



## Article

# Study of a Fractional Creep Problem with Multiple Delays in Terms of Boltzmann's Superposition Principle

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**Abstract:** We study a class of nonlinear fractional differential equations with multiple delays, which is represented by the Voigt creep fractional model of viscoelasticity. We discuss two Voigt models, the first being linear and the second being nonlinear. The linear Voigt model give us the physical interpretation and is associated with important results since the creep function characterizes the viscoelastic behavior of stress and strain. For the nonlinear model of Voigt, our theoretical study and analysis provides existence and stability, where time delays are expressed in terms of Boltzmann's superposition principle. By means of the Banach contraction principle, we prove existence of a unique solution and investigate its continuous dependence upon the initial data as well as Ulam stability. The results are illustrated with an example.

**Keywords:** fractional differential equations; creep function; Ulam stability; fixed-point theorems

**MSC:** 26A33; 34A12; 47H10



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## 1. Introduction

In recent years, fractional calculus has attracted the attention of many researchers [1–7]. Such a research axis is of great importance in many fields, especially when dealing with memory or hereditary properties, such as in viscoelastic phenomena: for example, stress–strain in polymeric materials [8–10].

To describe the behavior of materials and show their viscoelastic properties, one uses rheological models, which are of Voigt or Maxwell type or a combination of these basic models; see [11–15] and the references therein. On the other hand, there are many materials that are difficult to describe in rheological models that contain finite elements of elastic and viscous components. For that reason, it is common to resort to the use of fractional models, which give us the opportunity to use few elements and, at the same time, give us an accurate description. In order to understand the problem at hand, and as an initial idea, let us recall here the simplest form of the equation that governs the creep phenomenon:

$$\eta x'(t) + Ex(t) = \varphi(t), \quad x(t_0) = x_0, \quad (1)$$

where  $\eta$  is the viscosity coefficient and  $E$  is the modulus of the elasticity. In (1),  $t_0$  should be chosen in a way that for  $t < t_0$ , the material is at rest, without stress and strain. For a given stress history  $\varphi$ , the strain  $x$  is expressed by

$$x(t) = \int_{t_0}^t e^{-\frac{t-s}{\tau}} \varphi(s) ds, \quad \tau = \frac{\eta}{E}, \quad (2)$$

where the constant  $\tau$  is called the retardation time. The creep function associated with problem (1) is then given by

$$k(t - t_0) = \frac{1}{E} \left( 1 - \exp\left(-\frac{t - t_0}{\tau}\right) \right), \quad t \geq t_0, \tag{3}$$

which is, in general, a completely monotonic function; see, e.g., [12,14]. This means that the viscoelastic function  $k(t)$  must satisfy the inequalities

$$(-1)^n \frac{d^n}{dt^n} k(t) \geq 0, \quad n = 1, 2, \dots, \tag{4}$$

assuring its monotonicity. It is noticeable from Formulas (2) and (3) that for a given stress, the response is not instantaneous and the strain takes time. This is well apparent when dealing with viscoelastic phenomena. As fractional operators are in general integral operators with a singular kernel, involving a time delay makes those operators the best for modeling such rheological phenomena: see [16–18] and references therein.

Here, we study a more general creep problem, based on the Voigt model, that satisfies Boltzmann’s superposition as expressed in the second terms of the following nonlinear fractional differential equation:

$${}^C D_{0+}^\alpha x(t) + \lambda x(t) = \sum_{j=1}^n b_j(t) g_j(x(t - \tau_j)), \quad t \in [0, T], \tag{5}$$

where  ${}^C D_{0+}^\alpha$  denotes the Caputo derivative of order  $0 < \alpha < 1$ ;  $\lambda$  is a real positive constant;  $b_j, g_j$  are given functions; and  $\tau_j$  represents the time delays. We take

$$x(t) = \psi(t), \quad t \in [-v, 0], \tag{6}$$

where  $0 < v = \max_{j=1, \dots, n} \{\tau_j\} \leq T$  and  $x(0) = \psi(0) = 0$ .

