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Certain New Subclass of Multivalent Q-Starlike Functions Associated with Q-Symmetric Calculus

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Abstract: In our present investigation, we extend the idea of q-symmetric derivative operators to multivalent functions and then define a new subclass of multivalent q-starlike functions. For this newly defined function class, we discuss some useful properties of multivalent functions, such as the Hankel determinant, symmetric Toeplitz matrices, the Fekete–Szego problem, and upper bounds of the functional $\left|a_{p+1} - \mu a_{p+1}^2\right|$ and investigate some new lemmas for our main results. In addition, we consider the q-Bernardi integral operator along with q-symmetric calculus and discuss some applications of our main results.

Keywords: q-symmetric calculus; q-symmetric derivative operator; multivalent q-starlike functions; Hankel determinant; symmetric Toeplitz matrices

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1. Introduction and Definitions

The class of all analytic and p-valent functions ζ $(p \in \mathbb{N} = \{1, 2, \ldots\})$ in the open unit disk

$$U = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

denoted by $\mathcal{A}_p$ and the series expansion of $\zeta \in \mathcal{A}_p$ is given as:

$$\zeta(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n. \tag{1}$$

Clearly, $\mathcal{A}_p$ coincides with the set $\mathcal{A}$ of normalized analytic univalent (Schlicht) functions if $p = 1$. In addition, let us select the symbol $\mathcal{S}$ which represents all analytic functions that are univalent in U and satisfy the condition

$$\zeta(0) = 0 \text{ and } \zeta'(0) = 1.$$

A function $\zeta(z) \in \mathcal{A}_p$ is called a p-valently starlike $\left(\mathcal{S}_p^*\right)$ and convex $\left(\mathcal{K}_p\right)$, which are defined as:

$$\zeta(z) \in \mathcal{S}_p^* \iff \operatorname{Re}\left(\frac{z\zeta'(z)}{\zeta(z)}\right) > 0, \ (z \in U)$$

and

$$\zeta(z) \in \mathcal{K}_p \iff \operatorname{Re}\left(1 + \frac{z\zeta''(z)}{\zeta'(z)}\right) > 0, \ (z \in U).$$

A function $\zeta(z) \in \mathcal{A}_p$ is called a p-valently starlike function of order α, which are defined as:

$$\zeta(z) \in \mathcal{S}_p^*(\alpha) \iff \operatorname{Re}\left(\frac{z\zeta'(z)}{\zeta(z)}\right) > \alpha, \ (z \in U).$$

Further, p-valently convex functions of order α for $\zeta(z) \in \mathcal{A}_p$ are defined as follows:

$$\zeta(z) \in \mathcal{K}_p(\alpha) \Longleftrightarrow \mathrm{Re}\left(1 + \frac{z\zeta''(z)}{\zeta'(z)}\right) > \alpha, \ (z \in U),$$

for some $0 \leq \alpha < p$. For $\alpha = 0$, then

$$\mathcal{S}_p^*(0) = \mathcal{S}_p^*$$

and

$$\mathcal{K}_p(0) = \mathcal{K}_p.$$

The class of all analytic functions p which satisfy the condition $\mathrm{Re}(p(z)) > 0$ in the open unit disk U denoted by $\mathcal{P}$ and the series expansion of $p \in \mathcal{P}$ is given as:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{2}$$

Basic (or q-) calculus is used in many different areas of mathematics and other sciences. Jackson [1,2] used the idea of quantum (or q-) calculus and defined the q-analogue of derivative (D_q) and integral operators. In geometric function theory (GFT) of complex analysis, the usage of the operator (D_q) is quite significant. Ismail et al. [3] were among the first researchers to work on q-calculus along with GFT and have defined the q- extensions of starlike functions. However in the context of GFT, it was Srivastava [4] who used the operator D_q in GFT for the first time. Subsequently, a number of q-analogue operators have been defined. More in-depth information about operators in q-calculus, is available in [5–9]. In [10,11], Arif et al. defined and investigated some new subclasses of multivalent functions by implementing the concept of the q-derivative operator, while Zang et al. [12] used the basic concepts of q-calculus to define the generalized conic domain. For some recent investigations of q-function theory, we refer readers to [13–19].

The q-symmetric quantum calculus has proven to be useful in several fields, in particular in quantum mechanics [20]. As noted in [20], the q-symmetric derivative has important properties for the q-exponential function which turns out not to be true with the usual derivative. The application of q-symmetric calculus has been discussed in the field of mathematics and physics, particularly in quantum mechanics [21]. Recently, a number of researchers have started to study q-symmetric calculus in the field of GFT. Kanas et al. [22] used the basic concepts of q-symmetric calculus to define the symmetric operator of a q-derivative and studied its applications by defining some new subclasses of analytic univalent functions in open unit disc U. Khan et al. [23] made use of the q-symmetric calculus and have defined certain subclasses of analytic and starlike functions with the conic domains. Furthermore, in [24], using a q-symmetric operator, generalization of the conic domain was given, and, with the help of these conic domains, certain subclasses of starlike and convex functions were defined. Very recently, Khan et al. [25] used the q-symmetric operator with q-Chebyshev polynomials and defined certain subclasses of analytic and bi-univalent functions. Motivated by these recent studies, in this article, we define a new subclass of multivalent starlike functions and discuss some of their interesting properties.

