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(q_1, q_2)-Trapezium-Like Inequalities Involving Twice Differentiable Generalized $\mathfrak{m}$-Convex Functions and Applications

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Abstract: A new auxiliary result pertaining to twice (q_1, q_2)-differentiable functions is derived. Using this new auxiliary result, some new versions of Hermite–Hadamard's inequality involving the class of generalized $\mathfrak{m}$-convex functions are obtained. Finally, to demonstrate the significance of the main outcomes, some applications are discussed for hypergeometric functions, Mittag–Leffler functions, and bounded functions.

Keywords: Hermite–Hadamard inequality; Hölder's inequality; power mean inequality; generalized $\mathfrak{m}$-convex functions; post-quantum calculus; hypergeometric functions; Mittag–Leffler functions; bounded functions

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1. Introduction and Preliminaries

A set $C \subseteq \mathbb{R}$ is said to be convex if

$$(1-\tilde{\varpi})\mathfrak{y}_3 + \tilde{\varpi}\mathfrak{y}_4 \in C, \quad \forall \mathfrak{y}_3, \mathfrak{y}_4 \in C, \tilde{\varpi} \in [0,1].$$

A function $\mathfrak{B} : C \to \mathbb{R}$ is said to be convex if

$$(1-\tilde{\varpi})\mathfrak{B}(\mathfrak{y}_3) + \tilde{\varpi}\mathfrak{B}(\mathfrak{y}_4) \geqslant \mathfrak{B}((1-\tilde{\varpi})\mathfrak{y}_3 + \tilde{\varpi}\mathfrak{y}_4), \quad \forall \mathfrak{y}_3, \mathfrak{y}_4 \in C, \tilde{\varpi} \in [0,1].$$

Hermite–Hadamard's inequality (also known as the trapezium inequality) is a famous result pertaining to convex functions. It reads as follows:

Let $\mathfrak{B} : [\mathfrak{y}_3, \mathfrak{y}_4] \mapsto \mathbb{R}$ be a convex function; then,

$$\frac{\mathfrak{B}(\mathfrak{y}_3) + \mathfrak{B}(\mathfrak{y}_4)}{2} \geqslant \frac{1}{\mathfrak{y}_4 - \mathfrak{y}_3}\int_{\mathfrak{y}_3}^{\mathfrak{y}_4} \mathfrak{B}(\mathfrak{x})d\mathfrak{x} \geqslant \mathfrak{B}\left(\frac{\mathfrak{y}_3 + \mathfrak{y}_4}{2}\right),$$

Recently, a variety of different new approaches have been employed to obtain new refinements of trapezium's inequality. For details, see [1,2]. Tariboon and Ntouyas [3] obtained a q_2-analogue of trapezium's inequality using the concepts of quantum calculus (also known as calculus without limits) on finite intervals. In quantum calculus, basically, we establish the q_2-analogues of classical mathematical objects, which can be recaptured by taking $q_2 \to 1^-$ (for details, see [4]). Alp et al. [5] obtained a corrected q_2-analogue of trapezium's inequality. For more information, interested readers are referred to [6–11]. In recent years, the classical concepts of quantum calculus have also been extended and generalized in different directions. Chakarabarti and Jagannathan [12] introduced post-quantum calculus, which is a significant generalization of quantum calculus. In quantum calculus, we deal with q_2-numbers with one base q_2, while in post-quantum calculus, we deal with q_1 and q_2-numbers with two independent variables q_1 and q_2. Tunç and Gov [13] introduced the concepts of (q_1, q_2)-derivatives ${}_{\mathfrak{y}_3}D_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{x})$ and (q_1, q_2)-integrals on finite intervals

$$\int_{\mathfrak{y}_3}^{\mathfrak{x}} \tilde{\mathfrak{B}}(\tilde{\varpi})\, {}_{\mathfrak{y}_3}d_{q_1,q_2}\tilde{\varpi}$$

for all $\mathfrak{x} \neq \mathfrak{y}_3$, where $\mathfrak{x} \in K \subset \mathbb{R}$ as follows.

Definition 1 ([13])**.** *Let $\tilde{\mathfrak{B}} : K \subset \mathbb{R} \mapsto \mathbb{R}$ be a continuous function, and let $\mathfrak{x} \in K$, where $0 < q_2 < q_1 \leqslant 1$. Then, the (q_1, q_2)-derivative on K of function $\tilde{\mathfrak{B}}$ at $\mathfrak{x}$ is defined as*

$${}_{\mathfrak{y}_3}D_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{x}) = \frac{\tilde{\mathfrak{B}}(q_1\mathfrak{x} + (1-q_1)\mathfrak{y}_3) - \tilde{\mathfrak{B}}(q_2\mathfrak{x} + (1-q_2)\mathfrak{y}_3)}{(q_1 - q_2)(\mathfrak{x} - \mathfrak{y}_3)}, \quad \mathfrak{x} \neq \mathfrak{y}_3. \tag{1}$$

Definition 2 ([13])**.** *Let $\tilde{\mathfrak{B}} : K \subset \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. Then, (q_1, q_2)-integral on K is defined as*

$$\int_{\mathfrak{y}_3}^{\mathfrak{x}} \tilde{\mathfrak{B}}(\tilde{\varpi})\, {}_{\mathfrak{y}_3}d_{q_1,q_2}\tilde{\varpi} = (q_1 - q_2)(\mathfrak{x} - \mathfrak{y}_3) \sum_{n=0}^{\infty} \frac{{q_2}^n}{{q_1}^{n+1}} \tilde{\mathfrak{B}}\left(\frac{{q_2}^n}{{q_1}^{n+1}}\mathfrak{x} + \left(1 - \frac{{q_2}^n}{{q_1}^{n+1}}\right)\mathfrak{y}_3\right),$$

for $\mathfrak{x} \in K$ and $\mathfrak{x} \neq \mathfrak{y}_3$.

Since its appearance, several new variants of classical integral inequalities have been obtained using the concepts of post-quantum calculus. For example, see [14–21].

The classical concepts of convexity have also been extended and generalized in different directions using novel and innovative ideas. For example, Cortez et al. [22] presented a new generalization of the convexity class as follows:

Definition 3 ([22])**.** *Let $\tilde{\aleph}_1, \tilde{\aleph}_2 > 0$ and $\tilde{\aleph}_3 = (\tilde{\aleph}_3(0), \ldots, \tilde{\aleph}_3(k), \ldots)$ be a bounded sequence of positive real numbers. A non-empty set $I \subseteq \mathbb{R}$ is said to be generalized convex if*

$$\mathfrak{y}_3 + \tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{y}_3) \in I, \quad \forall\, \mathfrak{y}_3, \mathfrak{y}_4 \in I,\ \tilde{\varpi} \in [0,1].$$

Definition 4 ([22])**.** *Let $\tilde{\aleph}_1, \tilde{\aleph}_2 > 0$ and $\tilde{\aleph}_3 = (\tilde{\aleph}_3(0), \ldots, \tilde{\aleph}_3(k), \ldots)$ be a bounded sequence of positive real numbers. A function $\tilde{\mathfrak{B}} : I \subseteq \mathbb{R} \mapsto \mathbb{R}$ is said to be generalized convex if*

$$\tilde{\mathfrak{B}}(\mathfrak{y}_3 + \tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{y}_3)) \leqslant (1 - \tilde{\varpi})\tilde{\mathfrak{B}}(\mathfrak{y}_3) + \tilde{\varpi}\tilde{\mathfrak{B}}(\mathfrak{y}_4), \quad \forall\, \mathfrak{y}_3, \mathfrak{y}_4 \in I,\ \tilde{\varpi} \in [0,1].$$

Here, $\bar{\bar{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(z)$ is Raina's function introduced and studied by [23], as follows:

$$\bar{\bar{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(z) = \mathfrak{R}^{\tilde{\aleph}_3(0),\tilde{\aleph}_3(1),\ldots}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(z) := \sum_{k=0}^{\infty} \frac{\tilde{\aleph}_3(k)}{\Gamma(\tilde{\aleph}_1 k + \tilde{\aleph}_2)} z^k, \qquad z \in \mathbb{C}, \tag{2}$$

where $\tilde{\aleph}_1, \tilde{\aleph}_2 > 0$, with bounded modulus $|z| < M$, and $\tilde{\aleph}_3 = \{\tilde{\aleph}_3(0), \tilde{\aleph}_3(1), \ldots, \tilde{\aleph}_3(k), \ldots\}$ is a bounded sequence of positive real numbers. For more details, see [23].

Taking inspiration from the above definition, we now define the class of generalized $\mathfrak{m}$-convex sets.

Definition 5. *A set* $\mathsf{K} \subset \mathbb{R}$ *is said to be generalized* $\mathfrak{m}$*-convex if* $\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi}\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \in \mathsf{K}$ *for every* $\mathfrak{y}_3, \mathfrak{y}_4 \in \mathsf{K}$ *and* $\mathfrak{m} \in (0,1]$, $\tilde{\varpi} \in [0,1]$.

Now, we introduce the following class of generalized $\mathfrak{m}$-convex functions.

