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Strong Solution for Fractional Mean Field Games with Non-Separable Hamiltonians

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Abstract: In this paper, we establish the existence and uniqueness of a strong solution to a fractional mean field games system with non-separable Hamiltonians, where the fractional exponent $\sigma \in (\frac{1}{2}, 1)$. Our result is new for fractional mean field games with non-separable Hamiltonians, which generalizes the work of D.M. Ambrose for the integral case. The important step is to choose the new appropriate fractional order function spaces and use the Banach fixed-point theorem under stronger assumptions for the Hamiltonians.

Keywords: fractional Laplacian; mean field game; non-separable Hamiltonians

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1. Introduction

In this paper, we consider the following time-dependent fractional mean field games systems with non-separable Hamiltonians:

$$u_t - (-\Delta)^{\sigma} u + \mathcal{H}(t, x, Du, m) = 0 \quad (t.x) \in [0, T] \times \mathbb{T}^n, \tag{1}$$
$$m_t + (-\Delta)^{\sigma} m + \mathrm{div}\big(m\mathcal{H}_p(t, x, Du, m)\big) = 0 \quad (t.x) \in [0, T] \times \mathbb{T}^n, \tag{2}$$

where $\mathbb{T}^n$ is the tours $\mathbb{R}^n/\mathbb{Z}^n$, $n \geq 2$, u is a value function, m is a probability distribution, $\mathcal{H} = \mathcal{H}(t, x, Du, m)$ is a Hamiltonian, $\mathcal{H}_p$ denotes $\frac{\partial}{\partial p}\mathcal{H}(t, x, p, m)$, and $\sigma \in (\frac{1}{2}, 1)$. Here, we define the fractional Laplacian $(-\Delta)^{\sigma}$ by the Fourier decomposition. For any $u \in \mathbb{T}^n$, if

$$u(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}(k) e^{ikx} \qquad \text{with} \qquad \hat{u}(k) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} u(x) e^{ikx} dx,$$

where $i^2 = -1$, then its fractional Laplacian is defined by

$$(-\Delta)^{\sigma} u = \sum_{k \in \mathbb{Z}^n} k^{2\sigma} \hat{u}(k) e^{ikx}.$$

We especially consider the initial-terminal problem of Equations (1) and (2) (i.e., the initial value of m and the terminal value of u are prescribed functions):

$$m(0, x) = m_0(x), \qquad u(T, x) = u_T(x). \tag{3}$$

The mean field games (MFG) system describes systems with very large numbers of identical agents in noncooperative differential games. Assume each agent wants to control his or her own trajectory in the same state space, which is affected by a stochastic differential equation:

$$dX_t = \alpha_t dt + dW_t, \tag{4}$$

where W_t is a stochastic noise. Meanwhile, each agent is rational and aims to minimize the following cost functional:

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T L(X_s,\alpha_s,m(s))ds + G(X_T,m(T))\right],$$

where $T > 0$ is the finite horizon of the problem and L and G are given continuous maps. Originally, there were two different approaches to solving such a problem from two different points of view, which were proposed independently by Lasry and Lions [1–3] in the mathematics community and Huang, Malhamé, and Caines [4] in the engineering field almost at the same time. The main idea of MFG is to implement strategies based on the distribution of the other agents. In recent years, MFG theory has attracted more and more interest and been used in more fields widely. Lachapelle, Salomon, and Turinici [5] presented a model for the choice of insulation technology in households using an MFG model. They obtained an existing result for the associated optimization problem and gave a monotonic algorithm to find the mean field equilibria. Based on the SIR model, Lee et al. [6] introduced an effective mean-field game model for controlling the propagation of epidemics and provided fast numerical algorithms based on proximal primal-dual methods. There are more and more applications in engineering, finance, AI optimization problems, pandemic and vaccine control, etc. We refer the reader to [7–10] for a fairly large description of the current literature on the models and their applications.

If the stochastic noise $W(t)$ in Equation (4) is a Brownian motion, we define the value function u as

$$u(t,x) = \inf_{\alpha} J(t,x,\alpha).$$

By the dynamic programming principle, we formally obtain the integral order equation MFG system:

$$u_t + \Delta u + \mathcal{H}(t,x,Du,m) = 0, \tag{5}$$

$$m_t - \Delta m + \operatorname{div}\big(m\mathcal{H}_p(t,x,Du,m)\big) = 0, \tag{6}$$

where the Hamiltonian $H : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ is defined by

$$H(t,x,p,m) := \sup_{a\in A}[-L(x,a,m) - p \cdot b(x,a,m)].$$

The MFG system of partial differential equations takes the form of a backward Hamilton–Jacobi equation coupled with a forward Fokker–Planck equation. We say the system is separable if the Hamiltonian $\mathcal{H}$ can be written in the following form:

$$\mathcal{H}(t,x,p,m) = H(t,x,p) + F(t,x,m).$$

Otherwise, we say the system is non-separable. In the case of a separable Hamiltonian, MFG systems are well analyzed. The existence and uniqueness of smooth solutions for the second-order MFG system with non-local terms is given in [2,3]. The existence and uniqueness of weak solutions with local terms and different types of boundary conditions were proven in [11–13]. In particular, Porretta [11,12] obtained the existence and uniqueness of weak solutions to one kind of planning problem, where the equation is only prescribed with the initial and terminal conditions for the density m (i.e., $m(0) = m_0$, $m(T) = m_1$, where $m_0, m_1 > 0$ are smooth functions). In [13], Porretta also developed a complete weak framework for the well-posed nature of weak solutions. The key work of his was to find new results for Fokker–Planck equations under minimal assumptions on the drift. We also refer to the MFG system with standard diffusion terms [9,14], degenerated diffusion terms [15], and first-order systems [16,17].

Recently, fractional MFG systems have also interested many researchers. In such cases, the stochastic noise $W(t)$ in Equation (4) is modeled by a symmetric σ stable noise, which is a jump process or anomalous diffusion. In [18], the subcritical order of the fractional case $\sigma \in (\frac{1}{2}, 1)$ was studied in the case of local and non-local coupling between equations. In [19], the authors established the existence and uniqueness of solutions to the evolution fractional MFG system $\sigma \in (0, 1)$ with regularized coupling by the vanishing viscosity method. In [20], the authors studied the fractional and non-local parabolic MFG systems driven by jump Lévy processes in the whole space. They obtained the existence and uniqueness of classical solutions of MFG systems with local and non-local couplings for separable Hamiltonians.

In the case of non-separable Hamiltonians, many works are concerned with the standard diffusion term. The existence theorem of classical solutions by the Schauder fixed-point theorem is shown in [21]. The existence theorem of weak solutions of stationary problems was proven in [22,23]. In [24,25], the author proved the existence and uniqueness theorem for strong solutions to time-dependent problems, and in [26], the author also studied the existence and uniqueness theory for non-separable mean field games in Sobolev spaces. To the best of our knowledge, there are few theories involved fractional mean field games with non-separable Hamiltonians.

In this paper, we study the existence and uniqueness results for strong solutions of the fractional MFG system in the case of a non-separable Hamiltonian when $\sigma \in (\frac{1}{2}, 1)$. Our main ideas stem from [24,25]. We adopt the similar function space based on Wiener algebra and the Banach fixed-point theorem for a short time of existence. While different from [24,25], we find that when the value function u and the probability distribution m are in the same function space, it becomes tricky to extend the result to the fractional Laplacian. Therefore, we consider enhancing the regularity of the value function u to a higher case than the probability distribution m as in [26]. In addition, as long as the non-separable Hamiltonian satisfies the assumption **A1** in Section 3, we can draw a conclusion: if the initial measure m_0 is close enough to its uniform measure on $\mathbb{T}^n$, and the terminal value u_T is small enough, then the solutions to the problems in Equations (1)–(3) exists and are unique in a ball about the origin.