Here, we provide some basic concepts and definitions of the q-symmetric calculus. We suppose $0 < q < 1$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

In 1984, Biedenharn [26] defined a q-symmetric number $[n]_q$:

$$[n]_q = \frac{q^{-n} - q^n}{q^{-1} - q}, n \in \mathbb{N}$$

and for $n = 0$, then the q-symmetric number will be zero. For any $n \in \mathbb{Z}^+ \cup \{0\}$, the q-symmetric number shift factorial is defined by:

$$[n]_q! = [n]_q[n-1]_q[n-2]_q \ldots [2]_q[1]_q, \qquad n \geq 1,$$

and

$$[0]_q! = 1,$$

$$\lim_{q \to 1-} [n]_q! = n!.$$

Definition 1. *For $\zeta \in \mathcal{A}_p$, Kamel and Yosr [27] defined the q-symmetric derivative (q-difference) operator as follows:*

$$\widetilde{D}_q \zeta(z) = \frac{1}{z^p}\left(\frac{\zeta(qz) - \zeta(q^{-1}z)}{q - q^{-1}}\right), \; z \in U,$$
$$= [p]_q z^{p-1} + \sum_{n=p+1}^{\infty} [n]_q a_n z^{n-1}, \quad (z \neq 0, \; q \neq 1),$$

Note that

$$\lim_{q \to 1-} \widetilde{D}_q \zeta(z) = \zeta'(z).$$

By making use of the q-symmetric derivative operator $\zeta \in \mathcal{A}_p$, we now define new subclasses of q-starlike and q-convex functions of order α.

Definition 2. *A function $\zeta(z) \in \mathcal{A}_p$ is in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q)$ iff*

$$\mathrm{Re}\left(\frac{z\widetilde{D}_q\zeta(z)}{\zeta(z)}\right) > \alpha, \; (z \in U),$$

for some $0 \leq \alpha < p$.

Definition 3. *A function $\zeta(z) \in \mathcal{A}_p$ is in the class $\widetilde{\mathcal{K}}_p(\alpha, q)$ iff*

$$\mathrm{Re}\left(\frac{\widetilde{D}_q\left(z\widetilde{D}_q\zeta(z)\right)}{\widetilde{D}_q\zeta(z)}\right) > \alpha, \; (z \in U),$$

for some $0 \leq \alpha < p$.

Remark 1. *Let $\zeta(z) \in \mathcal{A}_p$, then it follows that*

$$\zeta(z) \in \widetilde{\mathcal{K}}_p(\alpha, q) \Leftrightarrow \frac{z\widetilde{D}_q\zeta(z)}{[p]_q} \in \widetilde{\mathcal{S}}_p^*(\alpha, q)$$

and

$$\zeta(z) \in \widetilde{\mathcal{S}}_p^*(\alpha, q) \Leftrightarrow \int_0^z \frac{[p]_q\zeta(\theta)}{\theta} d_q\theta \in \widetilde{\mathcal{K}}_p(\alpha, q).$$

Special cases: We can see that

$$\widetilde{\mathcal{S}}_p^*(q) = \widetilde{\mathcal{S}}_p^*(0, q), \;\; \widetilde{\mathcal{S}}^*(\alpha, q) = \widetilde{\mathcal{S}}_1^*(\alpha, q), \;\; \widetilde{\mathcal{K}}_p(q) = \widetilde{\mathcal{K}}_p(0, q) \text{ and } \widetilde{\mathcal{K}}(\alpha, q) = \widetilde{\mathcal{K}}_1(\alpha, q).$$

Remark 2. *By taking $q \to 1-$, then $\widetilde{\mathcal{S}}_p^*(\alpha, q) = \mathcal{S}_p^*(\alpha)$ and $\widetilde{\mathcal{K}}_p(\alpha, q) = \mathcal{K}_p(\alpha)$ as introduced by Mishra and Gochhayat in [28].*

In [29], Nonaan and Thomas introduced th jth Hankel determinant:

$$H_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+j-2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+j-1} & a_{n+j-2} & \cdots & a_{n+2j-2} \end{vmatrix} \tag{3}$$

where $n \in \mathbb{N}_0$, $a_1 = 1$, and $j \in \mathbb{N}$. For $j = 2$ and $n = 1$, then the Hankel determinant Equation (3) represents a Fekete–Szegö functional $H_2(1) = |a_3 - a_2^2|$ and for $j = 2$ and $n = 1$ then Equation (3) represents the functional $|a_2a_4 - a_3^2|$. For complex number μ then this functional is further generalized as $|a_3 - \mu a_2^2|$,(see [30]). The third Hankel determinant $H_3(1)$ was studied by Babalola [31].

The symmetric Toeplitz determinant $T_j(n)$ is defined as follows:

$$T_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+j-1} & \cdots & \cdots & a_n \end{vmatrix}, \tag{4}$$

so that

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \ T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \ T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}$$

and so on. The problem of finding the best possible bounds for $||a_{n+1}| - |a_n||$ has a long history (see [32]). A number of authors have studied the symmetric Toeplitz determinant $T_j(n)$ from different points of view (for details see [18,33]).

