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Article
Fractional-Order Interval Observer for Multiagent Nonlinear Systems

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Abstract: A framework of distributed interval observers is introduced for fractional-order multiagent systems in the presence of nonlinearity. First, a frame was designed to construct the upper and lower bounds of the system state. By using monotone system theory, the positivity of the error dynamics could be ensured, which implies that the bounds could trap the original state. Second, a sufficient condition was applied to guarantee the boundedness of distributed interval observers. Then, an extension of Lyapunov function in the fractional calculus field was the basis of the sufficient condition. An algorithm associated with the procedure of the observer design is also provided. Lastly, a numerical simulation is used to demonstrate the effectiveness of the distributed interval observer.

Keywords: distributed interval observer; fractional-order multiagent systems; monotone system theory

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1. Introduction

Fractional calculus is the definition of the differential and integral of arbitrary order systems. During the last 20 years, fractional calculus grew into a hot topic and attracted increasing interest [1,2]. It plays an important role when people are faced with natural dynamics problems. It works well when it is applied in describing the memory and hereditary properties of manifold materials. In addition, the applications of fractional calculus are rapidly expanding, such as non-Fickian dynamics [3], fractional boundary value problems [4], and variable-order thermostat models [5].

Due to the inaccessibility of the system state in many physical backgrounds and the availability of output information, research on observation problems is very significant. With the indepth study of observation problems, interval observers are widely accepted as an efficient tool in reconstructing the system state with nonlinearity or bounded uncertainty [6,7]. The concept of interval observers was first proposed by Gouzé, and it was successfully applied to uncertain biological systems [8]. After that, research on interval observers can be divided into two parts. The first is based on monotone system theory, and the second was developed from set-membership estimation. The monotone-system-theory-based method requires researchers to design suitable observer gains in order to ensure that the error dynamics is positive and bounded. In this case, the bounds from the interval observer converge to the original system. Inspired by this idea, there are many new techniques for various systems emerging. For linear time-invariant systems, the time-varying coordinate transformation technique was used in interval observer design [9]. Efimov et al. also extended the interval estimation technique to nonlinear time-varying systems [10]. An interval observer for switched systems was also introduced in [11]. On the other hand, a set-membership estimation-based method combined robust observer design with reachability analysis [12]. Interval observers in this framework obviously improved estimation accuracy. Additionally, scholars presented a new integrated version of interval observers that achieved an expected tradeoff between robust estimation conservatism and computational complexity [13]. In general, research on interval observers for integer-order systems is fruitful, which is instructive for us to design interval observers of fractional-order systems.

A tremendous expansion for multiagent systems (MASs) has been witnessed, which mainly regards output regulation [14], observer design [15] and consensus control [16]. It started from simple two-order systems and attracted more attention on various complex MASs, such as switched [17], nonlinear [18], and time-delaying [19] systems. The observers designed for MASs are called distributed observers if the communication topology is taken into account. A distributed observer was designed for leader-following MASs, and an actual vehicle model was used to verify the function of the distributed observer [20]. In [21], the existence condition of the distributed observer was established to recover the state of nonlinear MASs. There are some works for fractional-order MASs in [22] and [23]. The consensus control problem for fractional-order MASs under a fixed topology was studied by stability theory and the Lyapunov method [22]. For unknown nonlinear dynamics, an adaptive control protocol was introduced that was used in a trajectory tracking problem [23]. There were also some interesting results about the observation problem in [24–27]. Traditional observers and distributed controllers were both used in fractional-order MASs [24]. Compared with [24], the time-varying formation control was investigated, and the consensus protocol was designed to match a switching topology [25]. For the fractional-order heterogeneous nonlinear MASs, the consensus control problem was converted into the stabilisation problem using the distributed control technique [26]. In [27], the object was the fractional-order system with unknown orders, and a robust observer-based controller was formulated to solve the consensus problem.

For these works [24–27], the introduced observers were all Luenberger-like. When the original system suffers from uncertain disturbance or nonlinearity, Luenberger-like observers are unable to recover the system state. It is challenging to recover the system state when the general fractional-order system has a complex communication with its neighborhood. Therefore, a distributed interval observer is a better choice for fractional-order MASs. In many physical problems such as trajectory tracking or output regulation, distributed interval observers provide a more accurate state information, which is a great step forward for the consensus control protocol design. However, there is not any paper focusing on distributed interval observers for fractional-order MASs. In the field of fractional calculus, there are some interesting works about fractional-order systems, including state reconstruction and algorithm programming [28–30]. Motivated by the interval observer mentioned above, we developed a framework of distributed interval observers for the general fractional-order MASs. The specific contributions are outlined as follows:

1. By investigating the fractional differential problem, a novel fractional-order Lyapunov function is proposed for the boundedness problem in the observer deign, which is an approach to prove that a matrix is Hurwitz.
2. Different from linear MASs, nonlinear MASs are considered, and the solution for Lipschitz functions is combined with the distributed interval observer design.
3. For fractional-order MASs, the communication topology is applied to the observer design. Each observer of the corresponding agent could accept the information from its adjacent observers. A novel distributed interval observer was first designed for the fractional-order MASs.

The rest of the paper is structured as follows. Section 2 mainly presents some preparation work, including the fractional differential, graph theory and fractional-order systems. Section 3 proposes the distributed interval observer design method for fractional-order MASs. An illustrative example is given in Section 4 to verify the validity of the designed observer. Lastly, conclusions and future work are drawn in Section 5.

Notations: For two vectors $x \in R^n$ and $y \in R^n$, $x < (\leq) y$ are understood with $x_i < (\leq) y_i$, $i \in \{1, \ldots, n\}$. For $A \prec 0$ and $A \succ 0$, symbol $\prec$ $(\succ)$ means that matrix A is positive(negative) definite. For a matrix B, there exist three properties: $B^+ = max(B, 0)$, $B^- = B^+ - B$ and $|B| = B^+ + B^-$. Matrix B^T denotes the transpose of B, and $He(B)$ is defined as $He(B) = B + B^T$. $\lambda_{min}(A)$ represents the minimal nonzero eigenvalue of A. $\otimes$ is the Kronecker product used in MASs. 1_N means a N-order column vector, and all elements are 1.

2. Preliminaries

2.1. Fractional Calculus

There are several definitions used to regard the fractional differentiation operator, such as the Grünwald–Letnikov, Riemann–Liouville and Caputo definitions introduced in [31]. Among them, the Caputo definition is the most frequently used and has many applications in the engineering field. In the Caputo definition, the initial value of the differentiation is considered.