This paper is organized as follows. In Section 2, we show some preliminaries, including the property of the norm in the function space and the estimation of operators to the case of a fractional Laplacian. In Section 3, we give the main result and use the contraction principle to prove our first main theorem (Theorem 1). Finally, in Section 4, for the payoff boundary conditions, we make the assumption that the payoff function G satisfies the condition **A2** and obtain the second theorem of this paper.

2. Preliminaries

In this section, we state some necessary lemmas about function spaces as in [24,25] that will be used below. Indeed, our main work is to extend the integer order space to a fractional order function space.

Let $T > 0$ and $\alpha \in \left(0, \frac{T}{2}\right)$ be given. As in [24,25], the function $\beta : [0, T] \to [0, \alpha]$ is defined by

$$\beta(s) = \begin{cases} 2\alpha s/T, & s \in [0, T/2], \\ 2\alpha - 2\alpha s/T, & s \in [T/2, T]. \end{cases} \tag{7}$$

Let s be a positive real number. The space B^s consists of continuous functions f from $\mathbb{T}^n$ to $\mathbb{R}$ such that the norm $\|f\|_{B^s}$ is finite (i.e., the following is true):

$$\|f\|_{B^s} = \sum_{k \in \mathbb{Z}^n} (1 + |k|^s)|\hat{f}(k)| < \infty.$$

In addition, similar to the function space defined in [24,25], an extended space-time version $\mathcal{B}^s_\alpha$ consists of all functions in $C([0, T]; B^s)$ such that $\|f\|_{\mathcal{B}^s_\alpha}$ is finite (i.e., we have the following):

$$\|f\|_{\mathcal{B}^s_a} = \sum_{k\in\mathbb{Z}^n} \sup_{t\in[0,T]} (1+|k|^s)e^{\beta(t)|k|}|\hat{f}(t,k)| < \infty. \tag{8}$$

Before we prove some properties of $\mathcal{B}^j_\alpha$, we need the following algebra property:

Lemma 1. *For any $j\in\mathbb{N}$, there exists $c_j>0$ such that if $f\in\mathcal{B}^j_\alpha$ and $g\in\mathcal{B}^j_\alpha$, then $fg\in\mathcal{B}^j_\alpha$ with the estimate*

$$\|fg\|_{\mathcal{B}^j_a} \le c_j\|f\|_{\mathcal{B}^j_a}\|g\|_{\mathcal{B}^j_a}.$$

The proof of Lemma 1 follows along the same lines as that in [25], except for minor modifications which are omitted here.

Since the MFG system is a couple of backward and forward evolution equations, we introduce the operators I^+_σ and I^-_σ for the fractional nonhomogeneous heat equation, which are defined as follows:

$$(I^+_\sigma h)(t,\cdot) = \int_0^t e^{-(-\Delta)^\sigma(t-s)}h(s,\cdot)ds, \tag{9}$$

$$(I^-_\sigma h)(t,\cdot) = \int_t^T e^{-(-\Delta)^\sigma(s-t)}h(s,\cdot)ds. \tag{10}$$

For any $j\in\mathbb{N}$ and $\sigma\in(\frac{1}{2},1)$, we show that I^+_σ and I^-_σ are bounded linear operators from $\mathcal{B}^j_\alpha$ to $\mathcal{B}^{j+2\sigma}_\alpha$:

Lemma 2. *For any $\sigma\in(\frac{1}{2},1)$, if $h\in B^j_\alpha$ with $\int_{\mathbb{T}^d}h(t,x)dx=0$, then we have $(I^+_\sigma h)(t,\cdot)\in\mathcal{B}^{j+2\sigma}_\alpha$ and*

$$\|(I^+_\sigma h)(t,\cdot)\|_{\mathcal{B}^{j+2\sigma}_\alpha} \le C\|h(t,\cdot)\|_{\mathcal{B}^j_\alpha}. \tag{11}$$

Proof. Through Equation (9) and a convolution formula, we have

$$\widehat{(I^+_\sigma h)}(t,\cdot) = \int_0^t e^{-(t-s)|k|^{2\sigma}}\widehat{h}(s,\cdot)ds,$$

and

$$\|I^+_\sigma h\|_{\mathcal{B}^{j+2\sigma}_\alpha} = \sum_{k\in\mathbb{Z}^n}\left(1+|k|^{j+2\sigma}\right)\sup_{t\in[0,T]} e^{\beta(t)|k|}\left|\int_0^t e^{-(t-s)|k|^{2\sigma}}\hat{h}(s,\cdot)ds\right|.$$

Since $\int_{\mathbb{T}^d}h(t,x)dx=0$, $\hat{h}(s,0)=0$, then we have

$$\begin{aligned}
&\left\|I^+_\sigma h\right\|_{\mathcal{B}^{j+2\sigma}_a}\\
&\le 2\sum_{k\in\mathbb{Z}^n\setminus\{0\}}|k|^{j+2\sigma}\sup_{t\in[0,T]} e^{\beta(t)|k|}\left|\int_0^t e^{-|k|^{2\sigma}(t-s)}e^{-\beta(s)|k|}e^{\beta(s)|k|}\hat{h}(s,k)ds\right|\\
&\le 2\sum_{k\in\mathbb{Z}^n\setminus\{0\}}|k|^{j+2\sigma}\sup_{t\in[0,T]}\left[e^{\beta(t)|k|}\int_0^t e^{-|k|^{2\sigma}(t-s)}e^{-\beta(s)|k|}\sup_{\tau\in[0,T]}\left[e^{\beta(\tau)|k|}|\hat{h}(\tau,k)|\right]ds\right]\\
&\le 2\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\left[|k|^j\sup_{\tau\in[0,T]}e^{\beta(\tau)|k|}|\hat{h}(\tau,k)|\right][\sup_{t\in[0,T]}|k|^{2\sigma}e^{\beta(t)|k|}\int_0^t e^{-|k|^{2\sigma}(t-s)-\beta(s)|k|}ds]\\
&\le 2\left[\sup_{t\in[0,T]}\sup_{k\in\mathbb{Z}^n\setminus\{0\}}|k|^{2\sigma}e^{\beta(t)|k|}\int_0^t e^{-|k|^{2\sigma}(t-s)-\beta(s)|k|}ds\right]\|h\|_{\mathcal{B}^j_\alpha}.
\end{aligned}$$

Therefore, to verify that I_σ^+ is a bounded linear operator between $\mathcal{B}_\alpha^j$ and $\mathcal{B}_\alpha^{j+2\sigma}$, we have to prove that

$$\sup_{t\in[0,T]}\sup_{k\in\mathbb{Z}^n\backslash\{0\}}|k|^{2\sigma}e^{\beta(t)|k|-t|k|^{2\sigma}}\int_0^t e^{|k|^{2\sigma}s-\beta(s)|k|}ds<\infty.$$

First, when $t\in[0,T/2]$, we have $\beta(s)=2\alpha s/T$ by (7). Then, we obtain that

$$\int_0^t e^{|k|^{2\sigma}s-\beta(s)|k|}ds=\int_0^t e^{|k|^{2\sigma}s-2\alpha s|k|/T}ds=\frac{\exp\{|k|^{2\sigma}t-2\alpha t|k|/T\}-1}{|k|^{2\sigma}-2\alpha|k|/T}. \tag{12}$$

