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A Unified Inertial Iterative Approach for General Quasi Variational Inequality with Application

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Abstract: In this paper, we design two inertial iterative methods involving one and two inertial steps for investigating a general quasi-variational inequality in a real Hilbert space. We establish an existence result and a non-trivial example is furnished to substantiate our theoretical findings. We discuss the convergence of the inertial iterative algorithms to approximate the solution of a general quasi-variational inequality. Finally, we apply an inertial iterative scheme with two inertial steps to investigate a delay differential equation. The results presented herein can be seen as substantial generalizations of some known results.

Keywords: quasi-variational inequality; projection type inertial iterative algorithm; strong convergence; delay differential equation

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1. Introduction

Let $\mathcal{K}$ be a non-empty closed convex set in a real Hilbert space $\mathcal{H}$ and $\theta : \mathcal{H} \to \mathcal{H}$ be a non-linear mapping in $\mathcal{H}$. The variational inequality problem is to find a point $p^* \in \mathcal{K}$, such that:

$$\langle \theta(p^*), q^* - p^* \rangle \geq 0, \forall q^* \in \mathcal{K}. \quad (1)$$

It is well documented that the study of variational inequality, which was initiated by Stampacchia [1] becomes a very productive and fruitful tool to examine several problems arising in the natural sciences. Due to an application oriented nature, this field of research has been expanded and generalized in several directions, see [2–8]. One of the pronounced generalizations of variational inequality is quasi-variational inequality (QVI) which is to find $p^* \in \mathcal{K}(p^*)$, such that:

$$\langle \theta(p^*), q^* - p^* \rangle \geq 0, \forall q^* \in \mathcal{K}(p^*), \quad (2)$$

where $\mathcal{K}(p^*)$ is a closed convex-valued set in $\mathcal{H}$. The QVI (2) was coined for the first time by Bensoussan and Lions [9] to deal with impulse control problems. The quasi-variational inequalities are variational inequalities in which the admissible space or the involved potentials depend on the solution of the problem. Quasi-variational inequalities bring forth a consolidated platform for variation inequalities, as well as integrated modelling of various physical problems of significance. The resulting applications of quasi-variational inequalities include game theory [10], continuum and solid mechanics [11–13], transportation [14,15], superconductivity, thermoplasticity, or electrostatics [16–18].

It is well known that numerous physical problems occurring in non-linear analysis and related fields can be represented in the template of fixed point problem. Let $T : \mathcal{H} \to \mathcal{H}$ be a non-linear mapping. The fixed point of T is to locate a point $p^* \in \mathcal{H}$, such that $T(p^*) = p^*$. The set of fixed points is denoted as $Fix(T) = \{p^* \in \mathcal{H} : T(p^*) = p^*\}$. One of the most abundantly studied techniques for figuring out fixed points of non-expansive mappings in

Banach and Hilbert spaces is known as Mann iterative technique, proposed by Mann [19] as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, n \in \mathbb{N}, \tag{3}$$

where $\alpha_n \in [0,1]$ and $T : \mathcal{K} \to \mathcal{K}$ is a nonexpansive mapping on a closed convex subset $\mathcal{K}$ of a given Banach space. For some recently developed iterative methods, we refer [20–27]. Recently, Ali et al. [28] have constructed a new scheme. For initial $p_0 \in \mathcal{D}$ and $\{a_n\}$ in $(0,1)$, the sequence $\{p_n\}$ generated by this scheme is defined as:

$$\begin{cases} r_n = T[(1-a_n)p_n + a_n T p_n], \\ q_n = Tr_n, \\ p_{n+1} = Tq_n. \end{cases} \tag{4}$$

The authors demonstrated that their scheme converges faster than some noted iterative methods, such as S, Picard-S, Gursoy and Karakaya, and M-iteration schemes. To achieve augmented convergence rate of iterative methods for non-linear problems is fascinating for researchers. So far, numerous iterative techniques have been explored and examined for obtaining an incremental convergence rate. In this progression, many multi-step iterative algorithms are studied by adding initial term, see [29–32]. The inertial term is derived from the heavy ball with friction method due to Polyak [33] to examine optimization problems which were obtained by the discretizing of second order dynamical system for an oscillator with damping and conservative restoring force:

$$u^{''}(t) + \zeta u^{'}(t) + \nabla\omega(u(t)) = 0, \tag{5}$$

where $u(t), \omega(u(t))$ and $\zeta > 0$ represent time continuous trajectory, external gravitational field and friction, respectively, and $\omega : \mathcal{H} \to \mathbb{R}$ is differentiable. In fact, inertial type iterative methods are generalization of proximal point algorithm as they are produced by discretization of a second-order-in-time dissipative dynamical system. Alvarez [34] have shown that $\omega : \mathcal{H} \to \mathbb{R}$ is a smooth convex function then each trajectory $t \to u(t)$ converges weakly to a minimizer of ω. The relaxation method introduced by Richardson [35] for solving linear systems is also a technique for augmentation of convergence rate. Eckstein and Bertsekas [36] designed a relaxed proximal point scheme to accelerate the proximal point algorithm. They reported that the rate of convergence is enhanced by adding relaxation parameter. Alvarez [37] proposed an iterative scheme by combining relaxation techniques and inertial term to examine monotone inclusion and convex optimization problems. Maigne [38] added an inertial term to Krasnoselskii–Mann iteration and designed the inertial Mann iterative method for calculating fixed points of a non-expansive mapping in Hilbert spaces as following:

$$\begin{cases} y_n = x_n + \theta_n(x_n + x_{n-1}), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \forall n \in \mathbb{N}, \end{cases} \tag{6}$$

where θ_n is a damping type term and α_n is a relaxation factor.

Inspired and persuaded by the acknowledged facts in the above-mentioned references, we introduce two inertial iterative methods. The first is based on (4) which includes one inertial step and is defined as:

$$\begin{cases} \omega_n = p_n - \Theta_n(p_n - p_{n-1}), \\ r_n = T[(1-a_n)\omega_n + a_n T\omega_n], \\ q_n = Tr_n, \\ p_{n+1} = Tq_n, \end{cases} \tag{7}$$

where $\{a_n\}$ is a sequence in $(0,1)$. The second inertial scheme contains two inertial steps which is to define the sequence $\{p_n\}$ with initial points $p_0, p_1 \in \mathcal{H}$ as below:

$$\begin{cases} \omega_n = p_n + \mu_n(p_n - p_{n-1}), \\ q_n = p_n + \nu_n(p_n - p_{n-1}), \\ p_{n+1} = (1 - a_n - b_n)q_n + a_n T(q_n) + b_n\omega_n, \end{cases} \tag{8}$$

where $\varrho > 0$ is a constant and $\{a_n\}, \{b_n\}$ are sequences in $(0, 1)$. We deal with a class of general quasi-variational inequalities (GQVI) by implementing newly established inertial iterative methods. We prove an existence result and theoretical claims are verified by a non-trivial example. Additionally, we establish the convergence of inertial iterative algorithms involving one and two inertial steps. Finally, as an application of our proposed inertial method, we investigate a delay differential equation.

2. Preliminaries

Let $\mathcal{H}$ be a Hilbert space over the real numbers with norm $\|\cdot\|$ and inner-product $\langle\cdot,\cdot\rangle$, and let $\mathcal{C}(\mathcal{H})$ denote the collection of non-empty closed convex subsets of $\mathcal{H}$. In addition, let $\theta, \phi : \mathcal{H} \to \mathcal{H}$ be not necessarily linear mappings in $\mathcal{H}$, and suppose that the set-valued mapping $\mathcal{K} : \mathcal{H} \to \mathcal{C}(\mathcal{H})$ assigns to every $p^* \in \mathcal{H}$ a closed convex subset $\mathcal{K}(p^*)$ of $\mathcal{H}$.