By replacing $n = n + p - 1$, into Equation (4), then $T_j(n)$ can be written as:

$$T_j(n+p-1) = \begin{vmatrix} a_{n+p-1} & a_{n+p} & \cdots & a_{n+p+j-2} \\ a_{n+p} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+p+j-2} & \cdots & \cdots & a_{n+p-1} \end{vmatrix},$$

so that

$$T_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+1} \end{vmatrix}, \ T_2(p+2) = \begin{vmatrix} a_{p+2} & a_{p+3} \\ a_{p+3} & a_{p+2} \end{vmatrix},$$

and

$$T_3(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+1} & a_{p+2} \\ a_{p+3} & a_{p+2} & a_{p+1} \end{vmatrix}.$$

2. A Set of Lemmas

In this section, we provide some new and known lemmas to prove our main results.

Lemma 1 ([32])**.** *If a function* $u(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$, *then*

$$|c_n| \leq 2, \ \ n \geq 1.$$

The inequality is sharp for

$$\zeta(z) = \frac{1+z}{1-z}.$$

Lemma 2. *If a function* $u(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ *which satisfies* $\operatorname{Re}(u(z)) \geq \alpha$*, for some* α*,* $(0 \leq \alpha < p)$*, then*

$$|c_n| \leq 2\Big([p]_q - \alpha\Big),\ n \geq 1.$$

The inequality is sharp for

$$u(z) = \frac{1 + \Big([p]_q - 2\alpha\Big)z}{1-z} = 1 + \sum_{n=1}^{\infty} 2\Big([p]_q - \alpha\Big)z^n.$$

Remark 3. *The Lemma 2 is the generalization of lemmas which was introduced in [32,34].*

Lemma 3 ([35])**.** *If a function* $u(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ *and satisfies* $\operatorname{Re}(u(z)) > \alpha$*, for some* α $(0 \leq \alpha < 1)$*, then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - \left|x^2\right|)z,$$

for some $x, z \in \mathbb{C}$*, with* $|z| \leq 1$ *and* $|x| \leq 1$*.*

By virtue of Lemma 3, we have

Lemma 4. *If* $u(z) = [p]_q + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ *satisfy* $\operatorname{Re}(u(z)) > \alpha$*, for some* α $(0 \leq \alpha < p)$*, then*

$$2\Big([p]_q - \alpha\Big)c_2 = c_1^2 + \left\{4\Big([p]_q - \alpha\Big)^2 - c_1^2\right\}x$$

and

$$\begin{aligned} 4\Big([p]_q - \alpha\Big)^2 c_3 &= c_1^3 + 2\left\{4\Big([p]_q - \alpha\Big)^2 - c_1^2\right\}c_1 x - \left\{4\Big([p]_q - \alpha\Big)^2 - c_1^2\right\}c_1 x^2 + \\ &\quad 2\Big([p]_q - \alpha\Big)\left\{4\Big([p]_q - \alpha\Big)^2 - c_1^2\right\}(1 - \left|x^2\right|)z, \end{aligned}$$

for some $x, z \in \mathbb{C}$*, with* $|z| \leq 1$*, and* $|x| \leq 1$*.*

Proof. Let $l(z) = \frac{u(z)-\alpha}{[p]_q-\alpha} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{[p]_q-\alpha} z^n \in \mathcal{P}$, replacing c_2 and c_3 by $\frac{c_2}{[p]_q-\alpha}$ and $\frac{c_3}{[p]_q-\alpha}$ in Lemma 3, respectively, we immediately have the relations of Lemma 4. □

Remark 4. *For* $q \to 1-$ *, then the above Lemma 4, reduces to the lemma which was introduced by Hayami et al. [34] and* $p = 1$*,* $q \to 1-$ *, then it reduces to the lemma introduced by Singh in [35].*

Lemma 5 ([36])**.** *If* $u(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *satisfy* $\operatorname{Re}(u(z)) > 0$*, also let* $\mu \in \mathbb{C}$*, then*

$$|c_n - \mu c_k c_{n-k}| \leq 2\max(1, |2\mu - 1|),\ 1 \leq k \leq n-k.$$

3. Main Results

Theorem 1. *A function $\zeta(z) \in \mathcal{A}_p$ is in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q)$, then*

$$\begin{aligned} \left|a_{p+1}\right| &\leq \frac{2\left([p]_q - \alpha\right)}{\Upsilon_1}, \\ \left|a_{p+2}\right| &\leq \frac{2\left([p]_q - \alpha\right)}{\Upsilon_2}\left\{1 + \frac{2\left([p]_q - \alpha\right)}{\Upsilon_1}\right\}, \\ \left|a_{p+3}\right| &\leq \frac{2\left([p]_q - \alpha\right)}{\Upsilon_3}\left[1 + 2\left([p]_q - \alpha\right)\Lambda_2\left\{\rho_3 + 2\left([p]_q - \alpha\right)\right\}\right], \end{aligned}$$

where Λ_2, Υ_1, Υ_2, Υ_3 are given by Equations (10) and (12)–(14).