A large number of mathematicians obtained several results on the existence, uniqueness, and stability for equations with delays of the type described from (1) to (4) [19–26]. Motivated by these papers, we obtain here sufficient conditions for the stability of our fractional delayed differential problem (5) and (6). Before that, we prove the existence and uniqueness of the solution, which is crucial to give a physical meaning to our problem. This contrasts with previous works, which do not combine a comprehensive mathematical analysis with physical interpretation.

The manuscript is structured as follows. In Section 2, we give some fundamental results, which will be used throughout the paper. In particular, we obtain an integral representation of our problem and then extract the fractional creep function associated with the linear problem. In Section 3, we use the Banach contraction principle to show the existence and uniqueness of the solution and then establish its continuous dependence with the initial data. In Section 4, we study the Ulam stability, including an illustrative example. We end with conclusions in Section 5.

## 2. Fundamental Results on the Linear Problem

We begin by recalling the definition of the Mittag–Leffler function, which is an important tool in fractional calculus and will have an important role in our paper.

**Definition 1.** The two-parameter Mittag–Leffler function is defined by the series expansion

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z, \alpha, \beta \in \mathbb{C} \text{ with } \operatorname{Re} \alpha > 0, \tag{7}$$

where  $\Gamma(\cdot)$  is Euler’s gamma function. In particular,  $E_\alpha(z) = E_{\alpha, 1}(z)$ .

Schneider in [27] proved that the generalized Mittag–Leffler function  $E_{\alpha,\beta}(-t)$  with  $t \geq 0$  is completely monotonic if and only if  $0 < \alpha \leq 1$  and  $\beta \geq \alpha$ . In other words,

$$(-1)^n \frac{d^n}{dt^n} E_{\alpha,\beta}(-t) \geq 0, \quad (8)$$

for all  $n = 0, 1, 2, \dots$ . Note that this property generalizes (4).

From (8), we can see that for  $0 < \alpha \leq 1$  and  $\beta \geq \alpha$ ,

$$E_{\alpha,\beta}(-t) \leq \frac{1}{\Gamma(\beta)}, \quad t \geq 0, \quad (9)$$

and the above estimation (9) will enable us to establish our results.

In the sequel, we use the notation  $e_{\alpha}^{\lambda t}$  for the  $\alpha$ -exponential function:

$$e_{\alpha}^{\lambda t} := t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}). \quad (10)$$

**Lemma 1** (See [28]). *Let  $\alpha, \lambda \in \mathbb{C}$  and  $\operatorname{Re}(\alpha) > 0$ . Then,*

$$\int_0^t e_{\alpha}^{\lambda t} dt = t^{\alpha} E_{\alpha,\alpha+1}(\lambda t^{\alpha}). \quad (11)$$

Now, we consider our linear fractional problem as the fractional Voigt model

$$\begin{cases} {}^C D_{0+}^{\alpha} x(t) + \lambda x(t) = \varphi(t), & t > 0, \quad \lambda > 0, \\ x(0) = 0. \end{cases} \quad (12)$$

Viscoelastic phenomena can be expressed with integral equations or differential ones. While differential equations are related to rheological models, which provide a more direct physical interpretation of the viscoelastic behavior, the integral equations are more general and suitable for theoretical work. Indeed, using the Laplace transform, we can convert our linear problem (12) into a Volterra integral equation as follows:

$$x(t) = \int_0^t e_{\alpha}^{-\lambda(t-s)} \varphi(s) ds. \quad (13)$$

**Theorem 1.** *Suppose that the stress  $\varphi$  of the fractional initial value problem (12) is a continuous function on  $[0, T]$ . Then, (13) is the strain and the continuous solution of (12). Moreover, (12) and (13) are equivalent in  $C[0, T]$ .*

