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A Special Family of m-Fold Symmetric Bi-Univalent Functions Satisfying Subordination Condition

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Abstract: In this paper, we introduce a special family $\mathfrak{M}_{\sigma_m}(\tau, \nu, \eta, \varphi)$ of the function family σ_m of m-fold symmetric bi-univalent functions defined in the open unit disc $\mathfrak{D}$ and obtain estimates of the first two Taylor–Maclaurin coefficients for functions in the special family. Further, the Fekete–Szegö functional for functions in this special family is also estimated. The results presented in this paper not only generalize and improve some recent works, but also give new results as special cases.

Keywords: bi-univalent functions; coefficient estimates; Fekete–Szegö functional; m-fold symmetric bi-univalent functions

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Citation: Aldawish, I.; Swamy, S.R.; Frasin, B.A. A Special Family of m-Fold Symmetric Bi-Univalent Functions Satisfying Subordination Condition. *Fractal Fract.* **2022**, *6*, 271. https://doi.org/10.3390/fractalfract6050271

Academic Editors: Acu Mugur Alexandru and Shahram Najafzadeh

Received: 24 April 2022
Accepted: 15 May 2022
Published: 17 May 2022

1. Introduction

Let $\mathcal{A}$ be the set of functions s that are holomorphic in $\mathfrak{D} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, normalized by $s(0) = s'(0) - 1 = 0$ having the form

$$s(\varsigma) = \varsigma + \sum_{k=2}^{\infty} d_k \varsigma^k. \quad (1)$$

Let $\mathcal{S}$ stand for the subfamily of $\mathcal{A}$, which is univalent in $\mathfrak{D}$. The image of $\mathfrak{D}$ under every function $s \in \mathcal{S}$ contains a disc of radius $1/4$, which is known as the one-quarter theorem of Koebe [1]. According to this, every function $s \in \mathcal{S}$ has an inverse $g = s^{-1}$ satisfying $s^{-1}(s(\varsigma)) = \varsigma$, $\varsigma \in \mathfrak{D}$ and $s(s^{-1}(\omega)) = \omega$, $|\omega| < r_0(s)$, $r_0(s) \geq 1/4$ and is in fact given by

$$g(\omega) = s^{-1}(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \cdots \quad (2)$$

If a function s and its inverse s^{-1} are both univalent in $\mathfrak{D}$, then a member s of $\mathcal{A}$ is called bi-univalent in $\mathfrak{D}$. We symbolize by σ the family of bi-univalent functions in $\mathfrak{D}$ given by (1). Some functions in the family σ are given by $\frac{\varsigma}{1-\varsigma}$, $-log(1 - \varsigma)$ and $\frac{1}{2}log\left(\frac{1+\varsigma}{1-\varsigma}\right)$. However, the Koebe function does not belong to the set σ. Other functions in the class $\mathcal{S}$, such as $\varsigma - \frac{\varsigma^2}{2}$ and $\frac{\varsigma}{1-\varsigma^2}$, are not members of σ.

Lewin [2] examined the family σ and proved that $|d_2| < 1.51$ for elements of the family σ. Later, Brannan and Clunie [3] claimed that $|d_2| < \sqrt{2}$ for $s \in \sigma$. Subsequently, Tan [4] obtained some initial coefficient estimates of functions belonging to the class σ. Brannan and Taha in [5] proposed bi-convex and bi-starlike functions, which are similar to well-known subfamilies of $\mathcal{S}$. The research trend in the last decade was the study of subfamilies of σ. Generally, interest was shown to obtain the initial coefficient bounds for certain subfamilies of σ. In 2010, Srivastava et al. [6], introduced two interesting subfamilies of the function family σ and found bounds for $|d_2|$ and $|d_3|$ of functions belonging to these subfamilies.

In 2011, Frasin and Aouf [7] studied two new subfamilies of the function family σ and obtained bounds for $|d_2|$ and $|d_3|$ of functions belonging to these subfamilies. Deniz [8], in 2013, introduced four subfamilies of the family σ and investigated bounds for $|d_2|$ and $|d_3|$ of functions belonging to these four subfamilies. Tang et al. [9], in 2013, determined the coefficient estimates for new subfamilies of Ma-Minda bi-univalent functions. Frasin [10], in 2014, examined two more new subfamilies of σ. The recent research trend is the study of functions belonging to the class σ linked with certain polynomials, such as Lucas polynomials, Chebyshev polynomials, Legendrae polynomials, Horadam polynomials, Fibonacci polynomials and Gegenbauer polynomials. Interesting results related to initial coefficient estimates and Fekete–Szegö functional problem $|d_3 - \delta d_2^2|$ for some special subfamilies of σ associated with any of the above mentioned polynomials appeared, such as the ones in [11–15].

Let $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $\eta \in \mathbb{C}^* = \mathbb{C} - \{0\}$ throughout this paper. If a rotation of the domain $\mathfrak{E}$ about the origin with an angle $2\pi/m$ maps $\mathfrak{E}$ on itself, then $\mathfrak{E}$ is known as m-fold symmetric. A holomorphic function s in $\mathfrak{D}$ is called m-fold symmetric if $s\left(e^{\frac{2\pi i}{m}}\varsigma\right) = e^{\frac{2\pi i}{m}} s(\varsigma)$. For each function $f \in \mathcal{S}$, $s(\varsigma) = \sqrt[m]{f(\varsigma^m)}$ is univalent and maps $\mathfrak{D}$ into a region with m-fold symmetry. We symbolize the family of m-fold symmetric univalent functions in $\mathfrak{D}$ by $\mathcal{S}_m$. Clearly, $\mathcal{S}_1 = \mathcal{S}$. A function $s \in \mathcal{S}_m$ has a series expansion given by

$$s(\varsigma) = \varsigma + \sum_{k=1}^{\infty} d_{mk+1}\varsigma^{mk+1} \qquad (m \in \mathbb{N}; \varsigma \in \mathfrak{D}). \tag{3}$$

A natural extension of $\mathcal{S}_m$ was explored by Srivastava et al. [16] and they introduced the family σ_m of m-fold symmetric bi-univalent functions. The series expansion for $g = s^{-1}$ obtained by them is as follows:

$$\begin{aligned} g(\omega) = s^{-1}(\omega) = \omega - d_{m+1}\omega^{m+1} + [(m+1)d_{m+1}^2 - d_{2m+1}]\omega^{2m+1} \\ -\left[\frac{(m+1)(3m+2)}{2}d_{m+1}^3 - (3m+2)d_{m+1}d_{2m+1} + d_{3m+1}\right]\omega^{3m+1} + \cdots. \end{aligned} \tag{4}$$