Proof. Let $\zeta \in \widetilde{\mathcal{S}}_p^*(\alpha, q)$, then $u(z) = [p]_q + \sum_{n=1}^{\infty} c_n z^n$ such that

$$\operatorname{Re}(u(z)) > \alpha$$

and

$$\frac{z\left(\widetilde{D}_q \zeta\right)(z)}{\zeta(z)} = u(z).$$

Implies that

$$z\left(\widetilde{D}_q \zeta\right)(z) = u(z)\zeta(z).$$

Therefore, we get

$$\left([n]_q - [p]_q\right)a_n = \sum_{l=p}^{n-1} a_l c_{n-l}, \tag{5}$$

where $n \geq p+1$, $a_p = 1$, $c_0 = [p]_q$. From Equation (5), we have

$$\begin{aligned} a_{p+1} &= \frac{c_1}{\Upsilon_1}, & (6)\\ a_{p+2} &= \frac{1}{\Upsilon_2}\left\{c_2 + \frac{c_1^2}{\Upsilon_1}\right\}, & (7)\\ a_{p+3} &= \frac{1}{\Upsilon_3}\left\{c_3 + \Lambda_1 c_1 c_2 + \Lambda_2 c_1^3\right\}, & (8) \end{aligned}$$

where

$$\begin{aligned} \Lambda_1 &= \Lambda_2 \rho_3, & (9)\\ \Lambda_2 &= \frac{1}{\Upsilon_1(\Upsilon_2)}, & (10)\\ \rho_3 &= [p+1]_q + [p+2]_q - 2[p]_q, & (11) \end{aligned}$$

and

$$\Upsilon_1 = [p+1]_q - [p]_q, \tag{12}$$

$$\Upsilon_2 = \widetilde{[p+2]}_q - [p]_q, \tag{13}$$

$$\Upsilon_3 = [p+3]_q - [p]_q. \tag{14}$$

Now using the Lemma 2, we obtain our required result of Theorem 1. □

Theorem 2. *A function $\zeta(z) \in \mathcal{A}_p$ is in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q)$, then*

$$T_3((p+1) \leq \Lambda_3 \left[\Omega_4 + 4\left([p]_q - \alpha\right)^2 \Omega_5 + \Omega_7 + \Omega_8 \left| 1 - \frac{2\left([p]_q - \alpha\right)\Omega_6}{\Omega_8} \right| \right],$$

where

$$\Lambda_3 = 4\left([p]_q - \alpha\right)^2 [\Omega_1 + \Omega_2(1+\Omega_3)],\ \Omega_1 = \frac{2\left([p]_q - \alpha\right)}{\Upsilon_1}, \tag{15}$$

$$\Omega_2 = \frac{2\left([p]_q - \alpha\right)}{\Upsilon_3}, \tag{16}$$

$$\Omega_3 = 2\left([p]_q - \alpha\right)\Lambda_2\left\{\rho_3 + 2\left([p]_q - \alpha\right)\right\}, \tag{17}$$

$$\Omega_4 = \frac{1}{(\Upsilon_1)^2},\ \ \Omega_5 = 2\Lambda_2\Lambda_2 - \Lambda_2\rho_4, \tag{18}$$

$$\Omega_6 = \frac{4\Lambda_2}{\Upsilon_2} - \Lambda_2\rho_3\rho_4,$$

$$\Omega_7 = \frac{2}{(\Upsilon_2)^2}, \tag{19}$$

$$\Omega_8 = \rho_4 = \frac{1}{(\Upsilon_1)(\Upsilon_3)}. \tag{20}$$

Proof. The simple calculation of $T_3(p+1)$ is in the following order.

$$T_3(p+1) = \left(a_{p+1} - a_{p+3}\right)\left(a_{p+1}^2 - 2a_{p+2}^2 + a_{p+1}a_{p+3}\right),$$

where a_{p+1}, a_{p+2}, and a_{p+3} is given by Equations (6)–(8).

Now if $\zeta \in \widetilde{\mathcal{S}}^*(\alpha, q)$, then

$$\begin{aligned} \left|a_{p+1} - a_{p+3}\right| &\leq \left|a_{p+1}\right| + \left|a_{p+3}\right|, \\ &\leq \Omega_1 + \Omega_2(1+\Omega_3), \end{aligned} \tag{21}$$

where Ω_1, Ω_2, Ω_3, is given by Equations (16)–(18).

We need to maximize $\left|a_{p+1}^2 - 2a_{p+2}^2 + a_{p+1}a_{p+3}\right|$ for $\zeta \in \widetilde{\mathcal{S}}^*(\alpha, q)$, now we write a_{p+1}, a_{p+2}, a_{p+3} in terms of c_1, c_2, c_3. With the help of Equations (6)–(8), we get

$$\begin{aligned} &\left|a_{p+1}^2 - 2a_{p+2}^2 + a_{p+1}a_{p+3}\right| \\ \leq\ &\left|\Omega_4 c_1^2 - \Omega_5 c_1^4 - \Omega_6 c_1^2 c_2 - \Omega_7 c_2^2 + \Omega_8 c_1 c_3\right|, \\ \leq\ &\Omega_4 c_1^2 + \Omega_5 c_1^4 + \Omega_7 c_2^2 + \Omega_8 c_1 \left| c_3 - \frac{\Omega_6 c_1 c_2}{\Omega_8} \right|. \end{aligned} \tag{22}$$

Using the Lemmas 2 and 5, along with Equations (21) and (22), we obtain the required result. □

Taking $q \to 1-$, $\alpha = 0$, and $p = 1$, then we have the known corollary.