**Proof.** Since  $\varphi$  is a continuous function on  $[0, T]$ , then, from ([Theorem 3.24] of [29]), problem (12) is equivalent in  $C[0, 1]$  to the following Volterra integral equation of second kind:

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds. \quad (14)$$

Now, we apply the successive approximation method to solve (14). Let

$$x_0(t) = I^{\alpha} \varphi = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds. \quad (15)$$

Substituting (15) into (14), we obtain that

$$\begin{aligned} x_1 &= I^{\alpha} \varphi - \lambda I^{\alpha} x_0 \\ &= I^{\alpha} \varphi - \lambda I^{2\alpha} \varphi, \end{aligned}$$

and

$$\begin{aligned} x_2 &= I^\alpha \varphi - \lambda I^\alpha x_1 \\ &= I^\alpha \varphi - \lambda I^{2\alpha} \varphi + \lambda^2 I^{3\alpha} \varphi. \end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned} x_m(t) &= \frac{1}{\Gamma(k\alpha + \alpha)} \int_0^t (t-s)^{k\alpha + \alpha - 1} \sum_{k=0}^m (-\lambda)^k \varphi(s) ds \\ &= \int_0^t (t-s)^{\alpha-1} \sum_{k=0}^m \frac{(t-s)^{k\alpha}}{\Gamma(k\alpha + \alpha)} (-\lambda)^k \varphi(s) ds, \end{aligned}$$

and, passing to the limit, as  $m \rightarrow \infty$ ,

$$\begin{aligned} x(t) &= \int_0^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{(t-s)^{k\alpha}}{\Gamma(k\alpha + \alpha)} (-\lambda)^k \varphi(s) ds \\ &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \varphi(s) ds \\ &= \int_0^t e_\alpha^{-\lambda(t-s)} \varphi(s) ds. \end{aligned}$$

The proof is complete.  $\square$

**Remark 1.** We draw attention to the fact that the solution to Equation (1) is described by the exponential function while the solution to (12) is expressed in terms of the  $\alpha$ -exponential function, which is a generalization of (2) and plays an important role here.

Now, we look to the creep function  $k_\alpha(t)$  and give the following result.

**Theorem 2.** The creep function associated with the fractional problem (12) is given by

$$k_\alpha(t) = t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha). \tag{16}$$

**Proof.** We use the stress–strain relation

$$x(t) = \int_0^t k_\alpha(t-s) \varphi'(s) ds, \tag{17}$$

which is known as the Boltzmann–Volterra equation. Integrating by parts (13) and using (11), we obtain

$$\begin{aligned} x(t) &= \int_0^t e_\alpha^{-\lambda(t-s)} \varphi(s) ds \\ &= t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) \varphi(0) + \int_0^t (t-s)^\alpha E_{\alpha,\alpha+1}(-\lambda(t-s)^\alpha) \varphi'(s) ds, \end{aligned}$$

from which we can write that

$$k_\alpha(t) = t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha). \tag{18}$$

The proof is complete.  $\square$

Taking  $\lambda = \frac{E}{\eta}$  and evaluating the creep function (18), we obtain

$$\begin{aligned} k_\alpha(t) &= \frac{1}{\eta} t^\alpha E_{\alpha,\alpha+1} \left( -\frac{1}{\tau} t^\alpha \right) \\ &= \frac{1}{\eta} t^\alpha \sum_{n=0}^\infty \left( -\frac{1}{\tau} \right)^n \frac{t^{\alpha n}}{\Gamma(\alpha n + \alpha + 1)} \\ &= -\frac{\tau}{\eta} \sum_{n=1}^\infty \left( -\frac{1}{\tau} \right)^n \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)} \\ &= -\frac{1}{E} \left( E_\alpha \left( -\frac{1}{\tau} t^\alpha \right) - 1 \right), \end{aligned}$$

that is,

$$k_\alpha(t) = \frac{1}{E} \left( 1 - E_\alpha \left( -\frac{1}{\tau} t^\alpha \right) \right). \tag{19}$$

Note that for  $\alpha = 1$ , one obtains from (19) that

$$\begin{aligned} k_1(t) &= \frac{1}{E} \left( 1 - \exp \left( -\left( \frac{t}{\tau} \right) \right) \right) \\ &= k(t - t_0) \text{ for } t_0 = 0, \end{aligned}$$

which is the creep function (3) associated with problem (1).

