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Hermite–Hadamard Type Inequalities Involving (*k*-*p*) Fractional Operator for Various Types of Convex Functions

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Abstract: We establish various fractional convex inequalities of the Hermite–Hadamard type with addition to many other inequalities. Various types of such inequalities are obtained, such as (p, h) fractional type inequality and many others, as the (p, h)-convexity is the generalization of the other convex inequalities. As a consequence of the (h, m)-convexity, the fractional inequality of the (s, m)-type is obtained. Many consequences of such fractional inequalities and generalizations are obtained.

Keywords: Hermite–Hadamard inequality; (*h*,*m*)-convex function; Hölder inequality; (*p*,*h*)-convex function; fractional inequality

MSC: 26A51; 26D10; 26D15

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1. Introduction and Preliminaries

Convexity has been an important part of mathematics since the introduction of the first convex inequality by Jensen. Many inequalities were derived using convexity, see books [1,2]. Inequalities have various applications to analysis problems, optimization, probability theory, etc. For applications, we refer readers to the papers [3–9]. One of the most elegant results in the theory of convex inequalities is the Hermite–Hadamard inequality [10]. In the literature, the famous Hermite–Hadamard inequality, proved separately by Charles Hermite and Jacques Hadamard, has attracted the interest of many mathematicians who have used various types of convex functions to yield many generalizations of the said inequality. This inequality is stated as follows:

Let $f : \mathbb{I} \to \mathbb{R}$ be a convex function on $\mathbb{I}$ in $\mathbb{R}$ and $x, y \in \mathbb{I}$ with $x < y$, then

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt \leqslant \frac{f(x)+f(y)}{2}.$$

Lately, various types of Hermite–Hadamard type inequalities have been studied and generalized for different types of convex functions under different conditions and parameters. For more information, see [11–16] and the references therein. The fractional calculus used in the paper is an extension of the standard calculus, where we define the integral and the derivative for a fractional number. Different types of fractional integrals and derivatives were defined throughout the years, and we refer the interested reader to the following books for more information on the matter [17–19].

The goal of this paper is to provide various convex inequalities with the usage of the (h,m) and (p,h)-convexity in addition to the usage of the fractional calculus.

We start by defining various types of convex inequalities, from the Jensens inequality, which was the first inequality of its type, to the (h,m) and (p,h)-convexity used in the paper.

The motivation behind this paper is to establish various *k*-*p* and *k* Riemann–Liouville fractional inequalities. The obtained inequalities for special values of the parameters reduce to the Riemann–Liouville and Hadamard fractional inequalities paired with different variations of convexity. The *k*-*p* Riemann–Liouville fractional operator used in the paper is of interest because it generalizes previous types of fractional integral operators, as given in its definition.

Definition 1. *For an interval $\mathcal{I}$ in $\mathbb{R}$, a function $f:\mathcal{I}\to\mathbb{R}$ is said to be convex on $\mathcal{I}$ if*

$$f(\zeta x+(1-\zeta)y)\leqslant \zeta f(x)+(1-\zeta)f(y)$$

for all $x,y\in\mathcal{I}$ and $\zeta\in[0,1]$ holds and is said to be a concave function if the inequality is reversed.

The (s,m) convexity generalized the s convexity. J. Park asserted a new definition given in the following and gave some properties about this class of functions in [20].

Definition 2. *For some fixed $s\in(0,1]$ and $m\in[0,1]$, a mapping $f:[0,+\infty)\to\mathbb{R}$ is said to be (s,m)-convex in the second sense on $\mathcal{I}$ if*

$$f(tx+m(1-t)y)\leqslant t^s f(x)+m(1-t)^s f(y)$$

holds for all $x,y\in\mathcal{I}$ and $t\in[0,1]$.

The following definition was introduced by Zhong Fang, which generalizes the p-convexity. More about the property of the class of (p,h) convex functions can be found here [21].

Definition 3. *Let $h:J\to\mathbb{R}$ be a non-negative and non-zero function. We say that $f:\mathcal{I}\to\mathbb{R}$ is a (p,h)-convex function or that f belongs to the class $ghx(h,p,\mathcal{I})$ if f is non-negative and*

$$f([\alpha x^p+(1-\alpha)y^p]^{\frac{1}{p}})\leqslant h(\alpha)f(x)+h(1-\alpha)f(y)$$

for all $x,y\in\mathcal{I}$ and $\alpha\in(0,1)$. Similarly, if the inequality is reversed, then f is said to be a (p,h)-concave function or belong to the class $ghv(h,p,\mathcal{I})$.

The following definition [22] is due to M. Emin Ozdemir et al. and generalizes the definition of h convex functions.

Definition 4. *Let $J\subset\mathbb{R}$ be an interval containing $(0,1)$ and let $h:J\to\mathbb{R}$ be a non-negative function. We say that $f:[0,b]\to\mathbb{R}$ is an (h-m)-convex function if f is non-negative and, for all $x,y\in[0,b], m\in[0,1]$ and $\alpha\in(0,1)$, one has*

$$f(\alpha x+m(1-\alpha)y)\leqslant h(\alpha)f(x)+mh(1-\alpha)f(y).$$

If the inequality is reversed, then f is said to be an (h-m)-concave function on $[0,b]$.

For suitable choices of h and m, the class of (h-m)-convex functions is reduced to different known classes of convex and related functions defined on $[0,b]$ given in the following. remark.

Remark 1. *In the following cases, we fix various parameters in the (h-m)-convexity to obtain various other types of convexity.*

1. *If $m=1$, then we get an h-convex function.*
2. *If $h(\alpha)=\alpha$, then we get an m-convex function.*

3. *If $h(\alpha) = \alpha$ and $m = 1$, then we get a convex function.*
4. *If $h(\alpha) = 1$ and $m = 1$, then we get a p-function.*
5. *If $h(\alpha) = \alpha^s$ and $m = 1$, then we get an s-convex function in the second sense.*
6. *If $h(\alpha) = \frac{1}{\alpha}$ and $m = 1$, then we get a Godunova–Levin function.*
7. *If $h(\alpha) = \frac{1}{\alpha^s}$ and $m = 1$, then we get an s-Godunova–Levin function of the second kind.*

Before we introduce the fractional type integrals, we need the following definitions. The Pochammer k-symbol $(y)_{m,k}$ is defined as (see [23])

$$(y)_{m,k} = y(y+k)(y+2k)\dots y+(m-1)k,$$

where $m \in \mathbb{N} \cup 0, k > 0$.