Since $\alpha\in(0,T/2)$ and $\sigma\in(\frac{1}{2},1)$, by Equation (12), we have

$$\begin{aligned}
&\sup_{t\in[0,T/2]}\sup_{k\in\mathbb{Z}^n\backslash\{0\}}|k|^{2\sigma}e^{\beta(t)|k|-t|k|^{2\sigma}}\int_0^t e^{|k|^{2\sigma}s-\beta(s)|k|}ds\\
&=\sup_{t\in[0,T/2]}\sup_{k\in\mathbb{Z}^n\backslash\{0\}}|k|^{2\sigma}\exp\Big\{2\alpha t|k|/T-t|k|^{2\sigma}\Big\}\cdot\frac{\exp\{|k|^{2\sigma}t-2\alpha t|k|/T\}-1}{|k|^{2\sigma}-2\alpha|k|/T}\\
&=\sup_{t\in[0,T/2]}\sup_{k\in\mathbb{Z}^n\backslash\{0\}}\frac{1-\exp\{2\alpha t|k|/T-t|k|^{2\sigma}\}}{1-2\alpha(|k|^{1-2\sigma})/T}\\
&\le\frac{1}{1-2\alpha/T}.
\end{aligned} \tag{13}$$

Next, we consider the case of $t\in[T/2,T]$. From Equation (7), we know $\beta(s)=2\alpha-2\alpha s/T$. Therefore, we have

$$\int_0^t e^{|k|^{2\sigma}s-\beta(s)|k|}ds=\int_0^{T/2}e^{|k|^{2\sigma}s-\beta(s)|k|}ds+\int_{T/2}^t e^{|k|^{2\sigma}s-\beta(s)|k|}ds. \tag{14}$$

The second integral on the right side of (14) can be written as

$$\begin{aligned}
\int_{T/2}^t e^{|k|^{2\sigma}s-\beta(s)|k|}ds&=\int_{T/2}^t e^{|k|^{2\sigma}s-2\alpha|k|+2\alpha s|k|/T}ds\\
&=e^{-2\alpha|k|}\left(\frac{\exp\{|k|^{2\sigma}t+2\alpha t|k|/T\}-\exp\{|k|^{2\sigma}T/2+\alpha|k|\}}{|k|^{2\sigma}+2\alpha|k|/T}\right).
\end{aligned} \tag{15}$$

By combining Equation (12) with Equation (15), we have found that for $t\in[T/2,T]$, the following equality holds:

$$\begin{aligned}
\int_0^t e^{|k|^2s-\beta(s)|k|}ds=&\frac{\exp\{|k|^{2\sigma}t-2\alpha t|k|/T\}-1}{|k|^{2\sigma}-2\alpha|k|/T}\\
&+e^{-2\alpha|k|}\left(\frac{\exp\{|k|^{2\sigma}t+2\alpha t|k|/T\}-\exp\{|k|^{2\sigma}T/2+\alpha|k|\}}{|k|^{2\sigma}+2\alpha|k|/T}\right).
\end{aligned}$$

When $t\in[T/2,T]$, we set

$$\sup_{t\in[T/2,T]}\sup_{k\in\mathbb{Z}^n\backslash\{0\}}|k|^{2\sigma}e^{\beta(t)|k|-t|k|^{2\sigma}}\int_0^t e^{|k|^{2\sigma}s-\beta(s)|k|}ds\le I+II.$$

where I and II are given by

$$I = \sup_{t\in[T/2,T]} \sup_{k\in\mathbb{Z}^n\backslash\{0\}} |k|^{2\sigma} \exp(2\alpha|k| - 2\alpha|k|t/T - t|k|^{2\sigma}) \frac{\exp(|k|^{2\sigma}T/2 - \alpha|k|) - 1}{|k|^{2\sigma} - 2\alpha|k|/T},$$
$$II = \sup_{t\in[T/2,T]} \sup_{k\in\mathbb{Z}^n\backslash\{0\}} |k|^{2\sigma} \exp(-2\alpha|k|t/T - t|k|^{2\sigma}) \cdot \frac{\exp(|k|^{2\sigma}t + 2\alpha t|k|/T) - \exp(|k|^{2\sigma}T/2 + \alpha|k|)}{|k|^{2\sigma} + 2\alpha|k|/T}.$$

Finally, we will prove that I and II are bounded. It is obvious that

$$\begin{aligned} I &= \sup_{t\in[0,T/2]} \sup_{k\in\mathbb{Z}^n\backslash\{0\}} \frac{\exp(2\alpha|k| - 2\alpha|k|t/T - t|k|^{2\sigma})}{1 - 2\alpha|k|^{1-2\sigma}/T} \Big(\exp(|k|^{2\sigma}T/2 - \alpha|k|) - 1\Big) \\ &= \sup_{t\in[0,T/2]} \sup_{k\in\mathbb{Z}^n\backslash\{0\}} \frac{\exp(\alpha|k| - 2\alpha|k|t/T - (t - \frac{T}{2})|k|^{2\sigma}) - \exp(2\alpha|k| - 2\alpha|k|t/T - t|k|^{2\sigma})}{1 - 2\alpha|k|^{1-2\sigma}/T}. \end{aligned}$$

Using the conditions $t \geq T/2$, $\sigma \in (\frac{1}{2}, 1)$, $\alpha \in (0, T/2)$, and $k \in \mathbb{Z}^n\backslash\{0\}$, we have

$$\exp\Big\{|k|^{2\sigma}T/2 - \alpha|k|\Big\} - 1 > 0, \quad \alpha|k| - 2\alpha|k|t/T - (t - \frac{T}{2})|k|^{2\sigma} < 0.$$

Then, we have

$$I \leq \sup_{k\in\mathbb{Z}^n\backslash\{0\}} \frac{1}{1 - 2\alpha|k|^{1-2\sigma}/T} \leq \frac{1}{1 - 2\alpha/T} = \frac{T}{T - 2\alpha}. \tag{16}$$

Now, let us consider II. As in the above discussion, we have

$$1 - \exp\Big\{|k|^{2\sigma}\Big(\frac{T}{2} - t\Big) + \alpha|k|\Big(1 - \frac{2t}{T}\Big)\Big\} > 0.$$

and

$$\begin{aligned} II &= \sup_{t\in[T/2,T]} \sup_{k\in\mathbb{Z}^n} \frac{1}{1 + 2\alpha|k|^{1-2\sigma}/T} \Big(1 - \exp\Big\{|k|^{2\sigma}\Big(\frac{T}{2} - t\Big) + \alpha|k|\Big(1 - \frac{2t}{T}\Big)\Big\}\Big) \\ &\leq \sup_{k\in\mathbb{Z}^n\backslash\{0\}} \frac{1}{1 + 2\alpha|k|^{1-2\sigma}/T} = 1. \end{aligned} \tag{17}$$

Based on Equations (13), (16) and (17), we proved that I_σ^+ is a bounded linear operator and the operator norm satisfies

$$\|I_\sigma^+\|_{\mathcal{B}_\alpha^j \to \mathcal{B}_\alpha^{j+2\sigma}} \leq \Big(\frac{2T}{T - 2\alpha} + 2\Big).$$

□

Regarding I_σ^-, we have a similar result, and we omitted its proof.

