star\varpi_2^\star\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\breve{p}^\star\Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}\eta$$

$$+ \frac{\breve{r}^\star\vartheta_2^\star\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\breve{p}^\star\Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} + \frac{\breve{r}^\star\triangle_1^\star\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\breve{p}^\star\Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} + \triangle_2^\star.$$

After using some simple computations and substitutions, we get

$$|\mathbb{K}\sigma(w,u)| \leq \mathbf{U}_1\eta + \mathbf{U}_2 + \triangle_2^\star. \tag{10}$$

Further, it is easy to check that

$$|\vartheta_1(w,u,\sigma(w,u))| \leq \varpi_1^\star\eta + \vartheta_1^\star. \tag{11}$$

Hence, we get

$$|\mathbb{H}\sigma(w,u)| = |\vartheta_1(w,u,\sigma(w,u))||\mathbb{K}\sigma(w,u)| \leq (\varpi_1^\star\eta + \vartheta_1^\star)(\mathbf{U}_1\eta + \mathbf{U}_2 + \triangle_2^\star) \leq \eta.$$

Similarly we can prove that $|\mathbb{H}\tilde{\sigma}(w,u)| \leq \eta$. That is, $\alpha(\mathbb{H}\sigma, \mathbb{H}\tilde{\sigma}) \geq 1$. Then $\mathbb{H}$ is an α-admissible mapping. Note that,

$$|\mathbb{K}\sigma(w,u) - \mathbb{K}\tilde{\sigma}(w,u)| \leq \int_0^w \int_0^u \int_0^{\varrho_1} \int_0^{\varrho_2}$$

$$\frac{(\varrho_1-\varsigma_1)^{\lambda_{11}-1}(\varrho_2-\varsigma_2)^{\lambda_{12}-1}(w-\varrho_1)^{\lambda_{21}-1}(u-\varrho_2)^{\lambda_{22}-1}|\zeta(\varsigma_1,\varsigma_2,\sigma(\varsigma_1,\varsigma_2)) - \zeta(\varsigma_1,\varsigma_2,\tilde{\sigma}(\varsigma_1,\varsigma_2))|}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{p}(\varrho_1,\varrho_2)|} d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1$$

$$+ \int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\breve{r}(\imath,\jmath)||\sigma(\imath,\jmath) - \tilde{\sigma}(\imath,\jmath)|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{p}(\imath,\jmath)|} d\imath d\jmath$$

$$+ \int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\breve{r}(\imath,\jmath)||\vartheta_2(\imath,\jmath,\sigma(\imath,\jmath)) - \vartheta_2(\imath,\jmath,\tilde{\sigma}(\imath,\jmath))|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\breve{p}(\imath,\jmath)} d\imath d\jmath$$

$$\leq \int_0^w \int_0^u \int_0^{\varrho_1} \int_0^{\varrho_2}$$

$$\frac{(\varrho_1-\varsigma_1)^{\lambda_{11}-1}(\varrho_2-\varsigma_2)^{\lambda_{12}-1}(w-\varrho_1)^{\lambda_{21}-1}(u-\varrho_2)^{\lambda_{22}-1}\chi(\varsigma_1,\varsigma_2)|\sigma(\varsigma_1,\varsigma_2) - \tilde{\sigma}(\varsigma_1,\varsigma_2)|}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{p}(\varrho_1,\varrho_2)|} d\varsigma_2 d\varsigma_1 d\varrho_2 d\varrho_1$$

$$+ \int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\breve{r}(\imath,\jmath)||\sigma(\imath,\jmath) - \tilde{\sigma}(\imath,\jmath)|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})|\breve{p}(\imath,\jmath)|} d\imath d\jmath$$

$$+ \int_0^w \int_0^u \frac{(w-\jmath)^{\lambda_{21}-1}(u-\imath)^{\lambda_{22}-1}|\check{\mathbf{r}}(\imath,\jmath)|\varpi_2(\imath,\jmath)|\sigma(\imath,\jmath)-\tilde{\sigma}(\imath,\jmath)|}{\Gamma(\lambda_{21})\Gamma(\lambda_{22})\check{\mathbf{p}}(\imath,\jmath)} d\imath d\jmath$$

$$\leq \frac{\mathbf{T_1}^{\lambda_{11}+\lambda_{21}}\mathbf{T_2}^{\lambda_{12}+\lambda_{22}}\chi^\star\|\sigma-\tilde{\sigma}\|}{\check{\mathbf{p}}^\star\Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)} + \frac{\check{\mathbf{r}}^\star\mathbf{T_1}^{\lambda_{21}}\mathbf{T_2}^{\lambda_{22}}\|\sigma-\tilde{\sigma}\|}{\check{\mathbf{p}}^\star\Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} + \frac{\check{\mathbf{r}}^\star\mathbf{T_1}^{\lambda_{21}}\mathbf{T_2}^{\lambda_{22}}\varpi_2^\star\|\sigma-\tilde{\sigma}\|}{\check{\mathbf{p}}^\star\Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}$$

$$= \frac{\mathbf{T_1}^{\lambda_{21}}\mathbf{T_2}^{\lambda_{22}}}{\check{\mathbf{p}}^\star}\left[\frac{\mathbf{T_1}^{\lambda_{11}}\mathbf{T_2}^{\lambda_{12}}\chi^\star}{\Gamma(\lambda_{11}+\lambda_{21}+1)\Gamma(\lambda_{12}+\lambda_{22}+1)} + \frac{\check{\mathbf{r}}^\star(1+\varpi_2^\star)}{\Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}\right]\|\sigma-\tilde{\sigma}\|$$

$$= \mathbf{U}_1\|\sigma-\tilde{\sigma}\|.$$

Thus,

$$|(\mathbb{K}\sigma)(w,u) - (\mathbb{K}\tilde{\sigma})(w,u)| \leq \mathbf{U}_1\|\sigma-\tilde{\sigma}\|. \tag{12}$$