The $k-$gamma function Γ_k is given by (see [23]).

$$\Gamma_k(y) = \lim_{m \to +\infty} \frac{m! k^m (mk)^{\frac{y}{k}-1}}{(y)_{m,k}}$$

where $k > 0, y \in \mathbb{C} \setminus k\mathbb{Z}^- \cup 0$.

When $k = 1$, the above definitions reduce to the Pochammer symbol $(y)_m$

$$(y)_m = \begin{cases} \prod_{r=1}^{m}(y+r-1), m \in \mathbb{N} \\ 1, m = 0 \end{cases}$$

and Γ function defined as

$$\Gamma(t) = \int_0^{+\infty} e^{-z} z^{t-1} dz.$$

In the following, we introduce the fractional type integrals used throughout the paper, as well as the ones defined previously for educational and historical purposes.

The following definition represents the Riemann–Liouville k-fractional integral [24] which was defined by Mubeen and Habibullah.

Definition 5. *Let $g \in L_1[a,b]$. Then, the k-fractional integrals of order $\alpha, k > 0$ with $a \geqslant 0$ are defined as*

$$\mathcal{I}_{a+}^{\alpha,k} g(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} g(t) dt, x > a$$

and

$$\mathcal{I}_{b-}^{\alpha,k} g(x) = \frac{1}{\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} g(t) dt, x < b$$

where $\Gamma_k(.)$ is the k-Gamma function.

The following definition is due to Udita Katugampola [25] of Katugampola Fractional integrals, which generalizes the Riemann–Liouville fractional integrals.

Definition 6. *Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in [a,b]$ are defined by*

$${}^{\rho}I_{a+}^{\alpha} f(x) := \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{p-1}}{(x^p - t^p)^{1-\alpha}} f(t) dt$$

and

$${}^{\rho}I_{b-}^{\alpha} f(x) := \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{p-1}}{(t^p - x^p)^{1-\alpha}} f(t) dt$$

with $a < x < b$ and $p > 0$, if the integrals exist.

The following definition [26] generalizes all the fractional integrals.

Definition 7. *The (k-p) Riemann–Liouville fractional integral operator* ${}^{p}_{k}J^{\alpha}_{c}$ *of order* $\alpha > 0$ *for a real valued function* $g(t)$ *is defined as*

$$ {}^{p}_{k}J^{\alpha}_{c}g(x) = \frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_c^x [x^{p+1} - t^{p+1}]^{\frac{\alpha}{k}-1} t^p g(t)dt $$

where $k > 0, p \in \mathbb{R}, p \neq -1$.

The left and right-sided (k-p) Riemann–Liouville fractional integral operators are given by

$$ {}^{p}_{k}J^{\alpha}_{c^+}g(x) = \frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_{c^+}^x [x^{p+1} - t^{p+1}]^{\frac{\alpha}{k}-1} t^p g(t)dt $$

$$ {}^{p}_{k}J^{\alpha}_{d^-}g(x) = \frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^d [t^{p+1} - x^{p+1}]^{\frac{\alpha}{k}-1} t^p g(t)dt $$

The k-p Riemann–Liouville fractional operator fullfils the commutativity and the semigroup properties, and we have

$$ {}^{s}_{k}J^{\alpha}_{a}[{}^{s}_{k}J^{\beta}_{a}f(x)] = {}^{s}_{k}J^{\alpha+\beta}_{a}f(x) = {}^{s}_{k}J^{\beta}_{a}[{}^{s}_{k}J^{\alpha}_{a}f(x)]. $$

Special Cases

1. When $p = 0$ the (*k-p*) Riemann–Liouville fractional integral reduces to a *k*-Riemann–Liouville fractional integral.
2. When $k = 1$ the (*k-p*) Riemann–Liouville fractional integral reduces to a Katugampola fractional integral.
3. When $k = 1, p = 0$ the (*k-p*) Riemann–Liouville fractional integral reduces to a Riemann–Liouville fractional integral.

In our analysis, we need the integral version of the Hölder's inequality. If $f, g \in C([r,s], \mathbb{R})$ and $\lambda, \alpha \in \mathbb{R}$ with $\lambda > 1$ and $\frac{1}{\lambda} + \frac{1}{\alpha} = 1$, then

$$ \int_r^s |f(t)g(t)|dt \leqslant \left(\int_s^r |f(t)|^{\lambda} dt \right)^{\frac{1}{t}} \left(\int_r^s |g(t)|^{\alpha} dt \right)^{\frac{1}{\alpha}}. $$

This inequality is reversed if $0 < \lambda < 1$ and if $\lambda < 0$ or $\alpha < 0$.

In a recent paper [27] devoted to *k* Riemann–Liouville fractional integrals, the authors derived various fractional Hermite–Hadamard type inequalities. We list two of them (Theorems 2 and 3) since they are related to the Theorems from our paper; namely, Theorems 1 and 3 are variations of Theorems 2 and 3 in the paper.

Let $f : [x, y] \to \mathbb{R}$ be an (h-m)-convex function with $0 \leqslant x \leqslant y$, $m \in (0, 1]$. If $f \in L[x, y]$, then the following inequality for *k*-fractional integral holds:

$$ \frac{1}{h(\frac{1}{2})} f\left(\frac{x+my}{2}\right) \leqslant \frac{2^{\frac{\mu}{k}}\Gamma_k(\mu+k)}{(my-x)^{\frac{\mu}{k}}} [I^{\mu,k}_{(\frac{x+my}{2})^+} f(my) + m^{\frac{\mu}{k}+1} I^{\mu,k}_{(\frac{x+my}{2m})^-} f\left(\frac{x}{m}\right)] $$

$$ \leqslant \frac{\mu[f(x)+f(my)]}{k} \int_0^1 \zeta^{\frac{\mu}{k}-1} h\left(\frac{\zeta}{2}\right) d\zeta + \frac{\mu[mf(y)+m^2 f(\frac{x}{m})]}{k} \int_0^1 h\left(\frac{2-\zeta}{2}\right) \zeta^{\frac{\mu}{k}-1} d\zeta. $$

The other inequality from the paper related to the Theorem in our paper is as follows.