The Caputo fractional derivative for a function $f(t)$ is

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1} $$

where ${}_{t_0}^{C}D_t^{\alpha}$ is the fractional integration differentiation operator, t_0 represents the initial time, and $\alpha \in [n-1, n]$ is the order of the system. $\Gamma(\cdot)$ is a gamma function that is introduced in Definition 1. $f(t)$ is a differentiable function with the nth derivative. In this paper, we mainly focus on the fractional-order system with $n = 1$ and $t_0 = 0$, then we have $0 < \alpha < 1$. When $n = 1$, $f(t)$ is only required to have the first derivative. For the sake of simplicity, the differentiation operator ${}_{t_0}^{C}D_t^{\alpha}$ is replaced by D_t^{α}.

The Grünwald-Letnikov fractional derivative for a function $f(t)$ is

$$ {}_{t_0}^{GL}D_t^{\alpha} f(t) = \lim_{N\to\infty} [\frac{t-t_0}{N}]^{-\alpha} \textstyle\sum_{j=0}^{N-1} (-1)^j f(t - j[\frac{t-t_0}{N}]). \tag{2} $$

The Riemann-Liouville fractional derivative for a function $f(t)$ is

$$ {}_{t_0}^{RL}D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha-1} f(\tau) d\tau, & \alpha < 0 \\ f(t), & \alpha = 0 \\ D_t^n [{}_{t_0}^{RL}D_t^{\alpha-n} f(t)], & \alpha > 0 \end{cases} \tag{3} $$

with $n - 1 \leq \alpha < n$.

For a wide class of functions, the Grünwald–Letnikov and Riemann–Liouville definitions are equivalent [32]. However, it is difficult for the Grünwald-Letnikov definition to have a Laplace transform. The Laplace transform of the Riemann–Liouville definition is

$$ L[{}_0^{RL}D_t^{\alpha} f(t)] = \begin{cases} s^{\alpha} F(s), & q \leq 0 \\ s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k {}_0^{RL}D_t^{\alpha-k-1} f(0), & n-1 \leq q < n \end{cases} \tag{4} $$

where $F(s)$ is the Laplace transform of $f(t)$.

The Laplace transform of the Caputo definition is

$$ L[{}_0^{C}D_t^{\alpha}] = s^{\alpha} F(s) - \textstyle\sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0). \tag{5} $$

Comparing (4) and (5), it is obvious that the Riemann–Liouville fractional derivative is unsuitable for the Laplace transform technique because it requires the knowledge of noninteger-order derivatives of the function at $t = 0$, while the Caputo fractional derivative only requires the knowledge of integer-order derivatives of the function. This is why Caputo fractional derivative was chosen.

Definition 1 ([33]). *Gamma function is an important element for the Caputo fractional differentiation operator, which is defined by*

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt, \tag{6}$$

where value $\Gamma(a)$ is in the convergence of the right-hand side of the complex plane.

2.2. Graph Theory

A communication topology $\mathcal{G}(\mathcal{N}, \xi, \mathcal{A})$ contains three elements: $\mathcal{N} \in \{1, \ldots, N\}$ is the node set of MASs, $\xi \subset \mathcal{N} \times \mathcal{N}$ represents an edge set for nodes in MASs, and $\mathcal{A} \in R^{N\times N}$ is an adjacent matrix of the graph. For $\mathcal{A} = [a_{ij}] \in R^{N\times N}$, a_{ji} is the weight from node i to node j. In this paper, the graph has no self loops, i.e., $a_{ii} = 0, i \in \mathcal{N}$. $a_{ji} \neq 0$ if and only if $(i, j) \in \xi$. For node i, its neighborhood is denoted as $\mathcal{N}_i = \{j | j \in \mathcal{N}, a_{ji} \neq 0, j \neq i\}$. For $\mathcal{G}$, it is defined as a strongly connected graph if and only if any node in it is mutually reachable. For adjacent matrix $\mathcal{A}$, the corresponding Laplacian matrix is defined as $\mathcal{L} = [l_{ij}]_{N\times N}$, where $l_{ii} = \Sigma_{j=1}^N a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$. If a topology is call as a balanced topology, the edges of it must be balanced, which means that edge (i, j) belongs to set ξ if edge (j, i) belongs to set ξ.

Lemma 1 ([34]). *for a Laplacian matrix $\mathcal{L}$, zero is one of the eigenvalues and it has a fixed right eigenvector 1_N. The other nonzero eigenvalues are all positive. If there exists a directed spanning tree in $\mathcal{G}$, zero is a simple eigenvalue of $\mathcal{L}$.*

Lemma 2 ([35]). *considering $\mathcal{G}$ as a strongly connected graph, suppose that $r = [r_1, \ldots, r_N]$ is the left eigenvector connected with the eigenvalue zero. Then, we have $R\mathcal{L} + \mathcal{L}^T R \geq 0$, where $R = diag\{r_1, \ldots, r_N\}$. For a balanced graph, we have $r_1 = \cdots = r_N$.*

Lemma 3 ([35]). *If $\mathcal{G}$ is a strongly connected graph; its generalized algebraic connectivity is define as $a(\mathcal{L}) = \min\limits_{r^T x=0, x\neq 0} \frac{x^T(R\mathcal{L}+\mathcal{L}^T R)x}{2x^T Rx}$. Due to the topology in this paper being defined as balanced, matrix R can be converted into $R = r_1 I_N$. The generalized algebraic connectivity is equal to $a(\mathcal{L}) = \lambda_{min}(\frac{He(\mathcal{L})}{2})$.*

2.3. Fractional-Order Systems

Consider the following fractional-order system for the i-th agent.

$$\begin{cases} D_t^\alpha x_i(t) = Ax_i(t) + f(x_i(t)), \\ y_i(t) = Cx_i(t), \end{cases} \tag{7}$$

where $x_i \in R^n$ is the system state, $y_i \in R^m$ is the output, and $f(x_i) \in R^n$ is the Lipschitz function. $A \in R^{n\times n}$ and $C \in R^{m\times n}$ are matrices with suitable dimensions. Communication topology $\mathcal{G}$ for (7) is a strongly connected balanced graph.

Property 1. *Given a matrix $M \in R^{n\times n}$, M is a Metzler matrix if all its elements outside the main diagonal are non-negative. For example, the following matrix is Metzler:*

$$M = \begin{bmatrix} -1 & 1 \\ 2 & 0.5 \end{bmatrix}.$$

Property 2. *Given a matrix $N \in R^{n\times n}$, M is a Hurwitz matrix if its all real parts of the eigenvalues are negative. For example, the following matrix is Hurwitz:*

$$N = \begin{bmatrix} -3 & -2 \\ -2 & -3 \end{bmatrix}.$$

For nonlinearity $f(x_i)$, we began the analysis with the following properties.