Definition 6. *A function* $\tilde{\mathfrak{B}} : \mathsf{K} \mapsto \mathbb{R}$ *is said to be generalized* $\mathfrak{m}$*-convex if*

$$\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi}\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)\right) \leqslant \mathfrak{m}\,(1-\tilde{\varpi})\tilde{\mathfrak{B}}(\mathfrak{y}_3) + \tilde{\varpi}\tilde{\mathfrak{B}}(\mathfrak{y}_4),$$

$\forall\, \mathfrak{y}_3, \mathfrak{y}_4 \in \mathsf{K}, \quad \mathfrak{m} \in (0,1],\ \tilde{\varpi} \in [0,1]$.

Remark 1. *Note the following:*

I. *If we take* $\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) = \mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3$ *in Definition 6, then we have the definition of an* $\mathfrak{m}$*-convex function; see [24].*

II. *If we choose* $\mathfrak{m} = 1$ *in Definition 6, then we have Definition 4.*

The main motivation behind this paper is to derive a new post-quantum integral identity involving twice $(\mathsf{q}_1, \mathsf{q}_2)$-differentiable functions. Using this new identity as an auxiliary result, we obtain some new variants of Hermite–Hadamard's inequality, essentially via the class of generalized $\mathfrak{m}$-convex functions. In order to support our theoretical results, we also offer some applications to hypergeometric functions, Mittag–Leffler functions, and bounded functions. We hope that the ideas and techniques in this paper will inspire interested readers working in this field.

2. Main Results

In this section, we derive a new post-quantum integral identity. This result will be helpful in obtaining the main results of this paper. First, let us denote

$$\mathsf{R}^{\varepsilon\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) := \left[\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)\right]^{\varepsilon}.$$

Lemma 1. *Let* $\tilde{\mathfrak{B}} : \mathrm{I} \mapsto \mathbb{R}$ *be a twice* $(\mathsf{q}_1, \mathsf{q}_2)$*-differentiable function on* I° *(the interior of* I*) and* ${}_{\mathfrak{m}\mathfrak{y}_3}\mathsf{D}^2_{\mathsf{q}_1,\mathsf{q}_2}\tilde{\mathfrak{B}}$ *be continuous on* I*, where* $0 < \mathsf{q}_2 < \mathsf{q}_1 \leqslant 1$*. Then,*

$$\begin{aligned}
&\bar{\Xi}(\tilde{\mathfrak{B}}; \mathsf{q}_1, \mathsf{q}_2; \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}) \\
&= \frac{\mathsf{q}_2\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3) + \mathsf{q}_1\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + \mathsf{q}_1\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)\right)}{\mathsf{q}_1 + \mathsf{q}_2} \\
&- \frac{1}{\mathsf{q}_1{}^2\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)} \int_{\mathfrak{m}\mathfrak{y}_3}^{\mathfrak{m}\mathfrak{y}_3 + \mathsf{q}_1^2\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)} \tilde{\mathfrak{B}}(\mathfrak{x})\ {}_{\mathfrak{m}\mathfrak{y}_3}\mathsf{d}_{\mathsf{q}_1,\mathsf{q}_2}\mathfrak{x} \\
&= \frac{\mathsf{q}_1\mathsf{q}_2{}^2\mathsf{R}^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{\mathsf{q}_1 + \mathsf{q}_2} \int_0^1 \tilde{\varpi}(1 - \mathsf{q}_2\tilde{\varpi})\ {}_{\mathfrak{m}\mathfrak{y}_3}\mathsf{D}^2_{\mathsf{q}_1,\mathsf{q}_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi}\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)\right) {}_0\mathsf{d}_{\mathsf{q}_1,\mathsf{q}_2}\tilde{\varpi}.
\end{aligned}$$

Proof. Applying Definition 1, we have

$$\begin{aligned}
&{}_{\mathfrak{m}\mathfrak{y}_3}\mathsf{D}^2_{\mathsf{q}_1,\mathsf{q}_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi}\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)\right) \\
&= {}_{\mathfrak{m}\mathfrak{y}_3}\mathsf{D}_{\mathsf{q}_1,\mathsf{q}_2}\left({}_{\mathfrak{m}\mathfrak{y}_3}\mathsf{D}_{\mathsf{q}_1,\mathsf{q}_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi}\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{{}_{\mathfrak{m}\eta_3}D_{q_1,q_2}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right) - {}_{\mathfrak{m}\eta_3}D_{q_1,q_2}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)}{\tilde{\varpi}(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\\
&= \frac{1}{\tilde{\varpi}(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\left[\frac{\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1^2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right) - \mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1q_2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)}{\tilde{\varpi}p(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\right.\\
&\left.-\frac{\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1q_2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right) - \mathfrak{B}\left(\mathfrak{m}\eta_3 + q_2^2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)}{\tilde{\varpi}q(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\right]\\
&= \frac{\begin{array}{l} q_2\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1^2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)\\ \quad -(q_1+q_2)\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1q_2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right) + q_1\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_2^2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)\end{array}}{q_1q_2\tilde{\varpi}^2(q_1-q_2)^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}.
\end{aligned}
\tag{3}
$$

Now using Definition 2, we obtain

$$
\begin{aligned}
&\int_0^1 \tilde{\varpi}(1-q_2\tilde{\varpi})\,{}_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\mathfrak{B}\left(\mathfrak{m}\eta_3 + \tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right){}_0d_{q_1,q_2}\tilde{\varpi}\\
&= \int_0^1 \tilde{\varpi}(1-q_2\tilde{\varpi})\\
&\times \frac{q_2\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1^2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right) - (q_1+q_2)\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right) + q_1\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_2^2\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)}{\tilde{\varpi}^2q_1q_2(q_1-q_2)^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\,{}_0d_{q_1,q_2}\tilde{\varpi}\\
&= \frac{1}{q_1q_2(q_1-q_2)R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\\
&\times\left[q_2\sum_{n=0}^{\infty}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1{}^2\frac{q_2{}^n}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right) - (q_1+q_2)\sum_{n=0}^{\infty}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1\frac{q_2{}^{n+1}}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)\right.\\
&\quad\left.+q_1\sum_{n=0}^{\infty}\mathfrak{B}\left(\mathfrak{m}\eta_3 + \frac{q_2{}^{n+2}}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)\right]\\
&\quad -q_2\left\{\frac{q_2(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\sum_{n=0}^{\infty}\frac{q_2{}^n}{q_1{}^{n+1}}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1{}^2\frac{q_2{}^n}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)}{q_1q_2(q_1-q_2)^2R^{3\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\right.\\
&\quad -\frac{(q_1+q_2)(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\sum_{n=0}^{\infty}\frac{q_2{}^{n+1}}{q_1{}^{n+1}}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1\frac{q_2{}^{n+1}}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)}{q_1q_2{}^2(q_1-q_2)^2R^{3\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\\
&\quad \left.+\frac{q_1(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\sum_{n=0}^{\infty}\frac{q_2{}^{n+2}}{q_1{}^{n+1}}\mathfrak{B}\left(\mathfrak{m}\eta_3 + \frac{q_2{}^{n+2}}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)}{q_1q_2{}^3(q_1-q_2)^2R^{3\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\right\}\\
&= \frac{q_2\left[\sum_{n=0}^{\infty}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1{}^2\frac{q_2{}^n}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right) - \sum_{n=0}^{\infty}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1\frac{q_2{}^{n+1}}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)\right]}{q_1q_2(q_1-q_2)R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\\
&- \frac{q_1\left[\sum_{n=0}^{\infty}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1\frac{q_2{}^{n+1}}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right) - \sum_{n=0}^{\infty}\mathfrak{B}\left(\mathfrak{m}\eta_3 + \frac{q_2{}^{n+2}}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)\right]}{q_1q_2(q_1-q_2)R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\\
&-q_2\left\{\frac{q_2(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\sum_{n=0}^{\infty}\frac{q_2{}^n}{q_1{}^{n+1}}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1{}^2\frac{q_2{}^n}{q_1{}^{n+1}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)}{q_1q_2(q_1-q_2)^2R^{3\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}\right.\\
&- \frac{q_1{}^2(q_1+q_2)(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\sum_{n=0}^{\infty}\frac{q_2{}^{n+1}}{q_1{}^{n+2}}\mathfrak{B}\left(\mathfrak{m}\eta_3 + q_1{}^2\frac{q_2{}^{n+1}}{q_1{}^{n+2}}\bar{\bar{\mathfrak{R}}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)\right)}{q_1q_2{}^2(q_1-q_2)^2R^{3\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-\mathfrak{m}\eta_3)}
\end{aligned}
$$

$$+\frac{q_1{}^3(q_1-q_2)\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\sum_{n=0}^{\infty}\frac{q_2{}^{n+2}}{q_1{}^{n+3}}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3+q_1{}^2\frac{q_2{}^{n+2}}{q_1{}^{n+3}}\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\right)}{q_1q_2{}^3(q_1-q_2)^2R^{3\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}\Bigg\}$$

$$=\frac{q_2\left[\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3+q_1\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)-\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3)\right]-q_1\left[\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3+q_2\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3))-\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3)\right]}{q_1q_2(q_1-q_2)R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}$$

$$-\frac{q_1+q_2}{q_1{}^3q_2{}^2R^{3\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}\int_{\mathfrak{m}\mathfrak{y}_3}^{\mathfrak{m}\mathfrak{y}_3+q_1^2\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}\tilde{\mathfrak{B}}(\mathfrak{x})\,{}_0d_{q_1,q_2}\tilde{\varpi}-\frac{q_2{}^2+q_1q_2-q_1}{q_1q_2{}^2(q_1-q_2)R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3+q_1\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\right)$$

$$+\frac{\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3+q_2\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3))}{q_2(q_1-q_2)R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}$$

$$=\frac{\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3)}{q_1q_2R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}+\frac{\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3+q_1\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\right)}{q_2{}^2R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}-\frac{q_1+q_2}{q_1{}^3q_2{}^2R^{3\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}\int_{\mathfrak{m}\mathfrak{y}_3}^{\mathfrak{m}\mathfrak{y}_3+q_1^2\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}\tilde{\mathfrak{B}}(\mathfrak{x})\,{}_{\mathfrak{m}\mathfrak{y}_3}d_{q_1,q_2}\mathfrak{x}.$$

Multiplying both sides of the above equality by $\frac{q_1q_2{}^2R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}$, we acquire the required result. □

Using Lemma 1, we can obtain the following new results.