Lemma 3. *For any $\sigma \in (\frac{1}{2}, 1)$, if $h \in B_\alpha^j$ with $\int_{\mathbb{T}^d} h(t,x)dx = 0$, then we have $(I_\sigma^- h)(t,\cdot) \in \mathcal{B}_\alpha^{j+2\sigma}$ and*

$$\|(I_\sigma^- h)(t,\cdot)\|_{\mathcal{B}_\alpha^{j+2\sigma}} \leq C\|h(t,\cdot)\|_{\mathcal{B}_\alpha^j}. \tag{18}$$

At last, we have the following lemma:

Corollary 1. *For any* $\sigma \in (\frac{1}{2}, 1)$*, if* $f \in B^{2\sigma}$*, then*

$$\sup_{t\in[0,T]} \|e^{-(-\Delta)^{\sigma} t} f\|_{\mathcal{B}^{2\sigma}_{\alpha}} + \sup_{t\in[0,T]} \|e^{-(-\Delta)^{\sigma}(T-t)} f\|_{\mathcal{B}^{2\sigma}_{\alpha}} \le C\|f\|_{B^{2\sigma}}.$$

Proof. Using the definition $B^{2\sigma}$, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \|e^{-(-\Delta)^{\sigma} t} f\|_{\mathcal{B}^{2\sigma}_{\alpha}} &= \sup_{t\in[0,T]} \sum_{k\in\mathbb{Z}^n} \left(1 + |k|^{2\sigma}\right) e^{\beta(t)|k|} e^{-t|k|^{2\sigma}} \widehat{f} \\
&\le \sum_{k\in\mathbb{Z}^n} \left(1 + |k|^{2\sigma}\right) \widehat{f} \sup_{t\in[0,T]} e^{\beta(t)|k| - t|k|^{2\sigma}} \\
&\le C|f|_{B^{2\sigma}}.
\end{aligned}$$

Using similar methods, we can find $e^{-(-\Delta)^{\sigma}(T-t)} f \in \mathcal{B}^{2\sigma}_{\alpha}$. □

3. Main Result and Its Proof

In this section, we first state the definition of the strong solution for the problems in Equations (1)–(3). Then, we show the existence and uniqueness theorem and give its proof.

3.1. Strong Solution Formulation

Let $\mathbb{P}$ be the projection operator, which removes the mean value of the periodic function (i.e., the following is true):

$$\mathbb{P}f = f - \frac{1}{\mathrm{vol}(\mathbb{T}^n)} \int_{\mathbb{T}^n} f(x)\mathrm{d}x.$$

We define $w = \mathbb{P}u$ and $\mu = \mathbb{P}m = m - \bar{m}$, where $\bar{m} = 1/\mathrm{vol}(\mathbb{T}^n)$. Since m is a probability measure at each time, its integral in the spatial domain will always be equal to one. Then, from Equation (1), we can find the evolution equation of w:

$$w_t - (-\Delta)^{\sigma} w + \mathbb{P}\mathcal{H}(t, x, Dw, \mu) = 0,$$

Additionally, from Equation (2), we have the following evolution equation for μ:

$$\mu_t + (-\Delta)^{\sigma}\mu + \mathrm{div}\big(\mu \mathcal{H}_p(t, x, Dw, \mu)\big) + \bar{m}\,\mathrm{div}\big(\mathcal{H}_p(t, x, Dw, \mu)\big) = 0.$$

Of course, the initial-terminal condition is taken as $w(T, \cdot) = w_T := \mathbb{P}u_T$, $\mu(0, \cdot) = \mu_0 := m_0 - \bar{m}$.

Before writing the Duhamel formula for w and μ. We assume that $\mathbb{P}\mathcal{H}$ can be expressed as

$$\mathbb{P}\mathcal{H} = \mathbb{P}b(t, x)\mu + \mathbb{P}\Upsilon(t, x, Dw, \mu).$$

The conditions satisfied by b and Υ will be given below.

We then have the following Duhamel formula for μ:

$$\begin{aligned}
\mu(t, \cdot) =& e^{-(-\Delta)^{\sigma} t}\mu_0 \\
&+ I^{+}_{\sigma}\big(\mathrm{div}\big(\mu \mathcal{H}_p(\cdot, \cdot, Dw, \mu)\big)\big)(t, \cdot) + \bar{m}(I^{+}_{\sigma}(\mathrm{div}(\mathcal{H}_p(\cdot, \cdot, Dw, \mu))))(t, \cdot),
\end{aligned} \tag{19}$$

In addition, by integrating backward in time, we find

$$w(t, \cdot) = e^{-(-\Delta)^{\sigma}(T-t)} w_T - I^{-}_{\sigma}(\mathbb{P}\Upsilon(\cdot, \cdot, Dw, \mu)) - I^{-}_{\sigma}(\mathbb{P}(b\mu))(t, \cdot). \tag{20}$$

Indeed, by substituting Equation (19) into Equation (20), we obtain the following equation for w:

$$
\begin{aligned}
w(t,\cdot) =& e^{-(-\Delta)^{\sigma}(T-t)}w_T - I_{\sigma}^{-}(\mathbb{P}\Upsilon(\cdot,\cdot,Dw,\mu)) - I_{\sigma}^{-}(\mathbb{P}(be^{(-\Delta)^{\sigma}\cdot}\mu_0))(t) \\
&- I_{\sigma}^{-}\left(\mathbb{P}(bI_{\sigma}^{+}\operatorname{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot))\right)(t) \\
&- \bar{m}I_{\sigma}^{-}\left(\mathbb{P}(bI^{+}\operatorname{div}(\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot))\right)(t).
\end{aligned} \tag{21}
$$

We say (u,m) is a strong solution of the problems in Equations (1)–(3) if $(w,\mu) \in \mathcal{B}_{\alpha}^{2\sigma+1} \times \mathcal{B}_{\alpha}^{2\sigma}$ such that the integral equalities in Equations (21) and (19) hold.

3.2. Main Theorem

In order to obtain the existence of strong solutions for the problems in Equations (1)–(3), we assume that the Hamiltonian $\mathcal{H}$ satisfies the following assumption:

(A1) Let $\sigma \in (\frac{1}{2},1)$, $\Upsilon(\cdot,\cdot,0,0) = 0$ and $\mathcal{H}_p(\cdot,\cdot,0,0) = 0$. There exists a continuous function $\Phi_1 : \mathcal{B}_{\alpha}^{2\sigma+1} \times \mathcal{B}_{\alpha}^{2\sigma} \to \mathbb{R}$ such that as $(w_1,w_2,\mu_1,\mu_2) \to 0$, we have $\Phi_1(w_1,w_2,\mu_1,\mu_2) \to 0$ and

$$
\begin{aligned}
&\|\mathcal{H}_p(\cdot,\cdot,Dw_1,\mu_1) - \mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\|_{(\mathcal{B}_{\alpha}^{1})^n} \\
&\le \Phi_1(w_1,w_2,\mu_1,\mu_2)\left(\|Dw_1 - Dw_2\|_{(\mathcal{B}_{\alpha}^{2\sigma})^n} + \|\mu_1-\mu_2\|_{\mathcal{B}_{\alpha}^{2\sigma}}\right).
\end{aligned} \tag{22}
$$

There exists a continuous function $\Phi_2 : \mathcal{B}_{\alpha}^{2\sigma+1} \times \mathcal{B}_{\alpha}^{2\sigma} \to \mathbb{R}$ so that as $(w_1,w_2,\mu_1,\mu_2) \to 0$, we have $\Phi_2(w_1,w_2,\mu_1,\mu_2) \to 0$ and

$$
\begin{aligned}
&\|\mathbb{P}\Upsilon(\cdot,\cdot,Dw_1,\mu_1) - \mathbb{P}\Upsilon(\cdot,\cdot,Dw_2,\mu_2)\|_{\mathcal{B}_{\alpha}^{1}} \\
&\le \Phi_2(w_1,w_2,\mu_1,\mu_2)\left(\|Dw_1 - Dw_2\|_{(\mathcal{B}_{\alpha}^{2\sigma})^n} + \|\mu_1-\mu_2\|_{\mathcal{B}_{\alpha}^{2\sigma}}\right).
\end{aligned} \tag{23}
$$

From the preparations above, we now give the first main theorem. Our proof depends on the Banach fixed-point theorem as in [25]:

Theorem 1. *Let $T > 0$ and $\alpha \in (0,T/2)$ be given. Let assumption (**A1**) be satisfied, and assume $b \in \mathcal{B}_{\alpha}^{1}$. There exists $\delta > 0$ such that if u_T and the probability measure m_0 are such that $w_T = \mathbb{P}u_T$ and $\mu_0 = m_0 - \overline{m}$ satisfy $\|w_T\|_{B^{2\sigma+1}} + \|\mu_0\|_{B^{2\sigma}} < \delta$, then the system in Equations (1)–(3) has a strong, locally unique solution $(u,m) \in \mathcal{B}_{\alpha}^{2\sigma+1} \times \mathcal{B}_{\alpha}^{2\sigma}$.*

