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On Sharp Estimate of Third Hankel Determinant for a Subclass of Starlike Functions

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Abstract: In our present investigation, a subclass of starlike function $\mathcal{S}^*_{n-1,\mathcal{L}}$ connected with a domain bounded by an epicycloid with $n-1$ cusps was considered. The main work is to investigate some coefficient inequalities, and second and third Hankel determinants for functions belonging to this class. In particular, we calculate the sharp bounds of the third Hankel determinant for $f \in \mathcal{S}^*_{4\mathcal{L}}$ with $\frac{zf'(z)}{f(z)}$ bounded by a four-leaf shaped domain under the unit disk $\mathbb{D}$.

Keywords: univalent function; starlike function; cusps; four-leaf domain; Hankel determinant

MSC: 30C45; 30C80

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1. Introduction

Assuming that the class of analytic functions defined in the domain of open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be denoted by the notation $\mathcal{H}(\mathbb{D})$. Suppose that $\mathcal{A}$ is the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions f with the series expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{D}$. For two given functions, $g_1, g_2 \in \mathcal{H}(\mathbb{D})$, we say that g_1 is subordinated to g_2, if there exists a regular function v in $\mathbb{D}$ with the restrictions $v(0) = 0$ and $|v(z)| < 1$ such that

$$f(z) = g(v(z)). \tag{2}$$

Although the function theory was developed in 1851, the coefficient hypothesis proposed by Bieberbach [1] in 1916 made the field a hit as a potential new research field. De Branges [2] proved this conjecture in 1985. Between 1916 and 1985, several of the world's most famous scholars attempted to validate or refute this conjecture. As a result, they investigated a number of sub-families of the class $\mathcal{S}$ of univalent functions that are associated with various image domains. The most fundamental and significant subclasses of the set $\mathcal{S}$ are the families of starlike and convex functions, represented by $\mathcal{S}^*$ and $\mathcal{K}$, respectively. Ma and Minda [3] defined the general form of the family in 1992 as

$$\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}, \tag{3}$$

where ϕ is a holomorphic function with $\phi'(0) > 0$ and has a positive real part. In addition, the function ϕ maps $\mathbb{D}$ onto a star-shaped region with respect to $\phi(0) = 1$ and is symmetric about the real axis. Several sub-families of the set $\mathcal{A}$ have been explored as a special instance of the class $\mathcal{S}^*(\phi)$ in recent years. For example, If we select $\phi(z) = \frac{1+Lz}{1+Mz}$ with $-1 \leq M < L \leq 1$, then we obtain the class $\mathcal{S}^*[L, M] \equiv \mathcal{S}^*\left(\frac{1+Lz}{1+Mz}\right)$, which is defined as the class of Janowski starlike functions investigated in [4]. By selecting $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\phi)$ leads to the family $\mathcal{S}^*_{\sin}$, which was explored in [5] while $\mathcal{S}^*_e \equiv \mathcal{S}^*(e^z)$ has been produced in the article [6].

The Hankel determinant $\mathcal{H}_{q,m}(f)(q, m \in \mathbb{N})$ for a function $f \in \mathcal{S}$ was given by Pommerenke [7,8] as

$$\mathcal{H}_{q,m}(f) := \begin{vmatrix} a_m & a_{m+1} & \dots & a_{m+q-1} \\ a_{m+1} & a_{m+2} & \dots & a_{m+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{m+q-1} & a_{m+q} & \dots & a_{m+2q-2} \end{vmatrix}. \quad (4)$$

After that, many researchers were drawn to the problem of finding the sharp bounds of Hankel determinants in a given family of functions. For example, the sharp bound of $|\mathcal{H}_{2,2}(f)|$ for the class of convex and starlike functions were calculated by Janteng et al. [9,10].