Let $f : [x, y] \to \mathbb{R}$ be an $(h-m)$-convex function with $0 \leqslant x \leqslant y$, $m \in (0, 1]$. If $f \in L[x, y]$, then the following inequality for *k*-fractional integral holds:

$$ \frac{2^{\frac{\mu}{k}} k\Gamma(\mu)}{(y-x)^{\frac{\mu}{k}}} [I^{\mu,k}_{(\frac{x+y}{2})^+} f(y) + I^{\mu,k}_{(\frac{x+y}{2})^-} f(x)] $$

$$\leqslant [f(x)+f(y)]\int_0^1 \zeta^{\frac{\mu}{k}-1}h\left(\frac{\zeta}{2}\right)d\zeta + m[f\left(\frac{x}{m}\right)+f\left(\frac{y}{m}\right)]\int_0^1 \zeta^{\frac{\mu}{k}-1}h\left(\frac{2-\zeta}{2}\right)d\zeta$$

$$\leqslant \left(\frac{1}{\frac{\mu p}{k}-p+1}\right)^{\frac{1}{p}}\left([f(x)+f(y)]\left(\int_0^1\left(h\left(\frac{\zeta}{2}\right)\right)^q d\zeta\right)^{\frac{1}{q}}\right.$$

$$\left.+m[f\left(\frac{x}{m}\right)+f\left(\frac{y}{m}\right)]\left(\int_0^1\left(h\left(\frac{2-\zeta}{2}\right)\right)^q d\zeta\right)^{\frac{1}{q}}\right).$$

Check the cited paper and references therein for more convex fractional inequalities.

The following Theorem represents the new inequality of the Riemann–Liouville k-fractional integral type.

2. Main Results

We state our first Theorem in this paper.

Theorem 1. *Let $J \subset \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. If $f : [a,b] \to \mathbb{R}$ is an (h-m)-convex function, such that the Riemann–Liouville k-fractional integral is defined, $\zeta \in (0,1)$, $\Re(\frac{\alpha}{k}) > 0, \alpha \neq 0$, and in one of the cases, the following inequality holds:*

1. $a > 0, b > a, 0 < m < \frac{a}{b}$
2. $a > 0, b < a, 0 < m \leqslant 1$

$$\frac{f\left(\frac{a+bm}{2}\right)}{h(\frac{1}{2})} \leqslant \alpha\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(a)^-}f(mb)}{(a-bm)^{\frac{\alpha}{k}}} + \frac{mI^{\alpha,k}_{(b)^+}f(\frac{a}{m})}{(\frac{a}{m}-b)^{\frac{\alpha}{k}}}\right)$$

$$\leqslant \frac{\alpha(f(x)+mf(y))}{k}\int_0^1 t^{\frac{\alpha}{k}-1}h(t)dt + \frac{\alpha(f(x)+mf(y))}{k}\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t)dt.$$

Proof. Using the definition of (h,m)-convexity, we have the following inequality:

$$f(\alpha x + m(1-\alpha)y) \leqslant h(\alpha)f(x) + mh(1-\alpha)f(y).$$

From this, we get

$$f\left(\frac{x+ym}{2}\right) \leqslant h\left(\frac{1}{2}\right)(f(x)+mf(y)).$$

Using the following substitutions $x = m(1-t)b + ta, y = (1-t)\frac{a}{m} + bt$, we get the following:

$$\frac{f(\frac{a+bm}{2})}{h(\frac{1}{2})} \leqslant f(m(1-t)b+ta) + mf\left((1-t)\frac{a}{m}+bt\right).$$

Multiplying both sides by $t^{\frac{\alpha}{k}-1}$ and integrating with respect to t from 0 to 1, we get

$$f\left(\frac{a+bm}{2}\right)\int_0^1 \frac{t^{\frac{\alpha}{k}-1}}{h(\frac{1}{2})}dt \leqslant \int_0^1 f(m(1-t)b+ta)t^{\frac{\alpha}{k}-1}dt.$$

$$+m\int_0^1 f\left((1-t)\frac{a}{m}+bt\right)t^{\frac{\alpha}{k}-1}dt$$

The left-hand side is easy to integrate. Focusing onto the first integral on the right-hand side and introducing a substitution $m(1-t)b+ta=z, dt=\frac{dz}{a-mb}, t=\frac{z-mb}{a-mb}$, we get

$$\int_0^1 f(m(1-t)b+ta)t^{\frac{\alpha}{k}-1}dt=\int_{mb}^a f(z)\left(\frac{z-mb}{a-mb}\right)^{\frac{\alpha}{k}-1}\frac{dz}{a-mb}\cdot\frac{k\Gamma_k(\alpha)}{k\Gamma_k(\alpha)}$$

$$\int_0^1 f(m(1-t)b+ta)t^{\frac{\alpha}{k}-1}dt=\frac{k\Gamma_k(\alpha)}{(a-bm)^{\frac{\alpha}{k}}}I^{\alpha}_{(a)^-}f(mb).$$

Applying the similar procedure to the second integral, while making note that $\frac{a}{m}>b$, we find that the following holds:

$$m\int_0^1 f\Big((1-t)\frac{a}{m}+bt\Big)t^{\frac{\alpha}{k}-1}dt=\frac{mk\Gamma_k(\alpha)}{(\frac{a}{m}-b)^{\frac{\alpha}{k}}}I^{\alpha,k}_{(b)^+}f\Big(\frac{a}{m}\Big).$$

Adding the two integrals, we get

$$f\left(\frac{a+bm}{2}\right)\int_0^1\frac{t^{\frac{\alpha}{k}-1}}{h(\frac{1}{2})}dt\leqslant k\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(a)^-}f(mb)}{(a-bm)^{\frac{\alpha}{k}}}+\frac{mI^{\alpha,k}_{(b)^+}f(\frac{a}{m})}{(\frac{a}{m}-b)^{\frac{\alpha}{k}}}\right).$$

When solved for the left hand side and multiplied by the constants, this gives us the left-hand side inequality.