We consider the problem of finding $p^* \in \mathcal{H} : \phi(p^*) \in \mathcal{K}(p^*)$, such that:

$$\langle\theta(p^*), \phi(q^*) - \phi(p^*)\rangle \geq 0, \forall q^* \in \mathcal{H}, \phi(q^*) \in \mathcal{K}(p^*), \tag{9}$$

called the general quasi-variational inequality (GQVI). It is chronicle that quasi-variational inequalities are desperately applications oriented field of research. Several problems of practical applications, such as modeling of stochastic impulsive control problems, free boundary problems, mechanics, and economy, have been framed as a model of quasi-variational inequalities, see [10,39–41]. We take into account the following third order implicit obstacle boundary value problem ($OBVP$) of finding p, such that:

$$\begin{cases} -p'''(x) \geq g(x), \text{ on } \mathcal{D} = [a_1, a_2] \\ p(x) \geq C(p), \text{ on } \mathcal{D} = [a_1, a_2] \\ [-p'''(x) - g(x)][p - C(p)] = 0, \text{ on } \mathcal{D} = [a_1, a_2] \\ p(a_1) = 0, p'(a_1) = 0, p'(a_2) = 0, \end{cases} \tag{10}$$

where $g(x)$ is a continuous function and $C(p)$ stands for the cost (obstacle) function. A typical form of this function in (10) is:

$$C(p) = s + \inf_i\{p^i\}, \tag{11}$$

where s represents switching cost and the cost function C provides the coupling between unknowns $p = (p^1, p^2, \cdots, p^i)$. It is positive or zero, if the unit is turned on or off, respectively. To exhibit $OBVP$ (10) as a quasi-variational inequality, we define:

$$\mathcal{K}(p) = \{q : q \in \mathcal{H}_0^2(\mathcal{D}), q \geq C(p), \text{ on } \mathcal{D}\},$$

where $\mathcal{H}_0^2(\mathcal{D})$ is a Sobolev space, see [42] and $\mathcal{K}$ is a closed convex set in $\mathcal{H}_0^2(\mathcal{D})$. The $OBVP$ (10) can be imitated as the following energy functional:

$$\begin{aligned} E(q) &= -\int_{a_1}^{a_2} \left(\frac{d^2q}{dx^2}\right)\frac{dq}{dx}dx - 2\int_{a_1}^{a_2} g(x)\frac{dq}{dx}dx, \\ &= \int_{a_1}^{a_2} \left(\frac{dq}{dx}\right)^2 dx - 2\int_{a_1}^{a_2} g(x)\frac{dq}{dx}dx \\ &= \langle\theta(q), \frac{dq}{dx}\rangle - 2\langle g, \frac{dq}{dx}\rangle, \quad \forall\frac{dq}{dx} \in \mathcal{K}(p), \end{aligned} \tag{12}$$

where:

$$\begin{aligned}\langle\theta(p), q\rangle &= -\int_{a_1}^{a_2}\left(\frac{d^3p}{dx^3}\right)\left(\frac{dq}{dx}\right)dx = \int_{a_1}^{a_2}\frac{dp}{dx}\frac{dq}{dx}dx \\ \langle g, \frac{dq}{dx}\rangle &= \int_{a_1}^{a_2} g(x)\frac{dq}{dx}dx.\end{aligned} \tag{13}$$

Note that θ specified in (13) is linear, nonsymmetric, and g-positive. By implementing the approach as in [43], we see that the minimum of energy functional $E(q)$ on $\mathcal{K}(p)$ can be represented as:

$$\langle\theta(p), \phi(q) - \phi(p)\rangle \geq \langle g, \phi(q) - \phi(p)\rangle, \quad \forall q \in \mathcal{K}(p), \tag{14}$$

which is indeed GQVI (9), for more detail see [44]. The problem described in (9) is a unification of several others. Some special cases of GQVI (9) are listed below.

1. For $\mathcal{K}(p^*) = \mathcal{K}$, GQVI (9) reduces to the general variational inequality introduced by Noor [45] which is to find $p^* \in \mathcal{H} : \phi(p^*) \in \mathcal{K}(p)$, such that:

$$\langle\theta(p^*), \phi(q^*) - \phi(p^*)\rangle \geq 0, \quad \forall q^* \in \mathcal{H}, \phi(q^*) \in \mathcal{K}. \tag{15}$$

2. Let for $p_0^* \in \mathcal{H}$, the dual cone of $\mathcal{K}(p_0^*) \subset \mathcal{H}$ be represented by

$$\bar{\mathcal{K}}(p_0^*) = \{p^* \in \mathcal{H} : \langle p^*, q^*\rangle \geq 0, \quad \forall q^* \in \mathcal{K}(p_0^*)\}.$$

Then problem (15) becomes a general complementarity problem, that is, to find $p^* \in \mathcal{H}$, such that:

$$\langle\theta(p^*), \phi(p^*)\rangle \geq 0, \quad \phi(p^*) \in \mathcal{K}(p^*) \quad \text{and} \quad \theta(p^*) \in \bar{\mathcal{K}}(p^*). \tag{16}$$

3. For $\phi = I$, GQVI (9) reduces to the classical quasi-variational inequality (2) introduced in [9].
4. For $\phi = I$ and $\mathcal{K}(p^*) = \mathcal{K}$, GQVI (9) reduces to the classical variational inequality (1) introduced by Stampacchia [1].

Next, we list some handy tools to accomplish our results.

Definition 1. *A single-valued mapping $\theta : \mathcal{H} \to \mathcal{H}$ is called:*

(i) *η-strongly monotone if for some $\eta \geq 0$,*

$$\langle\theta(p^*) - \theta(q^*), p^* - q^*\rangle \geq \eta\|p^* - q^*\|^2, \quad \forall p^*, q^* \in \mathcal{H};$$

(ii) *relaxed (ς, τ)-cocoercive if for some $\varsigma, \tau > 0$,*

$$\langle\theta(p^*) - \theta(q^*), p^* - q^*\rangle \geq -\varsigma\|\theta(p^*) - \theta(q^*)\|^2 + \tau\|p^* - q^*\|^2, \quad \forall p^*, q^* \in \mathcal{H};$$

(iii) *σ-Lipschitz continuous if for some $\sigma > 0$,*

$$\|\theta(p^*) - \theta(q^*)\| \leq \sigma\|p^* - q^*\|, \quad \forall p^*, q^* \in \mathcal{H};$$

(iv) *g-positive if, and only if,*

$$\langle\theta(p^*), g(p^*)\rangle \geq 0, \quad \forall p^* \in \mathcal{H}.$$

Lemma 1 ([46])**.** *Let $\{p_n\}$ and $\{q_n\}$ be sequences of non-negative real numbers, such that there exists some $r \in [0, 1)$ with the property that for each $n \in \mathbb{N}$ the inequality $p_{n+1} \leq rp_n + q_n$ is satisfied. If $\lim_{n\to\infty} q_n = 0$, then $\lim_{n\to\infty} p_n = 0$.*

Lemma 2 ([47])**.** *Let $\{p_n\}$ be a sequence of non-negative real numbers, such that*

$$p_{n+1} \leq (1-\tau_n)p_n + \tau_n q_n + r_n, \text{ for all } n \in \mathbb{N},$$

where the sequences $\{\tau_n\}$, $\{q_n\}$ and $\{r_n\}$ accomplish the following conditions:

(i) *the sequence $\{\tau_n\}$ is in $[0,1]$ such that $\sum_{n=1}^{\infty} \tau_n = \infty$;*

(ii) $\limsup_{n\to\infty} q_n \leq 0$*;*

(iii) *$r_n \geq 0$ for all $n \in \mathbb{N}$, such that $\sum_n^{\infty} r_n < \infty$.*

Then, $\lim\limits_{n\to\infty} p_n = 0$*.*

Let $\mathcal{K}$ be a closed convex subset of $\mathcal{H}$. It is known that for each $p^* \in \mathcal{H}$ there exists a unique point $\Pi_{\mathcal{K}}(p^*)$ in $\mathcal{K}$, such that

$$\|p^* - \Pi_{\mathcal{K}}(p^*)\| = \min\{\|p^* - q^*\| : q^* \in \mathcal{K}\}.$$

Then, by definition, the surjective mapping $p^* \longmapsto \Pi_{\mathcal{K}}(p^*), p^* \in \mathcal{H}$, is the metric projection $\Pi_{\mathcal{K}}$ from $\mathcal{H}$ onto $\mathcal{K}$.

The following lemma is essential and plays a central role in achieving our goal.

Lemma 3. *For any given $r^* \in \mathcal{H}, p^* \in \mathcal{K}(p^*)$ and implicit projection $\Pi_{\mathcal{K}(p^*)}$ of $\mathcal{H}$ onto $\mathcal{K}(p^*) \subset \mathcal{H}$, we have $\langle p^* - r^*, q^* - p^* \rangle \geq 0, \forall q^* \in \mathcal{K}(p^*)$ if $p^* = \Pi_{\mathcal{K}(p^*)}(r^*)$.*

Note that the implicit projection mapping $\Pi_{\mathcal{K}(p^*)}$ is non-expansive, that is,

$$\|\Pi_{\mathcal{K}(p^*)}(q^*) - \Pi_{\mathcal{K}(p^*)}(r^*)\| \leq \|q^* - r^*\|, \forall q^*, r^* \in \mathcal{H}.$$

Assumption 1 ([48])**.** *For any $p^*, q^*, r^* \in \mathcal{H}$, the implicit projection mapping $\Pi_{\mathcal{K}(p^*)}$ satisfies following characteristic condition*

$$\|\Pi_{\mathcal{K}(p^*)}(r^*) - \Pi_{\mathcal{K}(q^*)}(r^*)\| \leq \kappa \|p^* - q^*\|,$$

where $\kappa > 0$ is a constant.