Property 3 ([36])**.** *the differentiable global Lipschitz function $f(x_i)$ can be divided into two increasing functions $a(x_i)$ and $b(x_i)$; their relationship is*

$$f(x_i) = a(x_i) - b(x_i). \tag{8}$$

Property 4 ([36])**.** *for a Lipschitz function $f(x_i)$, there exists a differentiable global Lipschitz function $\tilde{f}(x_i^{k_1}, x_i^{k_2})$, such that*

- $\tilde{f}(x_i^{k_1}, x_i^{k_2}) = a(x_i^{k_1}) - b(x_i^{k_2})$,
- $\tilde{f}(x_i, x_i) = f(x_i)$,
- $\frac{\partial \tilde{f}}{\partial a(x_i)} \geq 0$ *and* $\frac{\partial \tilde{f}}{\partial b(x_i)} \leq 0$.

Remark 1. *In Property 3, a Lipschitz function is transformed into two increasing functions, which is just for us to introduce the function $\tilde{f}(\cdot,\cdot)$. Because $a(x_i)$ and $b(x_i)$ are both increasing functions, one can deduce that $\frac{\partial \tilde{f}}{\partial a(x_i)} \geq 0$ and $\frac{\partial \tilde{f}}{\partial b(x_i)} \leq 0$ easily. For $\underline{x} \leq x \leq \overline{x}$, we have $\tilde{f}(\underline{x}, \overline{x}) \leq \tilde{f}(x, x) \leq \tilde{f}(\overline{x}, \underline{x})$, which are the upper and lower bounds of the Lipschitz function $f(x)$ in the structure of distributed interval observers.*

Assuming that the bounds of the system state satisfy $\underline{x}_i \leq x_i \leq \overline{x}_i$, on the basis of Properties 3 and 4, one can deduce that

$$\tilde{f}(\underline{x}_i, \overline{x}_i) \leq \tilde{f}(x_i, x_i) = f(x_i) \leq \tilde{f}(\overline{x}_i, \underline{x}_i). \tag{9}$$

By using the generalized Taylor formula, $\tilde{f}(\overline{x}_i, \underline{x}_i) - \tilde{f}(x_i, x_i)$ is written as

$$\tilde{f}(\overline{x}_i, \underline{x}_i) - \tilde{f}(x_i, x_i) = \int_0^1 \frac{\partial \tilde{f}}{\partial \delta_1}(\tau\delta_1 + (1-\tau)\delta_2)d\tau(\delta_1 - \delta_2), \tag{10}$$

where $\delta_1 = \begin{bmatrix} \overline{x} \\ \underline{x} \end{bmatrix}$ and $\delta_2 = \begin{bmatrix} x \\ x \end{bmatrix}$. It follows from Property 4 that

$$\begin{aligned} \tilde{f}(\overline{x}_i, \underline{x}_i) - \tilde{f}(x_i, x_i) &= [\int_0^1 \frac{\partial a}{\partial x}(\tau\overline{x} + (1-\tau)x)d\tau \\ &\quad - \int_0^1 \frac{\partial b}{\partial x}(\tau x + (1-\tau)\underline{x})d\tau](\delta_1 - \delta_2) \\ &= [\mathcal{F}_1(\overline{x}, x) - \mathcal{F}_2(x, \underline{x})](\delta_1 - \delta_2). \end{aligned} \tag{11}$$

Similarly, we have

$$\tilde{f}(x_i, x_i) - \tilde{f}(\overline{x}_i, \underline{x}_i) = [\mathcal{F}_3(x, \underline{x}) - \mathcal{F}_4(\overline{x}, x)](\delta_1 - \delta_2), \tag{12}$$

where matrices $\mathcal{F}_i, i \in \{1, \ldots, 4\}$ are non-negative and can be derived from (11). On the basis of the above discussion, the following property is presented.

Property 5 ([36])**.** *for $f(x_i)$ and $\tilde{f}(\cdot,\cdot)$ defined in Property 4, if Jacobian matrix $\frac{\partial \tilde{f}}{\partial \delta_1}$ is bounded, there exist matrices $F_i, i \in \{1,\dots,4\}$ such that*

$$\begin{cases} \tilde{f}(\overline{x}_i, \underline{x}_i) - \tilde{f}(x_i, x_i) \le F_1\overline{e}_i + F_2\underline{e}_i, \\ \tilde{f}(x_i, x_i) - \tilde{f}(\underline{x}_i, \overline{x}_i) \le F_3\overline{e}_i + F_4\underline{e}_i, \end{cases} \tag{13}$$

where $\overline{e}_i = \overline{x}_i - x_i$ and $\underline{e}_i = x_i - \underline{x}_i$.

Example 1. *For nonlinear function $f(x) = sinx$, corresponding functions $\tilde{f}(\overline{x}, \underline{x})$ and $\tilde{f}(\underline{x}, \overline{x})$ are defined as*

$$\begin{cases} \tilde{f}(\overline{x}, \underline{x}) = sin(\overline{x}) + \overline{x} - \underline{x}, \\ \tilde{f}(\underline{x}, \overline{x}) = sin(\underline{x}) + \underline{x} - \overline{x}, \end{cases}$$

where $f(x)$, $\tilde{f}(\underline{x}, \overline{x})$ and $\tilde{f}(\overline{x}, \underline{x})$ satisfy Properties 3 and 4.

The functions mentioned in Property 3 are defined as $a(x) = sin(x) + x$ and $b(x) = x$. $a(x)$ and $b(x)$ are both obviously increasing functions. Then, $\tilde{f}(\overline{x}, \underline{x}) - \tilde{f}(x, x)$ can be transformed into

$$\begin{aligned} &\tilde{f}(\overline{x}, \underline{x}) - \tilde{f}(x, x) \\ =\ &sin(\overline{x}) + \overline{x} - \underline{x} - sin(x) \\ =\ &(sin(\overline{x}) - sin(x)) + (\overline{x} - x) + (x - \underline{x}) \\ \le\ &2(\overline{x} - x) + (x - \underline{x}). \end{aligned} \tag{14}$$

From Property 5, we have $\tilde{f}(\overline{x}, \underline{x}) - \tilde{f}(x, x) \le F_1\overline{e} + F_2\underline{e}$. Combining (14) with Property 5, we chose $F_1 = 2I$ and $F_2 = I$. Similarly, we have $F_3 = I$ and $F_4 = 2I$.