Some functions in the family σ_m are

$$\left(\frac{\varsigma^m}{1-\varsigma^m}\right)^{1/m}, \ \left[\frac{1}{2}log\left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)\right]^{1/m}, \ [-log(1-\varsigma^m)]^{1/m}, \ \cdots$$

and the corresponding inverse functions are

$$\left(\frac{\omega^m}{1+\omega^m}\right)^{1/m}, \ \left(\frac{e^{2\omega^m}-1}{e^{2\omega^m}-1}\right)^{1/m}, \ \left(\frac{e^{\omega^m}-1}{e^{\omega^m}}\right)^{1/m}, \ \cdots.$$

The momentum on investigation of the family σ_m was gained in recent years, which is due to two papers [17,18] of Srivastava et al., and it has led to a large number of papers on subfamilies of σ_m. Note that $\sigma_1 = \sigma$. In 2018, Srivastava et al. [19] addressed initial coefficient estimations of the Taylor–Maclaurin series of functions in a new subfamily of σ_m. Sakar and Tasar [20] introduced new subfamilies of σ_m and obtained initial coefficient bounds for functions belonging to these families, coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions were determined in [21], a comprehensive subclass of σ_m using subordination principle was examined in [22], and a special family of m-fold symmetric bi-univalent functions satisfying subordination condition was examined very recently by Swamy et al. [23]. Interesting results related to initial coefficient estimates and Fekete–Szegö functional problem $|d_{2m+1} - \delta d_{m+1}^2|$ for certain subfamilies of σ_m appeared, such as the ones in [24–26].

Inspired substantially by the works of Ma and Minda [27] and Tang et al. [28], we define a special subfamily $\mathfrak{M}_{\sigma_m}(\tau,\nu,\eta,\varphi)$ $(0\le\nu\le 1,\ \tau\ge 1,\ \eta\in\mathbb{C}^*)$ of m-fold symmetric bi-univalent functions.

Definition 1. *A function $s\in\sigma_m$ is said to be in the class $\mathfrak{M}_{\sigma_m}(\tau,\nu,\eta,\varphi)$ $(0\le\nu\le 1,\tau\ge 1)$, if*

$$\left[1+\frac{1}{\eta}\left(\frac{\varsigma(s'(\varsigma))^{\tau}}{(1-\nu)\varsigma+\nu s(\varsigma)}-1\right)\right]\prec\varphi(\varsigma)$$

and

$$\left[1+\frac{1}{\eta}\left(\frac{\omega(g'(\omega))^{\tau}}{(1-\nu)\omega+\nu g(\omega)}-1\right)\right]\prec\varphi(\omega),$$

where $\varsigma,\ \omega\in\mathfrak{D}$, $g(\omega)=s^{-1}(\omega)$ is as stated in (4).

We observe that the certain choice of ν and τ lead the class $\mathfrak{M}_{\sigma_m}(\tau,\nu,\eta,\varphi)$ to the following few subfamilies:

(i) $H_{\sigma_m}(\tau,\eta,\varphi)\equiv\mathfrak{M}_{\sigma_m}(\tau,0,\eta,\varphi)$ $(\tau\ge 1)$ is the family of $s\in\sigma_m$ of the form (1) satisfying

$$\left[1+\frac{1}{\eta}\left(\left(s'(\varsigma)\right)^{\tau}-1\right)\right]\prec\varphi(\varsigma)$$

and

$$\left[1+\frac{1}{\eta}\left(\left(g'(\omega)\right)^{\tau}-1\right)\right]\prec\varphi(\varsigma),$$

where $\varsigma,\ \omega\in\mathfrak{D}$, $g(\omega)=s^{-1}(\omega)$ is as stated in (4).

(ii) $I_{\sigma_m}(\tau,\eta,\varphi)\equiv\mathfrak{M}_{\sigma_m}(\tau,1,\eta,\varphi)(\tau\ge 1)$ is the family of $s\in\sigma_m$ of the form (1) satisfying

$$\left[1+\frac{1}{\eta}\left(\left(\frac{\varsigma(s'(\varsigma))^{\tau}}{s(\varsigma)}\right)-1\right)\right]\prec\varphi(\varsigma)$$

and

$$\left[1+\frac{1}{\eta}\left(\left(\frac{\omega(g'(\omega))^{\tau}}{g(\omega)}\right)-1\right)\right]\prec\varphi(\varsigma),$$

where $\varsigma,\ \omega\in\mathfrak{D}$, $g(\omega)=s^{-1}(\omega)$ is as stated in (4).

In Section 2, we find bounds on first two coefficients in the Taylor–Maclaurin expansion and Fekete–Szegö [29] functional problem for functions belonging to the class $\mathfrak{M}_{\sigma_m}(\tau,\nu,\eta,\varphi)$. We also indicate interesting cases of the main results. In Section 3, we obtain bounds on $|d_{m+1}|$ and $|d_{2m+1}|$ in the Taylor–Maclaurin expansion and Fekete–Szegö functional problem for functions belonging to the class $\mathfrak{W}^{\varrho}_{\sigma_m}(\tau,\nu,\eta)=\mathfrak{M}_{\sigma_m}(\tau,\nu,\eta,\left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)^{\varrho})$, $0<\varrho\le 1$. In Section 4, we determine bounds on $|d_{m+1}|$ and $|d_{2m+1}|$ in the Taylor–Maclaurin expansion and Fekete–Szegö functional problem for functions belonging to the class $\mathfrak{X}^{\xi}_{\sigma_m}(\tau,\nu,\eta)=\mathfrak{M}_{\sigma_m}(\tau,\nu,\eta,\frac{1+(1-2\xi)\varsigma^m}{1-\varsigma^m})$, $0\le\xi<1$. We also indicate interesting cases of the main results. Relevant connections to the existing results are also mentioned.