By applying (10), (11) and (12), we conclude that

$$|\mathbb{H}\sigma(w,u) - \mathbb{H}\tilde{\sigma}(w,u)| = \left|\vartheta_1\big(w,u,\sigma(w,u)\big)\mathbb{K}\sigma(w,u) - \vartheta_1\big(w,u,\tilde{\sigma}(w,u)\big)\mathbb{K}\tilde{\sigma}(w,u)\right|$$

$$= \left|\vartheta_1\big(w,u,\sigma(w,u)\big)\mathbb{K}\sigma(w,u) - \vartheta_1\big(w,u,\tilde{\sigma}(w,u)\big)\mathbb{K}\sigma(w,u)\right.$$

$$\left. + \vartheta_1\big(w,u,\tilde{\sigma}(w,u)\big)\mathbb{K}\sigma(w,u) - \vartheta_1\big(w,u,\tilde{\sigma}(w,u)\big)\mathbb{K}\tilde{\sigma}(w,u)\right|$$

$$= \left|\mathbb{K}\sigma(w,u)\Big(\vartheta_1\big(w,u,\sigma(w,u)\big) - \vartheta_1\big(w,u,\tilde{\sigma}(w,u)\big)\Big)\right.$$

$$\left. + \vartheta_1\big(w,u,\tilde{\sigma}(w,u)\big)\Big(\mathbb{K}\sigma(w,u) - \mathbb{K}\tilde{\sigma}(w,u)\Big)\right|$$

$$\leq |\mathbb{K}\sigma(w,u)||\vartheta_1\big(w,u,\sigma(w,u)\big) - \vartheta_1\big(w,u,\tilde{\sigma}(w,u)\big)|$$

$$+ |\vartheta_1\big(w,u,\tilde{\sigma}(w,u)\big)||\mathbb{K}\sigma(w,u) - \mathbb{K}\tilde{\sigma}(w,u)|$$

$$\leq (\mathbf{U}_1\eta + \mathbf{U}_2 + \triangle_2^\star)\varpi_1^\star\|\sigma-\tilde{\sigma}\| + (\varpi_1^\star\eta + \vartheta_1^\star)\mathbf{U}_1\|\sigma-\tilde{\sigma}\|$$

$$= \big((\mathbf{U}_1\eta + \mathbf{U}_2 + \triangle_2^\star)\varpi_1^\star + (\varpi_1^\star\eta + \vartheta_1^\star)\mathbf{U}_1\big)\|\sigma-\tilde{\sigma}\|$$

and so $\|\mathbb{H}\sigma - \mathbb{H}\tilde{\sigma}\| \leq \big((\mathbf{U}_1\eta + \mathbf{U}_2 + \triangle_2^\star)\varpi_1^\star + (\varpi_1^\star\eta + \vartheta_1^\star)\mathbf{U}_1\big)\|\sigma-\tilde{\sigma}\|$. Now we define the map $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \big((\mathbf{U}_1\eta + \mathbf{U}_2 + \triangle_2^\star)\varpi_1^\star + (\varpi_1^\star\eta + \vartheta_1^\star)\mathbf{U}_1\big)t$. Then, $\psi \in \Psi$ and $\|\mathbb{H}\sigma - \mathbb{H}\tilde{\sigma}\| \leq \psi(\|\sigma-\tilde{\sigma}\|)$. On the other hand if $\alpha(\sigma,\tilde{\sigma}) = 0$, then $\alpha(\sigma,\tilde{\sigma})\|\mathbb{H}\sigma - \mathbb{H}\tilde{\sigma}\| = 0 \leq \psi(\|\sigma-\tilde{\sigma}\|)$. Thus, $\alpha(\sigma,\tilde{\sigma})\|\mathbb{H}\sigma - \mathbb{H}\tilde{\sigma}\| \leq \psi(\|\sigma-\tilde{\sigma}\|)$ for all $\sigma,\tilde{\sigma} \in \mathcal{Y}$. Hence, $\mathbb{H}$ is α-ψ-contractive map. Suppose that $\{\sigma_n\}$ is a sequence in $\mathcal{Y}$ such that $\alpha(\sigma_{n-1},\sigma_n) \geq 1$ for all $n \geq 1$ and $\sigma_n \to \sigma \in \mathcal{Y}$. Then, $\sigma_n(w,u) \leq \eta$ and so $\lim_{n\to\infty}\sigma_n(w,u) = \sigma \leq \eta$. This implies $\alpha(\sigma_n,\sigma) \geq 1$ for all $n \geq 1$. It is clear that there exists $\sigma_0 \in \mathcal{Y}$ such that $\alpha(\sigma_0,\mathbb{H}\sigma_0) \geq 1$. Thus all conditions of Lemma 1 hold and so $\mathbb{H}$ has a fixed point in $\mathcal{Y}$ which is a solution for the problem mentioned in (1). □

Now we are going to state and prove our second main result. The next Lemma plays a key role in the proof of our second main Theorem.

Lemma 3. *Let κ be a natural number and suppose that the pairs $\Lambda_1 = (\lambda_{11},\lambda_{12})$, $\Lambda_2 = (\lambda_{21},\lambda_{22})$, $Y_1 = (\epsilon_{11},\epsilon_{12})$, $Y_2 = (\epsilon_{21},\epsilon_{22}),\ldots,Y_\kappa = (\epsilon_{\kappa 1},\epsilon_{\kappa 2})$ belong to $(0,1] \times (0,1]$. Consider the problem*

$$\mathcal{D}_c^{\Lambda_1}\left(\frac{\check{\mathbf{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u) + \check{\mathbf{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{Y_1}\sigma(w,u),\mathcal{I}_c^{Y_2}\sigma(w,u),\ldots,\mathcal{I}_c^{Y_\kappa}\sigma(w,u)\big)}\right) = \ell\big(w,u,\sigma(w,u)\big) \tag{13}$$

with boundary conditions

$$
\begin{cases}
\left[\dfrac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}\right]_{w=0}=\Re_2(u), \\[4mm]
\left[\dfrac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}\right]_{u=0}=\Re_1(w), \\[4mm]
\sigma(0,u)=\hbar_2(u)\quad and \quad \sigma(w,0)=\hbar_1(w).
\end{cases}
\tag{14}
$$

Then the function $\sigma \in \mathcal{C}\left(\mathcal{I}_{\mathbf{T}_1}\times\mathcal{I}_{\mathbf{T}_2}\right)$ *is a solution of the problem* (13) *and* (14), *if*

$$
\sigma(w,u)
$$
$$
=\mathcal{I}_c^{\Lambda_2}\left[\frac{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)\big(\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u))+\Omega(w,u)\big)}{\mathbf{\check{p}}(w,u)}\right]
$$
$$
-\mathcal{I}_c^{\Lambda_2}\Big[\frac{\mathbf{\check{r}}(w,u)\sigma(w,u)}{\mathbf{\check{p}}(w,u)}\Big]+\Phi(w,u)
$$

where

$$
\begin{cases}
\Phi(w,u)=\hbar_2(u)+\hbar_1(w)-\hbar_1(0), \\
\Omega(w,u)=+\Re_2(u)+\Re_1(w)-\Re_1(0).
\end{cases}
$$

Proof. As we know, the equation in (13), can be rewritten as

$$
\mathcal{I}^{1-\Lambda_1}\left[\frac{\partial^2}{\partial w\partial u}\left(\frac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}\right)\right]=\ell\big(w,u,\sigma(w,u)\big).
$$

Thus

$$
\mathcal{I}^{1}\left[\frac{\partial^2}{\partial w\partial u}\left(\frac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}\right)\right]=\mathcal{I}^{\Lambda_1}\ell\big(w,u,\sigma(w,u)\big).
$$