From Example 1, Property 5 is feasible, and the result from Formulas (9)–(12) stands.

For System (7), nonlinear function $f(x_i(t))$ was assumed to satisfy Properties 3–5; then, the interval observer for (7) was designed with

$$\begin{cases} D_t^{\alpha}\overline{x}_i(t) = A\overline{x}_i(t) + L(y_i(t) - C\overline{x}_i(t)) + \gamma M \sum_{i=1}^{N} a_{ij}(\overline{x}_j(t) - \overline{x}_i(t)) + \tilde{f}(\overline{x}_i, \underline{x}_i), \\ D_t^{\alpha}\underline{x}_i(t) = A\underline{x}_i(t) + L(y_i(t) - C\underline{x}_i(t)) + \gamma M \sum_{i=1}^{N} a_{ij}(\underline{x}_j(t) - \underline{x}_i(t)) + \tilde{f}(\underline{x}_i, \overline{x}_i), \end{cases} \tag{15}$$

where M and L are interval observer gains.

The error dynamics of the interval observer is

$$\begin{cases} D_t^{\alpha}\overline{e}_i(t) = D_t^{\alpha}\overline{x}_i(t) - D_t^{\alpha}x_i(t) \\ \qquad = (A - LC - \gamma M \sum_{j=1}^{N} \mathcal{L}_{ij})\overline{e}_i(t) + \tilde{f}(\overline{x}_i, \underline{x}_i) - \tilde{f}(x_i, x_i), \\ D_t^{\alpha}\underline{e}_i(t) = D_t^{\alpha}x_i(t) - D_t^{\alpha}\overline{x}_i(t) \\ \qquad = (A - LC - \gamma M \sum_{j=1}^{N} \mathcal{L}_{ij})\underline{e}_i(t) + \tilde{f}(x_i, x_i) - \tilde{f}(\underline{x}_i, \overline{x}_i). \end{cases} \tag{16}$$

Then, $e(t)$, $\tilde{f}(\overline{x}, \underline{x})$, $\tilde{f}(\underline{x}, \overline{x})$ and $\tilde{f}(x, x)$ are defined as

$$e(t) = \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix}, \tilde{f}(\overline{x}, \underline{x}) = \begin{bmatrix} \tilde{f}(\overline{x}_1, \underline{x}_1) \\ \vdots \\ \tilde{f}(\overline{x}_N, \underline{x}_N) \end{bmatrix}, \tilde{f}(\underline{x}, \overline{x}) = \begin{bmatrix} \tilde{f}(\underline{x}_1, \overline{x}_1) \\ \vdots \\ \tilde{f}(\underline{x}_N, \overline{x}_N) \end{bmatrix}, \tilde{f}(x, x) = \begin{bmatrix} \tilde{f}(x_1, x_1) \\ \vdots \\ \tilde{f}(x_N, x_N) \end{bmatrix}.$$

System (16) can be written in the following form:

$$\begin{cases} D_t^{\alpha}\overline{e}(t) = (I_N \otimes (A - LC) - \gamma(\mathcal{L} \otimes M)\overline{e}(t) + \tilde{f}(\overline{x}, \underline{x}) - \tilde{f}(x, x), \\ D_t^{\alpha}\underline{e}(t) = (I_N \otimes (A - LC) - \gamma(\mathcal{L} \otimes M))\underline{e}(t) + \tilde{f}(x, x) - \tilde{f}(\underline{x}, \overline{x}). \end{cases} \tag{17}$$

3. Main Results

Proving the boundedness of the distributed interval observer is equal to proving the convergence of the error dynamics. The Lyapunov function is a great choice for it. For error dynamics $\dot{e}(t) = A_e e(t)$, the following Lyapunov function is constructed:

$$V(t) = e^T(t)Pe(t), \tag{18}$$

where P is a positive symmetric matrix with suitable dimensions.

In the integer-order system, $\dot{V}(t) < 0$ is the sufficient condition to prove the convergence of error dynamics. $\dot{V}(t) < 0$ is equivalent to

$$A_e^T P + PA_e \prec 0. \tag{19}$$

Equation (19) is the strict LMI that can be computed with MATLAB. Formula (19) can yield that A_e is a Hurwitz matrix. By configuring matrix A_e, the convergence of the system can be reached.

However, the above details are mainly about the integer-order system. There are no corresponding lemmas about the fractional-order system. Therefore, the fractional-order extension of a Lyapunov candidate function is introduced to demonstrate the convergence of the error dynamics by referring [37].

Lemma 4. *Consider error dynamics $e(t) \in R^n$ to be a continuous and derivable function. Then, for any time $t \geq 0$, the fractional derivative of the Lyapunov function is*

$$D_t^\alpha V(t) \leq (D_t^\alpha e^T(t))Pe(t) + e^T(t)P(D_t^\alpha e(t)), \tag{20}$$

where $V(t) = e^T(t)Pe(t)$ is the Lyapunov function connected with $e(t)$.

Proof. Denote that $J = (D_t^\alpha e^T)Pe + e^T P(D_t^\alpha e) - D_t^\alpha V$.

According to the definition of fractional calculus, $D_t^\alpha V(t)$ is equivalent to

$$\begin{aligned} D_t^\alpha V(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{V}(\tau)}{(t-\tau)^\alpha} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{e^T(\tau)P\dot{e}(\tau) + \dot{e}^T(\tau)Pe(\tau)}{(t-\tau)^\alpha} d\tau. \end{aligned} \tag{21}$$

Considering (21), J is rewritten as

$$\begin{aligned} J &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{e^T(t)P\dot{e}(\tau) + \dot{e}^T(\tau)Pe(t) - e^T(\tau)P\dot{e}(\tau) - \dot{e}^T(\tau)Pe(\tau)}{(t-\tau)^\alpha} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(e^T(t) - e^T(\tau))P\dot{e}(\tau) + \dot{e}^T(\tau)P(e(t) - e(\tau))}{(t-\tau)^\alpha} d\tau \\ &= -\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{z}^T(\tau)Pz(\tau) + z^T(\tau)P\dot{z}(\tau)}{(t-\tau)^\alpha} d\tau, \end{aligned} \tag{22}$$

where $z(\tau) = e(t) - e(\tau)$.