Corollary 1 ([37]). *A function $\zeta(z) \in \mathcal{A}$ is in the class $\widetilde{\mathcal{S}}^*$, then*

$$T_3(2) \leq 84.$$

Theorem 3. *A function $\zeta(z) \in \mathcal{A}_p$ is in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q)$, then*

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \frac{4}{([p]_q - \alpha)^2(\Upsilon_2)^2},$$

where Υ_2 is given by Equation (13).

Proof. Combining Equations (6)–(8), we have

$$\begin{aligned} & a_{p+1}a_{p+3} - a_{p+2}^2 \\ = \;& \rho_4 c_1 c_3 + \left(\Lambda_2\rho_3 - \widetilde{\mathfrak{B}}\right)c_1^2 c_2 - \widetilde{\mathcal{D}}c_2^2 + (\Lambda_2\rho_4 - \Lambda_2\Lambda_2)c_1^4, \end{aligned}$$

where

$$\widetilde{\mathcal{D}} = \frac{1}{(\Upsilon_2)^2}, \quad \widetilde{\mathfrak{B}} = \frac{2\Lambda_2}{\Upsilon_2}.$$

By using Lemma 3 we take

$$\Upsilon = 4\left([p]_q - \alpha\right)^2 - c_1^2$$

and $\mathcal{Z} = \left(1 - |x|^2\right)z$. We assume that

$$c = c_1, \quad \left(0 \leq c \leq 2\left([p]_q - \alpha\right)\right).$$

Therefore

$$a_{p+1}a_{p+3} - a_{p+2}^2 = \lambda_1 c^4 + \lambda_2 \Upsilon c^2 x - \lambda_3 \Upsilon c^2 x^2 - \lambda_4 \Upsilon^2 x^2 + \lambda_5 \Upsilon c \mathcal{Z}, \tag{23}$$

where

$$\begin{aligned} \lambda_1 &= \frac{\rho_4}{4([p]_q - \alpha)^2} + \frac{\Lambda_2\rho_3 - \widetilde{\mathfrak{B}}}{2([p]_q - \alpha)} - \frac{\widetilde{\mathcal{D}}}{4([p]_q - \alpha)^2} - \frac{\widetilde{\mathcal{D}}(\Lambda_2\rho_4 - \Lambda_2\Lambda_2)}{4([p]_q - \alpha)^2}, \\ \lambda_2 &= \frac{\rho_4}{2([p]_q - \alpha)^2} + \frac{\Lambda_2\rho_3 - \widetilde{\mathfrak{B}}}{2([p]_q - \alpha)} - \frac{\widetilde{\mathcal{D}}}{2([p]_q - \alpha)^2}, \\ \lambda_3 &= \frac{\rho_4}{4([p]_q - \alpha)^2}, \; \lambda_4 = \frac{\widetilde{\mathcal{D}}}{4([p]_q - \alpha)^2}, \; \lambda_5 = \frac{\rho_4}{2([p]_q - \alpha)}. \end{aligned}$$

Using the triangle inequality on Equation (23), we obtain

$$\begin{aligned} & \left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \\ \leq \;& |\lambda_1|c^4 + |\lambda_2|\Upsilon c^2|x| + |\lambda_3|\Upsilon c^2|x|^2 + |\lambda_4|\Upsilon^2|x|^2 + |\lambda_5|\left(1 - |x|^2\right)c\Upsilon \\ = \;& \mathcal{G}(c, |x|). \end{aligned}$$

Now, trivially, we have

$$\mathcal{G}'(c, |x|) > 0 \tag{24}$$

on $[0,1]$, in equality Equation (24) shows that $\mathcal{G}(c,|x|)$ is an increasing function in an interval $[0,1]$. So the maximum value obtained at $x=1$ and

$$Max(\mathcal{G}(c,|1|)) = \mathcal{G}(c),$$

and

$$\mathcal{G}(c) = |\lambda_1|c^4 + |\lambda_2|\Upsilon c^2 + |\lambda_3|\Upsilon c^2 + |\lambda_4|\Upsilon^2.$$

Hence, by putting $\Upsilon = 4 - c_1^2$, and after some simplification, we have

$$\mathcal{G}(c) = (|\lambda_1| - |\lambda_2| - |\lambda_3| + |\lambda_4|)c^4 + 4(|\lambda_2| + |\lambda_3| - 2|\lambda_4|)c^2 + 16|\lambda_4|.$$

We consider $\mathcal{G}'(c) = 0$, for the optimum value of $\mathcal{G}(c)$, which implies that $c = 0$.

Therefore, $\mathcal{G}(c)$ has a maximum value at $c = 0$ and the maximum value of $\mathcal{G}(c)$ is given by

$$16|\lambda_4|. \tag{25}$$

Which occurs at $c = 0$ or

$$c^2 = \frac{4(|\lambda_2| + |\lambda_3| - 2|\lambda_4|)}{|\lambda_1| - |\lambda_2| - |\lambda_3| + |\lambda_4|}.$$

Hence, by putting $\lambda_4 = \frac{\widetilde{\mathcal{D}}}{4([p]_q-\alpha)^2}$ and $\widetilde{\mathcal{D}} = \frac{1}{(\Upsilon_2)^2}$ in Equation (25), and after simplification, we obtain the desired result. □

Taking $q \to 1-$, $p = 1$ and $\alpha = 0$, we have the following known corollary.