### 3. Existence and Uniqueness Results

In this section, we consider the nonlinear problem (5) and (6) as a generalization of the classical creep problem. Here, we will rely on Boltzmann’s superposition principle [13]:

$$\varphi(t) = \sum_{j=1}^n b_j(t) g_j(x(t - \tau_j)),$$

which gives the stress as a sum of the delayed response to the strain.

We introduce the following assumptions:

**Hypothesis 1. (H1).**  $b_j : [0, T] \rightarrow \mathbb{R}$  are continuous functions with  $B_j = \sup_{t \in [0, T]} |b_j(t)|$ .

**Hypothesis 2. (H2).**  $g_j : C[-v, T] \rightarrow C[-v, T]$  are Lipschitz functions, i.e., there exists  $l_j > 0$  such that

$$\|g_j(x) - g_j(y)\|_{C[-v, T]} \leq l_j \|x - y\|_{C[-v, T]}, \quad x, y \in C[-v, T], \tag{20}$$

with  $g_j(0) \neq 0$ .

**Theorem 3.** Assume that the assumptions (H1) and (H2) hold. If

$$T^\alpha \sum_{j=1}^n B_j l_j < \Gamma(\alpha + 1), \tag{21}$$

then problem (5) and (6) has a unique solution  $x \in C([-v, T], \mathbb{R})$ .

**Proof.** The solution of problem (5) and (6) satisfies, for  $t \in [0, T]$ , the integral equation

$$x(t) = \int_0^t e_\alpha^{-\lambda(t-s)} \sum_{j=1}^n b_j(s) g_j(x(s - \tau_j)) ds, \tag{22}$$

and

$$x(t) = \psi(t) \text{ for } t \in [-v, 0] \text{ with } x(0) = \psi(0) = 0. \tag{23}$$

Let  $\psi$  be a continuous function and denote

$$X = \left\{ x \in C([-v, T], \mathbb{R}) : x|_{[-v, 0]} = \psi \right\},$$

which is a Banach space endowed with the sup-norm

$$\|\gamma\|_{C[-v, T]} = \sup_{t \in [-v, T]} |\gamma(t)|,$$

and where  $x|_{[-v, 0]}$  is the restriction of the function  $x$  on  $[-v, 0]$ . Taking into account Theorem 1 and the assumptions (H1)–(H2), we define the operator  $P : X \rightarrow X$  by

$$(Px)(t) = \begin{cases} \psi(t), & t \in [-v, 0], \\ 0, & t = 0, \\ \int_0^t e_\alpha^{-\lambda(t-s)} \sum_{j=1}^n b_j(s) g_j(x(s - \tau_j)) ds, & t \in [0, T]. \end{cases}$$

Now, using the contraction principle mapping of Banach, we investigate the existence and uniqueness of the fixed point of the operator  $P$  in  $X$ . In fact, for  $x, y \in X$ , one has

$$\begin{aligned} |Px(t) - Py(t)| &\leq \int_0^t e_\alpha^{-\lambda(t-s)} \sum_{j=1}^n |b_j(s)| l_j |x(s - \tau_j) - y(s - \tau_j)| ds \\ &\leq \sum_{j=1}^n \int_0^t e_\alpha^{-\lambda(t-s)} |b_j(s)| l_j |x(s - \tau_j) - y(s - \tau_j)| ds \\ &\leq \sum_{j=1}^n \int_{-\tau_j}^0 e_\alpha^{-\lambda(t-z-\tau_j)} |b_j(s)| l_j |x(z) - y(z)| dz \\ &\quad + \sum_{j=1}^n \int_0^{t-\tau_j} e_\alpha^{-\lambda(t-z-\tau_j)} |b_j(s)| l_j |x(z) - y(z)| dz. \end{aligned}$$