Then, the proof for Lemma 1 is equivalent to proving that

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(\dot{z}^T(\tau))Pz(\tau) + z^T(\tau)P\dot{z}(\tau)}{(t-\tau)^\alpha} d\tau \leq 0. \tag{23}$$

For $u = z^T(t)Pz(t)$, one can deduce that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \dot{z}^T Pz(t) + z^T(t)P\dot{z}(t). \tag{24}$$

Then, $v(\tau) = \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}$ is denoted, which yields

$$dv(\tau) = \frac{\alpha(t-\tau)^{-\alpha-1}}{\Gamma(1-\alpha)} d\tau. \tag{25}$$

On the basis of (24) and (25), (23) is equal to

$$\frac{z^T(\tau)Pz(\tau)}{\Gamma(1-\alpha)(t-\tau)^{\alpha}}\Big|_0^t - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{z^T(\tau)Pz(\tau)}{(t-\tau)^{\alpha}} d\tau \leq 0, \tag{26}$$

with

$$\frac{z^T(\tau)Pz(\tau)}{\Gamma(1-\alpha)(t-\tau)^{\alpha}}\Big|_0^t = \frac{z^T(\tau)Pz(\tau)}{\Gamma(1-\alpha)(t-\tau)^{\alpha}}\Big|_{\tau=t} - \frac{z^T(0)Pz(0)}{\Gamma(1-\alpha)(t)^{\alpha}}. \tag{27}$$

When $\tau = t$, the first term of (27), has nondeterminacy, and the analysis of its limitation is necessary:

$$\begin{aligned}\lim_{\tau\to t}\frac{z^T(\tau)Pz(\tau)}{\Gamma(1-\alpha)(t-\tau)^{\alpha}} &= \frac{-\dot{e}^T(\tau)Pe(t) + \dot{e}^T(\tau)Pe(\tau) - e^T(t)P\dot{e}(\tau) + e^T(\tau)P\dot{e}(\tau)}{-\Gamma(1-\alpha)\alpha(t-\tau)^{\alpha-1}} \\ &= \frac{-\dot{e}^T(\tau)Pe(t) + \dot{e}^T(\tau)Pe(\tau) - e^T(t)P\dot{e}(\tau) + e^T(\tau)P\dot{e}(\tau)}{-\Gamma(1-\alpha)\alpha}(t-\tau)^{1-\alpha}.\end{aligned} \tag{28}$$

Due to $\alpha \in (0,1)$, it is obvious that $\frac{z^T(\tau)Pz(\tau)}{\Gamma(1-\alpha)(t-\tau)^{\alpha}} \to 0$ when $\tau = t$.
Expression (26) can be simplified to

$$-\frac{z^T(0)Pz(0)}{\Gamma(1-\alpha)t^{\alpha}} - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{z^T(\tau)Pz(\tau)}{(t-\tau)^{\alpha}} d\tau \leq 0. \tag{29}$$

For $P \succ 0$ and $\Gamma(\cdot) > 0$, it is certain to stand. □

Theorem 1. *For System (15), there exist bounds $\overline{x}$ and $\underline{x}$, such that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ if the following conditions are satisfied:*

1. *$\Omega = I_N \otimes (A - LC) - \gamma(\mathcal{L} \otimes M)$ is Metzler;*
2. *The initial condition of (7) satisfies $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$;*
3. *Nonlinear function $f(x_i(t))$ possesses the features introduced in Properties 3–5.*

Proof. From Properties 3 and 4, the upper bound of the nonlinear function is equal to

$$\tilde{f}(\overline{x}_i, \underline{x}_i) = a(\overline{x}_i) - b(\underline{x}_i). \tag{30}$$

Then, it follows from (30) that

$$\tilde{f}(\overline{x}_i, \underline{x}_i) - \tilde{f}(x_i, x_i) = (a(\overline{x}_i) - a(x_i)) - (b(\underline{x}_i) - b(x_i)). \tag{31}$$

Functions $a(\cdot)$ and $b(\cdot)$ are all increasing, which yields

$$\tilde{f}(\overline{x}_i, \underline{x}_i) - \tilde{f}(x_i, x_i) \geq 0. \tag{32}$$

Similarly, one can deduce that

$$\tilde{f}(x_i, x_i) - \tilde{f}(\underline{x}_i, \overline{x}_i) \geq 0. \tag{33}$$

From $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, the initial value of the error dynamics satisfies $\overline{e}(0) \geq 0$ and $\underline{e}(0) \geq 0$. If matrix Ω is Metzler, and (32) and (33) are true, it follows that $\overline{e}(t) \geq 0$ and $\underline{e}(t) \geq 0$, i.e., $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, for any $t \geq 0$. □

Remark 2. *For Theorem 1, we constructed a frame for (7). The positivity of the error dynamics is guaranteed, which implies $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$. However, an interval observer not only requires the error dynamics be positive, but also that the upper and lower errors are in a convergence of zero. Then, we give the following theorem.*

Theorem 2. *On the basis of the result in Theorem 1, given constant $\tau > 0$ and positive matrix $P = P^T$, if there exists a solution such that*

$$\hat{\Pi} = \begin{bmatrix} He(PA - UC + PN_1) - 2\tau I & PN_2 + N_3^T P \\ PN_3 + N_2^T P & He(PA - UC + PN_4) - 2\tau I \end{bmatrix} \prec 0,$$

$$\gamma > \frac{\tau}{a(\mathcal{L})},$$

where γ is the coupling strength, $L = P^{-1}U$ and $M = P^{-1}$ are the observer gains, then System (15) is a distributed interval observer.

Proof. The Lyapunov function is chosen as follows:

$$V(t) = \sum_{i=1}^{N} r_i \overline{e}_i^T(t) P \overline{e}_i(t) + \sum_{i=1}^{N} r_i \underline{e}_i^T(t) P \underline{e}_i(t). \tag{34}$$

From Lemma 1, the fractional derivative of $V(t)$ is

$$\begin{aligned} D_t^\alpha V(t) \leq & \sum_{i=1}^{N} r_i (D_t^\alpha \overline{e}_i^T)(t) P \overline{e}_i(t) + \sum_{i=1}^{N} r_i \underline{e}_i^T(t) P (D_t^\alpha \underline{e}_i(t)) \\ & + \sum_{i=1}^{N} r_i (D_t^\alpha \underline{e}_i^T)(t) P \underline{e}_i(t) + \sum_{i=1}^{N} r_i \overline{e}_i^T(t) P (D_t^\alpha \overline{e}_i(t)). \end{aligned} \tag{35}$$