Corollary 2 ([30]). *A function $\zeta(z) \in \mathcal{A}$ is in the class $\widetilde{\mathcal{S}}_p^*$ then*

$$\left|a_2a_4 - a_3^2\right| \le 1.$$

3.1. Fekete–Szegö Problem

Theorem 4. *A function $\zeta(z) \in \mathcal{A}_p$ is in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q)$, $0 \le \alpha < p$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} \frac{2\left([p]_q-\alpha\right)}{\Upsilon_2}\{\rho_1 - \rho_2\mu\}, & \text{if } \mu \le \rho_5, \\ \frac{2\left([p]_q-\alpha\right)}{\Upsilon_2}, & \text{if } \rho_5 \le \mu \le \rho_6, \\ \frac{2\left([p]_q-\alpha\right)}{(\Upsilon_1)^2\Upsilon_2}\{\rho_2\mu - \rho_1\}, & \text{if } \mu \ge \rho_6, \end{cases}$$

where

$$\begin{aligned} \rho_1 &= \left\{2\left([p]_q - \alpha\right)(\Upsilon_1) + (\Upsilon_1)^2\right\}, \\ \rho_2 &= 2\left([p]_q - \alpha\right)(\Upsilon_2), \end{aligned}$$

$$\begin{aligned} \rho_5 &= \frac{(\Upsilon_1)\left\{2\left([p]_q - \alpha\right) + (\Upsilon_1)\right\} - 1}{2\left([p]_q - \alpha\right)(\Upsilon_2)}, \\ \rho_6 &= \frac{(\Upsilon_1)\left([p]_q - \alpha + (\Upsilon_1)\right)}{\left([p]_q - \alpha\right)(\Upsilon_2)}. \end{aligned}$$

where Υ_1 *and* Υ_2 *is given by Equations (12) and (13).*

Proof. Using Equations (6) and (7) and supposing that

$$c_1 = c\Big(0 \leq c \leq 2([p]_q - \alpha)\Big).$$

We derive

$$\begin{aligned} \left|a_{p+2} - \mu a_{p+1}^2\right| &= \frac{1}{\rho_7}\left|\{\rho_1 - \rho_2\mu\}c^2 + (\Upsilon_1)^2\left\{4\Big([p]_q - \alpha\Big)^2 - c^2\right\}\rho\right| \\ &= A(\rho), \end{aligned}$$

where

$$\rho_7 = 2\Big([p]_q - \alpha\Big)(\Upsilon_1)^2(\Upsilon_2).$$

Applying the triangle inequality, we deduce

$$\begin{aligned} A(\rho) &\leq \frac{1}{\rho_7}|\{\rho_1 - \rho_2\mu\}|c^2 + (\Upsilon_1)^2\left\{4\Big([p]_q - \alpha\Big)^2 - c^2\right\} \\ &= \begin{cases} \frac{1}{\rho_7}\left[\left\{2\Big([p]_q - \alpha\Big)\{\rho_{11} - \rho_{12}\mu\}\right\}c^2 + \rho_9\right], & \text{if } \mu \leq \rho_8, \\ \frac{1}{\rho_7}\left[2\left\{\Big([p]_q - \alpha\Big)(\Upsilon_2)\mu - \rho_{10}\right\}c^2 + \rho_9\right], & \text{if } \mu \geq \rho_8. \end{cases} \end{aligned}$$

where

$$\begin{aligned} \rho_8 &= \frac{2\Big([p]_q - \alpha\Big)(\Upsilon_1) + (\Upsilon_1)^2}{2\Big([p]_q - \alpha\Big)(\Upsilon_2)}, \\ \rho_8 &= \frac{2\Big([p]_q - \alpha\Big)(\Upsilon_1) + (\Upsilon_1)^2}{2\Big([p]_q - \alpha\Big)(\Upsilon_2)}, \end{aligned}$$

$$\begin{aligned} \rho_{10} &= (\Upsilon_1)\left\{\Big([p]_q - \alpha\Big) + (\Upsilon_1)\right\}, \\ \rho_{11} &= (\Upsilon_1),\ \rho_{12} = (\Upsilon_2), \\ \rho_{13} &= \frac{2\Big([p]_q - \alpha\Big)}{(\Upsilon_1)^2(\Upsilon_2)}. \end{aligned}$$

Now we have

$$\begin{aligned} &\left|a_{p+2} - \mu a_{p+1}^2\right| \\ &\leq \begin{cases} \frac{2\left([p]_q - \alpha\right)}{\Upsilon_2}\{\rho_1 - \rho_2\mu\}, & \text{if } \mu \leq \rho_5,\ c = 2\Big([p]_q - \alpha\Big), \\ \frac{2\left([p]_q - \alpha\right)}{\Upsilon_2}, & \text{if } \rho_5 \leq \mu \leq \rho_8,\ c = 0, \\ \frac{2\left([p]_q - \alpha\right)}{\Upsilon_2}, & \text{if } \rho_8 \leq \mu \leq \rho_6,\ c = 0, \\ \rho_{13}\left\{\rho_2\mu - \left\{2\Big([p]_q - \alpha\Big)\rho_{11} + \rho_{11}^2\right\}\right\}, & \text{if } \mu \geq \rho_6,\ c = 2\Big([p]_q - \alpha\Big). \end{cases} \end{aligned}$$