2. Coefficient Bounds for Function Family $\mathfrak{M}_{\sigma_m}(\tau,\nu,\eta,\varphi)$

We denote by P the family of holomorphic functions of the form:

$$p(\varsigma)=1+p_1\varsigma+p_2\varsigma^2+p_3\varsigma^3+\cdots,$$

with $\Re(P(\varsigma))>0$ $(\varsigma\in\mathfrak{D})$. In view of the study of Pommerenke [30], the m-fold symmetric function p in the family P is of the form:

$$p(\varsigma)=1+p_m\varsigma^m+p_{2m}\varsigma^{2m}+p_{3m}\varsigma^{3m}+\cdots.$$

In the sequel, it is assumed that $\varphi(\varsigma)$ is a holomorphic function having positive real part in $\mathfrak{D}$ satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(\mathfrak{D})$ is symmetric with respect to the real axis. Such a function has an infinite series expansion of the form

$$\varphi(\varsigma) = 1 + B_1\varsigma + B_2\varsigma^2 + B_3\varsigma^3 + \cdots (B_1 > 0).$$

Let $\mathfrak{h}(\varsigma)$ and $\mathfrak{p}(\omega)$ be two holomorphic functions in $\mathfrak{D}$ with $\mathfrak{h}(0) = \mathfrak{p}(0) = 0$ and $max\{|\mathfrak{h}(\varsigma)|; |\mathfrak{p}(\omega)|\} < 1$. We suppose that $\mathfrak{h}(\varsigma) = h_m\varsigma^m + h_{2m}\varsigma^{2m} + h_{3m}\varsigma^{3m} + \cdots$ and $\mathfrak{p}(\omega) = p_m\omega^m + p_{2m}\omega^{2m} + p_{3m}\omega^{3m} + \cdots$. Also we know that

$$|h_m| < 1; |h_{2m}| \leq 1 - |h_m|^2; |p_m| < 1; |p_{2m}| \leq 1 - |p_m|^2. \tag{5}$$

By simple calculations, we obtain

$$\varphi(\mathfrak{h}(\varsigma)) = 1 + B_1h_m\varsigma^m + (B_1h_{2m} + B_2h_m^2)\varsigma^{2m} + \ldots (|\varsigma| < 1) \tag{6}$$

and

$$\varphi(\mathfrak{p}(\omega)) = 1 + B_1p_m\omega^m + (B_1p_{2m} + B_2p_m^2)\omega^{2m} + \ldots (|\omega| < 1). \tag{7}$$

Theorem 1. *Let $\tau \geq 1$ and $0 \leq \nu \leq 1$. If a function s in $\mathcal{A}$ belongs to the class $\mathfrak{M}_{\sigma_m}(\tau, \nu, \eta, \varphi)$, then*

$$|d_{m+1}| \leq \frac{|\eta|B_1\sqrt{2B_1}}{\sqrt{|((m+1)(L+m\tau) - 2L\nu + \tau(\tau-1)(m+1)^2)\eta B_1^2 - 2L^2B_2| + 2L^2B_1}}, \tag{8}$$

$$|d_{2m+1}| \leq \begin{cases} \frac{|\eta|B_1}{L+m\tau} & ; B_1 < \frac{2L^2}{|\eta|(m+1)(L+m\tau)} \\ \frac{|\eta|B_1}{L+m\tau} + \left(\frac{m+1}{2} - \frac{L^2}{|\eta|B_1(L+m\tau)}\right)\frac{2\eta^2B_1^3}{|((m+1)(L+m\tau) - 2L\nu + \tau(\tau-1)(m+1)^2))\eta B_1^2 - 2L^2B_2| + 2L^2B_1} & \\ & ; B_1 \geq \frac{2L^2}{|\eta|(m+1)(L+m\tau)}, \end{cases} \tag{9}$$

and for δ a real number

$$|d_{2m+1} - \delta d_{m+1}^2| \leq \begin{cases} \frac{|\eta|B_1}{L+m\tau} & ; |m+1-2\delta| < J \\ \frac{|\eta|^2B_1^3|m+1-2\delta|}{|((m+1)(L+m\tau) - 2L\nu + \tau(\tau-1)(m+1)^2)\eta B_1^2 - 2L^2B_2|} & ; |m+1-2\delta| \geq J, \end{cases} \tag{10}$$

where

$$L = \tau(m+1) - \nu, \tag{11}$$

and

$$J = \left|\frac{((m+1)(L+m\tau) - 2L\nu + \tau(\tau-1)(m+1)^2)\eta B_1^2 - 2L^2B_2}{\eta(L+m\tau)B_1^2}\right|. \tag{12}$$

Proof. Let the function s given by (3) be in the family $\mathfrak{M}_{\sigma_m}(\tau, \nu, \eta, \varphi)$. Then, there are holomorphic functions $\mathfrak{h} : \mathfrak{D} \longrightarrow \mathfrak{D}$ and $\mathfrak{p} : \mathfrak{D} \longrightarrow \mathfrak{D}$ with $\mathfrak{h}(0) = \mathfrak{p}(0) = 0$ satisfying

$$1 + \frac{1}{\eta}\left(\frac{\varsigma(s'(\varsigma))^\tau}{(1-\nu)\varsigma + \nu s(\varsigma)} - 1\right) = \varphi(\mathfrak{h}(\varsigma)), \tag{13}$$

and

$$1 + \frac{1}{\eta}\left(\frac{\omega(g'(\omega))^\tau}{(1-\nu)\omega + \nu g(\omega)} - 1\right) = \varphi(\mathfrak{p}(\omega)), \tag{14}$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$.

Taylor–Maclaurin series expansions of the left-hand side of Equations (13) and (14) are, respectively

$$\begin{aligned}
&1+\frac{1}{\eta}\{(\tau(m\,+\,1)-\nu)d_{m+1}\varsigma^{m}+[(\tau(2m\,+\,1)-\nu)d_{2m+1}\\
&-\left((\tau(m\,+\,1)-\gamma)\gamma-\frac{\tau(\tau-1)}{2}(m\,+\,1^{2})\right)d_{m+1}^{2}\Big]\varsigma^{2m}+\cdots\Big\}
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&1+\frac{1}{\eta}\Big\{-(\tau(m\,+\,1)-\nu)d_{m+1}\omega^{m}+\Big[(\tau(2m\,+\,1)-\nu)((m\,+\,1)d_{m+1}^{2}-d_{2m+1})\\
&-\left((\tau(m\,+\,1)-\gamma)\gamma-\frac{\tau(\tau-1)}{2}(m\,+\,1^{2})\right)d_{m+1}^{2}\Big]\omega^{2m}+\cdots\Big\}.
\end{aligned} \tag{16}$$