Now, we prove the right-hand side inequality. Using the (h,m)-convexity, we get

$$f(ta+m(1-t)b)\leqslant h(t)f(a)+mh(1-t)f(b)$$

$$f\left(tb+\frac{1}{m}(1-t)a\right)\leqslant h(t)f(b)+\frac{1}{m}h(1-t)f(a).$$

Adding these two inequalities, multiplying with $t^{\frac{\alpha}{k}-1}$, and integrating with respect to t from 0 to 1, we get

$$k\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(a)^-}f(mb)}{(a-bm)^{\frac{\alpha}{k}}}+\frac{mI^{\alpha,k}_{(b)^+}f(\frac{a}{m})}{(\frac{a}{m}-b)^{\frac{\alpha}{k}}}\right)\leqslant$$

$$\leqslant(f(a)+mf(b))\int_0^1 t^{\frac{\alpha}{k}-1}h(t)dt+(f(a)+mf(a))\int_0^1 h(1-t)t^{\frac{\alpha}{k}-1}dt.$$

Adding the lower and upper bound together, integrating the lowest bound, and multiplying with the constants, we get the required inequality:

$$\frac{f\left(\frac{a+bm}{2}\right)}{h(\frac{1}{2})}\leqslant\alpha\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(a)^-}f(mb)}{(a-bm)^{\frac{\alpha}{k}}}+\frac{mI^{\alpha,k}_{(b)^+}f(\frac{a}{m})}{(\frac{a}{m}-b)^{\frac{\alpha}{k}}}\right)$$

$$\leqslant\frac{\alpha(f(a)+mf(b))}{k}\int_0^1 t^{\frac{\alpha}{k}-1}h(t)dt+\frac{\alpha(f(a)+mf(b))}{k}\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t)dt.$$

□

In the following Corollary, we show how our inequality can be used to get a potentially new k Riemann–Liouville type inequality.

Corollary 1. *Let f be an (h,m)-convex function such that $a,b\in[0,b]$. The following inequality holds in one of the cases :*

$$a>0,b>a,0<m<\frac{a}{b},\Re\left(\frac{\alpha}{k}\right)\ \textit{and}\ a>0,b<a,0<m\leqslant 1,\Re\left(\frac{\alpha}{k}\right)>0$$

$$f\left(\frac{a+b}{2}\right) \leqslant \alpha\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(a)^-}f(b)}{(a-b)^{\frac{\alpha}{k}}}+\frac{I^{\alpha,k}_{(b)^+}f(a)}{(a-b)^{\frac{\alpha}{k}}}\right) \leqslant \frac{f(a)+f(b)}{2}$$

Proof. Setting $h(\zeta) = \zeta,\ m = 1$ in the previously derived Theorem, we get the inequality. □

The following Corollary gives a result for P-functions as follows.

Corollary 2. *Let f be a (h,m)-convex function such that $a,b \in [0,b]$. The following inequality holds in one of the cases:*

$$a > 0, b > a, 0 < m < \frac{a}{b}, \Re\left(\frac{\alpha}{k}\right) \ \ and \ \ a > 0, b < a, 0 < m \leqslant 1, \Re\left(\frac{\alpha}{k}\right) > 0$$

$$\frac{f(\frac{a+b}{2})}{2} \leqslant \frac{\alpha\Gamma_k(\alpha)}{2}\left(\frac{I^{\alpha,k}_{(a)^-}f(b)}{(a-b)^{\frac{\alpha}{k}}}+\frac{I^{\alpha,k}_{(b)^+}f(a)}{(a-b)^{\frac{\alpha}{k}}}\right) \leqslant f(a)+f(b).$$

Proof. Setting $h(t) = m = 1$ in Theorem 1, we get the inequality. □

The following inequality gives a result on the (s,m)-convex functions in the second sense.

Corollary 3. *Let f be an (h,m)-convex function such that $a,b \in [0,b]$. The following inequality holds in one of the cases:*

$$a > 0, b > a, 0 < m < \frac{a}{b}, \Re\left(\frac{\alpha}{k}\right) \ \ and \ \ a > 0, b < a, 0 < m \leqslant 1, \Re\left(\frac{\alpha}{k}\right) > 0$$

$$\frac{f\left(\frac{a+bm}{2}\right)}{h(\frac{1}{2})} \leqslant \alpha\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(a)^-}f(mb)}{(a-bm)^{\frac{\alpha}{k}}}+\frac{mI^{\alpha,k}_{(b)^+}f(\frac{a}{m})}{(\frac{a}{m}-b)^{\frac{\alpha}{k}}}\right)$$
$$\leqslant \frac{\Gamma(s+1)\Gamma\left(\frac{k+\alpha}{k}\right)(mf(y)+f(x))}{\Gamma\left(s+\frac{\alpha}{k}+1\right)}+\frac{\alpha(f(x)+mf(y))}{\alpha+ks}$$

Proof. Setting $h(t) = t^s$ in Theorem 1, we get the inequality. □

The following Theorem gives an estimate for the right-sided Riemann–Liouville k-fractional integral.

Theorem 2. *Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. Let $f : [a^p, b^p] \to \mathbb{R}$ be a (p,h)-convex function, $p > 0, q \in \mathbb{R}^+, \zeta \in (0,1)$. Then, the following inequality holds:*

$$\frac{f\left([\frac{q(a^p+b^p)}{2}]^{\frac{1}{p}}\right)}{h(\frac{1}{2})}$$
$$\leqslant \frac{2^{\frac{\alpha}{k}}\alpha p^{\frac{\alpha}{k}}\Gamma_k(\alpha)}{(b^p-a^p)^{\frac{\alpha}{k}}}\left({}^{p-1}_{k}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{a^p}{2}-\frac{b^p}{2}\right)^{\frac{1}{p}}{}^+}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right)\right)$$
$$+\frac{2^{\frac{\alpha}{k}}\alpha p^{\frac{\alpha}{k}}\Gamma_k(\alpha)}{(b^p-a^p)^{\frac{\alpha}{k}}}\left({}^{p-1}_{k}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{b^p}{2}-\frac{a^p}{2}\right)^{\frac{1}{p}}{}^-}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right)\right)$$

$$\leqslant \frac{\alpha p}{k}(f(a)+f(b))\int_0^1\left(h\left(\frac{q-t^p}{2}\right)+h\left(\frac{q+t^p}{2}\right)\right)t^{\frac{\alpha p}{k}-1}dt$$

Proof. Since f is a (p,h)-convex function, we have the following inequality:

$$f([\zeta x^p+(1-\zeta)y^p]^{\frac{1}{p}})\leqslant h(\zeta)f(x)+h(1-\zeta)f(y).$$

Setting $\zeta=\frac{1}{2}$, we get

$$\frac{f([\frac{x^p+y^p}{2}]^{\frac{1}{p}})}{h(\frac{1}{2})}\leqslant f(x)+f(y).$$

Setting $x^p=\frac{q+t^p}{2}a^p+\frac{q-t^p}{2}b^p, y^p=\frac{q+t^p}{2}b^p+\frac{q-t^p}{2}a^p$, we get the following:

$$\frac{f([\frac{q(a^p+b^p)}{2}]^{\frac{1}{p}})}{h(\frac{1}{2})}\leqslant f\left([\frac{q+t^p}{2}a^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right)+f\left([\frac{q+t^p}{2}b^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right).$$