Proof. Let (a_0,b_0) be

$$
(a_0,b_0) = \left(e^{-(-\Delta)^{\sigma}(T-t)}w_T - I_{\sigma}^{-}(\mathbb{P}(be^{-(-\Delta)^{\sigma}\cdot}\mu_0))(t),\ e^{-(-\Delta)^{\sigma}t}\mu_0\right). \tag{24}
$$

We set X to be the closed ball in $\mathcal{B}_{\alpha}^{2\sigma+1} \times \mathcal{B}_{\alpha}^{2\sigma}$ centered at a point (a_0,b_0) with a radius r^*, where r^* will be determined later:

$$
X = \left\{(w,\mu) \in \mathcal{B}_{\alpha}^{2\sigma+1} \times \mathcal{B}_{\alpha}^{2\sigma} : \|w-a_0\|_{B^{2\sigma+1}} + \|\mu - b_0\|_{B^{2\sigma}} < r^*\right\}.
$$

We define a mapping $\mathcal{T}(w,\mu) = (\mathcal{T}_1(w,\mu), \mathcal{T}_2(w,\mu))$ on X based on Equations (21) and (20) (i.e., we have the following):

$$
\begin{aligned}
\mathcal{T}_1(w,\mu) =& e^{-(-\Delta)^{\sigma}(T-t)}w_T - I_{\sigma}^{-}(\mathbb{P}\Upsilon(\cdot,\cdot,Dw,\mu)) - I_{\sigma}^{-}(\mathbb{P}(be^{-(-\Delta)^{\sigma}\cdot}\mu_0))(t) \\
&- I_{\sigma}^{-}(\mathbb{P}(bI_{\sigma}^{+}\operatorname{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot)))(t) \\
&- \bar{m}I_{\sigma}^{-}(\mathbb{P}(bI_{\sigma}^{+}\operatorname{div}(\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot)))(t),
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\mathcal{T}_2(w,\mu) =& e^{-(-\Delta)^{\sigma}t}\mu_0 + I_{\sigma}^{+}(\operatorname{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu)))(t,\cdot) \\
&+ \bar{m}(I_{\sigma}^{+}(\operatorname{div}(\mathcal{H}_p(\cdot,\cdot,Dw,\mu))))(t,\cdot).
\end{aligned} \tag{26}
$$

First, we prove that $\mathcal{T}$ maps X to X. Because of $\mu_0 \in B^{2\sigma}$, it is easy to know that $b_0 = e^{-(-\Delta)^{\sigma} t}\mu_0 \in \mathcal{B}^{2\sigma}_{\alpha}$ from the inference in Corollary 1. By the definition of a_0 in Equation (24) and the property of I^{-}_{σ} given earlier in the previous section, we have

$$\begin{aligned}\|a_0\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} &= \|e^{-(-\Delta)^{\sigma}(T-t)}w_T - I^{-}_{\sigma}(\mathbb{P}(be^{-(-\Delta)^{\sigma}\cdot}\mu_0))(t)\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} \\ &\le \|e^{-(-\Delta)^{\sigma}(T-t)}w_T\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} + \|I^{-}_{\sigma}(\mathbb{P}(be^{-(-\Delta)^{\sigma}\cdot}\mu_0))(t)\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} \\ &\le \|e^{-(-\Delta)^{\sigma}(T-t)}w_T\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} + C\|b\|_{\mathcal{B}^{1}_{\alpha}}\|e^{-(-\Delta)^{\sigma}\cdot}\mu_0\|_{\mathcal{B}^{2\sigma}_{\alpha}}.\end{aligned}$$

Using $b \in \mathcal{B}^1_{\alpha}$ and $w_T \in \mathcal{B}^{2\sigma+1}$, it is easy to know that $a_0 \in \mathcal{B}^{2\sigma+1}_{\alpha}$.

To demonstrate that $\mathcal{T}$ maps X to X, we will only need to show that

$$\|\mathcal{T}_1(w,\mu) - a_0\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} \le r^*/2 \quad \text{and} \quad \|\mathcal{T}_2(w,\mu) - b_0\|_{\mathcal{B}^{2\sigma}_{\alpha}} \le r^*/2.$$

We begin with $\mathcal{T}_1$. Recalling the definition in Equation (25), we only need to prove the following inequality:

$$\|I^{-}_{\sigma}(\mathbb{P}\Upsilon(\cdot,\cdot,Dw,\mu))\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} \le r^*/6, \tag{27}$$

$$\|I^{-}_{\sigma}\left(\mathbb{P}(bI^{+}\operatorname{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot))\right)(t)\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} \le r^*/6, \tag{28}$$

$$\|\bar{m}I^{-}_{\sigma}\left(\mathbb{P}(bI^{+}\operatorname{div}(\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot))\right)(t)\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} \le r^*/6. \tag{29}$$

By assuming Equation (22) in (**A1**) and the property of I^{-}_{σ}, we have

$$\begin{aligned}\|I^{-}_{\sigma}(\mathbb{P}\Upsilon(\cdot,\cdot,Dw,\mu))\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} &\le C\|\mathbb{P}\Upsilon(\cdot,\cdot,Dw,\mu)\|_{\mathcal{B}^{1}_{\alpha}} \\ &\le C\Phi_2(w,0,\mu,0)(\|Dw\|_{\mathcal{B}^{2\sigma}_{\alpha}} + \|\mu\|_{\mathcal{B}^{2\sigma}_{\alpha}}) \\ &\le C\Phi_2(w,0,\mu,0)(\|w\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} + \|\mu\|_{\mathcal{B}^{2\sigma}_{\alpha}}).\end{aligned} \tag{30}$$

Since $w \in X$ and $\mu \in X$, we have

$$\|w\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} + \|\mu\|_{\mathcal{B}^{2\sigma}_{\alpha}} \le \|a_0\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} + \|b_0\|_{\mathcal{B}^{2\sigma}_{\alpha}} + r^*.$$

Let μ_0 and w_T be small enough that

$$\|a_0\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} + \|b_0\|_{\mathcal{B}^{2\sigma}_{\alpha}} \le r^*. \tag{31}$$

By combining Equations(30), (38) and (31), we have

$$\|I^{-}_{\sigma}(\mathbb{P}\Upsilon(\cdot,\cdot,Dw,\mu))\|_{\mathcal{B}^{2\sigma+1}_{\alpha}} \le 2Cr^*\Phi_2(w,0,\mu,0). \tag{32}$$

Since Φ_2 is continuous and $\Phi_2(0,0,0,0) = 0$, we may have μ_0, w_T and r^* at small enough values such that

$$\max_{(w,\mu)\in X}\Phi_2(w,0,\mu,0) \le \frac{1}{12Cr^*}.$$

Then, Equation (32) implies Equation (27).