Since

$$
\mathcal{I}^{1}\left[\frac{\partial^2}{\partial w\partial u}\left(\frac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}\right)\right]
$$
$$
=\frac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}
$$
$$
-\left(\frac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}\right)_{w=0}
$$
$$
-\left(\frac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}\right)_{u=0}
$$
$$
+\left(\frac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}\right)_{(w,u)=(0,0)}
$$
$$
=\frac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}-\Re_2(u)-\Re_1(w)+\Re_1(0)
$$

and therefore, we obtain

$$
\frac{\mathbf{\check{p}}(w,u)\mathcal{D}_c^{\Lambda_2}\sigma(w,u)+\mathbf{\check{r}}(w,u)\sigma(w,u)}{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)}
$$
$$
=\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u))+\Re_2(u)+\Re_1(w)-\Re_1(0)
$$
$$
=\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u))+\Omega(w,u).
$$

After doing some direct computations, we obtain

$$\mathcal{D}_c^{\Lambda_2}\sigma(w,u)$$
$$= \frac{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{Y_1}\sigma(w,u),\mathcal{I}_c^{Y_2}\sigma(w,u),...,\mathcal{I}_c^{Y_\kappa}\sigma(w,u)\big)\big(\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u))+\Omega(w,u)\big)}{\check{\mathbf{p}}(w,u)}$$
$$- \frac{\check{\mathbf{r}}(w,u)\sigma(w,u)}{\check{\mathbf{p}}(w,u)}.$$

As $\mathcal{D}_c^{\Lambda_2}\sigma(w,u) = \mathcal{I}_c^{1-\Lambda_2}\big[\frac{\partial}{\partial w \partial u}\sigma(w,u)\big]$, we get

$$\mathcal{I}_c^1 \frac{\partial}{\partial w \partial u}\sigma(w,u)$$
$$= \mathcal{I}_c^{\Lambda_2}\left[\frac{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{Y_1}\sigma(w,u),\mathcal{I}_c^{Y_2}\sigma(w,u),...,\mathcal{I}_c^{Y_\kappa}\sigma(w,u)\big)\big(\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u))+\Omega(w,u)\big)}{\check{\mathbf{p}}(w,u)}\right]$$
$$- \mathcal{I}_c^{\Lambda_2}\big[\frac{\check{\mathbf{r}}(w,u)\sigma(w,u)}{\check{\mathbf{p}}(w,u)}\big].$$

Eventually, we can write

$$\sigma(w,u)$$
$$= \mathcal{I}_c^{\Lambda_2}\left[\frac{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{Y_1}\sigma(w,u),\mathcal{I}_c^{Y_2}\sigma(w,u),...,\mathcal{I}_c^{Y_\kappa}\sigma(w,u)\big)\big(\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u))+\Omega(w,u)\big)}{\check{\mathbf{p}}(w,u)}\right]$$
$$- \mathcal{I}_c^{\Lambda_2}\big[\frac{\check{\mathbf{r}}(w,u)\sigma(w,u)}{\check{\mathbf{p}}(w,u)}\big] + \Phi(w,u).$$

Finally, note that

$$\mathcal{I}_c^1 \frac{\partial}{\partial w \partial u}\sigma(w,u) = \sigma(w,u) - \sigma(0,u) - \sigma(w,0) + \sigma(0,0)$$
$$= \sigma(w,u) - \hbar_2(u) - \hbar_1(w) + \hbar_1(0)$$
$$= \sigma(w,u) - \Phi(w,u).$$

This completes the proof. □

Now we establish and prove our second main theorem.

Theorem 2. *Assume that the following conditions are satisfied*

$(\mathcal{O}_1)$ *There exist $\beta,\omega \in \mathcal{C}(\mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}, \mathbb{R}^+)$, such that for all $(w,u,\mathbf{r}_1,\mathbf{r}_2) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2} \times \mathbb{R} \times \mathbb{R}$ and $(w,u,r_0,r_1,...,r_\kappa,s_0,s_1,...,s_\kappa) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2} \times \mathbb{R}^{2\kappa+2}$, we have the following functional inequalities hold true*

$$\begin{cases} |\ell(w,u,\mathbf{r}_1) - \ell(w,u,\mathbf{r}_2)| \leq \omega(w,u)|\mathbf{r}_1 - \mathbf{r}_1|, \\ \big|\Theta\big(w,u,r_0,r_1,...,r_\kappa\big) - \Theta\big(w,u,s_0,s_1,...,s_\kappa\big)\big| \leq \beta(w,u)\sum_{m=0}^{\kappa}|r_m - s_m| \end{cases}$$

$(\mathcal{O}_2)$ *there exists $\mu > 0$ such that*

$$\frac{\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\check{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}\left[\big(\frac{(\omega^\star \mu + \ell^\star)\mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)} + \Omega^\star\big)\big(\beta^\star \sum_{m=0}^{\kappa}\frac{\mathbf{T}_1^{\lambda_{m1}}\mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)}\mu + \Theta^\star\big) + \check{\mathbf{r}}^\star\right] + \Phi^\star \leq \mu$$

and

$$\frac{\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\check{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}\left[\frac{\mathbf{W}_2 \mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)} + \mathbf{W}_1 \beta^\star \sum_{m=0}^{\kappa}\frac{\mathbf{T}_1^{\lambda_{m1}}\mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)} + \check{\mathbf{r}}^\star\right] < 1,$$

where $\mathbf{W}_1 = \frac{(\omega^\star \mu + \ell^\star)\mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)} + \Omega^\star$, $\mathbf{W}_2 = \beta^\star \sum_{m=0}^{\kappa}\frac{\mathbf{T}_1^{\lambda_{m1}}\mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)}\mu + \Theta^\star$, $\ell^\star, \Theta^\star, \omega^\star, \Omega^\star,$
$\Phi^\star, \check{\mathbf{p}}^\star, \check{\mathbf{r}}^\star, \beta^\star$ *indicate the supremum of $\ell(w,u,0)$, $\Theta(w,u,0,...,0)$, $\omega(w,u,0)$, $\Omega(w,u)$, $\Phi(w,u)$,*
$\check{\mathbf{p}}(w,u)$, $\check{\mathbf{p}}(w,u)$, $\check{\mathbf{r}}(w,u)$, $\beta(w,u)$ *respectively.*
Then the problem mentioned in (3) *has a solution.*

Proof. Define $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, \infty)$ by

$$
\alpha(\sigma, \tilde{\sigma}) = \begin{cases} 1 & \text{if} \quad \sigma(w, u) \leq \eta, \quad \forall (w, u) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}, \\ 0 & \text{otherwise.} \end{cases}
$$