The calculation of $|\mathcal{H}_{3,1}(f)|$ is far more challenging compared with finding the bound of $|\mathcal{H}_{2,2}(f)|$. Babalola [11] investigated the bounds of third order Hankel determinant for the families of convex and starlike functions. Later, many authors [12–14] obtained their results regarding $|\mathcal{H}_{3,1}(f)|$ for certain sub-families of analytic and univalent functions. It needs to be pointed out that there are relatively few results on the sharp bounds of the third order Hankel determinant. In [15], Kowalczyk et al. and Lecko et al. [16] obtained a sharp bound of third Hankel determinant given by

$$|\mathcal{H}_{3,1}(f)| \leq \begin{cases} \frac{4}{135}, & f \in \mathcal{K}, \\ \frac{1}{9}, & f \in \mathcal{S}^*\left(\frac{1}{2}\right), \end{cases} \quad (5)$$

where $\mathcal{S}^*\left(\frac{1}{2}\right)$ is the starlike function family of order $\frac{1}{2}$. In [17], the authors obtained the sharp bounds of third Hankel determinant for the subclass of $\mathcal{S}^*_{\sin}$ and more sharp bounds of Hankel determinant for some interesting subclasses of univalent can be found in [18,19].

A curve with the parametric form of $(f(t), g(t))$ has a cusp at the point $(f(t_0), g(t_0))$ if $f'(t_0)$ and $g'(t_0)$ is zero but either $f''(t_0)$ or $g''(t_0)$ is not equal to zero; see [20]. It is noted that the special classes of $\mathcal{S}^*(\phi)$ with the function has no cusp under the unit disk have been widely studied, for example, by choosing ϕ to be equal to e^z, $1 + \sin z$ and $\frac{2}{1+e^{-z}}$, see [5,6,21]. In [22], Wani et al. studied the function of two cusps associated with a nephroid domain. The lune domain [23] and pental shaped domain [24] also have two cusps at the angles. In [25], Gandhi introduced a family of starlike functions connected with a three-leaf shaped domain defined by

$$\mathcal{S}^*_{3\mathcal{L}} := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4, \quad z \in \mathbb{D} \right\}. \quad (6)$$

For function belonging to this class, it maps $\frac{zf'(z)}{f(z)}$ onto a domain containing three cusps, one on the real axis and the other two at the angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ under the unit disk. Later, a more general function $\varphi_{n-1,\mathcal{L}} : \mathbb{D} \to \mathbb{C}$ defined by

$$\varphi_{n-1,\mathcal{L}} = 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n, \quad (z \in \mathbb{D}) \quad (7)$$

was introduced in [26] by Gandhi, Gupta, Nagpal, and Ravichandran. It is noted that, for $n \geq 4$, it maps the unit disk $\mathbb{D}$ onto a domain bounded by an epicycloid with $n - 1$ cusps, where an epicycloid is a plane curve produced by tracing the path of a chosen point on the circumference of a circle which rolls without slipping around a fixed circle, see [27].

Using the function $\varphi_{n-1,\mathcal{L}}(z)$, the authors introduced a subclass of starlike functions $\mathcal{S}^*_{n-1,\mathcal{L}}(n \geq 4)$ given by

$$\mathcal{S}^*_{n-1,\mathcal{L}} := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n, \quad z \in \mathbb{D} \right\}. \tag{8}$$

For $n = 4$, we obtain the function class $\mathcal{S}^*_{3\mathcal{L}}$ connected with a three-leaf shaped domain which has been studied in [28,29]. For $n = 5$, it reduces to

$$\mathcal{S}^*_{4\mathcal{L}} := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5, \quad z \in \mathbb{D} \right\}. \tag{9}$$

It is associated with a four-leaf shaped domain. If we take $n \to \infty$, it is observed that the class $\mathcal{S}^*_{n-1,\mathcal{L}}$ reduces to $\mathcal{S}^*(1+z)$. Gandhi et al. studied the sharp bounds for the first fifth coefficients for functions belonging to $\mathcal{S}^*_{n-1,\mathcal{L}}$ and some interesting properties such as various inclusion relations between the class $\mathcal{S}^*_{n-1,\mathcal{L}}$ and various subclasses of starlike functions. Some sharp radius results are also established.

In the present article, we obtain the Fekete–Szegö inequality, upper bounds for second and third Hankel determinants for the general class $\mathcal{S}^*_{n-1,\mathcal{L}}$. In particular, we calculate the sharp bounds of third Hankel determinants for the class $\mathcal{S}^*_{4\mathcal{L}}$.