Next, we remodel GQVI (9) into a fixed point problem by using the projection.

Lemma 4. *The function $p^* \in \mathcal{H} : \phi(p^*) \in \mathcal{K}(p^*)$ is a solution of GQVI (9) if, and only if, p^* is a fixed point of $I - \phi + \Pi_{\mathcal{K}(p^*)}[\phi - \varrho\theta]$, i.e.,*

$$p^* = p^* - \phi(p^*) + \Pi_{\mathcal{K}(p^*)}[\phi(p^*) - \varrho\theta(p^*)], \tag{17}$$

where $\Pi_{\mathcal{K}(p^*)}$ is the projection of $\mathcal{H}$ onto $\mathcal{K}(p^*)$ and $\varrho > 0$ is a constant.

Based on (17), we rewrite the algorithm (7) as under:

$$\begin{cases} \omega_n = p_n - \Theta_n(p_n - p_{n-1}), \\ q_n = \psi[(1-a_n)\omega_n + a_n\psi(\omega_n)], \\ r_n = \psi q_n, \\ p_{n+1} = \psi r_n, \end{cases} \tag{18}$$

where $\{a_n\}$ is a sequence in $(0,1)$ and $\psi : \mathcal{H} \to \mathcal{H}$ is an operator defined as

$$\psi := I - \phi + \Pi_{\mathcal{K}(p^*)}[\phi - \varrho\theta]. \tag{19}$$

3. Existence Result

Following theorem ensures the existence of unique solution of GQVI (9) which is followed by a demonstrative numerical example to verify our theoretical claims.

Theorem 1. *Let $\Pi_{\mathcal{K}(p^*)}: \mathcal{H} \to \mathcal{K}(p^*)$ be a projection and $\theta, \phi : \mathcal{H} \to \mathcal{H}$ be non-linear mappings, such that θ is relaxed (σ, τ)-cocoercive and ς-Lipschiz continuous and ϕ is γ-Lipschitz continuous and η-strongly monotone. Suppose that assumption C holds and there exists $\varrho > 0$ complying with the following condition*

$$\varrho\varsigma^2 < \frac{2\varrho\tau + \varphi(\varphi - 2)}{\varrho + 2\sigma}, \tag{20}$$

where $\varphi = 2\sqrt{1 - 2\eta + \gamma^2} + \kappa$. Then, GQVI (9) admits a unique solution.

Proof. From (19), assumption C and the non-expansiveness of $\Pi_{\mathcal{K}(p^*)}$, we acquire:

$$\begin{aligned}
&\|\psi(p^*) - \psi(q^*)\| \\
&= \|p^* - \phi(p^*) + \Pi_{\mathcal{K}(p^*)}[\phi(p^*) - \varrho\theta(p^*)] - \{q^* - \phi(q^*) + \Pi_{\mathcal{K}(p^*)}[\phi(q^*) - \varrho\theta(q^*)]\}\| \\
&\leq \|p^* - q^* - [\phi(p^*) - \phi(q^*)]\| + \|\Pi_{\mathcal{K}(p^*)}[\phi(p^*) - \varrho\theta(p^*)] - \Pi_{\mathcal{K}(q^*)}[\phi(q^*) - \varrho\theta(q^*)]\| \\
&\leq \|p^* - q^* - [\phi(p^*) - \phi(q^*)]\| + \|\Pi_{\mathcal{K}(p^*)}[\phi(p^*) - \varrho\theta(p^*)] - \Pi_{\mathcal{K}(p^*)}[\phi(q^*) - \varrho\theta(q^*)]\| \\
&\quad + \|\Pi_{\mathcal{K}(p^*)}[\phi(q^*) - \varrho\theta(q^*)] - \Pi_{\mathcal{K}(q^*)}[\phi(q^*) - \varrho\theta(q^*)]\| \\
&\leq \|p^* - q^* - [\phi(p^*) - \phi(q^*)]\| + \|\phi(p^*) - \phi(q^*) - \varrho[\theta(p^*) - \theta(q^*)]\| + \kappa\|p^* - q^*\| \\
&\leq 2\|p^* - q^* - [\phi(p^*) - \phi(q^*)]\| + \|p^* - q^* - \varrho[\theta(p^*) - \theta(q^*)]\| + \kappa\|p^* - q^*\|.
\end{aligned} \tag{21}$$

Using the γ-Lipschitz continuity and the η-strongly monotone property of ϕ, we obtain

$$\begin{aligned}
\|p^* - q^* - [\phi(p^*) - \phi(q^*)]\|^2 &= \|p^* - q^*\|^2 + \|\phi(p^*) - \phi(q^*)\|^2 - 2\langle p^* - q^*, \phi(p^*) - \phi(q^*)\rangle \\
&\leq \|p^* - q^*\|^2 + \gamma^2\|p^* - q^*\|^2 - 2\eta\|p^* - q^*\|^2 \\
&= [1 - 2\eta + \gamma^2]\|p^* - q^*\|^2,
\end{aligned}$$

which turns into

$$\|p^* - q^* - [\phi(p^*) - \phi(q^*)]\| \leq \sqrt{1 - 2\eta + \gamma^2}\|p^* - q^*\|. \tag{22}$$

Additionally, from the relaxed (σ, τ)-cocoercivity and ς-Lipschiz continuity of θ, we obtain:

$$\begin{aligned}
&\|p^* - q^* - \varrho[\theta(p^*) - \theta(q^*)]\|^2 \\
&\quad = \|p^* - q^*\|^2 + \varrho^2\|\theta(p^*) - \theta(q^*)\|^2 - 2\varrho\langle p^* - q^*, \theta(p^*) - \theta(q^*)\rangle \\
&\quad \leq \|p^* - q^*\|^2 + \varrho^2\varsigma^2\|p^* - q^*\|^2 + 2\varrho\sigma\|\theta(p^*) - \theta(q^*)\|^2 - 2\varrho\tau\|p^* - q^*\|^2 \\
&\quad \leq \|p^* - q^*\|^2 + \varrho^2\varsigma^2\|p^* - q^*\|^2 + 2\varrho\sigma\varsigma^2\|p^* - q^*\|^2 - 2\varrho\tau\|p^* - q^*\|^2 \\
&\quad = [1 - 2\varrho(\tau - \sigma\varsigma^2) + \varrho^2\varsigma^2]\|p^* - q^*\|^2,
\end{aligned}$$

which leads to:

$$\|p^* - q^* - \varrho[\theta(p^*) - \theta(q^*)]\| \leq \sqrt{1 - 2\varrho(\tau - \sigma\varsigma^2) + \varrho^2\varsigma^2}\|p^* - q^*\|. \tag{23}$$

Thus, from (22)–(23), (21) turns into

$$\|\psi(p^*) - \psi(q^*)\| \leq \Phi\|p^* - q^*\|, \tag{24}$$

where $\Phi = \varphi + \Delta(\varrho), \varphi = 2\sqrt{1-2\eta+\gamma^2}+\kappa$ and $\Delta(\varrho) = \sqrt{1-2\varrho(\tau-\sigma\varsigma^2)+\varrho^2\varsigma^2}$. From (20), it follows that $\Phi < 1$ and, hence, $\psi : \mathcal{H} \to \mathcal{H}$ is a contraction mapping. Therefore, $\psi(p^*) = p^*$ and hence $p^* \in \mathcal{H} : \phi(p^*) \in \mathcal{K}(p^*)$ is a unique solution of GQVI (9). □

4. Convergence Results

Now, we inspect the convergence of inertial iterative methods to figure out the approximate solution of GQVI (9). By means of (19), we can redesign (18) as below:

Next, we prove the following lemma, which plays a deciding role in establishing the convergence.