Next, we demonstrate Equation (28). By the properties of I^{-}_{σ} and I^{+}_{σ}, and the fact that the divergence is a first-order operator, combined with the algebra porperty of $\mathcal{B}^{j}_{\alpha}$ in Lemma 1, we have

$$\begin{aligned}
&\|I_\sigma^-(\mathbb{P}(bI^+\operatorname{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot)))(t)\|_{\mathcal{B}_\alpha^{2\sigma+1}}\\
&\le C\|(\mathbb{P}(bI^+\operatorname{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot)))(t)\|_{\mathcal{B}_\alpha^1}\\
&\le C\|b\|_{\mathcal{B}_\alpha^1}\|(I^+\operatorname{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot))(t)\|_{\mathcal{B}_\alpha^1}\\
&\le C\|b\|_{\mathcal{B}_\alpha^1}\|(I^+\operatorname{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot))(t)\|_{\mathcal{B}_\alpha^{2\sigma}}\\
&\le C\|b\|_{\mathcal{B}_\alpha^1}\|(\operatorname{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot))(t)\|_{\mathcal{B}_\alpha^0}\\
&\le C\|b\|_{\mathcal{B}_\alpha^1}\|\mu\|_{\mathcal{B}_\alpha^1}\|\mathcal{H}_p(\cdot,\cdot,Dw,\mu)(\cdot)(t)\|_{\mathcal{B}_\alpha^1}.
\end{aligned} \tag{33}$$

Since $\frac{1}{2}<\sigma<1$, $w\in X$, and $\mu\in X$, we have

$$\|\mu\|_{\mathcal{B}_\alpha^1}\le\|\mu\|_{\mathcal{B}_\alpha^{2\sigma}}\le\|a_0\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|b_0\|_{\mathcal{B}_\alpha^{2\sigma}}+r^*.$$

From assumption (23) in (**A1**), we have

$$\begin{aligned}
\|\mathcal{H}_p(\cdot,\cdot,Dw,\mu)(\cdot)(t)\|_{\mathcal{B}_\alpha^1}&\le\Phi_1(w,0,\mu,0)\Big(\|Dw\|_{\mathcal{B}_\alpha^{2\sigma}}+\|\mu\|_{\mathcal{B}_\alpha^{2\sigma}}\Big)\\
&\le\Phi_1(w,0,\mu,0)\Big(\|w\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|\mu\|_{\mathcal{B}_\alpha^{2\sigma}}\Big)\\
&\le\Phi_1(w,0,\mu,0)\Big(\|a_0\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|b_0\|_{\mathcal{B}_\alpha^{2\sigma}}+r^*\Big).
\end{aligned} \tag{34}$$

We again require that μ_0 and w_T are small enough that

$$\|a_0\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|b_0\|_{\mathcal{B}_\alpha^{2\sigma}}\le r^*. \tag{35}$$

By combining Equations (33) and (34) with Equation (35), we have

$$\begin{aligned}
&\|I_\sigma^-\big(\mathbb{P}\big(bI^+\operatorname{div}\big(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu)\big)(\cdot)\big)\big)(t)\|_{\mathcal{B}_\alpha^{2\sigma+1}}\\
\le&4C(r^*)^2\max_{(w,\mu)\in X}\Phi_1(w,0,\mu,0)\|b\|_{\mathcal{B}_\alpha^1}.
\end{aligned} \tag{36}$$

Thus, if r^* is chosen to be a value small enough that

$$4C(r^*)^2\max_{(w,\mu)\in X}\Phi_1(w,0,\mu,0)\|b\|_{\mathcal{B}_\alpha^1}\le\frac{r^*}{6},$$

then Equation (36) implies Equation (28).

Finally, we will demonstrate Equation (29). Similar to the proof of Equation (28), we have

$$\begin{aligned}
&\|\bar{m}I_\sigma^-(\mathbb{P}(bI^+\operatorname{div}(\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot)))(t)\|_{\mathcal{B}_\alpha^{2\sigma+1}}\\
\le&C\bar{m}\|(\mathbb{P}(bI^+\operatorname{div}(\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot)))(t)\|_{\mathcal{B}_\alpha^1}\\
\le&C\bar{m}\|b\|_{\mathcal{B}_\alpha^1}\|(I^+\operatorname{div}(\mathcal{H}_p(\cdot,\cdot,Dw,\mu))(\cdot))(t)\|_{\mathcal{B}_\alpha^1}\\
\le&C\bar{m}\|b\|_{\mathcal{B}_\alpha^1}\|\mathcal{H}_p(\cdot,\cdot,Dw,\mu)(\cdot)(t)\|_{\mathcal{B}_\alpha^1}\\
\le&C\bar{m}\|b\|_{\mathcal{B}_\alpha^1}\max_{(w,\mu)\in X}\Phi_1(w,0,\mu,0)(\|a_0\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|b_0\|_{\mathcal{B}_\alpha^{2\sigma}}+r^*).
\end{aligned} \tag{37}$$

If μ_0 and w_T are chosen to be values small enough that

$$\|a_0\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|b_0\|_{\mathcal{B}_\alpha^{2\sigma}}\le r^*\quad\text{and}\quad\max_{(w,\mu)\in X}\Phi_1(w,0,\mu,0)\le\frac{1}{12C\bar{m}\|b\|_{\mathcal{B}_\alpha^1}},$$

then Equation (37) implies Equation (29).

Next, we consider $\mathcal{T}_2$. Recalling the definition in Equation (26), we only need to prove the following inequalities:

$$\|I_\sigma^+\big(\mathrm{div}\big(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu)\big)\big)(t,\cdot)\|_{\mathcal{B}_\alpha^{2\sigma}} \le \frac{r^*}{4}, \tag{38}$$

$$\|\bar{m}(I_\sigma^+(\mathrm{div}(\mathcal{H}_p(\cdot,\cdot,Dw,\mu))))(t,\cdot)\|_{\mathcal{B}_\alpha^{2\sigma}} \le \frac{r^*}{4}. \tag{39}$$

We begin by establishing Equation (38). It is obvious that

$$\begin{aligned}\|I_\sigma^+(\mathrm{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu)))(t,\cdot)\|_{\mathcal{B}_\alpha^{2\sigma}} &\le C\|(\mathrm{div}(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu)))(t,\cdot)\|_{\mathcal{B}_\alpha^{0}} \\ &\le C\|\mu\|_{\mathcal{B}_\alpha^{1}}\|\mathcal{H}_p(\cdot,\cdot,Dw,\mu)(t,\cdot)\|_{\mathcal{B}_\alpha^{1}}.\end{aligned} \tag{40}$$

Adopting a method similar to that in Equation (28), we know if r^* is chosen to be small enough that

$$4C(r^*)^2 \max_{(w,\mu)\in X} \Phi_1(w,0,\mu,0) \le \frac{r^*}{4},$$

then Equation (40) implies Equation (38).

Then, we demonstrate Equation (39). It is immediate that

$$\begin{aligned}&\|\bar{m}(I_\sigma^+(\mathrm{div}(\mathcal{H}_p(\cdot,\cdot,Dw,\mu))))(t,\cdot)\|_{\mathcal{B}_\alpha^{2\sigma}} \\ \le& C\bar{m}\|\mathcal{H}_p(\cdot,\cdot,Dw,\mu)(t,\cdot)\|_{\mathcal{B}_\alpha^{1}} \\ \le& C\bar{m}\max_{(w,\mu)\in X}\Phi_1(w,0,\mu,0)(\|a_0\|_{\mathcal{B}_\alpha^{2\sigma+1}} + \|b_0\|_{\mathcal{B}_\alpha^{2\sigma}} + r^*).\end{aligned} \tag{41}$$

If r^*, μ_0 and w_T are chosen to be small enough that

$$\max_{(w,\mu)\in X}\Phi_1(w,0,\mu,0) \le \min\left\{\frac{1}{12C\bar{m}\|b\|_{\mathcal{B}_\alpha^{1}}}, \frac{1}{8C\bar{m}}\right\},$$

then Equation (41) implies Equation (39). Therefore, we proved that when choosing r^*, μ_0, and w_T appropriately, $\mathcal{T}$ maps X to X.