2. A Set of Lemmas

We say a function $p \in \mathcal{P}$ if and only if it has the series expansion

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \quad (z \in \mathbb{D}) \tag{10}$$

along with the $\Re p(z) \geq 0 (z \in \mathbb{D})$.

Lemma 1 (see [30])**.** *Assuming that $p \in \mathcal{P}$ with the series expansion of the form (10). Then, for $x, \delta, \rho \in \overline{\mathbb{D}}$, we have*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right)x, \tag{11}$$

$$4c_3 = c_1^3 + 2c_1 x\left(4 - c_1^2\right) - x^2 c_1\left(4 - c_1^2\right) + 2\left(1 - |x|^2\right)\left(4 - c_1^2\right)\delta, \tag{12}$$

$$8c_4 = c_1^4 + x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right](4 - c_1^2) - 4(4 - c_1^2)(1 - |x|^2) \left[c_1(x-1)\delta + \overline{x}\delta^2 - (1 - |\delta|^2)\rho\right]. \tag{13}$$

Lemma 2 (see [31])**.** *If $p \in \mathcal{P}$ has the series form (10), then*

$$|c_{m+k} - \mu c_m c_k| \leq 2\max(1, |2\mu - 1|), \tag{14}$$

$$|c_m| \leq 2 \text{ for } m \geq 1, \tag{15}$$

$$|c_{m+k} - \mu c_m c_k| \leq 2, \ 0 \leq \mu \leq 1. \tag{16}$$

Lemma 3 ([32])**.** *If $p \in \mathcal{P}$ and has the form (10), then*

$$\left| c_3 - 2Bc_1c_2 + Dc_1^3 \right| \leq 2, \tag{17}$$

if $B \in [0,1]$ and $B(2B-1) \leq D \leq B$.

3. Coefficient Inequalities for the Class $\mathcal{S}^*_{n-1,\mathcal{L}}$

We begin this section by finding the Fekete–Szegö inequality for the functions in the class $\mathcal{S}^*_{n-1,\mathcal{L}}$.

Theorem 1. *Let $f \in \mathcal{S}^*_{n-1,\mathcal{L}}$ be of the form (1). Then, for $\lambda \in \mathbb{C}$,*

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{n}{2(n+1)} \max\left\{1, \left|\frac{(2\lambda-1)n}{n+1}\right|\right\}.$$

This inequality is sharp.

Proof. Assuming that $f \in \mathcal{S}^*_{n-1,\mathcal{L}}$. Then, from the definition, we see that there is Schwarz function ω such that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{n}{n+1}\omega(z) + \frac{1}{n+1}[\omega(z)]^n = \varrho(z).$$

Define

$$p(z) := \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \tag{18}$$

It follows that

$$\begin{aligned} \omega(z) &= \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\right)z^3 \\ &\quad + \left(\frac{1}{2}c_4 - \frac{1}{2}c_1c_3 - \frac{1}{4}c_2^2 - \frac{1}{16}c_1^4 + \frac{3}{8}c_1^2c_2\right)z^4 + \cdots. \end{aligned} \tag{19}$$

Using (1), we obtain

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1 + a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(a_2^3 - 3a_2a_3 + 3a_4\right)z^3 \\ &\quad + \left(4a_5 - a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2\right)z^4 + \cdots. \end{aligned} \tag{20}$$

Utilizing the series expansion of (19) and some easy calculations, we obtain

$$\begin{aligned} \varrho(z) &= 1 + \frac{n}{2(n+1)}c_1 z + \left[\frac{n}{2(n+1)}c_2 - \frac{n}{4(n+1)}c_1^2\right]z^2 \\ &\quad + \left[\frac{n}{2(n+1)}c_3 - \frac{n}{2(n+1)}c_1c_2 + \frac{n}{8(n+1)}c_1^3\right]z^3 + \cdots. \end{aligned} \tag{21}$$