Lemma 5. *Under the assumptions of the Theorem 1, the sequence* $\{\zeta_n\|p_n - p_{n-1}\|\}$ *norm convergence in the Hilbert space* $\mathcal{H}$ *to* 0*. Here, the sequence* $\{\zeta_n\}_{n=1}^{\infty}$ *is given by:*

$$\zeta_n = \begin{cases} \min\left\{\dfrac{n-1}{n-1+c}, \dfrac{\epsilon_n}{\|p_n - p_{n-1}\|}\right\}, & \text{if } p_n \neq p_{n-1}, \\ \dfrac{n-1}{n-1+c}, & \text{if } p_n = p_{n-1}, \end{cases} \tag{25}$$

for all $n \in \mathbb{N}, c \geq 3$ *and* $\epsilon_n \in (0, \infty)$*, such that* $\lim_{n\to\infty} \epsilon_n = 0$.

Proof. We give a proof of Lemma 5 by considering four different cases.
Case (1). If $p_n = p_{n-1}$, then there is nothing to show because $\zeta_n\|p_n - p_{n-1}\|$ is zero.
Next, we consider the cases, when $p_n \neq p_{n-1}$.
Case (2). Suppose that for all $n \in \mathbb{N}$, $\zeta_n = \dfrac{n-1}{n-1+c}$, then from (25), we acquire $0 \leq \zeta_n = \dfrac{n-1}{n-1+c} \leq \dfrac{\epsilon_n}{\|p_n - p_{n-1}\|}$ and hence $0 \leq \zeta_n\|p_n - p_{n-1}\| \leq \epsilon_n$.
Case (3). Suppose that for all $n \in \mathbb{N}$, $\zeta_n = \dfrac{\epsilon_n}{\|p_n - p_{n-1}\|}$, then from (25), we obtain: $0 \leq \zeta_n = \dfrac{\epsilon_n}{\|p_n - p_{n-1}\|} \leq \dfrac{n-1}{n-1+c}$ and hence $0 \leq \zeta_n\|p_n - p_{n-1}\| = \epsilon_n$.
Case (4). Suppose that for some $n \in \mathbb{N}$, $\zeta_n = \dfrac{n-1}{n-1+c} = \dfrac{\epsilon_n}{\|p_n - p_{n-1}\|}$, then from (25), we obtain $0 \leq \zeta_n\|p_n - p_{n-1}\| = \epsilon_n$.

Thus, for all $n \in \mathbb{N}$, we have $0 \leq \zeta_n\|p_n - p_{n-1}\| \leq \epsilon_n$. Since $\lim_{n\to\infty} \epsilon_n = 0$ and, hence, $\{\zeta_n\|p_n - p_{n-1}\|\}$ converges to 0. □

Theorem 2. *Suppose that mappings* $\Pi_{\mathcal{K}(p^*)}, \theta$ *and* $\phi : \mathcal{H} \to \mathcal{H}$ *are the same and comply with all the assumptions of Theorem 1. Suppose the sequence* $\{p_n\}$ *initiated by Algorithm 1 with the updating parameter* ζ_n *represented by (25) with* $|\Theta_n| \leq \zeta_n$*, for all* $n \in \mathbb{N}$ *and* $\{a_n\}$ *is in* $(0,1)$*. Let* $\{\epsilon_n\}$ *be a sequence given in (25), such that* $\lim_{n\to\infty} \epsilon_n = 0$*. Then* $\{p_n\}$ *converges strongly to the unique solution* p^* *of GQVI (9).*

Algorithm 1. *Define the sequence* $\{p_n\}$ *with arbitrary initial point* p_0 *as below:*

$$\omega_n = p_n - \Theta_n(p_n - p_{n-1}), \tag{26}$$

$$z_n = (1-a_n)\omega_n + a_n\psi(\omega_n), \tag{27}$$

$$q_n = z_n - \phi(z_n) + \Pi_{\mathcal{K}(z_n)}[\phi(z_n) - \varrho\theta(z_n)], \tag{28}$$

$$r_n = q_n - \phi(q_n) + \Pi_{\mathcal{K}(q_n)}[\phi(q_n) - \varrho\theta(q_n)], \tag{29}$$

$$p_{n+1} = r_n - \phi(r_n) + \Pi_{\mathcal{K}(r_n)}[\phi(r_n) - \varrho\theta(r_n)], \tag{30}$$

where $\{a_n\}$ *is a sequence in* $(0,1)$.

Proof. It follows from (30), (17), assumption C and the non-expansiveness of $\Pi_{\mathcal{K}(p^*)}$ that:

$$\begin{aligned}\|p_{n+1}-p^*\| &= \|r_n-\phi(r_n)+\Pi_{\mathcal{K}(r_n)}[\phi(r_n)-\varrho\theta(r_n)] \\ &\quad -\{p^*-\phi(p^*)+\Pi_{\mathcal{K}(p^*)}[\phi(p^*)-\varrho\theta(p^*)]\}\| \\ &\leq \|r_n-p^*-[\phi(r_n)-\phi(p^*)]\| \\ &\quad +\|\Pi_{\mathcal{K}(r_n)}[\phi(r_n)-\varrho\theta(r_n)]-\Pi_{\mathcal{K}(p^*)}[\phi(p^*)-\varrho\theta(p^*)]\| \\ &\leq \|r_n-p^*-[\phi(r_n)-\phi(p^*)]\| \\ &\quad +\|\Pi_{\mathcal{K}(r_n)}[\phi(r_n)-\varrho\theta(r_n)]-\Pi_{\mathcal{K}(r_n)}[\phi(p^*)-\varrho\theta(p^*)]\| \\ &\quad +\|\Pi_{\mathcal{K}(r_n)}[\phi(p^*)-\varrho\theta(p^*)]-\Pi_{\mathcal{K}(p^*)}[\phi(p^*)-\varrho\theta(p^*)]\| \\ &\leq \|r_n-p^*-[\phi(r_n)-\phi(p^*)]\| \\ &\quad +\|\phi(r_n)-\phi(p^*)-\varrho[\theta(r_n)-\theta(p^*)]\|+\kappa\|r_n-p^*\| \\ &\leq 2\|r_n-p^*-[\phi(r_n)-\phi(p^*)]\| \\ &\quad +\|r_n-p^*-\varrho[\theta(r_n)-\theta(p^*)]\|+\kappa\|r_n-p^*\|.\end{aligned} \tag{31}$$

Utilizing the γ-Lipschitz continuity and η-strongly monotone property of ϕ, we obtain:

$$\begin{aligned}&\|r_n-p^*-[\phi(r_n)-\phi(p^*)]\|^2 \\ &\quad = \|r_n-p^*\|^2+\|\phi(r_n)-\phi(p^*)\|^2-2\langle r_n-p^*,\phi(r_n)-\phi(p^*)\rangle \\ &\quad \leq \|r_n-p^*\|^2+\gamma^2\|r_n-p^*\|^2-2\eta\|r_n-p^*\|^2 \\ &\quad = [1-2\eta+\gamma^2]\|r_n-p^*\|^2.\end{aligned} \tag{32}$$

It follows from the relaxed (σ,τ)-cocoercivity and ς-Lipschitz continuity of θ that:

$$\begin{aligned}&\|r_n-p^*-\varrho[\theta(r_n)-\theta(p^*)]\|^2 \\ &\quad = \|r_n-p^*\|^2+\varrho^2\|\theta(r_n)-\theta(p^*)\|^2-2\varrho\langle r_n-p^*,\theta(r_n)-\theta(p^*)\rangle \\ &\quad \leq \|r_n-p^*\|^2+\varrho^2\varsigma^2\|r_n-p^*\|^2+2\varrho\sigma\|\theta(r_n)-\theta(p^*)\|^2-2\varrho\tau\|r_n-p^*\|^2 \\ &\quad \leq \|r_n-p^*\|^2+\varrho^2\varsigma^2\|r_n-p^*\|^2+2\varrho\sigma\varsigma^2\|r_n-p^*\|^2-2\varrho\tau\|r_n-p^*\|^2 \\ &\quad = [1-2\varrho(\tau-\sigma\varsigma^2)+\varrho^2\varsigma^2]\|r_n-p^*\|^2.\end{aligned} \tag{33}$$

Thus, from (31)–(33), we obtain

$$\|p_{n+1}-p^*\| \leq \Phi\|r_n-p^*\|, \tag{34}$$

where $\Phi=[\varphi+\Delta(\varrho)]$, $\varphi=2\sqrt{1-2\eta+\gamma^2}+\kappa$ and $\Delta(\varrho)=\sqrt{1-2\varrho(\tau-\sigma\varsigma^2)+\varrho^2\varsigma^2}$. In a similar fashion to (29) and (17), we can obtain:

$$\begin{aligned}\|r_n-p^*\| &= \|q_n-\phi(q_n)+\Pi_{\mathcal{K}(q_n)}[\phi(q_n)-\varrho\theta(q_n)] \\ &\quad -\{p^*-\phi(p^*)+\Pi_{\mathcal{K}(p^*)}[\phi(p^*)-\varrho\theta(p^*)]\}\| \\ &\leq \Phi\|q_n-p^*\|.\end{aligned} \tag{35}$$