Additionally, define $\mathbb{M} : \mathcal{Y} \to \mathcal{Y}$ by

$$
\mathbb{M}\sigma(w, u)
$$

$$
= \mathcal{I}_c^{\Lambda_2} \left[\frac{\Theta\big(w, u, \sigma(w, u), \mathcal{I}_c^{\Upsilon_1}\sigma(w, u), \mathcal{I}_c^{\Upsilon_2}\sigma(w, u), ..., \mathcal{I}_c^{\Upsilon_\kappa}\sigma(w, u)\big)\big(\mathcal{I}^{\Lambda_1}\ell(w, u, \sigma(w, u)) + \Omega(w, u)\big)}{\check{\mathbf{p}}(w, u)} \right]
$$

$$
- \mathcal{I}_c^{\Lambda_2} \big[\frac{\check{\mathbf{r}}(w, u)\sigma(w, u)}{\check{\mathbf{p}}(w, u)} \big] + \Phi(w, u)
$$

According to Lemma 3, finding a fixed point for the operator $\mathbb{M}\sigma$ is equivalent to finding a solution to problem (3). By using Lemma 1, σ_0 solves problem (3), if and only if σ_0 is a fixed point of $\mathbb{M}$. Now we are going to prove that all conditions of Lemma 1 are holding for $\mathbb{M}$. At first, we have

$$
\big| \ell\big(w, u, \sigma(w, u)\big) \big| \leq \omega^\star \boldsymbol{\mu} + \ell^\star.
$$

Then, we can write

$$
\big| \mathcal{I}^{\Lambda_1} \ell\big(w, u, \sigma(w, u)\big) + \Omega(w, u) \big| \leq \mathcal{I}^{\Lambda_1} \big| \ell\big(w, u, \sigma(w, u)\big) \big| + |\Omega(w, u)|
$$

$$
\leq (\omega^\star \boldsymbol{\mu} + \ell^\star) \int_0^w \int_0^u \frac{(w - s)^{\lambda_{11}-1}(u - t)^{\lambda_{12}-1}}{\Gamma(\lambda_{11})\Gamma(\lambda_{12})} dt ds + \Omega^\star
$$

$$
\leq \frac{(\omega^\star \boldsymbol{\mu} + \ell^\star)\mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11} + 1)\Gamma(\lambda_{12} + 1)} + \Omega^\star.
$$

Thus,

$$
\big| \mathcal{I}^{\Lambda_1} \ell\big(w, u, \sigma(w, u)\big) + \Omega(w, u) \big| \leq \frac{(\omega^\star \boldsymbol{\mu} + \ell^\star)\mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11} + 1)\Gamma(\lambda_{12} + 1)} + \Omega^\star = \mathbf{W_1}. \tag{15}
$$

Furthermore

$$
\left| \Theta\big(w, u, \sigma(w, u), \mathcal{I}_c^{\Upsilon_1}\sigma(w, u), \mathcal{I}_c^{\Upsilon_2}\sigma(w, u), ..., \mathcal{I}_c^{\Upsilon_\kappa}\sigma(w, u)\big) \right|
$$

$$
= \left| \Theta\big(w, u, \sigma(w, u), \mathcal{I}_c^{\Upsilon_1}\sigma(w, u), \mathcal{I}_c^{\Upsilon_2}\sigma(w, u), ..., \mathcal{I}_c^{\Upsilon_\kappa}\sigma(w, u)\big) \right.
$$

$$
\left. - \Theta\big(w, u, 0, 0, 0, ..., 0, 0\big) + \Theta\big(w, u, 0, 0, 0, ..., 0, 0\big) \right|
$$

$$
\leq \beta(w, u) \sum_{m=0}^{\kappa} |\mathcal{I}_c^{\Upsilon_m}\sigma(w, u)| + |\Theta\big(w, u, 0, 0, 0, ..., 0, 0\big)|
$$

$$
\leq \beta^\star \boldsymbol{\mu} \sum_{m=0}^{\kappa} \mathcal{I}_c^{\Upsilon_m}(1) + \Theta^\star
$$

$$
= \beta^\star \sum_{m=0}^{\kappa} \frac{\mathbf{T}_1^{\lambda_{m1}}\mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1} + 1)\Gamma(\lambda_{m2} + 1)}\boldsymbol{\mu} + \Theta^\star.
$$

That is, we proved that

$$
\left| \Theta\big(w, u, \sigma(w, u), \mathcal{I}_c^{\Upsilon_1}\sigma(w, u), \mathcal{I}_c^{\Upsilon_2}\sigma(w, u), ..., \mathcal{I}_c^{\Upsilon_\kappa}\sigma(w, u)\big) \right| \leq \mathbf{W_2}. \tag{16}
$$

So by utilizing the Equations (4) and (16), we get

$$\left| \mathcal{I}_c^{\Lambda_2} \left[\frac{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)\big(\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u))+\Omega(w,u)\big)}{\check{\mathbf{p}}(w,u)} \right] \right|$$

$$\leq \mathcal{I}_c^{\Lambda_2} \left[\frac{\big|\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)\big|\big|\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u))+\Omega(w,u)\big|}{|\check{\mathbf{p}}(w,u)|} \right]$$

$$\leq \frac{\mathbf{W_1 W_2}}{\check{\mathbf{p}}^\star} \mathcal{I}_c^{\Lambda_2}(1) = \frac{\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}\mathbf{W_1 W_2}}{\check{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)}.$$

Further, we deduce that

$$\left| \mathcal{I}_c^{\Lambda_2}\big[\frac{\check{\mathbf{r}}(w,u)\sigma(w,u)}{\check{\mathbf{p}}(w,u)}\big] \right| + \big|\Phi(w,u)\big| \leq \frac{\check{\mathbf{r}}^\star \mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\check{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} + \Phi^\star.$$