Comparing the coefficients in (6) and (15), (7) and (16), we obtain

$$Ld_{m+1}=\eta B_{1}h_{m}\,, \tag{17}$$

$$(L\,+\,m\tau)d_{2m+1}-\left(L\nu-\frac{\tau(\tau-1)}{2}(m\,+\,1)^{2}\right)d_{m+1}^{2}=\eta[B_{1}h_{2m}+B_{2}h_{m}^{2}]\,, \tag{18}$$

$$-\,Ld_{m+1}=\eta B_{1}p_{m} \tag{19}$$

and

$$\left[(L\,+\,m\tau)((m\,+\,1)d_{m+1}^{2}-d_{2m+1})-\left(L\nu-\frac{\tau(\tau-1)}{2}(m\,+\,1)^{2}\right)d_{m+1}^{2}\right]=\eta[B_{1}p_{2m}+B_{2}p_{m}^{2}]\,, \tag{20}$$

where L is given by (11).
From (17) and (19), we obtain

$$h_{m}=-p_{m} \tag{21}$$

and

$$2L^{2}d_{m+1}^{2}=\eta^{2}B_{1}^{2}(h_{m}^{2}+p_{m}^{2}). \tag{22}$$

Using (22) in the addition of (18) and (20), we obtain

$$[((m\,+\,1)(L\,+\,m\tau)\;-\;2L\nu\;+\;\tau(\tau-1)(m\,+\,1)^{2}))\eta B_{1}^{2}\;-\;2L^{2}B_{2}]d_{m+1}^{2}=\eta^{2}B_{1}^{3}(h_{2m}+p_{2m}). \tag{23}$$

By using (5) and (17) in (23) for the coefficients h_{2m} and p_{2m}, we obtain

$$[|((m\,+\,1)(L\,+\,m\tau)\;-\;2L\nu\;+\;\tau(\tau-1)(m\,+\,1)^{2})\eta B_{1}^{2}\;-\;2L^{2}B_{2}|\;+\;2L^{2}B_{1}]|d_{m+1}|^{2}\leq 2\eta^{2}B_{1}^{3}, \tag{24}$$

which implies the assertion (8).
Subtracting (20) from (18), we obtain

$$d_{2m+1}=\frac{\eta B_{1}(h_{2m}-p_{2m})}{2(L\,+\,m\tau)}+\left(\frac{m\,+\,1}{2}\right)d_{m+1}^{2}. \tag{25}$$

In view of (17), (21), (25) and applying inequalities (5), it follows that

$$\begin{aligned}
|d_{2m+1}|\leq\frac{|\eta|B_{1}}{L\,+\,m\tau}+\left(\frac{m\,+\,1}{2}-\frac{L^{2}}{|\eta|B_{1}(L\,+\,m\tau)}\right)\\
\frac{2\eta^{2}B_{1}^{3}}{|((m\,+\,1)(L\,+\,m\tau)\,-\,2L\nu\,+\,\tau(\tau-1)(m\,+\,1)^{2})\eta B_{1}^{2}-2L^{2}B_{2}|\,+\,2L^{2}B_{1}},
\end{aligned} \tag{26}$$

which obtains the desired estimate (9).
It follows from (23) and (25) that

$$d_{2m+1} - \delta d_{m+1}^2 = \frac{\eta B_1}{2}\left[\left(T(\delta) + \frac{1}{L + m\tau}\right)h_{2m} + \left(T(\delta) - \frac{1}{L + m\tau}\right)p_{2m}\right],$$

where

$$T(\delta) = \frac{\eta B_1^2(m + 1 - 2\delta)}{((m + 1)(L + m\tau) - 2L\nu + \tau(\tau - 1)(m + 1)^2)\eta B_1^2 - 2L^2 B_2}.$$

In view of (5), we conclude that

$$|d_{2m+1} - \delta d_{m+1}^2| \leq \begin{cases} \frac{|\eta| B_1}{L + m\tau} & ; 0 \leq |T(\delta)| < \frac{1}{L + m\tau} \\ |\eta| B_1 |T(\delta)| & ; |T(\delta)| \geq \frac{1}{L + m\tau}, \end{cases}$$

which implies the assertion (10) with J as in (12). This complete the proof. □

Remark 1. *(i). For $\tau = 1$ in Theorem 1, we get Corollary 1 of Swamy et al. [23].*

(ii). For $\tau = \eta = 1$ and $\nu = 0$ in Theorem 1, we obtain Theorems 1 and 2 of Tang et al. [28]. Further, we obtain a result of Peng et al. [31] for the case of one-fold symmetric bi-univalent functions, if $m = 1$.

We note that for specializing the parameters, as mentioned in special cases (i) and (ii) of Definition 1, we deduce the following new results.

Corollary 1. *Let $\tau \geq 1$. If a function $s \in \mathcal{A}$ belongs to the family $H_{\sigma_m}(\tau, \eta, \varphi)$, then*

$$|d_{m+1}| \leq \frac{|\eta| B_1 \sqrt{2B_1}}{\sqrt{\tau(m + 1)[\,|(2m + 1 + (\tau - 1)(m + 1))\eta B_1^2 - 2\tau(m + 1)B_2| + 2\tau(m + 1)B_1\,]}},$$

$$|d_{2m+1}| \leq \begin{cases} \frac{|\eta| B_1}{\tau(2m + 1)} & ; B_1 < \frac{2\tau(m + 1)}{|\eta|(2m + 1)} \\ \frac{|\eta| B_1}{\tau(2m + 1)} + \left(\frac{m + 1}{2} - \frac{\tau(m + 1)^2}{|\eta|(2m + 1)B_1}\right)\frac{2\eta^2 B_1^3}{\tau(m + 1)[\,|(2m + 1 + \tau - 1(m + 1))\eta B_1^2 - 2\tau(m + 1)B_2| + 2\tau(m + 1)B_1\,]} \\ & ; B_1 \geq \frac{2\tau(m + 1)}{|\eta|(2m + 1)}, \end{cases}$$

and for δ a real number

$$|d_{2m+1} - \delta d_{m+1}^2| \leq \begin{cases} \frac{|\eta| B_1}{\tau(2m + 1)} \; ; \; |m + 1 - 2\delta| < (m + 1)\left|\frac{(2m + 1 + (\tau - 1)(m + 1))\eta B_1^2 - 2\tau(m + 1)B_2}{\eta(2m + 1)B_1^2}\right| \\ \frac{|\eta|^2 B_1^3 |m + 1 - 2\delta|}{\tau(m + 1)|(2m + 1 + \tau - 1(m + 1))\eta B_1^2 - 2\tau(m + 1)B_2|}; \\ \qquad |m + 1 - 2\delta| \geq (m + 1)\left|\frac{(2m + 1 + (\tau - 1)(m + 1))\eta B_1^2 - 2\tau(m + 1)B_2}{\eta(2m + 1)B_1^2}\right|. \end{cases}$$