Theorem 1. *Under the assumptions of Lemma 1, if* $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|$ *is a generalized* $\mathfrak{m}$*-convex function, then*

$$\left|\bar{\Xi}(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right|$$
$$\leqslant\frac{q_1q_2{}^2R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)(\mathfrak{m}(q_1{}^4-q_1{}^3+q_1{}^2q_2{}^2)|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|+q_1{}^3|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|)}{(q_1+q_2)^2(q_1{}^2+q_2{}^2)(q_2{}^2+q_1q_2+q_1{}^2)}.$$

Proof. Using Lemma 1, the property of the modulus and the given hypothesis, we have

$$\left|\bar{\Xi}(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right|$$
$$\leqslant\frac{q_1q_2{}^2R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}\int_0^1\tilde{\varpi}(1-q_2\tilde{\varpi})|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3+\tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\right)|\,{}_0d_{q_1,q_2}\tilde{\varpi}$$
$$\leqslant\frac{q_1q_2{}^2R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}\left(\mathfrak{m}|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|\int_0^1\tilde{\varpi}(1-\tilde{\varpi})(1-q_2\tilde{\varpi})\,{}_0d_{q_1,q_2}\tilde{\varpi}+|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|\int_0^1\tilde{\varpi}^2(1-q_2\tilde{\varpi})\,{}_0d_{q_1,q_2}\tilde{\varpi}\right)$$
$$=\frac{q_1q_2{}^2R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)(\mathfrak{m}(q_1{}^4-q_1{}^3+q_1{}^2q_2{}^2)|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|+q_1{}^3|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|)}{(q_1+q_2)^2(q_1{}^2+q_2{}^2)(q_2{}^2+q_1q_2+q_1{}^2)}.$$

This completes the proof. □

Theorem 2. *Under the assumptions of Lemma 1, if* $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* $\mathfrak{m}$*-convex function for* $r\geqslant 1$*, then*

$$\left|\bar{\Xi}(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right|$$
$$\leqslant\frac{q_1q_2{}^2R^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(\mathfrak{m}d_1|{}_{\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3)|^r+d_2|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\right)^{\frac{1}{r}},$$

where

$$d_1 := (q_1 - q_2) \sum_{n=0}^{\infty} \left(\frac{{q_2}^{2n}}{{q_1}^{2n+2}} - \frac{{q_2}^{3n}}{{q_1}^{3n+3}} \right) \left(1 - \frac{{q_2}^{n+1}}{{q_1}^{n+1}} \right)^r, \tag{4}$$

$$d_2 := (q_1 - q_2) \sum_{n=0}^{\infty} \frac{{q_2}^{3n}}{{q_1}^{3n+3}} \left(1 - \frac{{q_2}^{n+1}}{{q_1}^{n+1}} \right)^r. \tag{5}$$

Proof. Using Lemma 1, the power mean inequality, and the given hypothesis, we have

$$\begin{aligned}
&\left| \bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}) \right| \\
&\leqslant \frac{q_1 {q_2}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \int_0^1 \tilde{\varpi}(1 - q_2\tilde{\varpi}) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}} \left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) |\, {}_0d_{q_1,q_2}\tilde{\varpi} \\
&\leqslant \frac{q_1 {q_2}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \left(\int_0^1 \tilde{\varpi}\, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{1-\frac{1}{r}} \left(\int_0^1 \tilde{\varpi}(1 - q_2\tilde{\varpi})^r |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}} \left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) |^r\, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{r}} \\
&\leqslant \frac{q_1 {q_2}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \left(\frac{1}{q_1 + q_2} \right)^{1-\frac{1}{r}} \left(\mathfrak{m} |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r \int_0^1 \tilde{\varpi}(1 - \tilde{\varpi})(1 - q_2\tilde{\varpi})^r\, {}_0d_{q_1,q_2}\tilde{\varpi} \right. \\
&\quad \left. + |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r \int_0^1 \tilde{\varpi}^2 (1 - q_2\tilde{\varpi})^r\, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{r}} \\
&= \frac{q_1 {q_2}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1 + q_2)^{2-\frac{1}{r}}} \left(\mathfrak{m} d_1 |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r + d_2 |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r \right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Theorem 3. *Under the assumptions of Lemma 1, if* $|\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}|^r$ *is a generalized* $\mathfrak{m}$*-convex function for* $r > 1$ *and* $\frac{1}{s} + \frac{1}{r} = 1$*, then*

$$\begin{aligned}
&\left| \bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}) \right| \\
&\leqslant \frac{q_1 {q_2}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1 + q_2)} {d_3}^{\frac{1}{s}} \left(\frac{\mathfrak{m}({q_2}^2 + {q_1}^2 + q_1 q_2 - q_1 - q_2) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r + (q_1 + q_2) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1 + q_2)({q_2}^2 + q_1 q_2 + {q_1}^2)} \right)^{\frac{1}{r}},
\end{aligned}$$

where

$$d_3 := (q_1 - q_2) \sum_{n=0}^{\infty} \frac{{q_2}^{2n}}{{q_1}^{2n+2}} \left(1 - \frac{{q_2}^{n}}{{q_1}^{n+1}} \right)^s. \tag{6}$$

Proof. Using Lemma 1, Hölder's inequality, and the given hypothesis, we have

$$\begin{aligned}
&\left| \bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}) \right| \\
&\leqslant \frac{q_1 {q_2}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \int_0^1 \tilde{\varpi}(1 - q_2\tilde{\varpi}) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}} \left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) |\, {}_0d_{q_1,q_2}\tilde{\varpi} \\
&\leqslant \frac{q_1 {q_2}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \left(\int_0^1 \tilde{\varpi}(1 - q_2\tilde{\varpi})^s\, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{s}} \left(\int_0^1 \tilde{\varpi} |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}} \left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) |^r\, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{r}} \\
&\leqslant \frac{q_1 {q_2}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \left(\int_0^1 \tilde{\varpi}(1 - q_2\tilde{\varpi})^s\, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{s}}
\end{aligned}$$

$$\times\left(\mathfrak{m}|{}_{\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3)|^r\int_0^1\tilde{\varpi}(1-\tilde{\varpi})\,{}_0d_{q_1,q_2}\tilde{\varpi}+|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\int_0^1\tilde{\varpi}^2\,{}_0d_{q_1,q_2}\tilde{\varpi}\right)^{\frac{1}{r}}$$

$$=\frac{q_1{q_2}^2R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}d_3{}^{\frac{1}{s}}\left(\frac{\mathfrak{m}({q_2}^2+{q_1}^2+q_1q_2-q_1-q_2)|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r+(q_1+q_2)|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1+q_2)({q_2}^2+q_1q_2+{q_1}^2)}\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 4. *Under the assumptions of Lemma 1, if* $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* $\mathfrak{m}$*-convex function for* $r\geqslant 1$*, then*

$$\left|\bar{\Xi}(\tilde{\mathfrak{B}};q_1,q_2;\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2})\right|$$
$$\leqslant\frac{q_1{q_2}^2R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}\left(\mathfrak{m}d_4|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r+d_5|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\right)^{\frac{1}{r}},$$

where

$$d_4:=(q_1-q_2)\sum_{n=0}^{\infty}\left(\frac{{q_2}^n}{{q_1}^{n+1}}\right)^{r+1}\left(1-\frac{{q_2}^n}{{q_1}^{n+1}}\right)\left(1-\frac{{q_2}^{n+1}}{{q_1}^{n+1}}\right)^r \tag{7}$$

and

$$d_5:=(q_1-q_2)\sum_{n=0}^{\infty}\left(\frac{{q_2}^n}{{q_1}^{n+1}}\right)^{r+3}\left(1-\frac{{q_2}^{n+1}}{{q_1}^{n+1}}\right)^r. \tag{8}$$