□

Corollary 3 ([34]). *A function $\zeta(z) \in \mathcal{A}_p$ is in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q \to 1) = \widetilde{\mathcal{S}}_p^*(\alpha), 0 \leq \alpha < p$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} (p-\alpha)\{\{2(p-\alpha)+1\} - 4(p-\alpha)\mu\}, & \text{if } \mu \leq \frac{1}{2}, \\ (p-\alpha), & \text{if } \frac{1}{2} \leq \mu \leq \frac{p-\alpha+1}{2(p-\alpha)}, \\ (p-\alpha)\{4(p-\alpha)\mu - \{2(p-\alpha)+1\}\}, & \text{if } \mu \geq \frac{p-\alpha+1}{2(p-\alpha)}. \end{cases}$$

3.2. Applications

Definition 4 ([38]). *Let $\zeta \in \mathcal{A}_p$, let the q-analogue of the Benardi integral operator for multivalent functions defined by*

$$\mathcal{B}_{p,\beta}^q \zeta(z) = \frac{[p+\beta]_q}{z^\beta} \int_0^z t^{\beta-1} \zeta(t) d_q t, \ldots z \in U, \beta > -p, \tag{26}$$

$$= z^p + \sum_{n=1}^{\infty} \frac{[p+\beta]_q}{[n+\beta+p]_q} a_{n+p} z^{n+p},$$

$$= z^p + \sum_{n=1}^{\infty} \widetilde{\mathcal{B}}_{n+p} a_{n+p} z^{n+p}. \tag{27}$$

Remark 5. *For $q \to 1-$, then $\widetilde{\mathcal{B}}_{p,\beta}^q = \widetilde{\mathcal{B}}_{p,\beta}$, studied in [19].*

Remark 6. *For $p = 1$, then $\widetilde{\mathcal{B}}_{p,\beta}^q = \widetilde{\mathcal{B}}_{\beta}^q$, introduced in [39].*

Remark 7. *If $q \to 1-$ and $p = 1$, then $\widetilde{\mathcal{B}}_{p,\beta}^q = \widetilde{\mathcal{B}}_{\beta}$, studied in [40].*

Theorem 5. *A function $\zeta(z) \in \mathcal{A}_p$ is in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q)$, as $\zeta(z) \in \widetilde{\mathcal{S}}_p^*(\alpha, q),, 0 \leq \alpha < p$, and*

$$\widetilde{\mathcal{B}}_{p,\beta}^q \zeta(z) = z^p + \sum_{n=1}^{\infty} \widetilde{\mathcal{B}}_{n+p} a_{n+p} z^{n+p},$$

and $\widetilde{\mathcal{B}}_{p,\beta}^q$ is given by Equation (26), then

$$|a_{p+1}| \leq \frac{2(\widetilde{[p]_q} - \alpha)}{(\Upsilon_1)\widetilde{\mathcal{B}}_{p+1}},$$

$$|a_{p+2}| \leq \frac{2([p]_q - \alpha)}{(\Upsilon_2)\widetilde{\mathcal{B}}_{p+2}} \left\{1 + \frac{2([p]_q - \alpha)}{(\Upsilon_1)\widetilde{\mathcal{B}}_{p+1}}\right\},$$

$$|a_{p+3}| \leq \frac{2([p]_q - \alpha)}{(\Upsilon_3)\widetilde{\mathcal{B}}_{p+3}} \left[1 + \frac{2([p]_q - \alpha)\rho_{14}}{\rho_{15}}\right],$$

where

$$\rho_{14} = \left\{\left((\Upsilon_1)\widetilde{\mathcal{B}}_{p+1} + (\Upsilon_2)\widetilde{\mathcal{B}}_{p+2}\right) + 2([p]_q - \alpha)\right\},$$

$$\rho_{15} = (\Upsilon_1)(\Upsilon_2)\widetilde{\mathcal{B}}_{p+1}\widetilde{\mathcal{B}}p + 2,$$

and Υ_1, Υ_2 and Υ_3 are given by Equations (12)–(14).

Proof. The proof follows easily using Equation (27) and Theorem 1. □

Theorem 6. *Let an analytic function ζ given by Equation (1) be in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q)$, in addition $\widetilde{\mathcal{B}}_{p,\beta}^q$ is the integral operator defined by Equation (26) and is of the form Equation (27), then*

$$T_3((p+1) \leq \Upsilon_3 \left[\begin{array}{c} \frac{\Omega_4}{\widetilde{\mathcal{B}}_{p+1}^2} + 4\big([p]_q - \alpha\big)^2 \Omega_{10} + \frac{\Omega_7}{\widetilde{\mathcal{B}}_{p+2}^2} \\ + \frac{\Omega_8}{\widetilde{\mathcal{B}}_{p+1}\widetilde{\mathcal{B}}p+3} \left| 1 - \frac{2\big([p]_q - \alpha\big)\widetilde{\mathcal{B}}p+3\widetilde{\mathcal{B}}_{p+1}\Omega_{11}}{\Omega_8} \right| \end{array} \right],$$