Now, we are ready to demonstrate the contraction estimate (i.e., we will demonstrate that if r^*, μ_0, and w_T are sufficiently small, then there exists $\lambda \in (0,1)$ such that for all $(w_1,\mu_1) \in X$ and $(w_2,\mu_2) \in X$, we have the following):

$$\|\mathcal{T}(w_1,\mu_1) - \mathcal{T}(w_2,\mu_2)\|_{\mathcal{B}_\alpha^{2\sigma+1}\times\mathcal{B}_\alpha^{2\sigma}} \le \lambda(\|w_1 - w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}} + \|\mu_1 - \mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}). \tag{42}$$

Considering the definitions in Equations (25) and (26), we will demonstrate that

$$\|\mathcal{T}_1(w_1,\mu_1) - \mathcal{T}_1(w_2,\mu_2)\|_{\mathcal{B}_\alpha^{2\sigma+1}} \le \lambda_1\Big(\|w_1 - w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}} + \|\mu_1 - \mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}\Big),$$

$$\|\mathcal{T}_2(w_1,\mu_1) - \mathcal{T}_2(w_2,\mu_2)\|_{\mathcal{B}_\alpha^{2\sigma}} \le \lambda_2\Big(\|w_1 - w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}} + \|\mu_1 - \mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}\Big).$$

It can be seen that repeated use of the triangle inequality implies that it is sufficient to establish the following bounds:

$$\left\|I_\sigma^-(\mathbb{P}(\Upsilon(\cdot,\cdot,Dw_1,\mu_1)-\Upsilon(\cdot,\cdot(Dw_2,\mu_2))))\right\|_{\mathcal{B}_\alpha^{2\sigma+1}} \le \frac{1}{6}\Big(\|w_1-w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}\Big), \tag{43}$$

$$\begin{aligned}&\left\|I_\sigma^-\big(\mathbb{P}\big(bI^+\big(\operatorname{div}\big(\mu_1\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1)-\mu_2\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\big)\big)\big)\big)\right\|_{\mathcal{B}_\alpha^{2\sigma+1}}\\ &\qquad\le \frac{1}{6}\Big(\|w_1-w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}\Big),\end{aligned} \tag{44}$$

$$\begin{aligned}&\bar{m}\left\|I_\sigma^-\big(\mathbb{P}\big(bI_\sigma^+\big(\operatorname{div}\big(\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1)-\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\big)\big)\big)\big)\right\|_{\mathcal{B}^{2\sigma+1}}\\ &\qquad\le \frac{1}{6}\Big(\|w_1-w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}\Big),\end{aligned} \tag{45}$$

$$\begin{aligned}&\left\|I_\sigma^+\big(\operatorname{div}\big(\mu_1\mathcal{H}_p(\cdot,\cdot,Dw_1,\mu_1)-\mu_2\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\big)\big)\right\|_{\mathcal{B}_\alpha^{2\sigma}}\\ &\qquad\le \frac{1}{6}\Big(\|w_1-w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}\Big),\end{aligned} \tag{46}$$

$$\begin{aligned}&\bar{m}\left\|I_\sigma^+\big(\operatorname{div}\big(\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1)-\mathcal{H}_p(\cdot,\cdot Dw_2,\mu_2)\big)\big)\right\|_{\mathcal{B}_\alpha^{2\sigma}}\\ &\qquad\le \frac{1}{6}\Big(\|w_1-w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}\Big).\end{aligned} \tag{47}$$

We begin by establishing Equation (43). By following the assumptions and using the tools from before, such as triangle inequalities, the properties of $\mathcal{B}_\alpha^j$, and the mapping properties of I_σ^+ and I_σ^-, we have

$$\begin{aligned}&\|I_\sigma^-(\mathbb{P}(\Upsilon(\cdot,\cdot,Dw_1,\mu_1)-\Upsilon(\cdot,\cdot Dw_2,\mu_2)))\|_{\mathcal{B}_\alpha^{2\sigma+1}}\\ &\le C\|\mathbb{P}(\Upsilon(\cdot,\cdot,Dw_1,\mu_1)-\Upsilon(\cdot,\cdot Dw_2,\mu_2))\|_{\mathcal{B}_\alpha^1}\\ &\le C\Phi_2(w_1,w_2,\mu_1,\mu_2)(\|Dw_1-Dw_2\|_{\mathcal{B}_\alpha^{2\sigma}}+\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^{2\sigma}})\\ &\le C\Phi_2(w_1,w_2,\mu_1,\mu_2)(\|w_1-w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}}+\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}).\end{aligned} \tag{48}$$

Since Φ_2 is continuous and $\Phi_2(0,0,0,0) = 0$, if r^*, μ_0, and w_T are small enough such that

$$\max_{(w_1,\mu_1),(w_2,\mu_2)\in X}\Phi_2(w_1,w_2,\mu_1,\mu_2)\le\frac{1}{6C},$$

then Equation (48) implies Equation (43).

Now, we will demonstrate Equation (44). Similar to the above proof, we have the following facts:

$$\begin{aligned}&\|I_\sigma^-\big(\mathbb{P}\big(bI^+(\operatorname{div}(\mu_1\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1)-\mu_2\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)))\big)\big)\|_{\mathcal{B}_\alpha^{2\sigma+1}}\\ &\le C\|b\|_{\mathcal{B}_\alpha^1}\|(\mu_1\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1)-\mu_2\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2))\|_{\mathcal{B}_\alpha^1}\\ &\le C\|b\|_{\mathcal{B}_\alpha^1}\|(\mu_1\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1)-\mu_1\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2))\|_{\mathcal{B}_\alpha^1}\\ &\quad+C\|(\mu_1\mathcal{H}_p(\cdot,\cdot Dw_2,\mu_2)-\mu_2\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2))\|_{\mathcal{B}_\alpha^1}\\ &\le C\|b\|_{\mathcal{B}_\alpha^1}(\|\mu_1\|_{\mathcal{B}_\alpha^1}\|\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1)-\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\|_{\mathcal{B}_\alpha^1}\\ &\quad+\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^1}\|\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\|_{\mathcal{B}_\alpha^1})\\ &\le C_{r^*}\|b\|_{\mathcal{B}_\alpha^1}(\|\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1)-\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\|_{\mathcal{B}_\alpha^1}\\ &\quad+C_{r^*}\max_{(w,\mu)\in X}\Phi_1(w,0,\mu,0)\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^{2\sigma}})\\ &\le C_{r^*}\|b\|_{\mathcal{B}_\alpha^1}\max_{(w_1,\mu_1),(w_2,\mu_2)\in X}\Phi_1(w_1,w_2,\mu_1,\mu_2)\|w_1-w_2\|_{\mathcal{B}_\alpha^{2\sigma+1}}\\ &+C_{r^*}\|b\|_{\mathcal{B}_\alpha^1}\left(\max_{(w_1,\mu_1),(w_2,\mu_2)\in X}\Phi_1(w_1,w_2,\mu_1,\mu_2)+\max_{(w,\mu)\in X}\Phi_1(w,0,\mu,0)\right)\|\mu_1-\mu_2\|_{\mathcal{B}_\alpha^{2\sigma}}.\end{aligned} \tag{49}$$

Since Φ_1 is continuous and $\Phi_1(0,0,0,0) = 0$, if r^*, μ_0, and w_T are small enough, we can obtain

$$\max_{(w_1,\mu_1),(w_2,\mu_2)\in X} \Phi_1(w_1, w_2, \mu_1, \mu_2) + \max_{(w,\mu)\in X} \Phi_1(w, 0, \mu, 0) \le \frac{1}{6C_{r^*}\|b\|_{\mathcal{B}^1_\alpha}}.$$

Then, Equation (49) implies Equation (44).