In the same way as in (28) and (17), we obtain:

$$\begin{aligned}\|q_n-p^*\| &= \|z_n-\phi(z_n)+\Pi_{\mathcal{K}(z_n)}[\phi(z_n)-\varrho\theta(z_n)] \\ &\quad -\{p^*-\phi(p^*)+\Pi_{\mathcal{K}(p^*)}[\phi(p^*)-\varrho\theta(p^*)]\}\| \\ &\leq \Phi\|z_n-p^*\|.\end{aligned} \tag{36}$$

Similarly, from (27) and (17), we have

$$\begin{aligned}\|z_n - p^*\| &= \|\omega_n - \phi(\omega_n) + \Pi_{\mathcal{K}(\omega_n)}[\phi(\omega_n) - \varrho\theta(\omega_n)] \\ &\quad - \{p^* - \phi(p^*) + \Pi_{\mathcal{K}(p^*)}[\phi(p^*) - \varrho\theta(p^*)]\}\| \\ &\leq \Phi\|\omega_n - p^*\|.\end{aligned} \tag{37}$$

Now from (26) and (17), we obtain:

$$\begin{aligned}\|\omega_n - p^*\| &= \|p_n - \Theta_n(p_n - p_{n-1}) - p^*)\| \\ &\leq \|p_n - p^*\| + \Theta_n\|p_n - p_{n-1}\|.\end{aligned} \tag{38}$$

By using (38) and (37), we deduce that:

$$\|z_n - p^*\| \leq \Phi\|\omega_n - p^*\| \leq \Phi[\|p_n - p^*\| + \Theta_n\|p_n - p_{n-1}\|]. \tag{39}$$

Taking (39) and (36) into account, we have:

$$\|q_n - p^*\| \leq \Phi\|z_n - p^*\| \leq \Phi^2[\|p_n - p^*\| + \Theta_n\|p_n - p_{n-1}\|]. \tag{40}$$

By using (40) and (35) we can write:

$$\|r_n - p^*\| \leq \Phi\|q_n - p^*\| \leq \Phi^3[\|p_n - p^*\| + \Theta_n\|p_n - p_{n-1}\|]. \tag{41}$$

Additionally, from (41) and (34), we obtain:

$$\begin{aligned}\|p_{n+1} - p^*\| &\leq \Phi\|r_n - p^*\| \leq \Phi^4[\|p_n - p^*\| + \Theta_n\|p_n - p_{n-1}\|] \\ &\leq \Phi^4\|p_n - p^*\| + \zeta_n\|p_n - p_{n-1}\|\end{aligned} \tag{42}$$

From (20), we know that $\Phi < 1$ and from Lemma 5, $\zeta_n\|p_n - p_{n-1}\| \to 0$ as $n \to \infty$. Hence, from Lemma 1, we obtain that $\lim_{n\to\infty} p_n = p^*$. □

Next, we put forward a more prevalent inertial iterative algorithm for approximating GQVI (9), which contains two inertial terms. By making use of (19), (8) can be redesigned as follows:

Theorem 3. *Suppose that the mappings $\Pi_{\mathcal{K}(p^*)}, \theta$ and $\phi : \mathcal{H} \to \mathcal{H}$ are the same and comply with all the assumptions of Theorem 1. Suppose that the sequence $\{p_n\}$ initiated by Algorithm 2 with the updating parameter ζ_n represented by (24) with $|\nu_n| + |\mu_n| \leq \zeta_n$, for all $n \in \mathbb{N}$ and the sequences $\{a_n\}, \{b_n\}$ are in $(0,1)$, such that $0 < a_n + b_n < 1$, for all $n \in \mathbb{N}$. Let $\{\epsilon_n\}$ be a sequence given in (25), such that $\lim_{n\to\infty} \epsilon_n = 0$. Then $\{p_n\}$ converges strongly to the unique solution p^* of GQVI (9).*

Algorithm 2. *Define the sequence $\{p_n\}$ with initial points $p_0, p_1 \in \mathcal{H}$ as below:*

$$\omega_n = p_n + \mu_n(p_n - p_{n-1}), \tag{43}$$

$$q_n = p_n + \nu_n(p_n - p_{n-1}), \tag{44}$$

$$p_{n+1} = (1 - a_n - b_n)q_n + a_n(q_n - \phi(q_n) + \Pi_{\mathcal{K}(q_n)}[\phi(q_n) - \varrho\theta(q_n)]) + b_n\omega_n, \tag{45}$$

where $\varrho > 0$ is a constant and $\{a_n\}, \{b_n\}$ are sequences in $(0,1)$.

Proof. It is proved in Theorem 1 that GQVI (9) has a unique solution p^*. Next, it remains to substantiate that the sequence $\{p_n\}$ converges to p^* under the assumption of Algorithm 2. It follows from (17) that:

$$p^* = p^* - \phi(p^*) + \Pi_{\mathcal{K}(p^*)}[\phi(p^*) - \varrho\theta(p^*)].$$

By utilizing assumption C, the non-expansiveness of the mapping $\Pi_{\mathcal{K}(p^*)}$ and following the steps as in (21), we obtain:

$$\begin{aligned}
&\|q_n - \phi(q_n) + \Pi_{\mathcal{K}(q_n)}(\phi(q_n) - \varrho\theta(q_n)) - p^*\| \\
&\qquad = \|q_n - \phi(q_n) + \Pi_{\mathcal{K}(q_n)}(\phi(q_n) - \varrho\theta(q_n)) \\
&\qquad\quad - \{p^* - \phi(p^*) + \Pi_{\mathcal{K}(p^*)}[\phi(p^*) - \varrho\theta(p^*)]\}\| \\
&\qquad \leq 2\|q_n - p^* - [\phi(q_n) - \phi(p^*)]\| \\
&\qquad\quad + \|q_n - p^* - \varrho[\theta(q_n) - \theta(p^*)]\| + \kappa\|q_n - p^*\|.
\end{aligned} \tag{46}$$

Since ϕ is γ-Lipschitz continuous, η-strongly monotone and θ is relaxed (σ,τ)-cocoercive, ς-Lipschitz continuous, then adopting the approach as in (22)–(24), we can write:

$$\|q_n - \phi(q_n) + \Pi_{\mathcal{K}(q_n)}(\phi(q_n) - \varrho\theta(q_n)) - p^*\| \leq \Phi\|q_n - p^*\|. \tag{47}$$

From (45) and (47), we can conclude that:

$$\begin{aligned}
\|p_{n+1} - p^*\| &= \|(1 - a_n - b_n)q_n + a_n[q_n - \phi(q_n) \\
&\quad + \Pi_{\mathcal{K}(q_n)}(\phi(q_n) - \varrho\theta(q_n))] + b_n\omega_n - p^*\| \\
&\leq (1 - a_n - b_n)\|q_n - p^*\| + a_n\|q_n - \phi(q_n) \\
&\quad + \Pi_{\mathcal{K}(q_n)}(\phi(q_n) - \varrho\theta(q_n)) - p^*\| + b_n\|\omega_n - p^*\| \\
&\leq (1 - a_n - b_n + a_n\Phi)\|q_n - p^*\| + b_n\|\omega_n - p^*\|.
\end{aligned} \tag{48}$$

Since $\{a_n\}$ is in $(0,1)$, then for all $n \in \mathbb{N}$, one can find a constant $a \in \mathbb{R}$, such that $a \leq a_n$. It follows from (20) that $\Phi < 1$ and utilizing (43), (44), we can write:

$$\begin{aligned}
\|p_{n+1} - p^*\| &\leq (1 - a_n - b_n + a_n\Phi)\|q_n - p^*\| + b_n\|\omega_n - p^*\| \\
&\leq (1 - a_n - b_n + a_n\Phi)[\|p_n - p^*\| + |\nu_n|\|p_n - p_{n-1}\|] \\
&\quad + b_n\|p_n - p^*\| + |\mu_n|\|p_n - p_{n-1}\|] \\
&\leq (1 - a_n - b_n + a_n\Phi + b_n)\|p_n - p^*\| \\
&\quad + [(1 - a_n - b_n + a_n\Phi)|\nu_n| + b_n|\mu_n|]\|p_n - p_{n-1}\| \\
&\leq (1 - a_n(1 - \Phi))\|p_n - p^*\| + [(1 - a_n - b_n + a_n\Phi)|\nu_n| \\
&\quad + b_n|\mu_n|]\|p_n - p_{n-1}\| \\
&\leq (1 - a_n(1 - \Phi))\|p_n - p^*\| + [(1 - a_n - b_n + a_n)|\nu_n| \\
&\quad + b_n|\mu_n|]\|p_n - p_{n-1}\| \\
&\leq (1 - a(1 - \Phi))\|p_n - p^*\| + (|\nu_n| + |\mu_n|)\|p_n - p_{n-1}\| \\
&\leq (1 - a(1 - \Phi))\|p_n - p^*\| + \zeta_n\|p_n - p_{n-1}\|.
\end{aligned} \tag{49}$$