Multiplying the inequality with $t^{\frac{\alpha p}{k}-p}t^{p-1}$ and integrating with respect to t from 0 to 1, we get

$$\int_0^1 t^{\frac{\alpha p}{k}-p}t^{p-1}\frac{f([\frac{q(a^p+b^p)}{2}]^{\frac{1}{p}})}{h(\frac{1}{2})}dt\leqslant\int_0^1 t^{\frac{\alpha p}{k}-p}t^{p-1}f\left([\frac{q+t^p}{2}a^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right)dt$$

$$+\int_0^1 t^{\frac{\alpha p}{k}-p}t^{p-1}f\left([\frac{q+t^p}{2}b^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right)dt.$$

Let us focus on the first integral on the right hand side, introducing a substitution $\frac{q+t^p}{2}a^p+\frac{q-t^p}{2}b^p=z^p$, and noting that since $b>a$ we will have to swap the boundaries, we find that

$$\int_0^1 t^{\frac{\alpha p}{k}-p}t^{p-1}f\left([\frac{q+t^p}{2}a^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right)dt$$

$$=\frac{2^{\frac{\alpha}{k}}k\Gamma_k(\alpha)p^{1-\frac{\alpha}{k}}}{(b^p-a^p)^{\frac{\alpha}{k}}k\Gamma_k(\alpha)p^{1-\frac{\alpha}{k}}}\int_{\left(\frac{q(a^p+b^p)}{2}+\frac{a^p}{2}-\frac{b^p}{2}\right)^{\frac{1}{p}}}^{\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}}f(z)\left(q\frac{a^p+b^p}{2}-z\right)^{\frac{\alpha}{k}-1}z^{p-1}dz$$

This can be seen to be a k-p Riemann–Liouville fractional integral. This gives us the following equality:

$$\int_0^1 t^{\frac{\alpha p}{k}-p}t^{p-1}f\left([\frac{q+t^p}{2}a^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right)dt$$

$$=\frac{k\Gamma_k(\alpha)2^{\frac{\alpha}{k}}}{p^{1-\frac{\alpha}{k}}(b^p-a^p)^{\frac{\alpha}{k}}}\,{}_k^{p-1}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{a^p}{2}-\frac{b^p}{2})^{\frac{1}{p}}\right)^+}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right).$$

A similar method can be used to prove the following equality in terms of the second integral:

$$\int_0^1 t^{\frac{\alpha p}{k}-p}t^{p-1}f\left([\frac{q+t^p}{2}b^p+\frac{q-t^p}{2}a^p]^{\frac{1}{p}}\right)dt$$

$$=\frac{k\Gamma_k(\alpha)2^{\frac{\alpha}{k}}}{p^{1-\frac{\alpha}{k}}(b^p-a^p)^{\frac{\alpha}{k}}}\,{}_k^{p-1}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{b^p}{2}-\frac{a^p}{2})^{\frac{1}{p}}\right)^-}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right).$$

Now, we consider the following inequalities

$$f\left([\frac{q+t^p}{2}a^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right)+f\left([\frac{q+t^p}{2}b^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right).$$

This can be seen to give the following inequality when added, and when the (p,h) convexity is applied:

$$f\left([\frac{q+t^p}{2}a^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right)+f\left([\frac{q+t^p}{2}b^p+\frac{q-t^p}{2}b^p]^{\frac{1}{p}}\right)$$
$$\leqslant (f(a)+f(b))\left(h\left(\frac{q-t^p}{2}\right)+h\left(\frac{q+t^p}{2}\right)\right)$$

Multiplying both sides with $t^{\frac{\alpha p}{k}-p}t^{p-1}$ and integrating with respect to t from 0 to 1, we get the following:

$$\frac{k2^{\frac{\alpha}{k}}\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}(b^p-a^p)^{\frac{\alpha}{k}}}\left({}^{p-1}_{k}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{a^p}{2}-\frac{b^p}{2})^{\frac{1}{p}}\right)^+}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right)\right)$$

$$+\frac{k2^{\frac{\alpha}{k}}\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}(b^p-a^p)^{\frac{\alpha}{k}}}\left({}^{p-1}_{k}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{b^p}{2}-\frac{a^p}{2})^{\frac{1}{p}}\right)^-}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right)\right)$$
$$\leqslant (f(a)+f(b))\int_0^1\left(h\left(\frac{q-t^p}{2}\right)+h\left(\frac{q+t^p}{2}\right)\right)t^{\frac{\alpha p}{k}-1}dt.$$

Connecting the left-side inequality with the right-side inequality and multiplying everything with the integral of the left-hand side, we get the desired inequality. □

Corollary 4. *Setting $h(t)=t$ in the previously derived Theorem, we obtain a new p-type inequality for p-convex functions.*

$$f\left([\frac{q(a^p+b^p)}{2}]^{\frac{1}{p}}\right)$$
$$\leqslant \frac{2^{\frac{\alpha}{k}-1}\alpha p^{\frac{\alpha}{k}}\Gamma_k(\alpha)}{(b^p-a^p)^{\frac{\alpha}{k}}}\left({}^{p-1}_{k}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{a^p}{2}-\frac{b^p}{2})^{\frac{1}{p}}\right)^+}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right)\right)$$

$$+\frac{2^{\frac{\alpha}{k}-1}\alpha p^{\frac{\alpha}{k}}\Gamma_k(\alpha)}{(b^p-a^p)^{\frac{\alpha}{k}}}\left({}^{p-1}_{k}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{b^p}{2}-\frac{a^p}{2})^{\frac{1}{p}}\right)^-}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right)\right)$$
$$\leqslant \frac{q(f(a)+f(b))}{2}$$

Corollary 5. *Setting $h(t)=1$ in the previously derived Theorem, we obtain the following inequality*

$$f\left([\frac{q(a^p+b^p)}{2}]^{\frac{1}{p}}\right)$$

$$\leqslant \frac{2^{\frac{\alpha}{k}-1}\alpha p^{\frac{\alpha}{k}}\Gamma_k(\alpha)}{(b^p-a^p)^{\frac{\alpha}{k}}}\left({}_k^{p-1}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{a^p}{2}-\frac{b^p}{2}\right)^{\frac{1}{p}}}{}^{+}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right)\right)$$

$$+\frac{2^{\frac{\alpha}{k}-1}\alpha p^{\frac{\alpha}{k}}\Gamma_k(\alpha)}{(b^p-a^p)^{\frac{\alpha}{k}}}\left({}_k^{p-1}J^{\alpha}_{\left(q(\frac{a^p+b^p}{2})+\frac{b^p}{2}-\frac{a^p}{2}\right)^{\frac{1}{p}}}{}^{-}f\left(\left(\frac{q(a^p+b^p)}{2}\right)^{\frac{1}{p}}\right)\right)$$

$$\leqslant f(a)+f(b)$$

The following Theorem gives an estimate on the k Riemann–Liouville fractional integrals.