Proof. Using Lemma 1, the power mean inequality, and the given hypothesis, we have

$$\left|\bar{\Xi}(\tilde{\mathfrak{B}};q_1,q_2;\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2})\right|$$
$$\leqslant\frac{q_1{q_2}^2R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}\int_0^1\tilde{\varpi}(1-q_2\tilde{\varpi})|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3+\tilde{\varpi}\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\right)|\,{}_0d_{q_1,q_2}\tilde{\varpi}$$
$$\leqslant\frac{q_1{q_2}^2R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}\left(\int_0^1 1\,{}_0d_{q_1,q_2}\tilde{\varpi}\right)^{1-\frac{1}{r}}\left(\int_0^1\tilde{\varpi}^r(1-q_2\tilde{\varpi})^r|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3+\tilde{\varpi}\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\right)|^r\,{}_0d_{q_1,q_2}\tilde{\varpi}\right)^{\frac{1}{r}}$$
$$\leqslant\frac{q_1{q_2}^2R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}\left(\mathfrak{m}|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r\int_0^1\tilde{\varpi}^r(1-\tilde{\varpi})(1-q_2\tilde{\varpi})^r\,{}_0d_{q_1,q_2}\tilde{\varpi}\right.$$
$$\left.+|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\int_0^1\tilde{\varpi}^{r+2}(1-q_2\tilde{\varpi})^r\,{}_0d_{q_1,q_2}\tilde{\varpi}\right)^{\frac{1}{r}}$$
$$=\frac{q_1{q_2}^2R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}\left(\mathfrak{m}d_4|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r+d_5|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 5. *Under the assumptions of Lemma 1, if* $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* $\mathfrak{m}$*-convex function for* $r>1$ *and* $\frac{1}{s}+\frac{1}{r}=1$*, then*

$$\left|\bar{\Xi}(\tilde{\mathfrak{B}};q_1,q_2;\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2})\right|$$
$$\leqslant\frac{q_1{q_2}^2R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}d_6^{\frac{1}{s}}\left(\frac{\mathfrak{m}(q_1+q_2-1)|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r+|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1+q_2)}\right)^{\frac{1}{r}},$$

where

$$d_6 := (q_1 - q_2) \sum_{n=0}^{\infty} \left(\frac{q_2{}^n}{q_1{}^{n+1}} \right)^{s+1} \left(1 - \frac{q_2{}^{n+1}}{q_1{}^{n+1}} \right)^s. \tag{9}$$

Proof. Using Lemma 1, Hölder's inequality, and the given hypothesis, we have

$$\begin{aligned}
&\left| \bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}) \right| \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \int_0^1 \tilde{\varpi}(1 - q_2\tilde{\varpi}) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}} \left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) |\, {}_0d_{q_1,q_2}\tilde{\varpi} \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \left(\int_0^1 \tilde{\varpi}^s (1 - q_2\tilde{\varpi})^s \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{s}} \left(\int_0^1 |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}} \left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) |^r \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{r}} \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \\
&\quad \times \left(\int_0^1 \tilde{\varpi}^s (1 - q_2\tilde{\varpi})^s \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{s}} \left(\mathfrak{m} |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r \int_0^1 (1 - \tilde{\varpi}) \, {}_0d_{q_1,q_2}\tilde{\varpi} + |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r \int_0^1 \tilde{\varpi} \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{r}} \\
&= \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1 + q_2)} d_6^{\frac{1}{s}} \left(\frac{\mathfrak{m}(q_1 + q_2 - 1) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r + |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1 + q_2)} \right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Theorem 6. *Under the assumptions of Lemma 1, if* $|\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}|^r$ *is a generalized* $\mathfrak{m}$*-convex function for* $r > 1$ *and* $\frac{1}{s} + \frac{1}{r} = 1$*, then*

$$\begin{aligned}
&\left| \bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}) \right| \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1 + q_2)} \left(\frac{q_1 - q_2}{q_1{}^{s+1} - q_2{}^{s+1}} \right)^{\frac{1}{s}} \left(\mathfrak{m} d_7 |\, {}_{\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r + d_8 |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r \right)^{\frac{1}{r}},
\end{aligned}$$

where

$$d_7 := (q_1 - q_2) \sum_{n=0}^{\infty} \left(\frac{q_2{}^n}{q_1{}^{n+1}} - \frac{q_2{}^{2n}}{q_1{}^{2n+2}} \right) \left(1 - \frac{q_2{}^{n+1}}{q_1{}^{n+1}} \right)^r \tag{10}$$

and

$$d_8 := (q_1 - q_2) \sum_{n=0}^{\infty} \frac{q_2{}^{2n}}{q_1{}^{2n+2}} \left(1 - \frac{q_2{}^{n+1}}{q_1{}^{n+1}} \right)^r. \tag{11}$$

Proof. Using Lemma 1, Hölder's inequality, and the given hypothesis, we have

$$\begin{aligned}
&\left| \bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}) \right| \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \int_0^1 \tilde{\varpi}(1 - q_2\tilde{\varpi}) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}} \left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) |\, {}_0d_{q_1,q_2}\tilde{\varpi} \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \left(\int_0^1 \tilde{\varpi}^s \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{s}} \left(\int_0^1 (1 - q_2\tilde{\varpi})^r |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}} \left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) |^r \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{r}} \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \left(\frac{q_1 - q_2}{q_1{}^{s+1} - q_2{}^{s+1}} \right)^{\frac{1}{s}} \left(\mathfrak{m} |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r \int_0^1 (1 - \tilde{\varpi})(1 - q_2\tilde{\varpi})^r \, {}_0d_{q_1,q_2}\tilde{\varpi} \right.
\end{aligned}$$

$$+|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^{r}\int_0^1\tilde{\varpi}(1-q_2\tilde{\varpi})^{r}\,{}_0d_{q_1,q_2}\tilde{\varpi}\Bigg)^{\frac{1}{r}}$$

$$=\frac{q_1q_2{}^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}\left(\frac{q_1-q_2}{q_1{}^{s+1}-q_2{}^{s+1}}\right)^{\frac{1}{s}}\left(\mathfrak{m}d_7|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^{r}+d_8|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^{r}\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 7. *Under the assumptions of Lemma 1, if* $|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^{r}$ *is a generalized* $\mathfrak{m}$*-convex function for* $r>1$ *and* $\frac{1}{s}+\frac{1}{r}=1$*, then*

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2})\right|$$
$$\leqslant\frac{q_1q_2{}^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}d_9^{\frac{1}{s}}\left(\mathfrak{m}\left(\frac{q_1-q_2}{q_1{}^{r+1}-q_2{}^{r+1}}-\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}\right)|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^{r}+\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^{r}\right)^{\frac{1}{r}},$$

where

$$d_9:=(q_1-q_2)\sum_{n=0}^{\infty}\frac{q_2{}^{n}}{q_1{}^{n+1}}\left(1-\frac{q_2{}^{n+1}}{q_1{}^{n+1}}\right)^{s}. \tag{12}$$

Proof. Using Lemma 1, Hölder's inequality, and the given hypothesis, we have

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2})\right|$$
$$\leqslant\frac{q_1q_2{}^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}\int_0^1(1-q_2\tilde{\varpi})|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3+\tilde{\varpi}\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\right)|\,{}_0d_{q_1,q_2}\tilde{\varpi}$$
$$\leqslant\frac{q_1q_2{}^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}\left(\int_0^1(1-q_2\tilde{\varpi})^{s}\,{}_0d_{q_1,q_2}\tilde{\varpi}\right)^{\frac{1}{s}}\left(\int_0^1\tilde{\varpi}^{r}|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3+\tilde{\varpi}\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\right)|^{r}\,{}_0d_{q_1,q_2}\tilde{\varpi}\right)^{\frac{1}{r}}$$
$$\leqslant\frac{q_1q_2{}^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}\left(\int_0^1(1-q_2\tilde{\varpi})^{s}\,{}_0d_{q_1,q_2}\tilde{\varpi}\right)^{\frac{1}{s}}\Bigg(\mathfrak{m}|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^{r}\int_0^1\tilde{\varpi}^{r}(1-\tilde{\varpi})\,{}_0d_{q_1,q_2}\tilde{\varpi}$$
$$+|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^{r}\int_0^1\tilde{\varpi}^{r+1}\,{}_0d_{q_1,q_2}\tilde{\varpi}\Bigg)^{\frac{1}{r}}$$
$$=\frac{q_1q_2{}^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}d_9^{\frac{1}{s}}\left(\mathfrak{m}\left(\frac{q_1-q_2}{q_1{}^{r+1}-q_2{}^{r+1}}-\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}\right)|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^{r}+\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^{r}\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 8. *Under the assumptions of Lemma 1, if* $|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^{r}$ *is a generalized* $\mathfrak{m}$*-convex function for* $r\geqslant 1$*, then*

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2})\right|$$
$$\leqslant\frac{q_1{}^{2-\frac{1}{r}}q_2{}^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(\frac{q_1{}^2}{q_2{}^2+q_1q_2+q_1{}^2}\right)^{1-\frac{1}{r}}\left(\frac{\mathfrak{m}(q_1{}^4-q_1{}^3+q_1{}^2q_2{}^2)|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^{r}+q_1{}^3|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^{r}}{(q_1+q_2)(q_1{}^2+q_2{}^2)(q_2{}^2+q_1q_2+q_1{}^2)}\right)^{\frac{1}{r}}.$$

Proof. Using Lemma 1, the power mean inequality, and the given hypothesis, we have