From (20), we know that $\Phi < 1$ and, hence, $1 - a(1 - \Phi) < 1$ and from Lemma 5, $\zeta_n\|p_n - p_{n-1}\| \to 0$ as $n \to \infty$. Hence, from Lemma 1, we conclude that $\lim_{n\to\infty} p_n = p^*$. □

Remark 1. *It can be perceived that under the assumptions of Theorem 3, by following the approach as in Lemma 5,* $\sum_{n=1}^{\infty} \zeta_n\|p_n - p_{n-1}\|$ *is convergent.*

Corollary 1. *Suppose that the mappings* $\Pi_{\mathcal{K}(p^*)}, \theta$ *and* $\phi : \mathcal{H} \to \mathcal{H}$ *are the same and comply with all the assumptions of Theorem 1. Suppose that the sequence* $\{p_n\}$ *initiated by Algorithm 2 with the updating parameter* ζ_n *represented by (25) with* $|\mu_n| + |\nu_n| \leq \zeta_n$*, for all* $n \in \mathbb{N}$ *and the sequences* $\{a_n\}, \{b_n\}$ *are in* $(0,1)$*, such that* $0 < a_n + b_n < 1$*, for all* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} a_n = \infty$*. Let* $\{\epsilon_n\}$ *be a sequence given in (25) such that* $\lim_{n\to\infty} \epsilon_n = 0$*. Then, the sequence* $\{p_n\}$ *converges strongly to the unique solution* p^* *of GQVI (9).*

Proof. From (48) and (49), we have:

$$\|p_{n+1} - p^*\| \le (1 - a_n(1-\Phi))\|p_n - p^*\| + \zeta_n\|p_n - p_{n-1}\|. \tag{50}$$

It emanates from Remark 1 that $\sum_{n=1}^{\infty} \zeta_n \|p_n - p_{n-1}\| < \infty$ and $1 - a_n(1-\Phi) < 1$. From the assumption, we can write $\sum_{n=1}^{\infty} a_n(1-\Phi) = \infty$. Hence, from Lemma 2, we have $\lim_{n\to\infty} p_n = p^*$. □

Example 1. *Let* $l_2 = \Big\{p = (p_0, p_1, p_2, \cdots) : \sum_{n=0}^{\infty} |p_n|^2 < \infty, p_n \in \mathbb{R}, \forall n \in \mathbb{N} \cup \{0\}\Big\}$ *be a real Hilbert space equipped with norm* $\|p\|_2 = \Big(\sum_{n=0}^{\infty} |p_n|^2\Big)^{1/2}$. *Define the mappings* $\theta, \phi : \mathcal{H} \to \mathcal{H}$ *by*

$$\theta(p) = \frac{p}{2}, \phi(p) = \frac{2}{3}p, \forall p \in \mathcal{H}.$$

Then, for all $p, q \in \mathcal{H}$, *one can observe that:*

$$\langle p - q, \phi(p) - \phi(q)\rangle = \langle p - q, \frac{2}{3}p - \frac{2}{3}q\rangle = \frac{2}{3}\|p - q\|_2^2,$$

$$\|\phi(p) - \phi(q)\|_2 = \|\frac{2}{3}p - \frac{2}{3}q\|_2 = \frac{2}{3}\|p - q\|_2.$$

Thus, the mapping ϕ *is* $\frac{2}{3}$*-strongly monotone and* $\frac{2}{3}$*-Lipschitz continuous. Additionally,*

$$\langle p - q, \theta(p) - \theta(q)\rangle = \langle p - q, \frac{p}{2} - \frac{q}{2}\rangle \ge -\frac{1}{4}\|\theta(p) - \theta(q)\|_2^2 + \frac{1}{4}\|p - q\|_2^2,$$

$$\|\theta(p) - \theta(q)\|_2 = \Big\|\frac{p}{2} - \frac{p}{2}\Big\|_2 = \frac{1}{2}\|p - q\|_2.$$

Thus, the mapping θ *is relaxed* $\Big(\frac{1}{4}, \frac{1}{4}\Big)$*-cocoercive and* $\frac{1}{2}$*-Lipschitz continuous. Next, we define a set-valued mapping* $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ *by*

$$\mathcal{K}(p) = \mathcal{K}(\{p_n\}) = \Big\{u = \{u_n\} : u_0 \ge \frac{p_0}{15}, u_n = 0, \text{ for all } n \in \mathbb{N}\Big\}.$$

We claim that $\mathcal{K}(p)$ *is a closed convex set. Indeed, for any arbitrary* $\alpha \in [0,1]$ *and* $u_0, v_0 \in \mathcal{K}(p)$, *we have* $\alpha u_0 + (1-\alpha)v_0 \ge \frac{p_0}{15}$ *and, hence,* $\mathcal{K}(p)$ *is a convex set. Now, we define* $g : [\frac{p_0}{15}, \infty) \to \mathcal{K}(p)$ *by* $g(r) = (r, 0, 0, \cdots)$. *Evidently,* g *is well defined. In point of fact, for distinct* $u, v \in [\frac{p_0}{15}, \infty)$, *we have* $(u, 0, 0, \cdots) \ne (v, 0, 0, \cdots)$, *i.e.,* g *is injective. It is easy to see that there exist an* $u_0 \in [\frac{p_0}{15}, \infty)$ *so that* $g(u_0) = (u_0, 0, 0, \cdots)$ *for each* $u = (u_0, 0, 0, \cdots) \in \mathcal{K}(p)$, *i.e.,* g *is surjective. Let* (l_2, d_2) *and* $(\mathbb{R}, d)$ *be usual metric spaces, then for each* $u, v \in [\frac{p_0}{15}, \infty)$, *we obtain:*

$$d_2(g(u), g(v)) = d_2((u, 0, 0, \cdots), (v, 0, 0, \cdots)) = |u - v| = d(u, v).$$

Thus g *is continuous. Moreover,* g^{-1} *is also continuous and bijective and hence* g *is a homeomorphism. Being homeomorphic to a closed set* $[\frac{p_0}{15}, \infty)$, $\mathcal{K}(p)$ *is also closed. Define metric projection* $\Pi_{\mathcal{K}(p)} : \mathcal{H} \to \mathcal{K}(p)$ *by:*

$$\Pi_{\mathcal{K}(p)}(l_0, l_1, l_2, \cdots) = \begin{cases} (l_0, l_1, l_2, \cdots), & \text{if } (l_0, l_1, l_2, \cdots) \in \mathcal{K}(p) \\ (\frac{p_0}{15}, 0, 0, \cdots), & \text{if } (l_0, l_1, l_2, \cdots) \notin \mathcal{K}(p), l_0 < \frac{p_0}{15} \\ (l_0, 0, 0, \cdots), & \text{if } (l_0, l_1, l_2, \cdots) \notin \mathcal{K}(p), l_0 \ge \frac{p_0}{15}. \end{cases}$$

To show that the projection $\Pi_{\mathcal{K}(p)}$ *satisfies the assumption C, we discuss the following cases. Case* (a). *For arbitrary* $p = \{p_n\}, q = \{q_n\}, l = \{l_n\} \in \mathcal{H}$, *suppose that* $p_0 \le q_0$.