Since

$$|\mathbb{M}\sigma(w,u)|$$

$$\leq \left| \mathcal{I}_c^{\Lambda_2} \left[\frac{\Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big)\big(\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u))+\Omega(w,u)\big)}{\check{\mathbf{p}}(w,u)} \right] \right|$$

$$+ |\mathcal{I}_c^{\Lambda_2}\big[\frac{\check{\mathbf{r}}(w,u)\sigma(w,u)}{\check{\mathbf{p}}(w,u)}\big]| + |\Phi(w,u)|,$$

we have

$$|\mathbb{M}\sigma(w,u)| \leq \frac{\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\check{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \left[\frac{(\omega^\star \boldsymbol{\mu}+\ell^\star)\mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)} + \Omega^\star \right]$$

$$\times \left[\beta^\star \sum_{m=0}^{\kappa} \frac{\mathbf{T}_1^{\lambda_{m1}}\mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)}\boldsymbol{\mu} + \Theta^\star \right] + \frac{\check{\mathbf{r}}^\star \mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\check{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} + \Phi^\star$$

$$= \frac{\mathbf{T}_1^{\lambda_{21}}\mathbf{T}_2^{\lambda_{22}}}{\check{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \left[\Big(\frac{(\omega^\star \boldsymbol{\mu}+\ell^\star)\mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)} + \Omega^\star \Big) \right.$$

$$\left. \times \Big(\beta^\star \sum_{m=0}^{\kappa} \frac{\mathbf{T}_1^{\lambda_{m1}}\mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)}\boldsymbol{\mu} + \Theta^\star \Big) + \check{\mathbf{r}}^\star \right] + \Phi^\star$$

$$\leq \boldsymbol{\mu}.$$

Hence, $\mathbb{M}$ is α-admissible. Put

$$f_1\sigma(w,u) := \Theta\big(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u)\big),$$

and

$$f_2\sigma(w,u) := \mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u)) + \Omega(w,u).$$

So by applying Equations (15) and (16), we arrive at

$$|f_1\sigma(w,u)f_2\sigma(w,u) - f_1\tilde{\sigma}(w,u)f_2\tilde{\sigma}(w,u)|$$
$$= |f_1\sigma(w,u)f_2\sigma(w,u) - f_1\sigma(w,u)f_2\tilde{\sigma}(w,u)$$
$$+ f_1\sigma(w,u)f_2\tilde{\sigma}(w,u) - f_1\tilde{\sigma}(w,u)f_2\tilde{\sigma}(w,u)|$$
$$\leq |f_1\sigma(w,u)||f_2\sigma(w,u) - f_2\tilde{\sigma}(w,u)|$$
$$+ |f_2\sigma(w,u)||f_1\sigma(w,u) - f_1\tilde{\sigma}(w,u)|$$
$$\leq \left[\frac{\mathbf{W_2}\mathbf{T}_1^{\lambda_{11}}\mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)} \right.$$

$$+ \mathbf{W_1}\beta^\star \sum_{m=0}^{\kappa} \frac{\mathbf{T}_1^{\lambda_{m1}} \mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)} \Bigg] \|\sigma - \tilde{\sigma}\|.$$

This yields

$$\mathcal{I}_c^{\Lambda_2} \left| \frac{(f_1\sigma)(w,u)(f_2\sigma)(w,u)}{\breve{\mathbf{p}}(w,u)} - \frac{(f_1\tilde{\sigma})(w,u)(f_2\tilde{\sigma})(w,u)}{\breve{\mathbf{p}}(w,u)} \right|$$

$$\leq \frac{\mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \Bigg[\frac{\mathbf{W_2}\mathbf{T}_1^{\lambda_{11}} \mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)}$$

$$+ \mathbf{W_1}\beta^\star \sum_{m=0}^{\kappa} \frac{\mathbf{T}_1^{\lambda_{m1}} \mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)} \Bigg] \|\sigma - \tilde{\sigma}\|. \tag{17}$$

Furthermore,

$$\mathcal{I}_c^{\Lambda_2} \left| \frac{\breve{\mathbf{r}}(w,u)(\sigma(w,u) - \tilde{\sigma}(w,u))}{\breve{\mathbf{p}}(w,u)} \right| \leq \frac{\breve{\mathbf{r}}^\star \mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \|\sigma - \tilde{\sigma}\|. \tag{18}$$

Now, since

$$|\mathbb{M}\sigma(w,u) - \mathbb{M}\tilde{\sigma}(w,u)|$$

$$\leq \mathcal{I}_c^{\Lambda_2} \left| \frac{\Theta(w,u,\sigma(w,u),\mathcal{I}_c^{\Upsilon_1}\sigma(w,u),\mathcal{I}_c^{\Upsilon_2}\sigma(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\sigma(w,u))(\mathcal{I}^{\Lambda_1}\ell(w,u,\sigma(w,u)) + \Omega(w,u))}{\breve{\mathbf{p}}(w,u)} \right.$$

$$\left. - \frac{\Theta(w,u,\tilde{\sigma}(w,u),\mathcal{I}_c^{\Upsilon_1}\tilde{\sigma}(w,u),\mathcal{I}_c^{\Upsilon_2}\tilde{\sigma}(w,u),...,\mathcal{I}_c^{\Upsilon_\kappa}\tilde{\sigma}(w,u))(\mathcal{I}^{\Lambda_1}\ell(w,u,\tilde{\sigma}(w,u)) + \Omega(w,u))}{\breve{\mathbf{p}}(w,u)} \right|$$

$$+ \mathcal{I}_c^{\Lambda_2} \left| \frac{\breve{\mathbf{r}}(w,u)(\sigma(w,u) - \tilde{\sigma}(w,u))}{\breve{\mathbf{p}}(w,u)} \right|,$$

then by using Equations (17) and (18), we derive that

$$|\mathbb{M}\sigma(w,u) - \mathbb{M}\tilde{\sigma}(w,u)| \leq \frac{\mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \Bigg[\frac{\mathbf{W_2}\mathbf{T}_1^{\lambda_{11}} \mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)}$$

$$+ \mathbf{W_1}\beta^\star \sum_{m=0}^{\kappa} \frac{\mathbf{T}_1^{\lambda_{m1}} \mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)} \Bigg] \|\sigma - \tilde{\sigma}\|$$

$$+ \frac{\breve{\mathbf{r}}^\star \mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \|\sigma - \tilde{\sigma}\|$$

$$= \frac{\mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \Bigg[\frac{\mathbf{W_2}\mathbf{T}_1^{\lambda_{11}} \mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)}$$

$$+ \mathbf{W_1}\beta^\star \sum_{m=0}^{\kappa} \frac{\mathbf{T}_1^{\lambda_{m1}} \mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)} + \breve{\mathbf{r}}^\star \Bigg] \|\sigma - \tilde{\sigma}\|.$$

Thus $\|\mathbb{M}\sigma - \mathbb{M}\tilde{\sigma}\| \leq \mathfrak{g}\|\sigma - \tilde{\sigma}\|$, where

$$\mathfrak{g} := \frac{\mathbf{T}_1^{\lambda_{21}} \mathbf{T}_2^{\lambda_{22}}}{\breve{\mathbf{p}}^\star \Gamma(\lambda_{21}+1)\Gamma(\lambda_{22}+1)} \Bigg[\frac{\mathbf{W_2}\mathbf{T}_1^{\lambda_{11}} \mathbf{T}_2^{\lambda_{12}}}{\Gamma(\lambda_{11}+1)\Gamma(\lambda_{12}+1)} + \mathbf{W_1}\beta^\star \sum_{m=0}^{\kappa} \frac{\mathbf{T}_1^{\lambda_{m1}} \mathbf{T}_2^{\lambda_{m2}}}{\Gamma(\lambda_{m1}+1)\Gamma(\lambda_{m2}+1)} + \breve{\mathbf{r}}^\star \Bigg] < 1.$$