As  $x(t) = y(t) = \psi(t)$  for  $t \in [-v, 0]$ , we have

$$|Px(t) - Py(t)| \leq \sum_{j=1}^n \int_0^{t-\tau_j} e_\alpha^{-\lambda(t-z-\tau_j)} |b_j(s)| l_j |x(z) - y(z)| dz,$$

and, using (9) and (11), we obtain

$$\|Px - Py\|_X \leq \left( \frac{T^\alpha}{\Gamma(\alpha + 1)} \sum_{j=1}^n B_j l_j \right) \|x - y\|_X,$$

which shows by (21) that  $P$  is a contraction in  $X$ . Thus, from the Banach fixed point theorem, we conclude that  $P$  has a unique fixed point in  $C[-v, T]$ , which is the unique solution of (5) and (6) in  $C([-v, T], \mathbb{R})$ .  $\square$

As a consequence of Theorem 3, we prove the continuous dependence of the solution with respect to the initial data of the problem.

**Corollary 1.** *Under the conditions of Theorem 3, the unique solution of (5) and (6) depends continuously on function  $\psi(t)$ .*

**Proof.** Let  $x_1$  and  $x_2$  be solutions of Equations (5) and (6) corresponding to the initial data  $\psi_1(t)$  and  $\psi_2(t)$ , respectively. Then,

$$|x_1(t) - x_2(t)| = \int_0^t e^{-\lambda(t-s)} \sum_{j=1}^n |b_j(s)| l_j |x_1(s - \tau_j) - x_2(s - \tau_j)| ds.$$

Putting  $(s - \tau_j) = z$ , we obtain

$$\begin{aligned} |x_1(t) - x_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n B_j l_j \int_{-\tau_j}^{t-\tau_j} (t - \tau_j - z)^{\alpha-1} |x_1(z) - x_2(z)| dz \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n B_j l_j \int_{-\tau_j}^0 (t - \tau_j - z)^{\alpha-1} |x_1(z) - x_2(z)| dz \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n B_j l_j \int_0^{t-\tau_j} (t - \tau_j - z)^{\alpha-1} |x_1(z) - x_2(z)| dz \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n B_j l_j \sup_{z \in [-v, 0]} |x_1(z) - x_2(z)| \int_{-\tau_j}^0 (t - \tau_j - z)^{\alpha-1} dz \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n B_j l_j \sup_{z \in [0, T]} |x_1(z) - x_2(z)| \int_0^{t-\tau_j} (t - \tau_j - z)^{\alpha-1} dz \\ &\leq \frac{1}{\Gamma(\alpha + 1)} \sum_{j=1}^n (t^\alpha - (t - \tau_j)^\alpha) B_j l_j \|\psi_1 - \psi_2\|_{C[-v, 0]} \\ &\quad + \frac{1}{\Gamma(\alpha + 1)} \sum_{j=1}^n (t - \tau_j)^\alpha B_j l_j \|x_1 - x_2\|_{C[0, T]}. \end{aligned}$$

Consequently, we obtain that

$$\|x_1 - x_2\|_{C[0, T]} \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \sum_{j=1}^n B_j l_j \|x_1 - x_2\|_{C[0, T]} + \frac{T^\alpha}{\Gamma(\alpha + 1)} \sum_{j=1}^n B_j l_j \|\psi_1 - \psi_2\|_{C[-v, 0]}.$$

In view of (21), one has

$$\|x_1 - x_2\|_{C[0, T]} \leq \frac{T^\alpha \sum_{j=1}^n B_j l_j}{\Gamma(\alpha + 1) - T^\alpha \sum_{j=1}^n B_j l_j} \|\psi_1 - \psi_2\|_{C[-v, 0]}. \quad (24)$$