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2})\right|$$
$$\leqslant\frac{q_1q_2{}^2R^{2\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{q_1+q_2}\int_0^1\tilde{\varpi}(1-q_2\tilde{\varpi})|_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3+\tilde{\varpi}\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2}(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)\right)|\,{}_0d_{q_1,q_2}\tilde{\varpi}$$

$$\leqslant \frac{q_1 q_2{}^2 R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4 - m\eta_3)}{q_1+q_2}\left(\int_0^1 \tilde{\omega}(1-q_2\tilde{\omega})\,{}_0d_{q_1,q_2}\tilde{\omega}\right)^{1-\frac{1}{r}}\left(\int_0^1 \tilde{\omega}(1-q_2\tilde{\omega})|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(m\eta_3+\tilde{\omega}\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-m\eta_3)\right)|^r\,{}_0d_{q_1,q_2}\tilde{\omega}\right)^{\frac{1}{r}}$$

$$\leqslant \frac{q_1 q_2{}^2 R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4 - m\eta_3)}{q_1+q_2}\left(\int_0^1 \tilde{\omega}(1-q_2\tilde{\omega})\,{}_0d_{q_1,q_2}\tilde{\omega}\right)^{1-\frac{1}{r}}$$

$$\times\left(m|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r\int_0^1\tilde{\omega}(1-\tilde{\omega})(1-q_2\tilde{\omega})\,{}_0d_{q_1,q_2}\tilde{\omega}+|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r\int_0^1\tilde{\omega}^2(1-q_2\tilde{\omega})\,{}_0d_{q_1,q_2}\tilde{\omega}\right)^{\frac{1}{r}}$$

$$= \frac{q_1{}^{2-\frac{1}{r}} q_2{}^2 R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4 - m\eta_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(\frac{q_1{}^2}{q_2{}^2+q_1q_2+q_1{}^2}\right)^{1-\frac{1}{r}}\left(\frac{m(q_1{}^4-q_1{}^3+q_1{}^2q_2{}^2)|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r+q_1{}^3|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r}{(q_1+q_2)(q_1{}^2+q_2{}^2)(q_2{}^2+q_1q_2+q_1{}^2)}\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 9. *Under the assumptions of Lemma 1, if* $|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* m*-convex function for* $r \geqslant 1$*, then*

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2})\right|$$
$$\leqslant \frac{q_1{}^{2-\frac{1}{r}} q_2{}^2 R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4 - m\eta_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(m\left[\frac{q_1-q_2}{q_1{}^{r+1}-q_2{}^{r+1}}-\frac{(q_1-q_2)(1+q_2)}{q_1{}^{r+2}-q_2{}^{r+2}}+\frac{q_2(q_1-q_2)}{q_1{}^{r+3}-q_2{}^{r+3}}\right]|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r\right.$$
$$\left.+\left[\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}-\frac{q_2(q_1-q_2)}{q_1{}^{r+3}-q_2{}^{r+3}}\right]|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r\right)^{\frac{1}{r}}.$$

Proof. Using Lemma 1, the power mean inequality, and the given hypothesis, we have

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2})\right|$$
$$\leqslant \frac{q_1 q_2{}^2 R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4 - m\eta_3)}{q_1+q_2}\int_0^1 \tilde{\omega}(1-q_2\tilde{\omega})|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(m\eta_3+\tilde{\omega}\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-m\eta_3)\right)|\,{}_0d_{q_1,q_2}\tilde{\omega}$$
$$\leqslant \frac{q_1 q_2{}^2 R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4 - m\eta_3)}{q_1+q_2}\left(\int_0^1 (1-q_2\tilde{\omega})\,{}_0d_{q_1,q_2}\tilde{\omega}\right)^{1-\frac{1}{r}}\left(\int_0^1 \tilde{\omega}^r(1-q_2\tilde{\omega})|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(m\eta_3+\tilde{\omega}\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2}(\eta_4-m\eta_3)\right)|^r\,{}_0d_{q_1,q_2}\tilde{\omega}\right)^{\frac{1}{r}}$$
$$\leqslant \frac{q_1 q_2{}^2 R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4 - m\eta_3)}{q_1+q_2}\left(\int_0^1 (1-q_2\tilde{\omega})\,{}_0d_{q_1,q_2}\tilde{\omega}\right)^{1-\frac{1}{r}}$$
$$\times\left(m|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r\int_0^1\tilde{\omega}^r(1-\tilde{\omega})(1-q_2\tilde{\omega})\,{}_0d_{q_1,q_2}\tilde{\omega}+|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r\int_0^1\tilde{\omega}^{r+1}(1-q_2\tilde{\omega})\,{}_0d_{q_1,q_2}\tilde{\omega}\right)^{\frac{1}{r}}$$
$$= \frac{q_1{}^{2-\frac{1}{r}} q_2{}^2 R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4 - m\eta_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(m\left[\frac{q_1-q_2}{q_1{}^{r+1}-q_2{}^{r+1}}-\frac{(q_1-q_2)(1+q_2)}{q_1{}^{r+2}-q_2{}^{r+2}}+\frac{q_2(q_1-q_2)}{q_1{}^{r+3}-q_2{}^{r+3}}\right]|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r\right.$$
$$\left.+\left[\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}-\frac{q_2(q_1-q_2)}{q_1{}^{r+3}-q_2{}^{r+3}}\right]|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 10. *Under the assumptions of Lemma 1, if* $|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* m*-convex function for* $r > 1$ *and* $\frac{1}{s}+\frac{1}{r}=1$*, then*

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\mathfrak{R}}^{\aleph_3}_{\aleph_1,\aleph_2})\right|$$
$$\leqslant \frac{q_1 q_2{}^2 R^{2\aleph_3}_{\aleph_1,\aleph_2}(\eta_4 - m\eta_3)}{(q_1+q_2)}d_{10}^{\frac{1}{s}}\left(\frac{m(q_1{}^3-q_1{}^2+q_1q_2{}^2+q_1{}^2q_2)|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r+q_1{}^2|\,{}_{m\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r}{(q_1+q_2)(q_2{}^2+q_1q_2+q_1{}^2)}\right)^{\frac{1}{r}},$$

where

$$d_{10} := (q_1 - q_2) \sum_{n=0}^{\infty} \left(\frac{q_2{}^n}{q_1{}^{n+1}} \right)^{s+1} \left(1 - \frac{q_2{}^{n+1}}{q_1{}^{n+1}} \right)^s . \tag{13}$$

Proof. Using Lemma 1, Hölder's inequality, and the given hypothesis, we have

$$\begin{aligned}
&\left| \Xi(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}) \right| \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \int_0^1 \tilde{\varpi}(1 - q_2\tilde{\varpi}) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) | \, {}_0d_{q_1,q_2}\tilde{\varpi} \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \left(\int_0^1 \tilde{\varpi}^s (1 - q_2\tilde{\varpi}) \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{s}} \left(\int_0^1 (1 - q_2\tilde{\varpi}) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + \tilde{\varpi} \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right) |^r \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{r}} \\
&\leqslant \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{q_1 + q_2} \left(\int_0^1 \tilde{\varpi}^s (1 - q_2\tilde{\varpi})^s \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{s}} \\
&\quad \times \left(\mathfrak{m} |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r \int_0^1 (1 - \tilde{\varpi})(1 - q_2\tilde{\varpi}) \, {}_0d_{q_1,q_2}\tilde{\varpi} + |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r \int_0^1 \tilde{\varpi}(1 - q_2\tilde{\varpi}) \, {}_0d_{q_1,q_2}\tilde{\varpi} \right)^{\frac{1}{r}} \\
&= \frac{q_1 q_2{}^2 R^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1 + q_2)} d_{10}^{\frac{1}{s}} \left(\frac{\mathfrak{m}(q_1{}^3 - q_1{}^2 + q_1 q_2{}^2 + q_1{}^2 q_2) |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r + q_1{}^2 |\, {}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2} \tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1 + q_2)(q_2{}^2 + q_1 q_2 + q_1{}^2)} \right)^{\frac{1}{r}} .
\end{aligned}$$

This completes the proof □

3. Applications

In this section, we discuss some applications of our main results.

3.1. Applications to Hypergeometric Functions

Now, we derive some other inequalities involving hypergeometric functions and Mittag–Leffler functions.