where

$$\begin{aligned} \Upsilon_3 &= 4\big([p]_q - \alpha\big)^2 \left[\frac{\Omega_1}{\widetilde{\mathcal{B}}_{p+1}} + \frac{\Omega_2}{\widetilde{\mathcal{B}}_{p+3}} (1 + \Omega_9) \right], \\ \Omega_9 &= \Lambda_p \left(\frac{\rho_{14}}{\widetilde{\mathcal{B}}_{p+1}\widetilde{\mathcal{B}}_{p+2}} \right), \\ \Omega_{10} &= \Lambda_4 - \Lambda_5, \ \ \Omega_{11} = \Lambda_6 - \Lambda_7, \end{aligned}$$

$$\begin{aligned} \Lambda_4 &= \frac{2\Lambda_2\Lambda_2}{\widetilde{\mathcal{B}}_{p+1}^2 \widetilde{\mathcal{B}}_{p+2}^2}, \ \ \Lambda_5 = \frac{\Lambda_2\rho_4}{\widetilde{\mathcal{B}}_{p+1}^2 \widetilde{\mathcal{B}}_{p+2}\widetilde{\mathcal{B}}_{p+3}}, \\ \Lambda_6 &= \frac{4\Lambda_2}{([p+2]_q - [p]_q)\widetilde{\mathcal{B}}_{p+1}\widetilde{\mathcal{B}}_{p+2}^2}, \\ \Lambda_7 &= \frac{\Lambda_8\Lambda_2\rho_4}{\widetilde{\mathcal{B}}_{p+1}^2 \widetilde{\mathcal{B}}_{p+2}\widetilde{\mathcal{B}}_{p+3}}, \\ \Lambda_8 &= (\Upsilon_1)\widetilde{\mathcal{B}}_{p+1} + (\Upsilon_2)\widetilde{\mathcal{B}}_{p+2}. \end{aligned}$$

Proof. The proof follows easily using Equation (27) and Theorem 2. □

Theorem 7. *A function $\zeta \in \mathcal{A}_p$ be in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q)$, and $\widetilde{\mathcal{B}}_{p,\beta}^q$ is defined by Equation (26) then*

$$\left| a_{p+1}a_{p+3} - a_{p+2}^2 \right| \leq \frac{4}{([p]_q - \alpha)^2 (\Upsilon_2)^2 \widetilde{\mathcal{B}}_{p+2}}.$$

Theorem 8. *A function $\zeta \in \mathcal{A}_p$ be in the class $\widetilde{\mathcal{S}}_p^*(\alpha, q)$, and $\widetilde{\mathcal{B}}_{p,\beta}^q$ is defined by Equation (26) then*

$$\left| a_{p+2} - \mu a_{p+1}^2 \right| \leq \begin{cases} \frac{2\big([p]_q - \alpha\big)}{(\Upsilon_2)\widetilde{\mathcal{B}}_{p+2}} \Big\{ \rho_{16} - \rho_2 \widetilde{\mathcal{B}}_{p+2}\mu \Big\}, & \text{if } \mu \leq \rho_{17}, \\ \frac{2\big([p]_q - \alpha\big)}{(\Upsilon_2)\widetilde{\mathcal{B}}_{p+2}}, & \text{if } \rho_{17} \leq \mu \leq \rho_{18}, \\ \frac{2\Lambda_2\big([p]_q - \alpha\big)}{(\Upsilon_1)\widetilde{\mathcal{B}}_{p+1}^2 \widetilde{\mathcal{B}}_{p+2}} \Big\{ \rho_2 \widetilde{\mathcal{B}}_{p+2}\mu - \rho_{16} \Big\}, & \text{if } \mu \geq \rho_{18}, \end{cases}$$

where

$$\begin{aligned} \rho_{16} &= \Big\{ 2\big([p]_q - \alpha\big)(\Upsilon_1)\widetilde{\mathcal{B}}_{p+1} + (\Upsilon_1)^2 \widetilde{\mathcal{B}}_{p+1}^2 \Big\}, \\ \rho_{17} &= \frac{(\Upsilon_1)\widetilde{\mathcal{B}}_{p+1}\Big\{ 2\big([p]_q - \alpha\big) + (\Upsilon_1)\widetilde{\mathcal{B}}_{p+1} \Big\} - 1}{2\big([p]_q - \alpha\big)(\Upsilon_2)\widetilde{\mathcal{B}}_{p+2}}, \\ \rho_{18} &= \frac{(\Upsilon_1)\widetilde{\mathcal{B}}_{p+1}\Big([p]_q - \alpha + (\Upsilon_1)\widetilde{\mathcal{B}}_{p+1} \Big)}{\big([p]_q - \alpha\big)(\Upsilon_2)\widetilde{\mathcal{B}}_{p+2}}, \end{aligned}$$

and Λ_2 is given by Equation (10).

4. Conclusions

In this paper, we discussed the idea of a q-symmetric derivative operator to multivalent functions and then defined a new subclass of multivalent q-starlike functions. For this class, we investigated some useful properties of multivalent functions, with respect to the Hankel determinant, the symmetric Toeplitz matrices, and the Fekete–Szego problem. We used a q-Bernardi integral operator, along with a q-symmetric derivative operator for multivalent functions, and discussed some applications of our main results. Using the same approach, we applied a q-symmetric derivative operator to define a new subclass of meromorphic q-convex functions $\widetilde{\mathcal{K}}_p(\alpha, q)$ of order α in U and investigated the results for $\widetilde{\mathcal{K}}_p(\alpha, q)$, respectively.

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