Now by comparing (20) and (21), we find that

$$a_2 = \frac{n}{2(n+1)}c_1, \tag{22}$$

$$a_3 = \frac{n}{8(n+1)^2}\left[2(n+1)c_2 - c_1^2\right], \tag{23}$$

$$a_4 = \frac{n}{48(n+1)^3}\left[(n+2)c_1^3 - 2\left(n^2+5n+4\right)c_1c_2 + 8\left(n^2+2n+1\right)c_3\right], \tag{24}$$

$$\begin{aligned} a_5 &= \frac{n}{384(n+1)^4}\Big[48(n+1)^3c_4 - (n+2)(2n+3)c_1^4 - 12(n+2)(n+1)^2c_2^2 \\ &\quad -16(n+3)(n+1)^2c_1c_3 + 4(n^2+7n+9)(n+1)c_1^2c_2\Big]. \end{aligned} \tag{25}$$

From (22) and (23), we have

$$\begin{aligned} \left|a_3 - \lambda a_2^2\right| &= \frac{2n(n+1)}{8(n+1)^2}\left|c_2 - \left[\frac{2\lambda n^2 + n}{2n(n+1)}\right]c_1^2\right|, \\ &= \frac{n}{4(n+1)}\left|c_2 - \left[\frac{2\lambda n + 1}{2(n+1)}\right]c_1^2\right|. \end{aligned}$$

Implementation of (14) and using triangle inequality, we obtain

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{n}{2(n+1)} \max\left\{1, \left|\frac{(2\lambda-1)n}{n+1}\right|\right\}. \tag{26}$$

Equality is determined by using the function defined by

$$f_2(z) = z \exp\left(\int_0^z \frac{\left(\varphi_{n-1,\mathcal{L}}(t)^2\right) - 1}{t} dt\right) = z + \frac{n}{2(n+1)} z^3 + \cdots. \tag{27}$$

□

For $\lambda = 1$, we obtain the following corollary:

Corollary 1. *Let $f \in \mathcal{S}^*_{n-1,\mathcal{L}}$. Then,*

$$\left|a_3 - a_2^2\right| \leq \frac{n}{2(n+1)}. \tag{28}$$

This inequality is sharp and can be obtained by using (27).

Theorem 2. *Let $f \in \mathcal{S}^*_{n-1,\mathcal{L}}$ with the series expansion (1). Then,*

$$|a_4 - a_2 a_3| \leq \frac{n}{3(n+1)}. \tag{29}$$

This result is sharp.

Proof. From (22)–(24), we obtain

$$|a_4 - a_2 a_3| = \frac{1}{24(n+1)^3}\left|\left(2n^2 + n\right)c_1^3 - \left(4n^3 + 8n^2 + 4n\right)c_1 c_2 + \left(4n^3 + 8n^2 + 4n\right)c_3\right|.$$

After some easy calculations, it follows that

$$|a_4 - a_2 a_3| = \frac{n}{6(n+1)}\left|c_3 - 2\left(\frac{1}{2}\right)c_1 c_2 + \frac{2n+1}{4(n+1)^2} c_1^3\right|.$$

It is easy to be verified that

$$0 \leq B = \frac{1}{2} \leq 1,\ B = \frac{1}{2} \leq D = \frac{2n+1}{4(n+1)^2},$$

and

$$B(2B-1) = 0 \leq D = \frac{2n+1}{4(n+1)^2}.$$

For an application of Lemma 3, we obtain

$$|a_4 - a_2 a_3| \leq \frac{n}{3(n+1)}.$$

Equality is determined by using

$$f_3(z) = z \exp\left(\int_0^z \frac{\left(\varphi_{n-1,\mathcal{L}}(t)^3\right) - 1}{t} dt\right) = z + \frac{n}{3(n+1)} z^4 + \ldots. \tag{30}$$

□

Theorem 3. *If $f \in \mathcal{S}^*_{n-1,\mathcal{L}}$ is given by (1), then*

$$|\mathcal{H}_{2,2}(f)| = \left|a_2 a_4 - a_3^2\right| \leq \frac{n^2}{4(n+1)^2}. \quad (31)$$

This result is sharp.