Theorem 3. *Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function, f is (h,m) convex, $\theta \in (0,1), a > 0, 0 < b < a$, $\sqrt{\frac{b}{a}} < m \leqslant 1$. Then, the following inequality holds:*

$$k\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(b)^+}f(\frac{a}{m^2})}{(\frac{a}{m^2}-b)^{\frac{\alpha}{k}}}+\frac{I^{\alpha,k}_{(a)^-}f(\frac{b}{m^2})}{(a-\frac{b}{m^2})^{\frac{\alpha}{k}}}\right)\leqslant$$

$$\leqslant (f(a)+f(b))\int_0^1 h(\theta)\theta^{\frac{\alpha}{k}-1}d\theta+m\left(f\left(\frac{a}{m^3}\right)+f\left(\frac{b}{m^3}\right)\right)\int_0^1 h(1-\theta)\theta^{\frac{\alpha}{k}-1}d\theta$$

$$\leqslant\left(\frac{k}{\alpha q+k-kq}\right)^{\frac{1}{q}}(f(a)+f(b))\left(\int_0^1(h(\theta))^r d\theta\right)^{\frac{1}{r}}$$

$$+m\left(\frac{k}{\alpha q+k-kq}\right)^{\frac{1}{q}}\left(f\left(\frac{a}{m^3}\right)+f\left(\frac{b}{m^3}\right)\right)\left(\int_0^1(h(1-\theta))^r d\theta\right)^{\frac{1}{r}}.$$

Proof. We have the following inequality from the (h,m) convexity:

$$f\left(\theta a+m(1-\theta)\frac{b}{m^3}\right)\leqslant h(\theta)f(a)+mh(1-\theta)f\left(\frac{b}{m^3}\right)$$

$$f\left(b\theta+m(1-\theta)\frac{a}{m^3}\right)\leqslant h(\theta)f(b)+mh(1-\theta)f\left(\frac{a}{m^3}\right).$$

Adding the two inequalities, multiplying by $\theta^{\frac{\alpha}{k}-1}$, and integrating with respect to θ from 0 to 1, we get

$$\int_0^1 f\left(b\theta+m(1-\theta)\frac{a}{m^3}\right)\theta^{\frac{\alpha}{k}-1}d\theta+\int_0^1 f\left(\theta a+m(1-\theta)\frac{b}{m^3}\right)\theta^{\frac{\alpha}{k}-1}d\theta$$

$$\leqslant (f(a)+f(b))\int_0^1 h(\theta)\theta^{\frac{\alpha}{k}-1}d\theta+m\left(f\left(\frac{a}{m^3}\right)+f\left(\frac{b}{m^3}\right)\right)\int_0^1 h(1-\theta)\theta^{\frac{\alpha}{k}-1}d\theta.$$

Focusing on the left-hand side and introducing a substitution on the first integral $\theta b+m(1-\theta)\frac{a}{m^3}=z$, we get

$$\int_0^1 f\left(b\theta+m(1-\theta)\frac{a}{m^3}\right)\theta^{\frac{\alpha}{k}-1}d\theta$$

$$=\frac{k\Gamma_k(\alpha)}{(\frac{a}{m^2}-b)^{\frac{\alpha}{k}}}I^{\alpha,k}_{b^+}f\left(\frac{a}{m^2}\right).$$

After performing a similar procedure on the second integral and adding them, we get the inequality

$$\frac{k\Gamma_k(\alpha)}{(\frac{a}{m^2}-b)^{\frac{\alpha}{k}}} I^{\alpha,k}_{b^+} f\left(\frac{a}{m^2}\right) + \frac{k\Gamma_k(\alpha)}{(a-\frac{b}{m^2})^{\frac{\alpha}{k}}} I^{\alpha,k}_{a^-} f\left(\frac{b}{m^2}\right)$$

$$\leqslant (f(a)+f(b)) \int_0^1 h(\theta)\theta^{\frac{\alpha}{k}-1}d\theta + m\left(f\left(\frac{a}{m^3}\right)+f\left(\frac{b}{m^3}\right)\right)\int_0^1 h(1-\theta)\theta^{\frac{\alpha}{k}-1}d\theta.$$

Now, applying Hölder's inequality on the integrals, we get

$$\int_0^1 h(\theta)\theta^{\frac{\alpha}{k}-1}d\theta \leqslant \left(\int_0^1 (h(\theta))^r d\theta\right)^{\frac{1}{r}} \left(\int_0^1 (\theta^{\frac{\alpha}{k}-1})^q d\theta\right)^{\frac{1}{q}}$$

$$\int_0^1 h(1-\theta)\theta^{\frac{\alpha}{k}-1}d\theta \leqslant \left(\int_0^1 (h(1-\theta))^r d\theta\right)^{\frac{1}{r}} \left(\int_0^1 (\theta^{\frac{\alpha}{k}-1})^q d\theta\right)^{\frac{1}{q}}.$$

After combining all the inequalities, we recover the original inequality:

$$k\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(b)^+} f(\frac{a}{m^2})}{(\frac{a}{m^2}-b)^{\frac{\alpha}{k}}} + \frac{I^{\alpha,k}_{(a)^-} f(\frac{b}{m^2})}{(a-\frac{b}{m^2})^{\frac{\alpha}{k}}}\right) \leqslant$$

$$\leqslant (f(a)+f(b)) \int_0^1 h(\theta)\theta^{\frac{\alpha}{k}-1}d\theta + m\left(f\left(\frac{a}{m^3}\right)+f\left(\frac{b}{m^3}\right)\right)\int_0^1 h(1-\theta)\theta^{\frac{\alpha}{k}-1}d\theta$$

$$\leqslant \left(\frac{k}{\alpha q+k-kq}\right)^{\frac{1}{q}} (f(a)+f(b)) \left(\int_0^1 (h(\theta))^r d\theta\right)^{\frac{1}{r}}$$

$$+m\left(\frac{k}{\alpha q+k-kq}\right)^{\frac{1}{q}} \left(f\left(\frac{a}{m^3}\right)+f\left(\frac{b}{m^3}\right)\right)\left(\int_0^1 (h(1-\theta))^r d\theta\right)^{\frac{1}{r}}.$$