From the relation in (1), if we set $\tilde{\aleph}_1 = 1, \tilde{\aleph}_2 = 0$ and $\tilde{\aleph}_3(k) = \frac{(\phi)_k (\psi)_k}{(\eta)_k} \neq 0$, where ϕ, ψ, and η are parameters that may be real or complex values; $(\mathfrak{m})_k$ is defined as $(\mathfrak{m})_k = \frac{\Gamma(\mathfrak{m}+k)}{\Gamma(\mathfrak{m})}$, and its domain is restricted as $|\mathfrak{x}| \leqslant 1$, then we have the following hypergeometric function:

$$\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi, \psi; \eta, \mathfrak{x}) := \sum_{k=0}^{\infty} \frac{(\phi)_k (\psi)_k}{k! (\eta)_k} \mathfrak{x}^k .$$

Let us denote

$$\bar{\tilde{\mathfrak{R}}}^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi, \psi; \eta; \mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) := \left[\bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi, \psi; \eta; \mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right]^2 .$$

Lemma 2. *Let* $\tilde{\mathfrak{B}} : I \mapsto \mathbb{R}$ *be a twice* (q_1, q_2)*-differentiable function on* I° *(the interior of* I*) and* ${}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}$ *be continuous and* I*, where* $0 < q_2 < q_1 \leqslant 1$*. Then,*

$$\begin{aligned}
&\Xi_1(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}) \\
&= \frac{q_2 \tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3) + q_1 \tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + q_1 \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi, \psi; \eta; \mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) \right)}{q_1 + q_2} \\
&\quad - \frac{1}{q_1{}^2 \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi, \psi; \eta; \mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)} \int_{\mathfrak{m}\mathfrak{y}_3}^{\mathfrak{m}\mathfrak{y}_3 + q_1^2 \bar{\tilde{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi,\psi;\eta;\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)} \tilde{\mathfrak{B}}(\mathfrak{x}) \, {}_{\mathfrak{m}\mathfrak{y}_3}d_{q_1,q_2}\mathfrak{x}
\end{aligned}$$

$$= \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\phi,\psi;\eta;\mathfrak{y}_4 - m\mathfrak{y}_3)}{q_1+q_2} \int_0^1 \tilde{\varpi}(1-q_2\tilde{\varpi})\, {}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\left(m\mathfrak{y}_3 + \tilde{\varpi}\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\phi,\psi;\eta;\mathfrak{y}_4 - m\mathfrak{y}_3)\right) {}_0d_{q_1,q_2}\tilde{\varpi}.$$

Theorem 11. *Under the assumptions of Lemma 2, if* $|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|$ *is a generalized* m*-convex function, then*

$$\begin{aligned}&\left|\Xi_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\phi,\psi;\eta;\mathfrak{y}_4 - m\mathfrak{y}_3)\left(m(q_1{}^4 - q_1{}^3 + q_1{}^2 q_2{}^2)|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)| + q_1{}^3|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|\right)}{(q_1+q_2)^2(q_1{}^2+q_2{}^2)(q_2{}^2+q_1q_2+q_1{}^2)}.\end{aligned}$$

Theorem 12. *Under the assumptions of Lemma 2, if* $|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* m*-convex function for* $r \geqslant 1$*, then*

$$\begin{aligned}&\left|\Xi_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\phi,\psi;\eta;\mathfrak{y}_4 - m\mathfrak{y}_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(m d_1|{}_{\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(m\mathfrak{y}_3)|^r + d_2|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\right)^{\frac{1}{r}},\end{aligned}$$

where d_1 *and* d_2 *are given by (4) and (5), respectively.*

Theorem 13. *Under the assumptions of Lemma 2, if* $|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* m*-convex function for* $r > 1$ *and* $\frac{1}{s}+\frac{1}{r}=1$*, then*

$$\begin{aligned}&\left|\Xi_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\phi,\psi;\eta;\mathfrak{y}_4 - m\mathfrak{y}_3)}{(q_1+q_2)} d_3{}^{\frac{1}{s}}\left(\frac{m(q_2{}^2+q_1{}^2+q_1q_2-q_1-q_2)|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r + (q_1+q_2)|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1+q_2)(q_2{}^2+q_1q_2+q_1{}^2)}\right)^{\frac{1}{r}},\end{aligned}$$

where d_3 *is given by (6).*

Theorem 14. *Under the assumptions of Lemma 2, if* $|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* m*-convex function for* $r \geqslant 1$*, then*

$$\begin{aligned}&\left|\Xi_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\phi,\psi;\eta;\mathfrak{y}_4 - m\mathfrak{y}_3)}{(q_1+q_2)}\left(m d_4|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r + d_5|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\right)^{\frac{1}{r}},\end{aligned}$$

where d_4 *and* d_5 *are given by (7) and (8).*

Theorem 15. *Under the assumptions of Lemma 2, if* $|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* m*-convex function for* $r > 1$ *and* $\frac{1}{s}+\frac{1}{r}=1$*, then*

$$\begin{aligned}&\left|\Xi_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\phi,\psi;\eta;\mathfrak{y}_4 - m\mathfrak{y}_3)}{(q_1+q_2)} d_6^{\frac{1}{s}}\left(\frac{m(q_1+q_2-1)|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r + |{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1+q_2)}\right)^{\frac{1}{r}},\end{aligned}$$

where d_6 *is given by (9).*

Theorem 16. *Under the assumptions of Lemma 2, if* $|{}_{m\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* m*-convex function for* $r > 1$ *and* $\frac{1}{s}+\frac{1}{r}=1$*, then*

$$\left|\Xi_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right|$$

$$\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi,\psi;\eta;\eta_4-\mathfrak{m}\eta_3)}{(q_1+q_2)}\left(\frac{q_1-q_2}{q_1{}^{s+1}-q_2{}^{s+1}}\right)^{\frac{1}{s}}\left(\mathfrak{m}\,d_7|_{\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r + d_8|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r\right)^{\frac{1}{r}},$$

where d_7 *and* d_8 *are given by (10) and (11).*

Theorem 17. *Under the assumptions of Lemma 2, if* $|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* $\mathfrak{m}$*-convex function for* $r > 1$ *and* $\frac{1}{s}+\frac{1}{r}=1$*, then*

$$\begin{aligned}
&\left|\bar{\Xi}_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2})\right| \\
&\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi,\psi;\eta;\eta_4-\mathfrak{m}\eta_3)}{(q_1+q_2)} d_9^{\frac{1}{s}} \\
&\times\left(\mathfrak{m}\left(\frac{q_1-q_2}{q_1{}^{r+1}-q_2{}^{r+1}}-\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}\right)|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r+\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r\right)^{\frac{1}{r}},
\end{aligned}$$

where d_9 *is given by (12).*

Theorem 18. *Under the assumptions of Lemma 2, if* $|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* $\mathfrak{m}$*-convex function for* $r \geqslant 1$*, then*

$$\begin{aligned}
&\left|\bar{\Xi}_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2})\right| \\
&\leqslant \frac{q_1{}^{2-\frac{1}{r}} q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi,\psi;\eta;\eta_4-\mathfrak{m}\eta_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(\frac{q_1{}^2}{q_2{}^2+q_1q_2+q_1{}^2}\right)^{1-\frac{1}{r}} \\
&\times\left(\frac{\mathfrak{m}(q_1{}^4-q_1{}^3+q_1{}^2q_2{}^2)|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r+q_1{}^3|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r}{(q_1+q_2)(q_1{}^2+q_2{}^2)(q_2{}^2+q_1q_2+q_1{}^2)}\right)^{\frac{1}{r}}.
\end{aligned}$$

Theorem 19. *Under the assumptions of Lemma 2, if* $|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* $\mathfrak{m}$*-convex function for* $r \geqslant 1$*, then*

$$\begin{aligned}
&\left|\bar{\Xi}_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2})\right| \\
&\leqslant \frac{q_1{}^{2-\frac{1}{r}} q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi,\psi;\eta;\eta_4-\mathfrak{m}\eta_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(\mathfrak{m}\left[\frac{q_1-q_2}{q_1{}^{r+1}-q_2{}^{r+1}}-\frac{(q_1-q_2)(1+q_2)}{q_1{}^{r+2}-q_2{}^{r+2}}+\frac{q_2(q_1-q_2)}{q_1{}^{r+3}-q_2{}^{r+3}}\right]|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r\right. \\
&\left.+\left[\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}-\frac{q_2(q_1-q_2)}{q_1{}^{r+3}-q_2{}^{r+3}}\right]|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r\right)^{\frac{1}{r}}.
\end{aligned}$$

Theorem 20. *Under the assumptions of Lemma 2, if* $|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ *is a generalized* $\mathfrak{m}$*-convex function for* $r > 1$ *and* $\frac{1}{s}+\frac{1}{r}=1$*, then*

$$\begin{aligned}
&\left|\bar{\Xi}_1(\tilde{\mathfrak{B}};q_1,q_2;\bar{\bar{\mathfrak{R}}}^{\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2})\right| \\
&\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\tilde{\aleph}_3}_{\tilde{\aleph}_1,\tilde{\aleph}_2}(\phi,\psi;\eta;\eta_4-\mathfrak{m}\eta_3)}{(q_1+q_2)} d_{10}^{\frac{1}{s}}\left(\frac{\mathfrak{m}(q_1{}^3-q_1{}^2+q_1q_2{}^2+q_1{}^2q_2)|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_3)|^r+q_1{}^2|_{\mathfrak{m}\eta_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\eta_4)|^r}{(q_1+q_2)(q_2{}^2+q_1q_2+q_1{}^2)}\right)^{\frac{1}{r}},
\end{aligned}$$

where d_{10} *is given by (13).*

3.2. Applications to Mittag–Leffler Functions

Moreover, if we take $\tilde{\aleph}_3=(1,1,1,\ldots)$, $\tilde{\aleph}_2=1$, and $\tilde{\aleph}_1=\phi$ with $\Re(\phi)>0$, then we obtain the well-known Mittag–Leffler function:

$$\bar{\bar{\mathfrak{R}}}_\phi(\mathfrak{x}) := \sum_{k=0}^{\infty}\frac{1}{\Gamma(1+\phi k)}\mathfrak{x}^k.$$