This implies the continuous dependence of  $x$  on the initial data  $\psi$ .  $\square$

#### 4. Ulam–Hyers Stability

One of the main qualitative properties of solutions of differential equations is stability, which is studied by many methods [30,31]. Recently, Ulam-type stabilities have attracted more and more attention [32,33]. The classical concept of Ulam stability was posed by Ulam in 1940 and later obtained for functional equations by Hyers in 1941 [34]. Hyers' result was extended by replacing functional equations with differential equations, and this approach guarantees the existence of an  $\varepsilon$ -solution, which is quite useful in many applications where finding the exact solution is impossible. Now, let us give the definition of Ulam–Hyers stability in the fractional setting. For more details, we refer the reader to [22,25,34] and references therein.

**Definition 2.** The fractional differential equation

$$\Psi(t, y, f, {}^c D^{\alpha_1}, \dots, {}^c D^{\alpha_n}) = 0 \tag{25}$$

is Hyers–Ulam stable if, for a given  $\varepsilon > 0$  and a function  $y_\varepsilon$  such that

$$|\Psi(t, y_\varepsilon, f, {}^c D^{\alpha_1}, \dots, {}^c D^{\alpha_n})| \leq \varepsilon,$$

there exists a solution  $y_e$  of (25) and a positive constant  $K > 0$  such that

$$|y_\varepsilon(t) - y_e(t)| \leq K\varepsilon.$$

**Theorem 4.** If problem (5) and (6) has a unique solution, then it is Ulam–Hyers stable.

**Proof.** Let  $y_e$  be a unique solution in  $C([-v, T], \mathbb{R})$  satisfying (22) and (23) and  $y_\varepsilon$  be a solution of the following inequality:

$$\left| {}^c D_{0+}^\alpha y_\varepsilon(t) + \lambda y_\varepsilon(t) - \sum_{j=1}^n b_j(t) g_j(y_\varepsilon(t - \tau_j)) \right| \leq \varepsilon, \quad t \in [0, T], \tag{26}$$

which means there exists a function  $h(t)$  such that  $|h(t)| \leq \varepsilon$  for every  $t \in [0, T]$ .

Hence, for a given continuous function  $\psi$ , we have

$$\begin{cases} {}^c D_{0+}^\alpha y_\varepsilon(t) + \lambda y_\varepsilon(t) = \sum_{j=1}^n b_j(t) g_j(y_\varepsilon(t - \tau_j)) + h(t), & t \in [0, T], \\ y_\varepsilon(t) = \psi(t), & t \in [-v, 0]. \end{cases}$$

In addition, we have

$$\left| y_\varepsilon(t) - \int_0^t e^{-\lambda(t-s)} \sum_{j=1}^n b_j(t) g_j(y_\varepsilon(s - \tau_j)) ds \right| \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \varepsilon. \tag{27}$$

Keeping in mind the above inequality (27), we obtain

$$\begin{aligned} |y_\varepsilon(t) - y_e(t)| &\leq \left| y_\varepsilon(t) - \int_0^t e^{-\lambda(t-s)} \sum_{j=1}^n b_j(t) g_j(y_e(s - \tau_j)) ds \right| \\ &\leq \left| y_\varepsilon(t) - \int_0^t e^{-\lambda(t-s)} \sum_{j=1}^n b_j(t) g_j(y_\varepsilon(s - \tau_j)) ds \right| \\ &\quad + \left| \int_0^t e^{-\lambda(t-s)} \sum_{j=1}^n b_j(t) |g_j(y_\varepsilon(s - \tau_j)) - g_j(y_e(s - \tau_j))| ds \right| \\ &\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \varepsilon + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n B_j l_j \int_{-\tau_j}^{t-\tau_j} (t - \tau_j - z)^{\alpha-1} |y_\varepsilon(z) - y_e(z)| dz. \end{aligned}$$