Remark 2. *For $\tau = \eta = 1$ in Corollary 1, we obtain Theorems 1 and 2 of Tang et al. [28]. Further, we obtain a result of Peng et al. [31] for the case of the one-fold symmetric bi-univalent function, when $m = 1$.*

Corollary 2. *Let $\tau \geq 1$. If the function s in $\mathcal{A}$ belongs to the family $I_{\sigma_m}(\tau, \eta, \varphi)$, then*

$$|d_{m+1}| \leq \frac{|\eta| B_1 \sqrt{2B_1}}{\sqrt{|((m + 1)(L_1 + m\tau) - 2L_1 + \tau(\tau - 1)(m + 1)^2)\eta B_1^2 - 2L_1^2 B_2| + 2L_1^2 B_1}},$$

$$|d_{2m+1}| \leq$$

$$\begin{cases} \dfrac{|\eta|B_1}{L_1 + m\tau} & ;\ B_1 < \dfrac{2L_1^2}{|\eta|(m+1)(L_1+m\tau)} \\ \dfrac{|\eta|B_1}{L_1+m\tau} + \left(\dfrac{m+1}{2} - \dfrac{L_1^2}{|\eta|B_1(L_1+m\tau)}\right)\dfrac{2\eta^2 B_1^3}{|((m+1)(L_1+m\tau) - 2L_1 + \tau(\tau-1)(m+1)^2)\eta B_1^2 - 2L_1^2 B_2| + 2L_1^2 B_1}; \\ & B_1 \geq \dfrac{2L_1^2}{|\eta|(m+1)(L_1+m\tau)}, \end{cases}$$

and for δ a real number

$$|d_{2m+1} - \delta d_{m+1}^2| \leq \begin{cases} \dfrac{|\eta|B_1}{L_1+m\tau} & ;\ |m+1-2\delta| < J_1 \\ \dfrac{|\eta|^2 B_1^3 |m+1-2\delta|}{|((m+1)(L_1+m\tau) - 2L_1 + \tau(\tau-1)(m+1)^2)\eta B_1^2 - 2L_1^2 B_2|} & ;\ |m+1-2\delta| \geq J_1, \end{cases}$$

where

$$L_1 = \tau(m+1) - 1, \tag{27}$$

and

$$J_1 = \left|\frac{((m+1)(L_1+m\tau) - 2L_1 + \tau(\tau-1)(m+1)^2)\eta B_1^2 - 2L_1^2 B_2}{\eta(L_1+m\tau)B_1^2}\right|.$$

Remark 3. *For $\tau = \eta = 1$ in Corollary 2, we get Corollary 2.2 and Corollary 2.11 [32]. Further, we obtain Corollary 2.6 and Corollary 2.13 of [32] for the case of one-fold symmetric bi-univalent functions, when $m = 1$.*

3. Coefficient Bounds for Function Family $\mathfrak{W}^{\varrho}_{\sigma_m}(\tau, \nu, \eta)$

If $\varphi(\varsigma) = \left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)^{\varrho}$ $(0 < \varrho \leq 1)$, in the Definition 1, then we have $\mathfrak{W}^{\varrho}_{\sigma_m}(\tau, \nu, \eta) = \mathfrak{M}_{\sigma_m}(\tau, \nu, \eta, \left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)^{\varrho})$, the subclass of functions $s \in \sigma_m$ satisfying the conditions

$$\left|arg\left[1 + \frac{1}{\eta}\left(\frac{\varsigma(s'(\varsigma))^{\tau}}{(1-\nu)\varsigma + \nu s(\varsigma)} - 1\right)\right]\right| < \frac{\varrho\pi}{2}$$

and

$$\left|arg\left[1 + \frac{1}{\eta}\left(\frac{\omega(g'(\omega))^{\tau}}{(1-\nu)\omega + \nu g(\omega)} - 1\right)\right]\right| < \frac{\varrho\pi}{2},$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

We observe that the certain choice of the parameter ν leads the class $\mathfrak{W}^{\varrho}_{\sigma_m}(\tau, \nu, \eta)$ to the following few subfamilies:

(i) $B^{\varrho}_{\sigma_m}(\tau, \eta) \equiv \mathfrak{W}^{\varrho}_{\sigma_m}(\tau, 0, \eta)$ $(0 < \varrho \leq 1,\ \tau \geq 1)$ is the family of $s \in \sigma_m$ of the form (1) satisfying

$$\left|arg\left[1 + \frac{1}{\eta}\left((s'(\varsigma))^{\tau} - 1\right)\right]\right| < \frac{\varrho\pi}{2}$$

and

$$\left|arg\left[1 + \frac{1}{\eta}\left((g'(\omega))^{\tau} - 1\right)\right]\right| < \frac{\varrho\pi}{2},$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

(ii) $C^{\varrho}_{\sigma_m}(\mu, \eta) \equiv \mathfrak{W}^{\varrho}_{\sigma_m}(\mu, 1, \eta)(0 < \varrho \leq 1,\ \mu \geq 0)$ is the family of $s \in \sigma_m$ of the form (1) satisfying

$$\left|arg\left[1 + \frac{1}{\eta}\left(\frac{\varsigma(s'(\varsigma))^{\tau}}{s(\varsigma)} - 1\right)\right]\right| < \frac{\varrho\pi}{2}$$

and

$$\left|arg\left[1 + \frac{1}{\eta}\left(\frac{\omega(g'(\omega))^{\tau}}{g(\omega)} - 1\right)\right]\right| < \frac{\varrho\pi}{2},$$

where $\varsigma, \omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4). If we take $\varphi(\varsigma) = \left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)^{\varrho}$ in Theorem 1, we get