1. *If* $l = \{l_n\} \in \mathcal{K}(q)$*, then* $l = \{l_n\} \in \mathcal{K}(p)$ *and, hence:*

$$\begin{aligned}\|\Pi_{\mathcal{K}(p)}(l) - \Pi_{\mathcal{K}(q)}(l)\|_2 &= \|(l_0, l_1, l_2, \cdots) - (l_0, l_1, l_2, \cdots)\|_2 \\ &= 0 \le \frac{1}{15}\|p - q\|_2.\end{aligned}$$

2. *If* $l = (l_0, l_1, l_2, \cdots) \notin \mathcal{K}(q)$ *and* $l = (l_0, l_1, l_2, \cdots) \in \mathcal{K}(p)$*, then either* $l_0 < \frac{q_0}{15}$ *or* $l_0 \ge \frac{q_0}{15}$*. For* $l_0 < \frac{q_0}{15}$*, we have:*

$$\begin{aligned}\|\Pi_{\mathcal{K}(p)}(l) - \Pi_{\mathcal{K}(q)}(l)\|_2 &= \|(l_0, l_1, l_2, \cdots) - \left(\frac{p_0}{15}, 0, 0, \cdots\right)\|_2 \\ &= \|(l_0, 0, 0, \cdots) - \left(\frac{p_0}{15}, 0, 0, \cdots\right)\|_2 \\ &= \left|l_0 - \frac{p_0}{15}\right| = l_0 - \frac{p_0}{15} \\ &\le \frac{1}{15}\|p - q\|_2.\end{aligned}$$

For $l_0 \ge \frac{q_0}{15}$*, we have:*

$$\begin{aligned}\|\Pi_{\mathcal{K}(p)}(l) - \Pi_{\mathcal{K}(q)}(l)\|_2 &= \|(l_0, l_1, l_2, \cdots) - (l_0, 0, 0, \cdots)\|_2 \\ &= \|(l_0, 0, 0, \cdots) - (l_0, 0, 0, \cdots)\|_2 \\ &= 0 \le \frac{1}{15}\|p - q\|_2.\end{aligned}$$

3. *If* $l = (l_0, l_1, l_2, \cdots) \notin \mathcal{K}(q)$ *and* $l = (l_0, l_1, l_2, \cdots) \notin \mathcal{K}(p)$*, then either* $l_0 < \frac{q_0}{15}$ *or* $l_0 \ge \frac{q_0}{15}$ *and* $l_0 < \frac{p_0}{15}$ *or* $l_0 \ge \frac{p_0}{15}$*. For* $l_0 < \frac{p_0}{15}$*, we have* $l_0 < \frac{q_0}{15}$*. Therefore, we can conclude that*

$$\begin{aligned}\|\Pi_{\mathcal{K}(p)}(l) - \Pi_{\mathcal{K}(q)}(l)\|_2 &= \left\|\left(\frac{p_0}{15}, 0, 0, \cdots\right) - \left(\frac{q_0}{15}, 0, 0, \cdots\right)\right\|_2 \\ &= \left|\frac{p_0}{15} - \frac{q_0}{15}\right| \le \frac{1}{15}\|p - q\|_2.\end{aligned}$$

For $l_0 \ge \frac{p_0}{15}$ *and* $l_0 \ge \frac{q_0}{15}$*, we have*

$$\begin{aligned}\|\Pi_{\mathcal{K}(p)}(l) - \Pi_{\mathcal{K}(q)}(l)\|_2 &= \|(l_0, 0, 0, \cdots) - (l_0, 0, 0, \cdots)\|_2 \\ &= 0 \le \frac{1}{15}\|p - q\|_2.\end{aligned}$$

For $l_0 \ge \frac{p_0}{15}$ *and* $l_0 < \frac{q_0}{15}$*, we have*

$$\begin{aligned}\|\Pi_{\mathcal{K}(p)}(l) - \Pi_{\mathcal{K}(q)}(l)\|_2 &= \|(l_0, 0, 0, \cdots) - \left(\frac{q_0}{10}, 0, 0, \cdots\right)\|_2 \\ &= \left|l_0 - \frac{q_0}{15}\right| = \frac{q_0}{15} - l_0 \le \frac{q_0}{15} - \frac{p_0}{15} \le \frac{1}{15}\|p - q\|_2.\end{aligned}$$

Case (b)*. Similarly, for arbitrary* $p = \{p_n\}, q = \{q_n\}, r = \{l_n\} \in \mathcal{H}$ *with* $p_0 > q_0$*, we can verify that*

$$\|\Pi_{\mathcal{K}(p)}(l) - \Pi_{\mathcal{K}(q)}(l)\|_2 \le \frac{1}{15}\|p - q\|_2.$$

Thus, we conclude that the projection $\Pi_{\mathcal{K}(p)}$ *satisfies the assumption C with constant* $\kappa = \frac{1}{15}$*. Additionally, the condition (20) is satisfied for* $\eta = \gamma = \frac{2}{3}, \sigma = \tau = \frac{1}{4}, \varsigma = \frac{1}{2}$*, and* $\varrho = 1$*. Finally,* *we shall find a unique point* $p^* \in \mathcal{K}(p^*)$ *which solves GQVI (9). Consider* $p^* = (p_0^*, 0, 0, \cdots) :$ $p_0^* \geq 0$, *then for* $p^* > 0$, *we have:*

$$\begin{aligned}\langle \theta(p^*), \phi(q^*) - \phi(p^*) \rangle &= \langle \frac{p^*}{2}, \frac{2}{3}(q^*) - \frac{2}{3}(p^*) \rangle \\ &= \frac{1}{3} \langle (p_0^*, 0, 0, \cdots), (q_0^* - p_0^*, 0, 0, \cdots) \rangle \\ &< 0, \forall q^* = (q_0^*, 0, 0, \cdots) \in \mathcal{K}(p^*).\end{aligned}$$

However, for $p^* = (0, 0, 0, \cdots)$, *we have:*

$$\langle \theta(p^*), \phi(q^*) - \phi(p^*) \rangle = \langle (0, 0, 0, \cdots), (q_0^* - p_0^*, 0, 0, \cdots) \rangle = 0, \forall q^* = (q_0^*, 0, 0, \cdots) \in \mathcal{K}(p^*).$$

Hence, $p^* = (0, 0, 0, \ \cdots) \in \mathcal{K}(p^*)$ *is a unique solution of GQVI (9). Consider a sequence* $\{p^{(n)}\} \in l_2$ *and for all* $n \in \mathbb{N}$, *take* $\mu_n = \frac{n}{6(n+2)}, \nu_n = \frac{n}{5(n+1)}, a_n = \frac{n+7}{2(n+5)}, b_n = \frac{n-1}{4(n+6)}$ *with initial points* $p^{(0)} = (\frac{1}{2}, \frac{1}{6}, 0, 0, \ \cdots), p^{(1)} = (1, \frac{1}{3}, 0, 0, \ \cdots)$ *in Algorithm 2. Table 1 shows that the scheme 2 converges faster with initial terms than that of with* $\mu_n = \nu_n = 0$.

Table 1. Numerical comparison of iterative Algorithm 2 with inertial and non-inertial terms.

No. Iter.	x_n(*with* $\mu_n = \frac{n}{6(n+2)}$, $v_n = \frac{n}{5(n+n)}$)	$\|x_n\|_2$	x_n (*with* $\mu_n = v_n = 0$)	$\|x_n\|_2$
0	$(\frac{1}{2}, \frac{1}{6}, 0, 0, \ldots)$	0.527046268	$(\frac{1}{2}, \frac{1}{6}, 0, 0, \ldots)$	0.52704268
1	$(1, \frac{1}{3}, 0, 0, \ldots)$	1.054092481	$(1, \frac{1}{3}, 0, 0, \ldots)$	1.054092481
2	(0.411111101, 0.114197529, 0, 0, ...)	0.426676275	(0.411111221, 0.114199632, 0, 0, ...)	0.586937317
⋮	⋮	⋮	⋮	⋮
25	$(1.56798 \times 10^{-11}, 2.80294 \times 10^{-16}, 0, 0, \ldots)$	3.53429×10^{-12}	$(9.40659 \times 10^{-11}, 1.99635 \times 10^{-14}, 0, 0, \ldots)$	9.40267×10^{-11}
⋮	⋮	⋮	⋮	⋮
110	$(5.23109 \times 10^{-91}, 6.59583 \times 10^{-118}, 0, 0, \ldots)$	2.40527×10^{-47}	$(2.14532 \times 10^{-43}, 1.40867 \times 10^{-57}, 0, 0, \ldots)$	2.10462×10^{-43}
⋮	⋮	⋮	⋮	⋮
220	$(3.1594 \times 10^{-147}, 2.0037 \times 10^{-192}, 0, 0, \ldots)$	8.19276×10^{-92}	$(1.08984 \times 10^{-84}, 1.10363 \times 10^{-86}, 0, 0, \ldots)$	1.09486×10^{-84}

5. Application to Delay Differential Equation

In this section, we make use of the inertial iterative Algorithm 2 to find an approximate solution of the delay differential equation stated as under:

$$\begin{cases} \omega'(s) = h(s, \omega(s), \omega(s-\tau)), & s \in [s_0, a_1], \\ \omega(s) = \vartheta(s), & s \in [s_0 - \tau, s_0]. \end{cases} \tag{51}$$

Let $C[a_0, a_1]$ denotes the space of all continuous real-valued functions defined on $[a_0, a_1]$ equipped with the Chebyshev norm $\|p\|_\infty = \max_{s \in [a_0, a_1]} |p(s)|$. From classical analysis, observe that $(C[a_0, a_1], \| \cdot \|_\infty)$ is a Banach space. Suppose the following assumptions are fulfilled.