As  $y_\varepsilon(z) = y_e(z) = \psi(z)$ , for  $z \in [-\tau_j, 0]$ , we obtain

$$\begin{aligned} |y_\varepsilon(t) - y_e(t)| &\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \varepsilon + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n B_j l_j \int_0^{t-\tau_j} (t - \tau_j - z)^{\alpha-1} |y_\varepsilon(z) - y_e(z)| dz \\ &\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \varepsilon + \frac{T^\alpha \sum_{j=1}^n B_j l_j}{\Gamma(\alpha + 1)} \|y_\varepsilon - y_e\|_{C[0, T]}, \end{aligned}$$



and hence

$$\|y_\epsilon - y_e\|_{C[0,T]} < \frac{T^\alpha}{\Gamma(\alpha + 1) - T^\alpha \sum_{j=1}^n B_j l_j} \epsilon.$$

In this way, we find a constant  $K = \frac{T^\alpha}{\Gamma(\alpha+1) - T^\alpha \sum_{j=1}^n B_j l_j}$ , which is well defined taking into account condition (21). □

We end with an example of application of our results.

**Example 1.** Consider the following creep fractional problem:

$$\begin{cases} {}^C D_{0+}^{\frac{1}{2}} x(t) + x(t) = \sum_{j=1}^n t^j \frac{x(t-\tau_j)^{j+1}}{j+3}, & t \in [0, 1], \\ x(t) = t, & t \in [-1, 0], \end{cases} \tag{28}$$

where  $\alpha = \frac{1}{2}$ ,  $\lambda = 1$ ,  $b_j(t) = t^j$  and  $g_j(x(t - \tau_j)) = \frac{x(t-\tau_j)^{j+1}}{j+3}$  with  $\tau_j = \frac{1}{j}$  for  $j = \overline{1, 3}$ .

First, we know from Theorem 2 that the creep function  $k_\alpha(t)$  associated with the linear problem of (28) is given by

$$k_{\frac{1}{2}}(t) = \left(1 - E_{\frac{1}{2}}\left(-t^{\frac{1}{2}}\right)\right).$$

Hence,

$$k_{\frac{1}{2}}(t) = 1 - e^t \left(1 + \operatorname{erf}\left(-t^{\frac{1}{2}}\right)\right),$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

We have that  $g_j(x(t - \tau_j)) = \frac{x(t-\tau_j)^{j+1}}{j+3}$  are Lipschitz functions with  $l_j = \frac{1}{j+3}$ . Then, for  $T = 1$ ,  $\alpha = \frac{1}{2}$ ,  $n = 3$ ,  $l_1 = \frac{1}{4}$ ,  $l_2 = \frac{1}{5}$ ,  $l_3 = \frac{1}{6}$  and  $B_j = 1$  for  $j = \overline{1, 3}$ , we obtain

$$T^\alpha \sum_{j=1}^n B_j l_j = 0.62 \leq \Gamma\left(\frac{3}{2}\right) \approx 0.87,$$

and condition (21) holds. It follows from our Theorem 3 that problem (28) has a unique solution in  $C([-1, 1], \mathbb{R})$ . Finally, by Theorem 4, we conclude that problem (28) is Ulam–Hyers stable.

### 5. Conclusions

In this paper, we have studied a class of nonlinear fractional differential equations with multiple delays as a Voigt model, which are expressed by Boltzmann’s superposition principle. We first took into account the Voigt linear model, because it expresses the physical meaning, giving strength to our problem and explaining its connection with real-life models where stability analysis is one of the most important investigation topics. A new form to our model is then given as a Volterra integral equation involving a creep kernel as a generalized exponential function. Such a kind of integral equation has shown to be very appropriate to our theoretical work, allowing the existence and stability analysis. In particular, using the fixed point method and Banach’s contraction mapping principle, we succeeded to give a sufficient condition for establishing the important result of existence. Then, we investigated the continuous dependence upon the initial data and Ulam’s stability.

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