Define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \mathfrak{g}t$. Then $\psi \in \Psi$ and $\mathbb{M}$ is α-ψ-contractive mapping. Suppose that $\{\sigma_n\}$ is a sequence in $\mathcal{Y}$ such that $\alpha(\sigma_{n-1},\sigma_n) \geq 1$ for all $n \geq 1$ and $\sigma_n \to \sigma \in \mathcal{Y}$. Hence, $\sigma_n(w,u) \leq \eta$. So $\lim_{n\to\infty} \sigma_n(w,u) = \sigma \leq \eta$. That is, $\alpha(\sigma_n,\sigma) \geq 1$ for all $n \geq 1$. Further evidently, there exists $\sigma_0 \in \mathcal{Y}$ such that $\alpha(\sigma_0, \mathbb{M}\sigma_0) \geq 1$. Then all conditions of Lemma 1 are valid and $\mathbb{M}$ has a fixed point in $\mathcal{Y}$ which is a solution for the problem (3). □

3. Examples

Example 1. *Let $(w, u) \in [0, \frac{1}{2}] \times [0, 1]$. Consider the fractional partial Sturm-Lioville problem*

$$
\mathcal{D}_c^{(\frac{1000}{1001}, \frac{2000}{2001})} \left[102 e^{\sqrt[5]{w^4 + u^2}} \mathcal{D}_c^{(\frac{1}{2}, \frac{1}{3})} \left(\frac{\sigma(w, u)}{\frac{1}{1 + w^2 + u^2 + \ln(wu+1)} + \frac{wu}{6} |\sigma(w, u)|} \right) \right.
$$
$$
\left. + e^{\sin^2(\frac{\pi wu}{1 + w + u})} \left(\sigma(w, u) - e^{-\frac{\pi^2}{1 + \frac{1}{1 + \sin^2(w+u)}}} - \frac{e^{-w^2 u^6} |\sigma(w, u))|}{1 + |\sigma(w, u))|} \right) \right]
$$
$$
= \cos(\pi wu) + \frac{1}{10 + \ln(w^4 + u^8 + 1)} + \frac{e^{\frac{w^2 + u^2}{2}} + 1}{8} \sigma(w, u),
$$

(19)

with boundary conditions

$$
\begin{cases}
\left[102 e^{-\sqrt[5]{w^4 + u^2}} \mathcal{D}_c^{(\frac{1}{2}, \frac{1}{3})} \left(\frac{|\sigma(w,u)|}{\frac{1}{1 + w^2 + u^2 + \ln(wu+1)} + \frac{wu}{6} |\sigma(w,u)|} \right) + e^{\sin^2(\frac{\pi wu}{1 + w + u})} \left(\sigma(w, u) \right. \right. \\
\left. \left. - e^{-\frac{\pi^2}{1 + \frac{1}{1 + \sin^2(w+u)}}} - \frac{e^{-w^2 u^6} |\sigma(w,u))|}{1 + |\sigma(w,u))|} \right) \right]_{w=0} = e^u, \\[3mm]
\left[102 e^{\sqrt[5]{w^4 + u^2}} \mathcal{D}_c^{(\frac{1}{2}, \frac{1}{3})} \left(\frac{\sigma(w,u)}{\frac{1}{1 + w^2 + u^2 + \ln(wu+1)} + \frac{wu}{6} |\sigma(w,u)|} \right) + e^{\sin^2(\frac{\pi wu}{1 + w + u})} \left(\sigma(w, u) \right. \right. \\
\left. \left. - e^{-\frac{\pi^2}{1 + \frac{1}{1 + \sin^2(w+u)}}} - \frac{e^{-w^2 u^6} |\sigma(w,u))|}{1 + |\sigma(w,u))|} \right) \right]_{u=0} = \cos(w), \\[3mm]
\left[\frac{\sigma(w,u)}{\frac{1}{1 + w^2 + u^2 + \ln(wu+1)} + \frac{wu}{6} |\sigma(w,u)|} \right]_{w=0} = \ln(u + 1), \\[3mm]
\left[\frac{\sigma(w,u)}{\frac{1}{1 + w^2 + u^2 + \ln(wu+1)} + \frac{wu}{6} |\sigma(w,u)|} \right]_{u=0} = \arcsin(w)
\end{cases}
$$

(20)

Put, $\eta = 1$, $\mathbf{T}_1 = \frac{1}{2}$, $\mathbf{T}_2 = 1$, $\Lambda_1 = (\lambda_{11}, \lambda_{12}) = (\frac{1000}{1001}, \frac{2000}{2001})$, $\Lambda_2 = (\lambda_{21}, \lambda_{22}) = (\frac{1}{2}, \frac{1}{3})$, $\wp_2(u) = \frac{1}{2} e^u$, $\wp_1(w) = \frac{1}{1 + \cos^4(\pi w)}$, $\Im_2(u) = \frac{1}{2} \ln(u + 1)$, $\Im_1(w) = \frac{1}{10} \sin(\pi w)$, $\mathbf{\check{p}}(w, u) = 102 e^{\sqrt[5]{w^4 + u^2}}$, $\mathbf{\check{r}}(w, u) = e^{\sin^2(\frac{\pi wu}{1 + w + u})}$, $\vartheta_1(w, u, r) = \frac{1}{1 + w^2 + u^2 + \ln(wu+1)} + \frac{wu}{6} |r|$, $\vartheta_2(w, u, r) = e^{-\frac{\pi^2}{1 + \frac{1}{1 + \sin^2(w+u)}}} + \frac{e^{-w^2 u^6} |r|}{1 + |r|}$, $\zeta(w, u, r) = \cos(\pi wu) + \frac{1}{10 + \ln(w^4 + u^8 + 1)} + \frac{e^{\frac{w^2 + u^2}{2}} + 1}{8} r$.