□

Corollary 6. *Setting $h(\theta)=\theta^l, l>0$ in the previously derived Theorem, we obtain the following inequality:*

$$k\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(b)^+} f(\frac{a}{m^2})}{(\frac{a}{m^2}-b)^{\frac{\alpha}{k}}} + \frac{I^{\alpha,k}_{(a)^-} f(\frac{b}{m^2})}{(a-\frac{b}{m^2})^{\frac{\alpha}{k}}}\right) \leqslant$$

$$\leqslant (f(a)+f(b))\frac{k}{\alpha+kl} + m\left(f\left(\frac{a}{m^3}\right)+f\left(\frac{b}{m^3}\right)\right)\frac{\Gamma(l+1)\Gamma(\frac{\alpha}{k})}{\Gamma(l+\frac{\alpha}{k}+1)}$$

$$\leqslant \left(\frac{k}{\alpha q+k-kq}\right)^{\frac{1}{q}} (f(a)+f(b)) \left(\frac{1}{lr+1}\right)^{\frac{1}{r}}$$

$$+m\left(\frac{k}{\alpha q+k-kq}\right)^{\frac{1}{q}} \left(f\left(\frac{a}{m^3}\right)+f\left(\frac{b}{m^3}\right)\right)\left(\frac{1}{lr+1}\right)^{\frac{1}{r}}.$$

Theorem 4. *Let $h: J\to\mathbb{R}$ be a non-negative and non-zero function. Let $f:[a^p,b^p]\to\mathbb{R}$ be a (p,h)-convex function, $p>0, \zeta\in(0,1)$. Then, the following inequality holds*

$$\frac{f\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)}{h(\frac{1}{2})} \leqslant \frac{2^{\frac{\alpha}{k}} p^{\frac{\alpha}{k}} \alpha\Gamma_k(\alpha)}{(b^p-a^p)^{\frac{\alpha}{k}}}\left({}^{p-1}_{k}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+} f(b) + {}^{p-1}_{k}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-} f(a)\right)$$

$$\leqslant \frac{\alpha p}{k}\Big(f(a)+f(b)\Big)\left(\int_0^1 t^{\frac{\alpha p}{k}-1}\left(h\left(\frac{t^p}{2}\right)+h\left(1-\frac{t^p}{2}\right)\right)dt\right).$$

Proof. Since f is (p,h)-convex function, we have the following inequality:

$$f([\zeta x^p + (1-\zeta)y^p]^{\frac{1}{p}}) \leqslant h(\zeta)f(x) + h(1-\zeta)f(y).$$

Setting $\zeta = \frac{1}{2}$, we get

$$\frac{f([\frac{x^p+y^p}{2}]^{\frac{1}{p}})}{h(\frac{1}{2})} \leqslant f(x) + f(y).$$

Now, setting $x^p = \frac{t^p a^p}{2} + \frac{(2-t^p)}{2}b^p, y^p = \frac{2-t^p}{2}a^p + \frac{t^p b^p}{2}$, we get the following:

$$\frac{f([\frac{a^p+b^p}{2}]^{\frac{1}{p}})}{h(\frac{1}{2})} \leqslant f\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right) + f\left([\frac{2-t^p}{2}a^p + \frac{t^p b^p}{2}]^{\frac{1}{p}}\right).$$

Multiplying both sides of the inequality with $\int_0^1 t^{\frac{\alpha p}{k}-p} \cdot t^{p-1}dt$, we get

$$\int_0^1 t^{\frac{\alpha p}{k}-p} t^{p-1} \frac{f([\frac{a^p+b^p}{2}]^{\frac{1}{p}})}{h(\frac{1}{2})} dt$$

$$\leqslant \int_0^1 \left(f\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right) + f\left([\frac{2-t^p}{2}a^p + \frac{t^p b^p}{2}]^{\frac{1}{p}}\right)\right) t^{\frac{\alpha p}{k}-1} t^{p-1} dt.$$

The left-hand side is trivial; therefore, we focus onto the right-hand side—the first integral.

$$\int_0^1 t^{\frac{\alpha p}{k}-p} t^{p-1} f\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right) dt.$$

Introducing a substitution $\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p = z^p$, we get the following:

$$\int_0^1 t^{\frac{\alpha p}{k}-p} t^{p-1} f\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right) dt$$

$$= \frac{2^{\frac{\alpha}{k}}}{(b^p-a^p)^{\frac{\alpha}{k}}} \int_{\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}}^{b} f(z)(b^p-a^p)^{\frac{\alpha}{k}-1} z^{p-1} dz.$$

which is a $(k-p)$ Riemann–Liouville fractional integral.

$$\int_0^1 t^{\frac{\alpha p}{k}-p} t^{p-1} f\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right) dt = \frac{2^{\frac{\alpha}{k}} k\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}(b^p-a^p)^{\frac{\alpha}{k}}} \cdot {}_k^{p-1}J^{\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+} f(b)$$

Now, focusing onto the second integral on the right hand side

$$\int_0^1 t^{\frac{\alpha p}{k}-p} t^{p-1} f\left([\frac{2-t^p}{2}a^p + \frac{t^p b^p}{2}]^{\frac{1}{p}}\right) dt.$$

Introducing a substitution $\frac{2-t^p}{2}a^p + \frac{t^p b^p}{2} = z^p$, we get

$$\int_0^1 t^{\frac{\alpha p}{k}-p} t^{p-1} f\left([\frac{2-t^p}{2}a^p + \frac{t^p b^p}{2}]^{\frac{1}{p}}\right) dt$$

$$= \int_a^{\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}} f(z)\left(2\frac{z^p-a^p}{b^p-a^p}\right)^{\frac{\alpha}{k}} \frac{b^p-a^p}{2(z^p-a^p)} \frac{z^{p-1}dz}{b^p-a^p}.$$

When multiplied with the necessary constants and factored, we get

$$\int_0^1 t^{\frac{\alpha p}{k}-p} t^{p-1} f\left([\frac{2-t^p}{2}a^p + \frac{t^p b^p}{2}]^{\frac{1}{p}} \right) dt$$

$$= \frac{2^{\frac{\alpha}{k}} k\Gamma_k(\alpha) p^{1-\frac{\alpha}{k}}}{(b^p - a^p)^{\frac{\alpha}{k}} k\Gamma_k(\alpha) p^{1-\frac{\alpha}{k}}} \int_a^{\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{\alpha}}} f(z)(z^p - a^p)^{\frac{\alpha}{k}-1} z^{p-1} dz.$$