Next, we will demonstrate Equation (45). Similar to the above proof, we have the following facts:

$$\begin{aligned}
&\bar{m}\big\|I^-_\sigma(\mathbb{P}(bI^+_\sigma(\mathrm{div}(\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1) - \mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)))))\big\|_{\mathcal{B}^{2\sigma+1}_\alpha} \\
&\le C\bar{m}\|b\|_{\mathcal{B}^1_\alpha}\big\|(\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1) - \mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2))\big\|_{\mathcal{B}^1_\alpha} \\
&\le C\bar{m}\|b\|_{\mathcal{B}^1_\alpha} \max_{(w_1,\mu_1),(w_2,\mu_2)\in X} \Phi_1(w_1,w_2,\mu_1,\mu_2)\Big(\|w_1 - w_2\|_{\mathcal{B}^{2\sigma+1}_\alpha} + \|\mu_1 - \mu_2\|_{\mathcal{B}^{2\sigma}_\alpha}\Big).
\end{aligned} \tag{50}$$

If r^*, μ_0, and w_T are small enough such that

$$\max_{(w_1,\mu_1),(w_2,\mu_2)\in X} \Phi_1(w_1,w_2,\mu_1,\mu_2) \le \min\left\{\frac{1}{6C\bar{m}\|b\|_{\mathcal{B}^1_\alpha}}, \frac{1}{12C_{r^*}\|b\|_{\mathcal{B}^1_\alpha}}\right\},$$

then Equation (50) implies Equation (45).

Our task now moves to proving Equation (46). It is immediate that

$$\begin{aligned}
&\big\|I^+_\sigma(\mathrm{div}(\mu_1\mathcal{H}_p(\cdot,\cdot,Dw_1,\mu_1) - \mu_2\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)))\big\|_{\mathcal{B}^{2\sigma}_\alpha} \\
&\le C\big\|\mu_1\mathcal{H}_p(\cdot,\cdot,Dw_1,\mu_1) - \mu_2\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\big\|_{\mathcal{B}^1_\alpha} \\
&\le C\|\mu_1\|_{\mathcal{B}^1_\alpha}\big\|\mathcal{H}_p(\cdot,\cdot,Dw_1,\mu_1) - \mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\big\|_{\mathcal{B}^1_\alpha} \\
&\quad + C\|\mathcal{H}_p(\cdot,\cdot,Dw_2,\mu_2)\|_{\mathcal{B}^1_\alpha}\|\mu_1 - \mu_2\|_{\mathcal{B}^1_\alpha} \\
&\le C_{r^*} \max_{(w_1,\mu_1),(w_2,\mu_2)\in X} \Phi_1(w_1,w_2,\mu_1,\mu_2)\|w_1 - w_2\|_{\mathcal{B}^{2\sigma+1}} \\
&\quad + C_{r^*}\Big(\max_{(w_1,\mu_1),(w_2,\mu_2)\in X} \Phi_1(w_1,w_2,\mu_1,\mu_2) + \max_{(w,\mu)\in X} \Phi_1(w,0,\mu,0)\Big)\|\mu_1 - \mu_2\|_{\mathcal{B}^{2\sigma}_\alpha}.
\end{aligned} \tag{51}$$

If r^*, μ_0, and w_T are small enough, we will have

$$\max_{(w_1,\mu_1),(w_2,\mu_2)\in X} \Phi_1(w_1,w_2,\mu_1,\mu_2) \le \min\left\{\frac{1}{6C\bar{m}\|b\|_{\mathcal{B}^1_\alpha}}, \frac{1}{12C_{r^*}\|b\|_{\mathcal{B}^1_\alpha}}, \frac{1}{12C_{r^*}}\right\}.$$

Then, Equation (51) implies Equation (46).

Finally we have to show Equation (47). It is easy to find that

$$\begin{aligned}
&\bar{m}\|I^+_\sigma(\mathrm{div}(\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1) - \mathcal{H}_p(\cdot,\cdot Dw_2,\mu_2)))\|_{\mathcal{B}^{2\sigma}_\alpha} \\
&\le \bar{m}\big\|(\mathcal{H}_p(\cdot,\cdot Dw_1,\mu_1) - \mathcal{H}_p(\cdot,\cdot Dw_2,\mu_2))\big\|_{\mathcal{B}^1_\alpha} \\
&\le C\bar{m} \max_{(w_1,\mu_1),(w_2,\mu_2)\in X} \Phi_1(w_1,w_2,\mu_1,\mu_2)(\|w_1 - w_2\|_{\mathcal{B}^{2\sigma+1}_\alpha} + \|\mu_1 - \mu_2\|_{\mathcal{B}^{2\sigma}_\alpha}).
\end{aligned} \tag{52}$$

If r^*, μ_0 and w_T are small enough, we have

$$\max_{(w_1,\mu_1),(w_2,\mu_2)\in X} \Phi_1(w_1,w_2,\mu_1,\mu_2) \le \min\left\{\frac{1}{6C\bar{m}\|b\|_{\mathcal{B}^1_\alpha}}, \frac{1}{12C_{r^*}\|b\|_{\mathcal{B}^1_\alpha}}, \frac{1}{12C_{r^*}}, \frac{1}{6C\bar{m}}\right\}.$$

Then, Equation (52) implies Equation (47).

We established the estimates in Equations (43)–(47) and successfully proved the constant $\lambda = \frac{5}{6}$ in Equaiton (42). This completes the proof of Theorem 1. □

4. The Payoff Problem

The above formulation and existence theorem can be readily adapted to the payoff problem, in which Equation (3) is replaced by

$$m(0,x) = m_0(x), \qquad u(T,x) = G(m(T,\cdot)). \tag{53}$$

The needed modification in the formulation is that Equation (21) is replaced with

$$\begin{aligned} w(t,\cdot) =& e^{-(-\Delta)^{\sigma}(T-t)}\mathbb{P}G(\mu(T,\cdot)) - I_\sigma^-(\mathbb{P}\Upsilon(\cdot,\cdot,Dw,\mu)) - I_\sigma^-\Big(\mathbb{P}\Big(be^{(-\Delta)^{\sigma}\cdot}\mu_0\Big)\Big)(t) \\ &- I_\sigma^-\big(\mathbb{P}\big(bI_\sigma^+ \operatorname{div}\big(\mu\mathcal{H}_p(\cdot,\cdot,Dw,\mu)\big)(\cdot)\big)\big)(t) \\ &- \bar{m}I_\sigma^-\big(\mathbb{P}\big(bI_\sigma^+ \operatorname{div}\big(\mathcal{H}_p(\cdot,\cdot,Dw,\mu)\big)(\cdot)\big)\big)(t). \end{aligned}$$

In addition, we need to make assumptions about the payoff function $\mathbb{P}G$:

(A2) $\mathbb{P}G(0) = 0$, and $\mathbb{P}G$ is in the neighborhood of the origin in $B^{2\sigma+1}$. Specifically, we assume that there exists $c > 0$ and $\epsilon > 0$ such that for all a_1, a_2 satisfies $|a_1|_{B^{2\sigma}} < \epsilon$ and $|a_2|_{B^{2\sigma}} < \epsilon$:

$$|\mathbb{P}G(a_1) - \mathbb{P}G(a_2)|_{B^{2\sigma+1}} \le c|a_1 - a_2|_{B^{2\sigma}}.$$

For example, $G(a) = (-\Delta)^{-\frac{1}{2}}a$ and thus certainly satisfies this assumption:

Theorem 2. *Let $T > 0$ and $\alpha \in (0, T/2)$ be given. Let assumptions (**A1**) and (**A2**) be satisfied, and assume $b \in \mathcal{B}_\alpha^1$. There exists $\delta > 0$ such that if u_T and the probability measure m_0 are such that $w_T = \mathbb{P}u_T$ and $\mu_0 = m_0 - \overline{m}$ satisfy $|\mu_0|_{B^{2\sigma}} < \delta$, then the system in Equations (1), (2) and (53) has a strong, locally unique solution $(u, m) \in \mathcal{B}_\alpha^{2\sigma+1} \times \mathcal{B}_\alpha^{2\sigma}$.*

The proof of this result is quite similar to that given earlier for Theorem 1, and thus it was omitted.

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