The graphs and numerical results for these functions are presented in Figures 1, 2 and Table 1. Hence,

$$
\begin{cases}
|\zeta(w, u, \mathbf{r}_1) - \zeta(w, u, \mathbf{r}_2)| \leq \frac{e^{\frac{w^2 + u^2}{2}} + 1}{8} |\mathbf{r}_1 - \mathbf{r}_1| \\[2mm]
|\vartheta_1(w, u, \mathbf{r}_1) - \vartheta_1(w, u, \mathbf{r}_2)| \leq \frac{wu}{6} |\mathbf{r}_1 - \mathbf{r}_1| \\[2mm]
|\vartheta_2(w, u, \mathbf{r}_1) - \vartheta_2(w, u, \mathbf{r}_2)| \leq e^{-w^2 u^6} |\mathbf{r}_1 - \mathbf{r}_1|
\end{cases}
$$

and so

$$
\begin{cases}
\chi(w, u) = \frac{e^{\frac{w^2 + u^2}{2}} + 1}{8}, \\[2mm]
\wp_1(w, u) = \frac{wu}{6} \\[2mm]
\wp_2(w, u) = e^{-w^2 u^6}.
\end{cases}
$$

Also, we have $\chi^\star = \sup_{(w,u) \in [0, \frac{1}{2}] \times [0,1]} \chi(w, u) = \sup_{(w,u) \in [0, \frac{1}{2}] \times [0,1]} \frac{e^{\frac{w^2 + u^2}{2}} + 1}{8} = \frac{e^{\frac{5}{8}} + 1}{8}$, $\wp_1^\star = \frac{1}{12}$, $\wp_2^\star = 1$, $\zeta^\star = \frac{11}{10}$, $\vartheta_1^\star = 1$, $\vartheta_2^\star = e^{-\frac{\pi^2}{2}}$, $\mathbf{\check{p}}^\star = 102$, $\mathbf{\check{r}}^\star = e$, $\triangle_1^\star = \frac{1}{2}(e + 1)$ and $\triangle_2^\star = \frac{1}{2} \ln 2 + \frac{1}{10}$. Then, we get

$$\begin{cases} \mathbf{U}_1 = \dfrac{(\frac{1}{2})^{\frac{1}{2}}}{102} \left[\dfrac{(\frac{1}{2})^{\frac{1}{2}}(\frac{e^{\frac{5}{8}}+1}{8})}{\Gamma(\frac{5003}{2002})\Gamma(\frac{14004}{6003})} + \dfrac{2e}{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})} \right] \approx 0.0489462698139 \\[4mm] \mathbf{U}_2 = \dfrac{(\frac{1}{2})^{\frac{1}{2}}}{102} \left[\dfrac{(\frac{1}{2})^{\frac{1000}{1001}}(\frac{11}{10})}{\Gamma(\frac{5003}{2002})\Gamma(\frac{14004}{6003})} + \dfrac{e(\frac{1}{2}(e+1)+e^{-\frac{\pi^2}{2}})}{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})} \right] \approx 0.0468538853756 \end{cases}$$

and so, we find

$$\eta = 1 \geq 0.5875715575919$$

$$\approx (\frac{1}{12} \times 1 + 1)(0.0489462698139 \times 1 + 0.0468538853756 + \frac{1}{2}\ln 2 + \frac{1}{10})$$

and

$$(0.0489462698139 \times 1 + 0.0468538853756 + \frac{1}{2}\ln 2 + \frac{1}{10}) \times \frac{1}{12}$$

$$+ (\frac{1}{12} \times 1 + 1) \times 0.0489462698139 \approx 0.0982229377542 < 1.$$

Thus, all conditions of Theorem 1 hold and the problem (19) *and* (20) *has a solution.*

Table 1. Numerical resualts for some functions in Example 1.

w	$\wp_1(w)$	$\Im_1(w)$	u	$\wp_2(u)$	$\Im_2(u)$
0	0.5000	0	0	0.5000	0
0.1	0.5500	0.0309	0.2	0.6107	0.0911
0.2	0.7000	0.0587	0.4	0.7459	0.1682
0.3	0.8933	0.0809	0.6	0.9110	0.2350
0.4	0.9909	0.0951	0.8	1.1127	0.2938
0.5	1	0.1	1	1.3591	0.3465
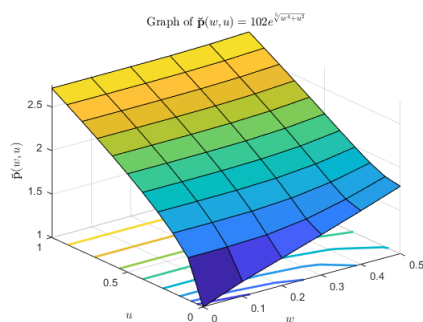
Figure 1. The graph of $\check{\mathbf{p}}(w,u)$ in Example 1.
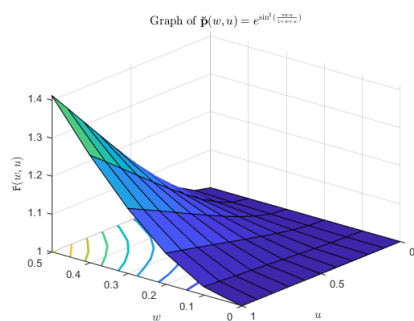

Figure 2. The graph of $\check{\mathbf{r}}(w,u)$ in Example 1.

Example 2. *Let* $(w, u) \in [0, 1] \times [0, \frac{1}{3}]$. *Consider the fractional partial Sturm-Lioville problem*

$$\mathcal{D}_c^{(\frac{1}{100}, \frac{1}{200})} \left(\frac{96 \ln(e + \sqrt[9]{w^2 + u^5}) \mathcal{D}_c^{(\frac{1}{7}, \frac{1}{11})} \sigma(w, u) + \sqrt{1 + w^2 + u^3} \sigma(w, u)}{\Theta(w, u, \sigma(w, u), \mathcal{I}_c^{(\frac{1}{101}, \frac{1}{202})} \sigma(w, u))} \right) = e^{\frac{1}{1 + w^2 + u^2} - 1} + \frac{|\sigma(w, u)|}{1 + |\sigma(w, u)|} \quad (21)$$

with boundary conditions

$$\begin{cases} \left[\dfrac{96 \ln(e + \sqrt[9]{w^2 + u^5}) \mathcal{D}_c^{(\frac{1}{7}, \frac{1}{11})} \sigma(w, u) + \sqrt{1 + w^2 + u^3} \sigma(w, u)}{\Theta(w, u, \sigma(w, u), \mathcal{I}_c^{(\frac{1}{101}, \frac{1}{202})} \sigma(w, u))} \right]_{w=0} = \dfrac{1}{4} u^2, \\[4mm] \left[\dfrac{96 \ln(e + \sqrt[9]{w^2 + u^5}) \mathcal{D}_c^{(\frac{1}{7}, \frac{1}{11})} \sigma(w, u) + \sqrt{e + w^2 + u^3} \sigma(w, u)}{\Theta(w, u, \sigma(w, u), \mathcal{I}_c^{(\frac{1}{101}, \frac{1}{202})} \sigma(w, u))} \right]_{u=0} = \dfrac{1}{6} w, \\[4mm] \sigma(0, u) = e^{-u} \quad and \quad \sigma(w, 0) = e^{-w}, \end{cases} \quad (22)$$