Proof. From (22)–(24), we have

$$\begin{aligned} \mathcal{H}_{2,2}(f) &= \frac{1}{192(n+1)^4}\Big|\left(2n^3+n^2\right)c_1^4 - \left(4n^4+8n^3+4n^2\right)c_1^2c_2 + \left(16n^4+32n^3+16n^2\right)c_1c_3 \\ &\quad -\left(12n^4+24n^3+12n^2\right)c_2^2\Big|. \end{aligned}$$

Using (11) and (12) to express c_2 and c_3 in terms of c_1 and assuming that $c_1 = c$, with $0 \leq c \leq 2$, we obtain

$$\begin{aligned} |\mathcal{H}_{2,2}(f)| &= \frac{1}{192(n+1)^4}\Big|-\left(3n^4+6n^3+3n^2\right)x^2\left(4-c^2\right)^2 + \left(8n^4+16n^3+8n^2\right) \\ &\quad \left(4-c^2\right)\left(1-|x|^2\right)c\delta - \left(4n^4+8n^3+4n^2\right)x^2c^2\left(4-c^2\right) - c^4n^4\Big|. \end{aligned}$$

Let $|x| = b$ with $b \leq 1$. By invoking $|\delta| \leq 1$ and the triangle inequality, we see that

$$\begin{aligned} |\mathcal{H}_{2,2}(f)| &\leq \frac{1}{192(n+1)^4}\Big\{\left(3n^4+6n^3+3n^2\right)b^2\left(4-c^2\right)^2 + \left(8n^4+16n^3+8n^2\right) \\ &\quad \left(4-c^2\right)\left(1-b^2\right)c + \left(4n^4+8n^3+4n^2\right)b^2c^2\left(4-c^2\right) + c^4n^4\Big\} := \Xi(c,b). \end{aligned}$$

Since $\Xi(c,b)$ is an increasing function with respect to b, $\Xi(c,b) \leq \Xi(c,1)$. This leads to

$$\begin{aligned} |\mathcal{H}_{2,2}(f)| &\leq \frac{1}{192(n+1)^4}\Big\{\left(3n^4+6n^3+3n^2\right)\left(4-c^2\right)^2 + c^4n^4 \\ &\quad +\left(4n^4+8n^3+4n^2\right)c^2\left(4-c^2\right)\Big\} := G(c). \end{aligned}$$

It is clear that $G(c)$ attains its maximum at $c = 0$. Thus, we obtain

$$|\mathcal{H}_{2,2}(f)| \leq \frac{48n^4+96n^3+48n^2}{192(n+1)^4} = \frac{n^2}{4(n+1)^2}.$$

The equality holds for the extremal function given by

$$f_2(z) = z\exp\left(\int_0^z \frac{\left(\varphi_{n-1,\mathcal{L}}(t)^2\right)-1}{t}dt\right) = z + \frac{n}{2(n+1)}z^3 + \cdots. \quad (32)$$

This completes the proof of Theorem 3. □

Corollary 2. *If $f \in \mathcal{S}^*_{4\mathcal{L}}$ is given by (1), then*

$$|\mathcal{H}_{2,2}(f)| \leq \frac{25}{144}.$$

This inequality is sharp.

Theorem 4. *If $f \in \mathcal{S}^*_{n-1,\mathcal{L}}$ is given by (1), then*

$$|\mathcal{H}_{3,1}(f)| \le \frac{n^2(26n+17)}{72(n+1)^3}.$$

Proof. The third order Hankel determinant can be written as

$$|\mathcal{H}_{3,1}(f)| \le |a_3|\left|a_2a_4 - a_3^2\right| + |a_4||a_4 - a_2a_3| + |a_3|\left|a_3 - a_2^2\right|.$$

Applying the bounds

$$|a_3| \le \frac{n}{2(n+1)},\ |a_4| \le \frac{n}{3(n+1)},\ |a_5| \le \frac{n}{4(n+1)},$$

which was proved by Gandhi et al. in [26] along with (28), (29) and (31), we obtained the required result. □

Corollary 3. *If $f \in \mathcal{S}^*_{3\mathcal{L}}$ is given by (1), then*

$$|\mathcal{H}_{3,1}(f)| \le \frac{242}{1125} \approx 0.215.$$

Corollary 4. *If $f \in \mathcal{S}^*_{4\mathcal{L}}$ is given by (1), then*

$$|\mathcal{H}_{3,1}(f)| \le \frac{1225}{5184} \approx 0.2363.$$

4. Sharp Bounds of Third Hankel Determinant for the Class $\mathcal{S}^*_{4\mathcal{L}}$

It is seen that the upper bound of the third Hankel determinant for $f \in \mathcal{S}^*_{4\mathcal{L}}$ is less than $\frac{1225}{5184}$ from Corollary 4. However, this bound is not sharp. In this section, we aim to give a sharp bound of third Hankel determinant for the class of $\mathcal{S}^*_{4\mathcal{L}}$.