Substituting the error dynamics into (35)

$$\begin{aligned} D_t^\alpha V(t) = & \sum_{i=1}^{N} r_i (\Omega_i \overline{e}_i + \overline{\Delta f}(x_i))^T P \overline{e}_i + \sum_{i=1}^{N} r_i \underline{e}_i^T P (\Omega_i \underline{e}_i + \underline{\Delta f}(x_i)) \\ & + \sum_{i=1}^{N} r_i (\Omega_i \underline{e}_i + \underline{\Delta f}(x_i))^T P \overline{e}_i + \sum_{i=1}^{N} r_i \overline{e}_i^T P (\Omega_i \overline{e}_i + \overline{\Delta f}(x_i)) \\ \leq & \sum_{i=1}^{N} r_i (\Omega_i \overline{e}_i + F_1 \overline{e}_i + F_2 \underline{e}_i)^T P \overline{e}_i + \sum_{i=1}^{N} r_i \overline{e}_i^T P (\Omega_i \overline{e}_i + F_1 \overline{e}_i + F_2 \underline{e}_i) \\ & + \sum_{i=1}^{N} r_i \underline{e}_i^T P (\Omega_i \overline{e}_i + F_3 \overline{e}_i + N_F \underline{e}_i) + \sum_{i=1}^{N} r_i (\Omega_i \overline{e}_i + F_3 \overline{e}_i + F_4 \underline{e}_i)^T P \underline{e}_i \\ = & ((R \otimes (A - LC) - \gamma R\mathcal{L} \otimes M)\overline{e} + R \otimes F_1 \overline{e} + R \otimes F_2 \underline{e})^T (I \otimes P) \overline{e} \\ & + \overline{e}^T (I \otimes P)((R \otimes (A - LC) - \gamma R\mathcal{L} \otimes M)\overline{e} + R \otimes F_1 \overline{e} + R \otimes F_2 \underline{e} \\ & + ((R \otimes (A - LC) - \gamma R\mathcal{L} \otimes M)\underline{e} + R \otimes N_3 \overline{e} + R \otimes F_4 \underline{e})^T (I \otimes P) \overline{e} \\ & + \overline{e}^T (I \otimes P)((R \otimes (A - LC) - \gamma R\mathcal{L} \otimes M)\overline{e} + R \otimes F_3 \overline{e} + R \otimes N_4 \underline{e}) \\ = & \overline{e}^T (R \otimes (He(PA - PLC + PF_1)) - \gamma (R\mathcal{L} + \mathcal{L}^T R) \otimes PM) \overline{e} \\ & + \underline{e}^T (R \otimes (He(PA - PLC + PF_4)) - \gamma (R\mathcal{L} + \mathcal{L}^T R) \otimes PM) \underline{e} \\ & + \overline{e}^T (PF_2 + F_3^T P) \underline{e} + \underline{e}^T (PF_3 + F_2^T P) \overline{e}, \end{aligned} \tag{36}$$

where $\Omega_i = A - LC - \gamma M \sum_{j=1}^{N} \mathcal{L}_{ij}$, $\overline{\Delta f}(x_i) = \tilde{f}(\overline{x}_i, \underline{x}_i) - \tilde{f}(x_i, x_i)$ and $\underline{\Delta f}(x_i) = \tilde{f}(x_i, x_i) - \tilde{f}(\underline{x}_i, \overline{x}_i)$.

According to Lemma 3, $R\mathcal{L} + \mathcal{L}^T R$ could be simplified as

$$2a(\mathcal{L})\bar{e}^T R\bar{e} \leq \bar{e}^T(R\mathcal{L} + \mathcal{L}^T R)\bar{e}, \tag{37}$$

$$2a(\mathcal{L})\underline{e}^T R\underline{e} \leq \underline{e}^T(R\mathcal{L} + \mathcal{L}^T R)\underline{e}. \tag{38}$$

Taking $M = P^{-1}$, (37) and (38) into account, it follows from (36) that

$$\begin{aligned} D_t^\alpha V(t) \leq\ & \bar{e}^T(R \otimes (He(PA - PLC + PF_1)) - 2\gamma a(\mathcal{L})R \otimes I_n)\bar{e} \\ & + \underline{e}^T(R \otimes (He(PA - PLC + PF_4)) - 2\gamma a(\mathcal{L})R \otimes I_n\underline{e} \\ & + \bar{e}^T(PF_2 + F_3^T P)\underline{e} + \underline{e}^T(PF_3 + F_2^T P)\bar{e}. \end{aligned} \tag{39}$$

Considering $\gamma > \frac{\tau}{a(\mathcal{L})}$, we have

$$\begin{aligned} D_t^\alpha V(t) \leq\ & \bar{e}^T(R \otimes (He(PA - PLC + PF_1)) - 2\tau R \otimes I_n\bar{e} \\ & + \underline{e}^T(R \otimes (He(PA - PLC + PF_4)) - 2\tau R \otimes I_n\underline{e} \\ & + \bar{e}^T(PF_2 + F_3^T P)\underline{e} + \underline{e}^T(PF_3 + F_2^T P)\bar{e} \\ =\ & \epsilon^T(t)R \otimes \Pi\epsilon(t), \end{aligned}$$

where $\epsilon(t) = [\bar{e}^T(t), \underline{e}^T(t)]^T$ and

$$\Pi = \begin{bmatrix} He(PA - PLC + PF_1) - 2\tau I_n & PF_2 + F_3^T P \\ PF_3 + F_2^T P & He(PA - PLC + PF_4) - 2\tau I_n \end{bmatrix}.$$

To satisfy the LMI toolbox, $U = PL$ is applied in Π, which leads to

$$\hat{\Pi} = \begin{bmatrix} He(PA - UC + PF_1) - 2\tau I_n & PF_2 + F_3^T P \\ PF_3 + F_2^T P & He(PA - UC + PF_4) - 2\tau I_n \end{bmatrix}.$$

Then, matrix $\hat{\Pi} \prec 0$ is equal to $D_t^\alpha V(t) < 0$, which implies that $\lim_{t\to\infty} \bar{e}(t) = 0$ and $\lim_{t\to\infty} \underline{e}(t) = 0$. The boundedness of the error dynamics could be guaranteed. □

Remark 3. *Constant τ is simple without an actual effect. However, it is a parameter connected with γ. If we just compute γ, the LMI toolbox just gives one feasible solution. Nevertheless, if we compute τ, $\gamma > \frac{\tau}{a(\mathcal{L})}$ would have more regions to select.*

On the basis of Theorem 2, an algorithm was constructed to design distributed interval observers for fractional-order MASs.