(C_1) $s_0, a_1 \in \mathbb{R}, \tau > 0$;
(C_2) $h \in C([s_0, a_1] \times \mathbb{R}^2, \mathbb{R})$;
(C_3) $\vartheta \in C([s_0 - \tau, a_1], \mathbb{R})$;
(C_4) there exists $L_h > 0$ so that
$|h(s, m_1, m_2) - h(s, n_1, n_2)| \leq L_h \sum_{i=1}^2 |m_i - n_i|$, for all $m_i, n_i \in \mathbb{R}, s \in [s_0, a_1]$;
(C_5) $2L_h(a_1 - s_0) < 1$.

Suppose that the solution $p^* \in C([s_0 - \tau, a_1], \mathbb{R}) \cap C^1([s_0, a_1], \mathbb{R})$ of the problem (51) exists. Then it can be modelled as being a solution of the following integral equation.

$$\omega(s) = \begin{cases} \vartheta(s), & s \in [s_0 - \tau, s_0], \\ \vartheta(s_0) + \int_{s_0}^{s} h(r, \omega(r), \omega(r - \tau)) dr, & s \in [s_0, a_1]. \end{cases} \tag{52}$$

Next, we present convergence of our inertial iterative algorithm to look over the solution of Problem (51).

Theorem 4. *Suppose the assumptions* (C_1)–(C_5) *are fulfilled. Let* $\{\epsilon_n\}$ *be a sequence given in (25), such that* $\lim_{n\to\infty} \epsilon_n = 0$. *Let* $\{p_n\}$ *be a sequence initiated by (43)–(45) with the updating parameter* ζ_n *represented by (25) with* $|\mu_n| + |\nu_n| \le \zeta_n$, *for all* $n \in \mathbb{N}$ *and* $\{a_n\}, \{b_n\} \in (0,1)$, *such that* $0 < a_n + b_n < 1$, *for all* $n \in \mathbb{N}$. *Then,* $\{p_n\}$ *converges to the unique solution* $p^* \in C([s_0 - \tau, a_1], \mathbb{R}) \cap C^1([s_0, a_1], \mathbb{R})$ *of Problem (51).*

Proof. Define the operator $\psi(\omega(s)) = \begin{cases} \vartheta(s), & s \in [s_0 - \tau, s_0], \\ \vartheta(s_0) + \int_{s_0}^{s} h(r, \omega(r), \omega(r-\tau))dr, & s \in [s_0, a_1], \end{cases}$
where $\{a_n\}$ and $\{b_n\}$ are sequences in (0,1), such that $\sum_{k=0}^{\infty} a_k b_k = \infty$. From (17) and (19), it follows that p^* is a fixed point of ψ. In addition, let $\{p_n\}$ be a sequence generated by (45). Then, for $s \in [s_0, a_1]$, we have:

$$\begin{aligned}
&\|p_{n+1} - p^*\|_\infty \\
&= \|(1 - a_n - b_n)q_n + a_n\psi(q_n) + b_n\omega_n - p^*\|_\infty \\
&\le (1 - a_n - b_n)\|q_n - p^*\|_\infty + a_n\|\psi(q_n) - \psi(p^*)\|_\infty + b_n\|\omega_n - p^*\|_\infty \\
&= (1 - a_n - b_n)\|q_n - p^*\|_\infty + a_n \max_{s\in[s_0-\tau,a_1]} |\psi q_n(s) - \psi p^*(s)| + b_n\|\omega_n - p^*\|_\infty \\
&\le (1 - a_n - b_n)\|q_n - p^*\|_\infty + a_n \max_{s\in[s_0-\tau,a_1]} \Big|\vartheta(s_0) + \int_{s_0}^{s} h(r, q_n(r), q_n(r-\tau))dr \\
&\quad - \vartheta(s_0) - \int_{s_0}^{s} h(r, p^*(r), p^*(r-\tau))dr\Big| + b_n\|\omega_n - p^*\|_\infty \\
&\le (1 - a_n - b_n)\|q_n - p^*\|_\infty + a_n \max_{s\in[s_0-\tau,a_1]} \int_{s_0}^{s} |h(r, q_n(r), q_n(r-\tau)) \\
&\quad - h(r, p^*(r), p^*(r-\tau))|dr + b_n\|\omega_n - p^*\|_\infty \\
&\le (1 - a_n - b_n)\|q_n - p^*\|_\infty + a_n \int_{s_0}^{s} L_h\Big(\max_{s\in[s_0-\tau,a_1]} |q_n(r) - p^*(r)| \\
&\quad + \max_{s\in[s_0-\tau,a_1]} |q_n(r-\tau) - p^*(r-\tau)|\Big)dr + b_n\|\omega_n - p^*\|_\infty \\
&\quad \le (1 - a_n - b_n)\|q_n - p^*\|_\infty + a_n L_h\Big(\max_{s\in[s_0-\tau,a_1]} |q_n(r) - p^*(r)| \\
&\qquad + \max_{s\in[s_0-\tau,a_1]} |q_n(r-\tau) - p^*(r-\tau)|\Big) \int_{s_0}^{s} dr + b_n\|\omega_n - p^*\|_\infty \\
&\quad \le (1 - a_n - b_n)\|q_n - p^*\|_\infty + 2a_n L_h(a_1 - s_0)\|q_n - p^*\|_\infty + b_n\|\omega_n - p^*\|_\infty \\
&\quad = (1 - a_n - b_n + 2a_n L_h(a_1 - s_0))\|q_n - p^*\|_\infty + b_n\|\omega_n - p^*\|_\infty.
\end{aligned} \tag{53}$$

Since $\{a_n\}, \{b_n\}$ are sequences in $(0,1)$ and from assumption (C_5), we know that $2L_h(a_1 - s_0) < 1$. Then, by utilizing (43) and (44), (53) becomes:

$$\begin{aligned}
\|p_{n+1} - p^*\| &\le (1 - a_n - b_n + 2a_n L_h(a_1 - s_0))\|q_n - p^*\|_\infty + b_n\|\omega_n - p^*\|_\infty \\
&\le (1 - a_n - b_n + 2a_n L_h(a_1 - s_0))[\|p_n - p^*\| + |\nu_n|\|p_n - p_{n-1}\|] \\
&\quad + b_n\|p_n - p^*\| + |\mu_n|\|p_n - p_{n-1}\|] \\
&\le (1 - a_n - b_n + 2a_n L_h(a_1 - s_0) + b_n)\|p_n - p^*\| \\
&+ [(1 - a_n - b_n + 2a_n L_h(a_1 - s_0))|\nu_n| + b_n|\mu_n|]\|p_n - p_{n-1}\| \\
&\le (1 - a_n + 2a_n L_h(a_1 - s_0))\|p_n - p^*\| \\
&\quad + [(1 - a_n - b_n + a_n)|\nu_n| + b_n|\mu_n|]\|p_n - p_{n-1}\| \\
&\le (1 - a_n(1 - 2L_h(a_1 - s_0)))\|p_n - p^*\| + (|\nu_n| + |\mu_n|)\|p_n - p_{n-1}\| \\
&\le (1 - a_n(1 - 2L_h(a_1 - s_0)))\|p_n - p^*\| + \zeta_n\|p_n - p_{n-1}\|.
\end{aligned} \tag{54}$$

From (C_5), we know that $2L_h(a_1 - s_0) < 1$ and hence $1 - a_n(1 - 2L_h(a_1 - s_0)) < 1$ and from Lemma 5, $\zeta_n \|p_n - p_{n-1}\| \to 0$ as $n \to \infty$. Hence, from Lemma 1, we conclude that $\lim_{n\to\infty} p_n = p^*$. $\square$

6. Concluding Remarks

In this paper, new inertial iterative algorithms have been constructed and their convergence analysis is considered in order to approximate solutions of general quasi-variational inequalities. The existence result of vectors satisfying a general quasi-variational inequality is proved and verified by illustrative example. Finally, as an application of the two steps inertial iterative algorithm, we examined a delay differential equation.

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