Theorem 5. *Let $f \in \mathcal{S}^*_{4\mathcal{L}}$. Then,*

$$H_{2,3}(f) = \left|a_3a_5 - a_4^2\right| \le \frac{25}{324} \approx 0.0772. \tag{33}$$

This result is sharp.

Proof. By using (23)–(25) for $n = 5$, along with $c_1 = c \in [0,2]$, we have

$$\begin{aligned}\left|a_3a_5 - a_4^2\right| &= \frac{1}{M_0}\Big(A_1c^6 + A_2c^4c_2 + A_3c^3c_3 + A_4c^2c_2^2 \\ &\quad + A_5c_2c_4 + A_6c^2c_4 + A_7cc_2c_3 + A_8c_2^3 + A_9c_3^2\Big),\end{aligned} \tag{34}$$

where $A_1 = 1925$, $A_2 = -54{,}900$, $A_3 = -57{,}600$, $A_4 = 550{,}800$, $A_5 = 9{,}331{,}200$, $A_6 = -777{,}600$, $A_7 = 2{,}073{,}600$, $A_8 = -2{,}721{,}600$, $A_9 = -8{,}294{,}400$ and $M_0 = 429{,}981{,}696$.

Applying Lemma 1, it can be obtained that

$$\left|a_3a_5 - a_4^2\right| = \frac{1}{M_0}\Big(\tau_1(c,x) + \tau_2(c,x)\delta + \tau_3(c,x)\delta^2 + \Psi(c,x,\delta)\rho\Big), \tag{35}$$

where

$$
\begin{aligned}
\tau_1(c,x) &= -450\left(4-c^2\right)x\left[18\left(4-c^2\right)x\left(-8x^2c^2-50xc^2+85c^2-120x\right)\right. \\
&\quad \left.-1080c^4x^2+1480c^4x+125c^4-4320xc^2\right]-15{,}625c^6, \\
\tau_2(c,x) &= -14{,}400\left(4-c^2\right)\left(1-|x|^2\right)c\left[\left(18x^2+90x\right)\left(4-c^2\right)+135xc^2-25c^2\right], \\
\tau_3(c,x) &= -129{,}600\left(4-c^2\right)\left(1-|x|^2\right)\left[\left(2|x|^2+16\right)\left(4-c^2\right)+15\bar{x}c^2\right], \\
\Psi(c,x,\delta) &= 388{,}800\left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\delta|^2\right)\left[6\left(4-c^2\right)x+5c^2\right].
\end{aligned}
$$

Now, by utilizing $|\delta| = y$, $|x| = x$ and taking $|\rho| \leq 1$, we achieve

$$
\begin{aligned}
\left|a_3a_5-a_4^2\right| &\leq \frac{1}{M_0}\left(|\tau_1(c,x)|+|\tau_2(c,x)|y+|\tau_3(c,x)|y^2+|\Phi(c,x,y)|\right), \\
&\leq \frac{1}{M_0}\Gamma(c,x,y), \qquad (36)
\end{aligned}
$$

where

$$
\Gamma(c,x,y) = \chi_1(c,x)+\chi_2(c,x)y+\chi_3(c,x)y^2+\chi_4(c,x)\left(1-y^2\right) \qquad (37)
$$

and

$$
\begin{aligned}
\chi_1(c,x) &= 450\left(4-c^2\right)x\left[18\left(4-c^2\right)x\left(8c^2x^2+50c^2x+120x+85c^2\right)\right. \\
&\quad \left.+1080c^4x^2+1480c^4x+4320c^2x+125c^4\right]+15{,}625c^6, \\
\chi_2(c,x) &= 14{,}400\left(4-c^2\right)\left(1-x^2\right)c\left[\left(18x^2+90x\right)\left(4-c^2\right)+135xc^2+25c^2\right], \\
\chi_3(c,x) &= 129{,}600\left(4-c^2\right)\left(1-x^2\right)\left[\left(2x^2+16\right)\left(4-c^2\right)+15xc^2\right], \\
\chi_4(c,x) &= 388{,}800\left(4-c^2\right)\left(1-x^2\right)\left[6x\left(4-c^2\right)+5c^2\right].
\end{aligned}
$$