Corollary 3. *Let $\tau \geq 1$, $0 \leq \nu \leq 1$ and $0 < \varrho \leq 1$. If a function s in $\mathcal{A}$ belongs to the class $\mathfrak{W}^{\varrho}_{\sigma_m}(\tau,\nu,\eta)$, then*

$$|d_{m+1}| \leq \frac{2|\eta|\varrho}{\sqrt{\varrho|((m+1)(L+m\tau) - 2L\nu + \tau(\tau-1)(m+1)^2)\eta - L^2| + L^2}},$$

$$|d_{2m+1}| \leq$$

$$\begin{cases} \frac{2|\eta|\varrho}{L+m\tau} & ; \varrho < \frac{L^2}{|\eta|(m+1)(L+m\tau)} \\ \frac{2|\eta|\varrho}{L+m\tau} + \left(m+1-\frac{L^2}{|\eta|\varrho(L+m\tau)}\right)\frac{2\eta^2\varrho^2}{\varrho|((m+1)(L+m\tau) - 2L\nu + \tau(\tau-1)(m+1)^2)\eta - L^2| + L^2} & ; \varrho \geq \frac{L^2}{|\eta|(m+1)(L+m\tau)}, \end{cases}$$

and for δ a real number

$$|d_{2m+1} - \delta d^2_{m+1}| \leq \begin{cases} \frac{2|\eta|\varrho}{L+m\tau} & ; |m+1-2\delta| < J_2 \\ \frac{2|\eta|^2\varrho|m+1-2\delta|}{|((m+1)(L+m\tau) - 2L\nu + \tau(\tau-1)(m=1)^2)\eta - L^2|} & ; |m+1-2\delta| \geq J_2, \end{cases}$$

where L is as in (11) and

$$J_2 = \left|\frac{((m+1)(L+m\tau) - 2L\nu + \tau(\tau-1)(m+1)^2)\eta - L^2}{\eta(L+m\tau)}\right|.$$

Remark 4. *(i) We obtain Corollary 5 of Swamy et al. [23] from Corollary 3 when $\tau = 1$.*
(ii) For $m = \tau = \eta = 1$ and $\nu = 0$, Corollary 3 agrees with Corollary 2 of Tang et al. [28].
(iii) For $\tau = \eta = \nu = 1$ in Corollary 3, bound on $|d_{m+1}|$ reduce to the bound given in Corollary 6 of [33]. Further, if $m = 1$, we obtain a result of [34].
(iv) For $\tau = \eta = \nu = 1$ in Corollary 3, the result shown on $|d_{2m+1}|$ is better than the bound given in Corollary 6 of [33], in terms of ranges of ϱ as well as the bounds.

We note that for specializing the parameters, as mentioned in special cases (i) and (ii) of the class $\mathfrak{W}^{\varrho}_{\sigma_m}(\tau,\nu,\eta)$, we deduce the following new results.

Corollary 4. *Let $\tau \geq 1$ and $0 < \varrho \leq 1$. If a function s in $\mathcal{A}$ belongs to the class $B^{\varrho}_{\sigma_m}(\tau,\eta)$, then*

$$|d_{m+1}| \leq \frac{2|\eta|\varrho}{\sqrt{\tau(m+1)[\varrho|((2m+1+(\tau-1)(m+1))\eta - \tau(m+1)| + \tau(m+1)]}},$$

$$|d_{2m+1}| \leq \begin{cases} \frac{2|\eta|\varrho}{\tau(2m+1)} & ; \varrho < \frac{\tau(m+1)}{|\eta|(2m+1)} \\ \frac{2|\eta|\varrho}{\tau(2m+1)} + \left(m+1-\frac{\tau(m+1)^2}{|\eta|(2m+1)\varrho}\right)\frac{2\eta^2\varrho^2}{\tau(m+1)[\varrho|(2m+1+(\tau-1)(m+1))\eta - \tau(m+1)| + \tau(m+1)]}; \\ \varrho \geq \frac{\tau(m+1)}{|\eta|(2m+1)}, \end{cases}$$

and for δ a real number

$$|d_{2m+1} - \delta d^2_{m+1}| \leq \begin{cases} \frac{2|\eta|\varrho}{\tau(2m+1)} & ; |m+1-2\delta| < (m+1)\left|\frac{(2m+1+(\tau-1)(m+1))\eta - \tau(m+1)}{\eta(2m+1)}\right| \\ \frac{2|\eta|^2\varrho|m+1-2\delta|}{\tau(m+1)|(2m+1+(\tau-1)(m+1))\eta - \tau(m+1)|}; \\ |m+1-2\delta| \geq (m+1)\left|\frac{(2m+1+(\tau-1)(m+1))\eta - \tau(m+1)}{\eta(2m+1)}\right|. \end{cases}$$

Remark 5. *We obtain Corollary 2 of Tang et al. [28] from Corollary 4, when $m = \tau = \eta = 1$.*

Corollary 5. *Let $\tau \geq 1$ and $0 < \varrho \leq 1$. If a function s in $\mathcal{A}$ belongs to the class $C^{\varrho}_{\sigma_m}(\tau,\eta,\varphi)$, then*

$$|d_{m+1}| \leq \frac{2|\eta|\varrho}{\sqrt{\varrho|((m+1)(L_1+m\tau)-2L_1+\tau(\tau-1)(m+1)^2)\eta - L_1^2| + L_1^2}},$$

$$|d_{2m+1}| \leq \begin{cases} \frac{2|\eta|\varrho}{L_1+m\tau} & ; \varrho < \frac{L_1^2}{|\eta|(m+1)(L_1+m\tau)} \\ \frac{2|\eta|\varrho}{L_1+m\tau} + \left(m+1-\frac{L_1^2}{|\eta|\varrho(L_1+m\tau)}\right)\frac{2\eta^2\varrho^2}{\varrho|((m+1)(L_1+m\tau)-2L_1+\tau(\tau-1)(m+1)^2)\eta - L_1^2| + L_1^2} & \\ & ; \varrho \geq \frac{L_1^2}{|\eta|(m+1)(L_1+m\tau)}, \end{cases}$$

and for δ a real number

$$|d_{2m+1} - \delta d_{m+1}^2| \leq \begin{cases} \frac{2|\eta|\varrho}{L_1+m\tau} & ; |m+1-2\delta| < J_3 \\ \frac{2|\eta|^2\varrho|m+1-2\delta|}{|((m+1)(L_1+m\tau)-2L_1+\tau(\tau-1)(m+1)^2)\eta - L_1^2|} & ; |m+1-2\delta| \geq J_3, \end{cases}$$

where L_1 is as in (27) and

$$J_3 = \left|\frac{((m+1)(L_1+m\tau)-2L_1+\tau(\tau-1)(m+1)^2)\eta - L_1^2}{\eta(L_1+m\tau)}\right|.$$

Remark 6. *(i) For $\tau = \eta = 1$ in Corollary 5, bound on $|d_{m+1}|$ reduce to the bound given in Corollary 6 of [33]. Further, if $m = 1$ we obtain a result of [34].*
(ii) For $\tau = \eta = 1$ in Corollary 5, result shown on $|d_{2m+1}|$ is better than the bound given in Corollary 6 of [33], in terms of ranges of ϱ as well as the bounds.