Let us denote

$$\bar{\bar{\mathfrak{R}}}_{\phi}^{2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3) := \left[\bar{\bar{\mathfrak{R}}}_{\phi}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)\right]^2.$$

Lemma 3. *Let $\tilde{\mathfrak{B}} : \mathrm{I} \mapsto \mathbb{R}$ be a twice $(\mathrm{q}_1, \mathrm{q}_2)$-differentiable function on I° (the interior of I) and ${}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}$ be continuous and I, where $0 < \mathrm{q}_2 < \mathrm{q}_1 \leqslant 1$. Then,*

$$\begin{aligned}
&\Xi(\tilde{\mathfrak{B}}; \mathrm{q}_1, \mathrm{q}_2; \bar{\bar{\mathfrak{R}}}_{\phi}) \\
&= \frac{\mathrm{q}_2\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3) + \mathrm{q}_1\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + \mathrm{q}_1\bar{\bar{\mathfrak{R}}}_{\phi}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)\right)}{\mathrm{q}_1 + \mathrm{q}_2} \\
&- \frac{1}{\mathrm{q}_1{}^2\bar{\bar{\mathfrak{R}}}_{\phi}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)} \int_{\mathfrak{m}\mathfrak{y}_3}^{\mathfrak{m}\mathfrak{y}_3 + \mathrm{q}_1^2\bar{\bar{\mathfrak{R}}}_{\phi}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)} \tilde{\mathfrak{B}}(\mathfrak{x})\, {}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{d}_{\mathrm{q}_1,\mathrm{q}_2}\mathfrak{x} \\
&= \frac{\mathrm{q}_1\mathrm{q}_2{}^2\bar{\bar{\mathfrak{R}}}_{\phi}^{2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{\mathrm{q}_1 + \mathrm{q}_2} \int_0^1 \tilde{\varpi}(1 - \mathrm{q}_2\tilde{\varpi})\, {}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}\left(\mathfrak{m}\mathfrak{y}_3 + \mathfrak{m}\tilde{\varpi}\bar{\bar{\mathfrak{R}}}_{\phi}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)\right) {}_0\mathrm{d}_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\varpi}.
\end{aligned}$$

Theorem 21. *Under the assumptions of Lemma 3, if $|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}|$ is a generalized $\mathfrak{m}$-convex function, then*

$$\begin{aligned}
&\left|\Xi(\tilde{\mathfrak{B}}; \mathrm{q}_1, \mathrm{q}_2; \bar{\bar{\mathfrak{R}}}_{\phi})\right| \\
&\leqslant \frac{\mathrm{q}_1\mathrm{q}_2{}^2\bar{\bar{\mathfrak{R}}}_{\phi}^{2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)(\mathfrak{m}(\mathrm{q}_1{}^4 - \mathrm{q}_1{}^3 + \mathrm{q}_1{}^2\mathrm{q}_2{}^2)|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)| + \mathrm{q}_1{}^3|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|)}{(\mathrm{q}_1 + \mathrm{q}_2)^2(\mathrm{q}_1{}^2 + \mathrm{q}_2{}^2)(\mathrm{q}_2{}^2 + \mathrm{q}_1\mathrm{q}_2 + \mathrm{q}_1{}^2)}.
\end{aligned}$$

Theorem 22. *Under the assumptions of Lemma 3, $|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}|^{\mathrm{r}}$ is a generalized $\mathfrak{m}$-convex function for $\mathrm{r} \geqslant 1$, then*

$$\begin{aligned}
&\left|\Xi(\tilde{\mathfrak{B}}; \mathrm{q}_1, \mathrm{q}_2; \bar{\bar{\mathfrak{R}}}_{\phi})\right| \\
&\leqslant \frac{\mathrm{q}_1\mathrm{q}_2{}^2\bar{\bar{\mathfrak{R}}}_{\phi}^{2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(\mathrm{q}_1 + \mathrm{q}_2)^{2-\frac{1}{\mathrm{r}}}} \left(\mathfrak{m}\mathrm{d}_1|{}_{\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}(\mathfrak{m}\mathfrak{y}_3)|^{\mathrm{r}} + \mathrm{d}_2|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}},
\end{aligned}$$

where d_1 and d_2 are given by (4) and (5), respectively.

Theorem 23. *Under the assumptions of Lemma 3, $|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}|^{\mathrm{r}}$ is a generalized $\mathfrak{m}$-convex function for $\mathrm{r} > 1$ and $\frac{1}{\mathrm{s}} + \frac{1}{\mathrm{r}} = 1$. Then,*

$$\begin{aligned}
&\left|\Xi(\tilde{\mathfrak{B}}; \mathrm{q}_1, \mathrm{q}_2; \bar{\bar{\mathfrak{R}}}_{\phi})\right| \\
&\leqslant \frac{\mathrm{q}_1\mathrm{q}_2{}^2\bar{\bar{\mathfrak{R}}}_{\phi}^{2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(\mathrm{q}_1 + \mathrm{q}_2)} \mathrm{d}_3{}^{\frac{1}{\mathrm{s}}} \left(\frac{\mathfrak{m}(\mathrm{q}_2{}^2 + \mathrm{q}_1{}^2 + \mathrm{q}_1\mathrm{q}_2 - \mathrm{q}_1 - \mathrm{q}_2)|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^{\mathrm{r}} + (\mathrm{q}_1 + \mathrm{q}_2)|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^{\mathrm{r}}}{(\mathrm{q}_1 + \mathrm{q}_2)(\mathrm{q}_2{}^2 + \mathrm{q}_1\mathrm{q}_2 + \mathrm{q}_1{}^2)}\right)^{\frac{1}{\mathrm{r}}},
\end{aligned}$$

where d_3 is given by (6).

Theorem 24. *Under the assumptions of Lemma 3, $|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}|^{\mathrm{r}}$ is a generalized $\mathfrak{m}$-convex function for $\mathrm{r} \geqslant 1$. Then*

$$\begin{aligned}
&\left|\Xi(\tilde{\mathfrak{B}}; \mathrm{q}_1, \mathrm{q}_2; \bar{\bar{\mathfrak{R}}}_{\phi})\right| \\
&\leqslant \frac{\mathrm{q}_1\mathrm{q}_2{}^2\bar{\bar{\mathfrak{R}}}_{\phi}^{2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(\mathrm{q}_1 + \mathrm{q}_2)} \left(\mathfrak{m}\mathrm{d}_4|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^{\mathrm{r}} + \mathrm{d}_5|{}_{\mathfrak{m}\mathfrak{y}_3}\mathrm{D}^2_{\mathrm{q}_1,\mathrm{q}_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}},
\end{aligned}$$

where d_4 and d_5 are given by (7) and (8).

Theorem 25. *Under the assumptions of Lemma 3, $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ is a generalized $\mathfrak{m}$-convex function for $r > 1$ and $\frac{1}{s}+\frac{1}{r}=1$. Then,*

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\tilde{\mathfrak{K}}}_\phi)\right| \leqslant \frac{q_1q_2{}^2\bar{\tilde{\mathfrak{K}}}^2_\phi(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}d_6^{\frac{1}{s}}\left(\frac{\mathfrak{m}(q_1+q_2-1)|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r+|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1+q_2)}\right)^{\frac{1}{r}}, \quad (14)$$

where d_6 is given by (9).

Theorem 26. *Under the assumptions of Lemma 3, $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ is a generalized $\mathfrak{m}$-convex function for $r > 1$ and $\frac{1}{s}+\frac{1}{r}=1$. Then,*

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\tilde{\mathfrak{K}}}_\phi)\right| \leqslant \frac{q_1q_2{}^2\bar{\tilde{\mathfrak{K}}}^2_\phi(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}\left(\frac{q_1-q_2}{q_1{}^{s+1}-q_2{}^{s+1}}\right)^{\frac{1}{s}}\left(\mathfrak{m}d_7|{}_{\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r+d_8|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\right)^{\frac{1}{r}},$$

where d_7 and d_8 are given by (10) and (11).

Theorem 27. *Under the assumptions of Lemma 3, $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ is a generalized $\mathfrak{m}$-convex function for $r > 1$ and $\frac{1}{s}+\frac{1}{r}=1$. Then,*

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\tilde{\mathfrak{K}}}_\phi)\mathfrak{r}\right| \leqslant \frac{q_1q_2{}^2\bar{\tilde{\mathfrak{K}}}^2_\phi(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}d_9^{\frac{1}{s}}\left(\mathfrak{m}\left(\frac{q_1-q_2}{q_1{}^{r+1}-q_2{}^{r+1}}-\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}\right)|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r+\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\right)^{\frac{1}{r}},$$

where d_9 is given by (12).