Let the closed cuboid be $\Delta : [0,2]\times[0,1]\times[0,1]$. It can easily be observed that $\Gamma(0,0,1) = 33{,}177{,}600$. Denote $m_0 = 3.31776\times 10^7$. Thus, we know

$$
\max\Gamma(c,x,y) \geq m_0, \quad (c,x,y)\in\Delta. \qquad (38)
$$

Now, we will prove that

$$
\max\Gamma(c,x,y) = m_0, \quad (c,x,y)\in\Delta. \qquad (39)
$$

To do this, we first show that $\Gamma(c,x,y)$ is sure to obtain its global maximum value with $y=1$.

For $x=1$, $\Gamma(c,x,y)$ is a function independent of y defined by

$$
\varsigma(c) = 450\left(-111c^6-12{,}012c^4+41{,}184c^2+34{,}560\right). \qquad (40)
$$

It is not hard to calculate that $\varsigma(c)$ achieved its maximum value 1.5552×10^7 at $c_0 = 0$. Thus, it is impossible for Γ to obtain its global maxima on the face of $x=1$. On the face of $c=2$, it is seen that $\Gamma(c,x,y)\equiv 10^6$. Therefore, we can assume $x\in[0,1)$ and $c\in[0,2)$ in the following discussion.

Let $(c,x,y)\in[0,2)\times[0,1)\times(0,1)$. Taking the partial derivative, it follows that

$$
\begin{aligned}
\frac{\partial\Gamma}{\partial y} &= 14{,}400\left(4-c^2\right)\left(1-x^2\right)\left\{18y(x-1)\left[(2x-16)\left(4-c^2\right)+15c^2\right]\right. \\
&\quad \left.+108cx\left[\left(4-c^2\right)+c^2(135x+25)\right]\right\}. \qquad (41)
\end{aligned}
$$

Assume $\frac{\partial\Gamma}{\partial y} = 0$ yields

$$y = \frac{c\left[108\left(4-c^2\right)x + c^2(135x+25)\right]}{18(1-x)[(4-c^2)(2x-16)+15c^2]}. \tag{42}$$

If y_0 is a critical point inside Δ, then $y_0 \in (0,1)$, which is possible only if

$$18\left(4-c^2\right)\left[16+6cx-18x+2x^2\right] + c^3(135x+25) < 270(1-x)c^2 \tag{43}$$

and

$$c^2 > \frac{4(16-2x)}{31-2x}. \tag{44}$$

For the existence of critical points, we must now find solutions that satisfy both inequality (43) and (44). Letting

$$\varpi(x) = \frac{4(16-2x)}{31-2x}. \tag{45}$$

Since $\varpi'(x) < 0$ for $x \in (0,1)$, we see that $\varpi(x)$ is a decreasing function in $(0,1)$. Hence, $c^2 > \frac{56}{29}$. A simple exercise shows that (43) does not hold in this case for all values of $x \in [\frac{1}{2}, 1)$. This means that there are no critical points of Γ in $[0,2) \times [\frac{1}{2}, 1) \times (0,1)$.