4. Coefficient Bounds for Function Family $\mathfrak{X}_{\sigma_m}^{\xi}(\tau,\nu,\eta)$

If $\varphi(\varsigma) = \frac{1+(1-2\xi)\varsigma^m}{1-\varsigma^m}$ $(0 \leq \xi < 1)$ in the Definition 1, then we obtain $\mathfrak{X}_{\sigma_m}^{\xi}(\tau,\nu,\eta) = \mathfrak{M}_{\sigma_m}(\tau,\nu,\eta,\left(\frac{1+(1-2\xi)\varsigma^m}{1-\varsigma^m}\right)$, a subclass of functions $s \in \sigma_m$ satisfying

$$\Re\left[1 + \frac{1}{\eta}\left(\frac{\varsigma(s'(\varsigma))^\tau}{(1-\nu)\varsigma + \nu s(\varsigma)} - 1\right)\right] > \xi$$

and

$$\Re\left[1 + \frac{1}{\eta}\left(\frac{\omega(g'(\omega))^\tau}{(1-\nu)\omega + \nu g(\omega)} - 1\right)\right] > \xi,$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

We observe that certain values of the parameter ν lead the class $\mathfrak{X}_{\sigma_m}^{\xi}(\tau,\nu,\eta)$ to the following few subfamilies:

(i) $E_{\sigma_m}^{\xi}(\tau,\eta) \equiv \mathfrak{X}_{\sigma_m}^{\xi}(\tau,0,\eta)$ $(0 \leq \xi < 1,\ \tau \geq 1)$, is the class of functions $s \in \sigma_m$ of the form (1) satisfying

$$\Re\left[1 + \frac{1}{\eta}\left((s'(\varsigma))^\tau - 1\right)\right] > \xi$$

and

$$\Re\left[1 + \frac{1}{\eta}\left((g'(\omega))^\tau - 1\right)\right] > \xi,$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

(ii) $F_{\sigma_m}^{\xi}(\tau,\eta) \equiv \mathfrak{X}_{\sigma_m}^{\xi}(\tau,1,\eta)$ $(0 \leq \xi < 1,\ \tau \geq 1)$, is the family of functions $s \in \sigma_m$ of the form (1) satisfying

$$\Re\left[1 + \frac{1}{\eta}\left(\frac{\varsigma(s'(\varsigma))^\tau}{s(\varsigma)} - 1\right)\right] > \xi$$

and

$$\Re\left[1+\frac{1}{\eta}\left(\frac{\omega(g'(\omega))^{\tau}}{g(\omega)}-1\right)\right]>\xi,$$

where $\varsigma,\,\omega\in\mathfrak{D}$, $g(\omega)=s^{-1}(\omega)$ is as stated in (4).

If we take $\varphi(\varsigma)=\frac{1+(1-2\xi)\varsigma^m}{1-\varsigma^m}$ in Theorem 1, we obtain

Corollary 6. *Let $\tau\geq 1$, $0\leq\nu\leq 1$ and $0\leq\xi<1$. If a function s in $\mathcal{A}$ belongs to the class $\mathfrak{X}^{\xi}_{\sigma_m}(\tau,\nu,\eta)$, then*

$$|d_{m+1}|\leq\frac{2|\eta|(1-\xi)}{\sqrt{|((m+1)(L+m\tau)-2L\nu+\tau(\tau-1)(m+1)^2)\eta(1-\xi)-L^2|+L^2}},$$

$$|d_{2m+1}|\leq$$

$$\begin{cases}\frac{2(1-\xi)|\eta|}{L+m\tau} & ;\,(1-\xi)<\frac{L^2}{|\eta|(m+1)(L+m\tau)}\\ \frac{2(1-\tau)|\eta|}{L+m\tau}+\left(m+1-\frac{L^2}{|\eta|(1-\xi)(L+m\tau)}\right)\frac{2|\eta|^2(1-\xi)^2}{|((m+1)(L+m\tau)-2L\nu+\tau(\tau-1)(m+1)^2)(1-\xi)\eta-L^2|+L^2} & \\ & ;\,(1-\xi)\geq\frac{L^2}{|\eta|(m+1)(L+m\tau)}\end{cases}$$

and for δ a real number

$$|d_{2m+1}-\delta d_{m+1}^2|\leq\begin{cases}\frac{2|\eta|(1-\xi)}{L+m\tau} & ;\,|m+1-2\delta|<J_4\\ \frac{2|\eta|^2(1-\xi)^2|m+1-2\delta|}{|((m+1)(L+m\tau)-2L\nu+\tau(\tau-1)(m+1)^2)(1-\xi)\eta-L^2|} & ;\,|m+1-2\delta|\geq J_4,\end{cases}$$

where L is as in (11) and

$$J_4=\left|\frac{((m+1)(L+m\tau)-2L\nu+\tau(\tau-1)(m+1)^2)\eta(1-\xi)-L^2}{\eta(L+m\tau)(1-\xi)}\right|.$$

Remark 7. *(i). For $\tau=1$, Corollary 6 match with Corollary 9 of Swamy et al. [23].*
(ii). For $\tau=\eta=\nu=1$ in Corollary 6, bound on $|d_{m+1}|$ reduce to the bound given in Corollary 7 of [33]. Further, if $m=1$, we obtain a result of [34].
(iii). For $\tau=\eta=\nu=1$ in Corollary 6, the result proved on $|d_{2m+1}|$ is better than the bound given in Corollary 7 of [33], in terms of ranges of ξ as well as the bounds.

We note that for specializing the parameter ν, as mentioned in special cases (i) and (ii) of the class $\mathfrak{X}^{\tau}_{\sigma_m}(\mu,\nu,\eta)$, we deduce the following new results.