4. Simulation

Considering a fractional-order MAS with nonlinearity, the state-space model is similar to (7), where

$$A = \begin{bmatrix} -0.5 & 2 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad f(x_i) = sin(x_i).$$

For nonlinear function $f(x_i)$, corresponding function $\tilde{f}(\bar{x}_i, \underline{x}_i)$ is defined as

$$\tilde{f}(\bar{x}_i, \underline{x}_i) = sin(\bar{x}_i) + \bar{x}_i - \underline{x}_i.$$

For functions $a(x_i) = sin(x_i) + x_i$ and $b(x_i) = x_i$, it is obvious that $a(x_i)$ and $b(x_i)$ are increasing functions. $a(x_i) = sin(x_i) + x_i$ and $b(x_i) = x_i$ are substituted into (13); then,

$$\begin{aligned} & sin(\overline{x}_i) + \overline{x}_i - \underline{x}_i - sin(x_i) \\ &= (sin(\overline{x}_i) - sin(x_i)) + (\overline{x}_i - x_i) + (x_i - \underline{x}_i) \\ &\leq 2(\overline{x}_i - x_i) + (x_i - \underline{x}_i) \\ &= F_1\overline{e}_i + F_2\underline{e}_i. \end{aligned}$$

We chose $F_1 = 2I$ and $F_2 = I$. Similarly, we have $F_3 = I$ and $F_4 = 2I$.
Laplacian matrix $\mathcal{L}$ is

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

By using Lemma 3, we obtained generalized algebraic connectivity $a(\mathcal{L}) = 1$. By solving the LMI in Theorem 2, the observer gains can be computed:

$$L = \begin{bmatrix} -1.1771 \\ -0.2282 \end{bmatrix}, \ M = \begin{bmatrix} 0.1906 & 0.0757 \\ 0.0757 & 0.1470 \end{bmatrix}, \tau = 2.9151.$$

Then we chose $\gamma = 3$. The initial value of the original system state is defined as $x(0) = [1\ 2\ 3\ 5\ 2\ -1\ -1\ 4]^T$. The initial value of the distributed interval observer is defined as $\overline{x}(0) = [6\ 7\ 8\ 10\ 7\ 4\ 4\ 9]^T$ and $\underline{x}(0) = [-4\ -3\ -2\ 0\ -4\ -6\ -6\ -1]^T$.

By performing Steps 5–8 in Algorithm 1, we can obtain Figures 1–6. Then, Figures 1 and 2 show the original system state of the four agents. Figures 3 and 4 display the bounds from a distributed interval observer. v_{ij} means the upper bound of the jth state of the ith agent, while u_{ij} means the lower bound of the jth state of the ith agent. From Figures 3 and 4, the bounds of the distributed observer trap the state of the original system. Define that $e_{ij}^+ = \overline{x}_{ij} - x_{ij}$ and $e_{ij}^+ = x_{ij} - \underline{x}_{ij}$. For Figures 5 and 6, the error between the observer and the original system could be reduced to a bounded value, which implies that the distributed interval observer is feasible.

Algorithm 1 Distributed interval estimation for fractional-order MASs.

Step 1: Given the constant matrix $\mathcal{L}$, compute the generalized algebraic connectivity $a(\mathcal{L})$;
Step 2: For given matrices $A \in R^{n\times n}$, $C \in R^{m\times n}$, compute the observer gains L and M and the constant τ from Theorem 2;
Step 3: Select appropriate constant $\gamma > \frac{\tau}{a(\mathcal{L})}$;
Step 4: Ensure that the matrix Ω is a Metzler matrix;
Step 5: For the total time $t = 10$, choosing the step size $h = 0.001$, the step $N = T/h$ can be calculated.
Step 6: Based on Step 5, construct two loops. The first loop is for the original fractional-order system and the other loop is for the distributed interval observer.
Step 7: Construct an array with $N + 1$ volumes to store the output of the original system in step 6.
Step 8: Establish a distributed interval observer based on the output in Step 7 and the other loop proposed in Step 6.

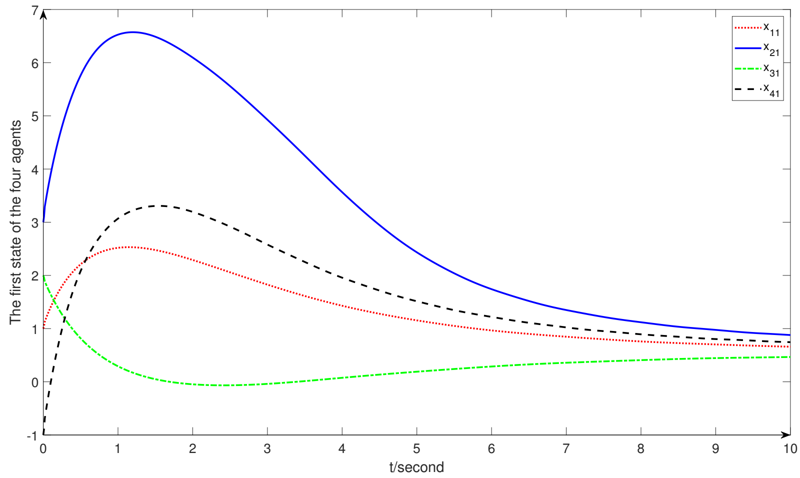

Figure 1. The first state of the four-agents.

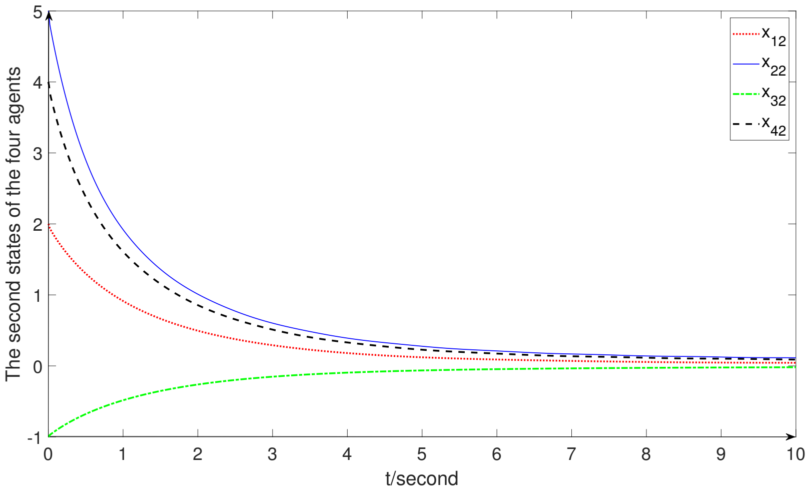

Figure 2. The second state of the four-agents.