where $\Theta(w, u, r_0, s_0) = e^{-w^2 - u^2} + \frac{e^{-w^2 u^2}}{80}(r_0 + s_0)$. *In this case we put,* $\boldsymbol{\mu} = 0.9$, $\mathbf{T}_1 = 1$, $\mathbf{T}_2 = \frac{1}{3}$, $\Lambda_1 = (\lambda_{11}, \lambda_{12}) = (\frac{1}{100}, \frac{1}{200})$, $\Lambda_2 = (\lambda_{21}, \lambda_{22}) = (\frac{1}{7}, \frac{1}{11})$, $Y_1 = (\epsilon_1, \epsilon_2) = (\frac{1}{101}, \frac{1}{202})$, $\Re_2(u) = \frac{1}{4} u^2$, $\Re_1(w) = \frac{1}{8} w^2$, $\hbar_2(u) = \frac{e^{-u}}{602}$, $\hbar_1(w) = \frac{e^{-w}}{301(1 + e^{-w})}$, $\check{\mathbf{p}}(w, u) = 96 \ln(e + \sqrt[9]{w^2 + u^5})$, $\check{\mathbf{r}}(w, u) = \sqrt{1 + w^2 + u^3}$, $\ell(w, u, r) = \frac{e^{\frac{1}{1 + w^2 + u^2} - 1} |r|}{1 + |r|}$, *and the graphs and numerical results for these functions are presented in Figures 3, 4 and in Table 2. So we can deduce that* $\ell^\star = \sup_{(w, u) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}} \ell(w, u, 0) = \sup_{(w, u) \in \mathcal{I}_{\mathbf{T}_1} \times \mathcal{I}_{\mathbf{T}_2}} e^{\frac{1}{1 + w^2 + u^2} - 1} = 1$, $\Theta^\star = 1$, $\Phi^\star = \frac{3}{602}$, $\check{\mathbf{p}}^\star = 96$, $\check{\mathbf{r}}^\star = \sqrt{\frac{55}{27}}$, *and* $\Omega^\star = \frac{5}{12}$. *Also, note that* $|\ell(w, u, r_1) - \ell(w, u, r_2)| \leq e^{\frac{1}{1 + w^2 + u^2} - 1} |r_1 - r_2|$ *and*

$$|\Theta(w, u, r_0, s_0) - \Theta(w, u, r_1, s_1)| \leq \frac{e^{-w^2 u^2}}{80}(|r_1 - r_0| + |s_1 - s_0|).$$

Thus, we can obtain that $\omega^\star = \frac{1}{80}$ *and* $\beta^\star = 1$. *Hence,*

$$\begin{cases} \mathbf{W}_1 = \dfrac{(\frac{1}{80} \times 0.9 + 1) \times \frac{1}{3^{\frac{1}{200}}}}{\Gamma(\frac{101}{100}) \Gamma(\frac{201}{200})} + \dfrac{5}{12} \approx 1.4310186844 \\[5mm] \mathbf{W}_2 = 0.9 + \dfrac{\frac{1}{3^{\frac{1}{202}}}}{\Gamma(\frac{102}{101}) \Gamma(\frac{203}{202})} \times 0.9 + 1 \approx 2.903038092981 \end{cases}$$

and so

$$\frac{(\frac{1}{3})^{\frac{1}{11}}}{96 \Gamma(\frac{8}{7}) \Gamma(\frac{12}{11})} \left[\left(\frac{(\frac{1}{80} \times 0.9 + 1) \times \frac{1}{3^{\frac{1}{200}}}}{\Gamma(\frac{101}{100}) \Gamma(\frac{201}{200})} + \frac{15}{12} \right) \left(\frac{\frac{1}{3^{\frac{1}{202}}}}{\Gamma(\frac{102}{101}) \Gamma(\frac{203}{202})} \times 0.9 \right. \right.$$
$$\left. \left. + 1 \right) + \sqrt{\frac{55}{27}} \right] + \frac{3}{602} \approx 0.0655017025002 \leq 0.9$$

*and*

$$\frac{(\frac{1}{3})^{\frac{1}{11}}}{96 \Gamma(\frac{8}{7}) \Gamma(\frac{12}{11})} \left[\frac{2.903038092981 \times \frac{1}{3^{\frac{1}{200}}}}{\Gamma(\frac{101}{100}) \Gamma(\frac{201}{200})} + 1.4310186844 \times \frac{\frac{1}{3^{\frac{1}{202}}}}{\Gamma(\frac{102}{101}) \Gamma(\frac{203}{202})} \right.$$
$$\left. + \sqrt{\frac{55}{27}} \right] \approx 0.0609282177553 < 1.$$

Thus all conditions of Theorem 2 hold and the problem (21) *and* (22) *has a solution.*

Table 2. Numerical resualts for some functions in Example 2.

w	$\Re_1(w)$	$\hbar_1(w)$	u	$\Re_2(u)$	$\hbar_2(u)$
0	0	1.6020	0	0	0.00166
0.3	9.8000	0.0014	0.1	0.0025	0.00150
0.5	1.3200	0.0012	0.15	0.0056	0.00142
0.7	49.8000	0.0011	0.2	0.01	0.00136
0.9	81.8000	0.0009	0.25	0.0156	0.00129
1	0.1250	0.0008	0.3	0.0225	0.00123
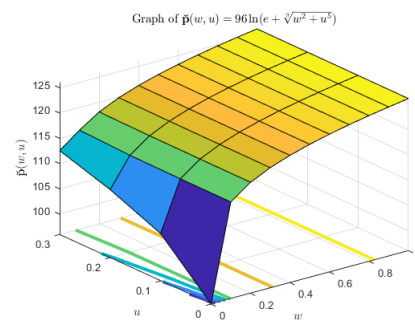
Figure 3. The graph of $\check{\mathbf{p}}(w, u)$ in Example 2.
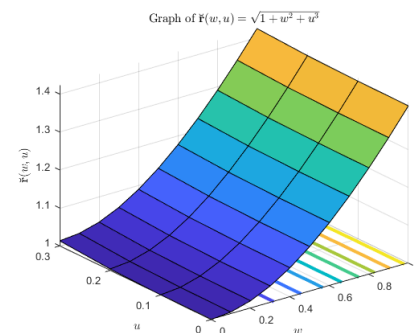
Figure 4. The graph of $\check{\mathbf{r}}(w, u)$ in Example 2.

4. Conclusions

For a better understanding and interpretation of physical phenomena, we must undoubtedly increase our ability to solve fractional differential equations. In this way, examining the properties and solutions of important equations is of particular importance. In this study, we investigated the partial fractional hybrid version of generalized Sturm–Liouville–Langevin equations. The existing derivation operators in this work are of the Caputo type. One of the recent techniques of fixed point theory, namely α-ψ-contraction, has been used to prove the existence of solutions. Also, we provided two examples with some figures and tables to illustrate our main results.

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