Corollary 7. *Let $\tau\geq 1$ and $0\leq\xi<1$. If a function s in $\mathcal{A}$ belongs to the class $E^{\xi}_{\sigma_m}(\tau,\eta)$, then*

$$|d_{m+1}|\leq\frac{2|\eta|(1-\xi)}{\sqrt{\tau(m+1)\left[|(2m+1+(\tau-1)(m+1))\eta(1-\xi)-\tau(m+1)|+\tau(m+1)\right]}},$$

$$|d_{2m+1}|\leq$$

$$\begin{cases}\frac{2(1-\xi)|\eta|}{\tau(2m+1)} & ;\,(1-\xi)<\frac{\tau(m+1)}{|\eta|(2m+1)}\\ \frac{2|\eta|(1-\xi)}{\tau(2m+1)}+\left(m+1-\frac{\tau(m+1)^2}{|\eta|(1-\xi)(2m+1)}\right)\frac{2\eta^2(1-\xi)^2}{\tau(m+1)\left[|(2m+1+\tau(\tau-1)(m+1)\eta(1-\xi)-\tau(m+1)|+\tau(m+1)\right]} & \\ & ;\,(1-\xi)\geq\frac{\tau(m+1)}{|\eta|(2m+1)},\end{cases}$$

and for δ a real number

$$|d_{2m+1}-\delta d_{m+1}^2|\leq\begin{cases}\frac{2|\eta|(1-\xi)}{\tau(2m+1)} & ;\,|m+1-2\delta|<J_5\\ \frac{2|\eta|^2(1-\xi)^2|m+1-2\delta|}{\tau(m+1)\,|(2m+1+\tau(\tau-1)(m+1))\eta(1-\xi)-\tau(m+1)|} & ;\,|m+1-2\delta|\geq J_5,\end{cases}$$

where

$$J_5 = (m + 1)\left|\frac{(2m + 1 + (\tau - 1)(m + 1))\eta(1-\xi) - \tau(m + 1)}{\eta(1-\xi)(2m + 1)}\right|.$$

Remark 8. *(i) For $\tau = 1$ and $\eta = 1$ in Corollary 7, we obtain the Corollary 11 of Swamy et al. [23]. (ii) For $m = \tau = \eta = 1$, Corollary 7 would lead us to Corollary 12 of Swamy et al. [23].*

Corollary 8. *Let $\tau \geq 1$ and $0 \leq \xi < 1$. If a function s in $\mathcal{A}$ belongs to the class $F^{\xi}_{\sigma_m}(\tau, \eta)$, then*

$$|d_{m+1}| \leq \frac{2|\eta|(1-\xi)}{\sqrt{|((m + 1)(L_1 + m\tau) - 2L_1 + \tau(\tau-1)(m + 1)^2)\eta(1-\xi) - L_1^2| + L_1^2}},$$

$$|d_{2m+1}| \leq$$

$$\begin{cases} \frac{2(1-\xi)|\eta|}{L_1 + m\tau} & ; (1-\xi) < \frac{L^2}{|\eta|(m + 1)(L_1 + m\tau)} \\ \frac{2(1-\tau)|\eta|}{L_1 + m\tau} + \left(m + 1 - \frac{L_1^2}{|\eta|(1-\xi)(L + m\tau)}\right)\frac{2|\eta|^2(1-\xi)^2}{|((m + 1)(L_1 + m\tau) - 2L_1 + \tau(\tau-1)(m + 1)^2)(1-\xi)\eta - L_1^2| + L_1^2} & \\ & ; (1-\xi) \geq \frac{L^2}{|\eta|(m + 1)(L_1 + m\tau)} \end{cases}$$

and for δ a real number

$$|d_{2m+1} - \delta d_{m+1}^2| \leq \begin{cases} \frac{2|\eta|(1-\xi)}{L_1 + m\tau} & ; |m + 1 - 2\delta| < J_6 \\ \frac{2|\eta|^2(1-\xi)^2|m + 1 - 2\delta|}{|((m + 1)(L_1 + m\tau) - 2L_1 + \tau(\tau-1)(m + 1)^2)(1-\xi)\eta - L_1^2|} & ; |m + 1 - 2\delta| \geq J_6, \end{cases}$$

where L_1 is as in (27) and

$$J_6 = \left|\frac{((m + 1)(L_1 + m\tau) - 2L_1 + \tau(\tau-1)(m + 1)^2)\eta(1-\xi) - L_1^2}{\eta(L_1 + m\tau)(1-\xi)}\right|.$$

Remark 9. *For $\tau = \eta = 1$ in Corollary 8, the bound on $|d_{m+1}|$ reduces to the bound given in Corollary 7 of [33]. Further, if $m = 1$, we obtain a result of [34]. For $\tau = \eta = 1$ in Corollary 8, the result proved on $|d_{2m+1}|$ is better than the bound given in Corollary 7 of [33], in terms of the ranges of ξ as well as the bounds.*

5. Conclusions

In this study, we introduced a special family $\mathfrak{M}_{\sigma_m}(\tau, \nu, \eta, \varphi)$ of m-fold symmetric bi-univalent functions in the disc $\{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ and studied coefficient problems associated with the defined family. For functions belonging to this family, we determined the upper bounds for $|d_{m+1}|$ and $|d_{2m+1}|$. The Fekete–Szegö functional problem for functions in this family was also considered. Various cases of the special family $\mathfrak{M}_{\sigma_m}(\tau, \nu, \eta, \varphi)$ were discussed. Our results generalize many results of Swamy et al. [23], Tang et al. [28] and Akgul [32].

A special family examined in this paper could inspire further research related to some aspects, such as certain special families of bi-univalent functions using (i) the Hohlov operator associated with the Legendre polynomial [35], (ii) the integro-differential operator [36], (iii) the q-derivative operator [37] and so on.

Author Contributions: Conceptualization and methodology, S.R.S. and B.A.F.; software, S.R.S.; validation and formal analysis, I.A.; investigation, I.A., S.R.S. and B.A.F.; resources and data curation, I.A.; writing—original draft preparation, S.R.S.; writing—review and editing, I.A. and B.A.F.; visualization, I.A., S.R.S. and B.A.F.; supervision, S.R.S. and B.A.F.; project administration, I.A. and B.A.F.; funding, I.A., S.R.S. and B.A.F.; acquisition, I.A., S.R.S. and B.A.F. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees for their helpful comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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