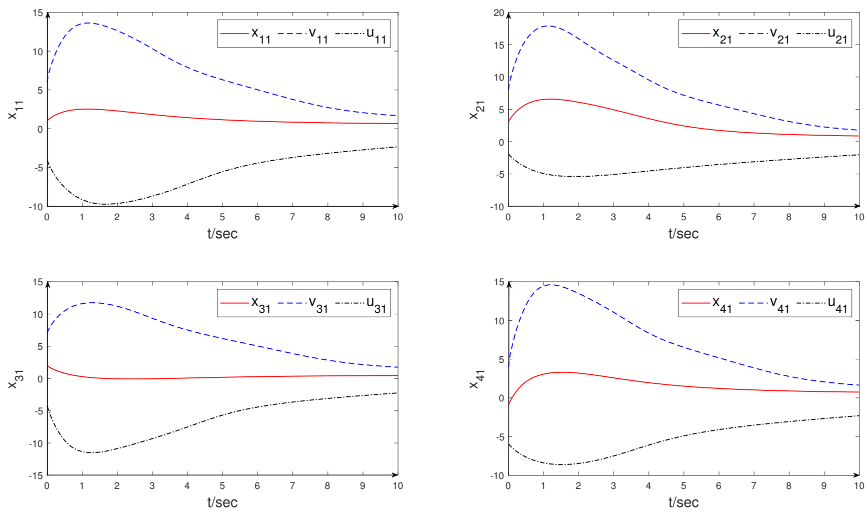

Figure 3. The bounds of the first state for the four-agents.

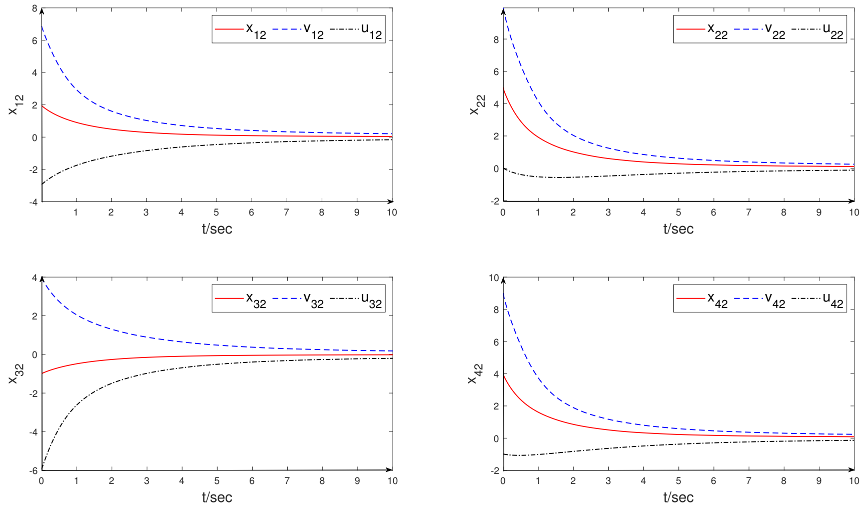

Figure 4. The bounds of the second state for the four-agents.

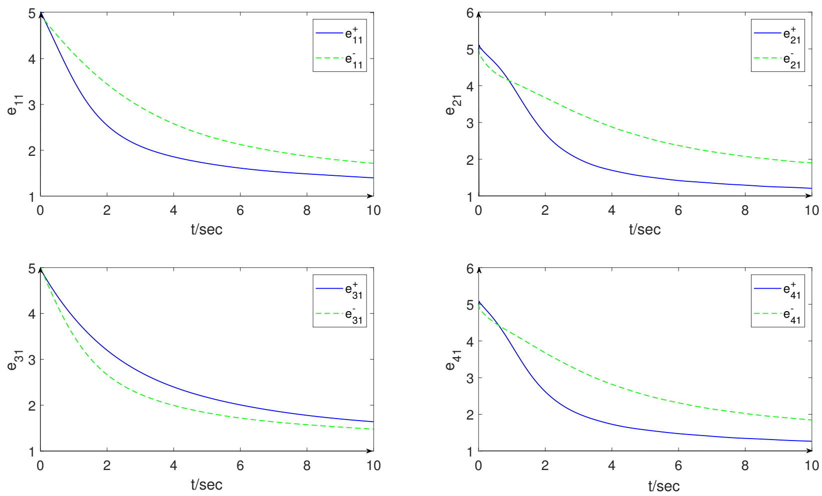

Figure 5. The error of the first state for the four-agents.

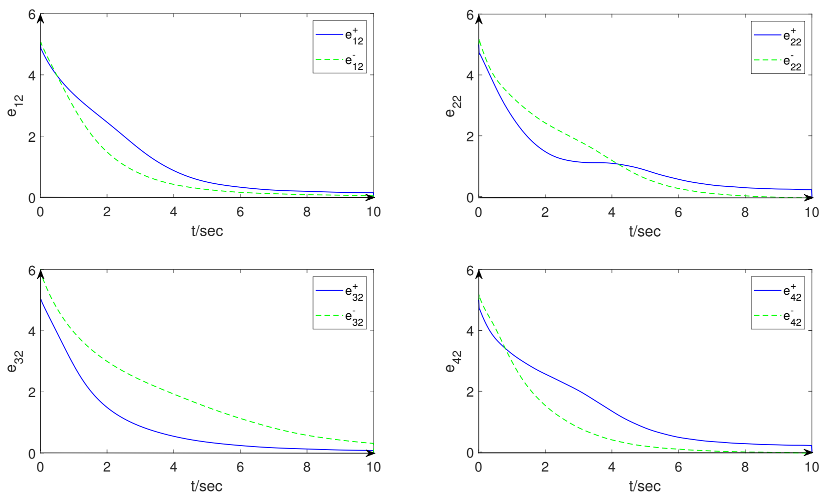

Figure 6. The error of the second state for the four-agents.

5. Conclusions

In this paper, a distributed interval observer design methodology for linear fractional-order MASs with nonlinearity was proposed. A Lyapunov method that is useful for observer and controller design was first introduced for general fractional-order systems. For MASs, graph theory was applied to fractional-order systems, and the strict LMI and an effective algorithm were presented for observer design. Lastly, an example was given to demonstrate the effectiveness of the proposed method. In the future, by using the H_∞ technique, we aim to focus on research regarding the consensus control or formation control of fractional-order MASs.

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