Theorem 28. *Under the assumptions of Lemma 3, $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ is a generalized $\mathfrak{m}$-convex function for $r \geqslant 1$. Then,*

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\tilde{\mathfrak{K}}}_\phi)\right| \leqslant \frac{q_1{}^{2-\frac{1}{r}}q_2{}^2\bar{\tilde{\mathfrak{K}}}^2_\phi(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(\frac{q_1{}^2}{q_2{}^2+q_1q_2+q_1{}^2}\right)^{1-\frac{1}{r}}\left(\frac{\mathfrak{m}(q_1{}^4-q_1{}^3+q_1{}^2q_2{}^2)|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r+q_1{}^3|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1+q_2)(q_1{}^2+q_2{}^2)(q_2{}^2+q_1q_2+q_1{}^2)}\right)^{\frac{1}{r}}.$$

Theorem 29. *Under the assumptions of Lemma 3, $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ is a generalized $\mathfrak{m}$-convex function for $r \geqslant 1$. Then,*

$$\begin{aligned}&\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\tilde{\mathfrak{K}}}_\phi)\right| \\ &\leqslant \frac{q_1{}^{2-\frac{1}{r}}q_2{}^2\bar{\tilde{\mathfrak{K}}}^2_\phi(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)^{2-\frac{1}{r}}}\left(\mathfrak{m}\left[\frac{q_1-q_2}{q_1{}^{r+1}-q_2{}^{r+1}}-\frac{(q_1-q_2)(1+q_2)}{q_1{}^{r+2}-q_2{}^{r+2}}+\frac{q_2(q_1-q_2)}{q_1{}^{r+3}-q_2{}^{r+3}}\right]|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r\right.\\ &\left.+\left[\frac{q_1-q_2}{q_1{}^{r+2}-q_2{}^{r+2}}-\frac{q_2(q_1-q_2)}{q_1{}^{r+3}-q_2{}^{r+3}}\right]|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r\right)^{\frac{1}{r}}.\end{aligned}$$

Theorem 30. *Under the assumptions of Lemma 3, $|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}|^r$ is a generalized $\mathfrak{m}$-convex function for $r > 1$ and $\frac{1}{s}+\frac{1}{r}=1$. Then,*

$$\left|\Xi(\tilde{\mathfrak{B}};q_1,q_2;\bar{\tilde{\mathfrak{K}}}_\phi)\right| \leqslant \frac{q_1q_2{}^2\bar{\tilde{\mathfrak{K}}}^2_\phi(\mathfrak{y}_4-\mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}d_{10}^{\frac{1}{s}}\left(\frac{\mathfrak{m}(q_1{}^3-q_1{}^2+q_1q_2{}^2+q_1{}^2q_2)|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_3)|^r+q_1{}^2|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}(\mathfrak{y}_4)|^r}{(q_1+q_2)(q_2{}^2+q_1q_2+q_1{}^2)}\right)^{\frac{1}{r}},$$

where d_{10} is given by (13).

3.3. Applications to Bounded Functions

We suppose that the following condition is satisfied:

$$\left|{}_{\mathfrak{m}\mathfrak{y}_3}D^2_{q_1,q_2}\tilde{\mathfrak{B}}\right| \leqslant \mathcal{M},$$

which means that the twice (q_1, q_2)-differentiable function $\tilde{\mathfrak{B}}$ is in an absolute value bounded from the positive real number $\mathcal{M}$.

Proposition 1. *Under the conditions of Theorem 1, we have*

$$\begin{aligned}&\left|\bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)(\mathfrak{m}(q_1{}^4 - q_1{}^3 + q_1{}^2 q_2{}^2) + q_1{}^3)\mathcal{M}}{(q_1+q_2)^2(q_1{}^2+q_2{}^2)(q_2{}^2+q_1q_2+q_1{}^2)}.\end{aligned}$$

Proposition 2. *Under the conditions of Theorem 2, we have*

$$\begin{aligned}&\left|\bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)^{2-\frac{1}{r}}}(\mathfrak{m}d_1 + d_2)^{\frac{1}{r}}\mathcal{M}.\end{aligned}$$

Proposition 3. *Under the conditions of Theorem 3, we have*

$$\begin{aligned}&\left|\bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)} d_3{}^{\frac{1}{s}}\left(\frac{\mathfrak{m}(q_2{}^2+q_1{}^2+q_1q_2-q_1-q_2)+q_1+q_2}{(q_1+q_2)(q_2{}^2+q_1q_2+q_1{}^2)}\right)^{\frac{1}{r}}\mathcal{M}.\end{aligned}$$

Proposition 4. *Under the conditions of Theorem 4, we have*

$$\begin{aligned}&\left|\bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}(\mathfrak{m}d_4 + d_5)^{\frac{1}{r}}\mathcal{M}.\end{aligned}$$

Proposition 5. *Under the conditions of Theorem 5, we have*

$$\begin{aligned}&\left|\bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)} d_6^{\frac{1}{s}}\left(\frac{\mathfrak{m}(q_1+q_2-1)+1}{(q_1+q_2)}\right)^{\frac{1}{r}}\mathcal{M}.\end{aligned}$$

Proposition 6. *Under the conditions of Theorem 6, we have*

$$\begin{aligned}&\left|\bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\bar{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)}\left(\frac{q_1-q_2}{q_1{}^{s+1}-q_2{}^{s+1}}\right)^{\frac{1}{s}}(\mathfrak{m}d_7 + d_8)^{\frac{1}{r}}\mathcal{M}.\end{aligned}$$

Proposition 7. *Under the conditions of Theorem 7, we have*

$$\left|\bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\bar{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2})\right|$$

$$\leqslant \frac{q_1 q_2{}^2 \bar{\tilde{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)} d_9^{\frac{1}{s}} \left(\mathfrak{m} \left(\frac{q_1-q_2}{q_1{}^{r+1} - q_2{}^{r+1}} - \frac{q_1-q_2}{q_1{}^{r+2} - q_2{}^{r+2}} \right) + \frac{q_1-q_2}{q_1{}^{r+2} - q_2{}^{r+2}} \right)^{\frac{1}{r}} \mathcal{M}.$$

Proposition 8. *Under the conditions of Theorem 8, we have*

$$\begin{aligned} &\left| \bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}) \right| \\ &\leqslant \frac{q_1{}^{2-\frac{1}{r}} q_2{}^2 \bar{\tilde{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)^{2-\frac{1}{r}}} \left(\frac{q_1{}^2}{q_2{}^2 + q_1 q_2 + q_1{}^2} \right)^{1-\frac{1}{r}} \left(\frac{\mathfrak{m}(q_1{}^4 - q_1{}^3 + q_1{}^2 q_2{}^2) + q_1{}^3}{(q_1+q_2)(q_1{}^2+q_2{}^2)(q_2{}^2+q_1q_2+q_1{}^2)} \right)^{\frac{1}{r}} \mathcal{M}. \end{aligned}$$

Proposition 9. *Under the conditions of Theorem 9, we have*

$$\begin{aligned} &\left| \bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}) \right| \\ &\leqslant \frac{q_1{}^{2-\frac{1}{r}} q_2{}^2 \bar{\tilde{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)^{2-\frac{1}{r}}} \left(\mathfrak{m} \left[\frac{q_1-q_2}{q_1{}^{r+1} - q_2{}^{r+1}} - \frac{(q_1-q_2)(1+q_2)}{q_1{}^{r+2} - q_2{}^{r+2}} + \frac{q_2(q_1-q_2)}{q_1{}^{r+3} - q_2{}^{r+3}} \right] \right. \\ &\quad \left. + \left[\frac{q_1-q_2}{q_1{}^{r+2} - q_2{}^{r+2}} - \frac{q_2(q_1-q_2)}{q_1{}^{r+3} - q_2{}^{r+3}} \right] \right)^{\frac{1}{r}} \mathcal{M}. \end{aligned}$$

Proposition 10. *Under the conditions of Theorem 10, we have*

$$\begin{aligned} &\left| \bar{\Xi}(\tilde{\mathfrak{B}}; q_1, q_2; \bar{\tilde{\mathfrak{R}}}^{\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}) \right| \\ &\leqslant \frac{q_1 q_2{}^2 \bar{\tilde{\mathfrak{R}}}^{2\breve{\aleph}_3}_{\breve{\aleph}_1,\breve{\aleph}_2}(\mathfrak{y}_4 - \mathfrak{m}\mathfrak{y}_3)}{(q_1+q_2)} d_{10}^{\frac{1}{s}} \left(\frac{\mathfrak{m}(q_1{}^3 - q_1{}^2 + q_1 q_2{}^2 + q_1{}^2 q_2) + q_1{}^2}{(q_1+q_2)(q_2{}^2+q_1q_2+q_1{}^2)} \right)^{\frac{1}{r}} \mathcal{M}. \end{aligned}$$

Remark 2. *Interested readers can derive several new inequalities from our main results using special means of positive real numbers. We omit their proofs here.*

4. Conclusions

We established a new integral identity utilizing twice (q_1, q_2)-differentiable functions. Some new variants of Hermite–Hadamard's inequality essentially involving the class of generalized $\mathfrak{m}$-convex functions were derived. In order to illustrate the efficiency and significance of our main outcomes, some applications regarding hypergeometric functions, Mittag–Leffler functions, and twice (q_1, q_2)-differentiable functions that are bounded were discussed in detail. To the best of our knowledge, these results are new in the literature. Note that these (q_1, q_2)-trapezium inequalities involving twice differentiable generalized $\mathfrak{m}$-convex functions are connected to theoretical methods and have applications related to fractional operators. It is also worth mentioning here that one can extend these results using fractional quantum integrals. This will be an interesting problem for future research.

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