This can be seen to be a *(k-p)* Riemann–Liouville fractional integral.

$$\int_0^1 t^{\frac{\alpha p}{k}-p} t^{p-1} f\left([\frac{2-t^p}{2}a^p + \frac{t^p b^p}{2}]^{\frac{1}{p}} \right) dt = \frac{2^{\frac{\alpha}{k}} k\Gamma_k(\alpha)}{(b^p - a^p)^{\frac{\alpha}{k}} p^{1-\frac{\alpha}{k}}} {}^{p-1}_{k} J^{\alpha}_{\left((\frac{a^p+b^p}{2})^{\frac{1}{p}} \right)^-} f(a).$$

Adding the two inequalities, we get the following:

$$\int_0^1 \frac{t^{\frac{\alpha}{k}-p} t^{p-1} f\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}} \right)}{h(\frac{1}{2})} dt$$

$$\leqslant \frac{2^{\frac{\alpha}{k}} k\Gamma_k(\alpha)}{(b^p - a^p)^{\frac{\alpha}{k}} p^{1-\frac{\alpha}{k}}} \left({}^{p-1}_{k} J^{\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \right)^+} f(b) + {}^{p-1}_{k} J^{\alpha}_{\left((\frac{a^p+b^p}{2})^{\frac{1}{p}} \right)^-} f(a) \right).$$

Now, we focus on obtaining the right-hand side inequality. Using the (p,h)-convexity on the following functions, we obtain the inequalities

$$f\left([\frac{t^p a^p}{2} + (1 - \frac{t^p}{2}) b^p]^{\frac{1}{p}} \right) \leqslant h\left(\frac{t^p}{2}\right) f(a) + h\left(1 - \frac{t^p}{2}\right) f(b)$$

$$f\left([\frac{t^p b^p}{2} + (1 - \frac{t^p}{2}) a^p]^{\frac{1}{p}} \right) \leqslant h\left(\frac{t^p}{2}\right) f(b) + h\left(1 - \frac{t^p}{2}\right) f(a).$$

Adding the two inequalities, multiplying with $t^{\frac{\alpha p}{k}-1} t^{p-1}$ and integrating with respect to t from 0 to 1, we get

$$\frac{2^{\frac{\alpha}{k}} k\Gamma_k(\alpha)}{(b^p - a^p)^{\frac{\alpha}{k}} p^{1-\frac{\alpha}{k}}} \left({}^{p-1}_{k} J^{\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \right)^+} f(b) + {}^{p-1}_{k} J^{\alpha}_{\left((\frac{a^p+b^p}{2})^{\frac{1}{p}} \right)^-} f(a) \right)$$

$$\leqslant \left(f(a) + f(b) \right) \left(\int_0^1 t^{\frac{\alpha p}{k}-1} \left(h\left(\frac{t^p}{2}\right) + h\left(1 - \frac{t^p}{2}\right) \right) dt \right).$$

When multiplied with the constants which come from the lowest integral bound, we get the desired inequality.

$$\frac{f\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}} \right)}{h(\frac{1}{2})} \leqslant \frac{2^{\frac{\alpha}{k}} p^{\frac{\alpha}{k}} \alpha \Gamma_k(\alpha)}{(b^p - a^p)^{\frac{\alpha}{k}}} \left({}^{p-1}_{k} J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+} f(b) + {}^{p-1}_{k} J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-} f(a) \right)$$

$$\leqslant \frac{\alpha p}{k} \left(f(a) + f(b) \right) \left(\int_0^1 t^{\frac{\alpha p}{k}-1} \left(h\left(\frac{t^p}{2}\right) + h\left(1 - \frac{t^p}{2}\right) \right) dt \right).$$

□

Corollary 7. *Setting $h(t) = t$, we get the following result for p-convex functions:*

$$f\left([\frac{a^p + b^p}{2}]^{\frac{1}{p}}\right) \leqslant \frac{2^{\frac{\alpha}{k}-1} p^{\frac{\alpha}{k}} \alpha \Gamma_k(\alpha)}{(b^p - a^p)^{\frac{\alpha}{k}}} \left({}_k^{p-1}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+} f(b) + {}_k^{p-1}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-} f(a) \right)$$

$$\leqslant \frac{f(a) + f(b)}{2}.$$

Corollary 8. *Setting $h(t) = 1$ in the previously derived Theorem, we get the following inequality:*

$$\frac{f\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)}{2} \leqslant \frac{2^{\frac{\alpha}{k}-1} p^{\frac{\alpha}{k}} \alpha \Gamma_k(\alpha)}{(b^p - a^p)^{\frac{\alpha}{k}}} \left({}_k^{p-1}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+} f(b) + {}_k^{p-1}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-} f(a) \right)$$

$$\leqslant f(a) + f(b).$$

Corollary 9. *Setting $h(t) = t^l, l > 0$ in the previously derived Theorem, we obtain the inequality under the conditions $2^{\frac{1}{p}} \geqslant 1$ and $\Re\left(\frac{\alpha p}{k}\right) > 0$ and $\Re\left(p\left(\frac{\alpha}{k} + l\right)\right) > 0$*

$$f\left([\frac{a^p + b^p}{2}]^{\frac{1}{p}}\right) \leqslant \frac{2^{\frac{\alpha}{k}-l} p^{\frac{\alpha}{k}} \alpha \Gamma_k(\alpha)}{(b^p - a^p)^{\frac{\alpha}{k}}} \left({}_k^{p-1}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+} f(b) + {}_k^{p-1}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-} f(a) \right)$$

$$\leqslant {}_2F_1\left(-l, \frac{\alpha}{k}; \frac{k+\alpha}{k}; \frac{1}{2}\right) + \frac{\alpha 2^{-l}}{\alpha + kl}.$$

3. Conclusions and Outlook

1. In this paper, various new fractional inequalities have been obtained. The paper utilized the (h, m) and (p, h) convexity to produce results involving fractional operators. Various inequalities in corollaries have been obtained as a consequence of the generalized convexity of (h, m) and (p, h) types.
2. Questions arise whether further generalizations of the obtained convex fractional inequalities are obtainable.
3. As a possible open problem, considering there are various types of convexity definitions, it is natural to ask whether other types of convexity could be used to produce a fractional integral inequality using the *k-p* fractional operator. Perhaps research into using (Ψ, h)-convexity with Raina's function withthe *k-p* fractional operator could produce results.

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