If there is a critical point $(\tilde{c}, \tilde{x}, \tilde{y})$ with $\tilde{y} \in (0,1)$ of Γ existing in Δ, it is clear that $\tilde{x} < \frac{1}{2}$ and $\tilde{c}^2 > \varpi(1/2) = 2$. Now, we will prove that $\Gamma(\tilde{c}, \tilde{x}, \tilde{y}) < m_0$. In fact, for $(c,x,y) \in \left(\sqrt{2}, 2\right) \times (0, \frac{1}{2}) \times (0,1)$, by invoking $x < \frac{1}{2}$ and $1 - x^2 < 1$, it is not hard to be seen that

$$\chi_1(c,x) \le \chi_1\left(c, \frac{1}{2}\right) = \phi_1(c) \tag{46}$$

and

$$\chi_j(c,x) \le \frac{4}{3}\chi_j\left(c, \frac{1}{2}\right) := \phi_j(c), \quad j = 2,3,4. \tag{47}$$

Therefore, we have

$$\Gamma(c,x,y) \le \phi_1(c) + \phi_2(c)y + \phi_3(c)y^2 + \phi_4(c)\left(1-y^2\right) := \Upsilon(c,y). \tag{48}$$

Obviously, it can be seen that

$$\frac{\partial\Upsilon}{\partial y} = \phi_2(c) + 2[\phi_3(c) - \phi_4(c)]y \tag{49}$$

and

$$\frac{\partial^2\Upsilon}{\partial y^2} = 2[\phi_3(c) - \phi_4(c)] = 3{,}888{,}000(4-c^2)(-c^2+2). \tag{50}$$

Since $\phi_3(c) - \phi_4(c) \le 0$ for $c \in (\sqrt{2}, 2)$, we obtain that $\frac{\partial^2\Upsilon}{\partial y^2} \le 0$ for $y \in (0,1)$ and thus it follows that

$$\frac{\partial\Upsilon}{\partial y} \ge \frac{\partial\Upsilon}{\partial y}|_{y=1} = 14{,}400(4-c^2)(540+198c-270c^2+43c^3) \ge 0. \tag{51}$$

Therefore, we have

$$\Upsilon(c,y) \le \Upsilon(c,1) = \phi_1(c) + \phi_2(c) + \phi_3(c) := \iota(c). \tag{52}$$

It is calculated that $\iota(c)$ attains its maximum value 3.042114×10^7 at $c = \sqrt{2}$. Thus, we have

$$\Gamma(c,x,y) < m_0, \quad (c,x,y) \in \left(\sqrt{2}, 2\right) \times (0, \frac{1}{2}) \times (0,1). \tag{53}$$

Hence, $\Gamma(\tilde{c}, \tilde{x}, \tilde{y}) < m_0$. Thus, we conclude that $\Gamma(c, x, y) < m_0$ for all $y \in (0,1)$.

Second, we will show that

$$\Gamma(c, x, 0) \leq m_0, \quad (c, x) \in [0,2) \times [0,1), \tag{54}$$

which implies that we can only consider the global maximum value of Γ on the face of $y = 1$.

It is noted that

$$\Gamma(c, x, 0) = \chi_1(c, x) + \chi_4(c, x) := \Omega(c, x) \tag{55}$$

For $x \leq \frac{1}{2}$, we have

$$\chi_1(c, x) \leq \chi_1\left(c, \frac{1}{2}\right) \tag{56}$$

and

$$\chi_4(c, x) \leq \frac{4}{3}\chi_4\left(c, \frac{1}{2}\right). \tag{57}$$

Thus, we know

$$\Omega(c, x) \leq \chi_1\left(c, \frac{1}{2}\right) + \frac{4}{3}\chi_4\left(c, \frac{1}{2}\right) := \varrho(c). \tag{58}$$

A basic calculation shows that $\varrho(c)$ attains its maximum value 2.208844×10^7 at $c \approx 1.109813$. Now, we assume that $x \in \left(\frac{1}{2}, 1\right]$. As it is observed that

$$\Gamma(c, x, 1) = \chi_1(c, x) + \chi_2(c, x) + \chi_3(c, x), \tag{59}$$

we see

$$\Gamma(c, x, 1) - \Gamma(c, x, 0) = \chi_2(c, x) + \chi_3(c, x) - \chi_4(c, x). \tag{60}$$

A simple calculation shows that

$$\chi_3(c, x) - \chi_4(c, x) = 129{,}600\left(4 - c^2\right)\left(1 - x^2\right)U(c, x), \tag{61}$$

where

$$U(c, x) = \left(4 - c^2\right)\left(2x^2 - 18x + 16\right) + 15c^2(x - 1). \tag{62}$$

Written in another form, we have

$$U(c, x) = 8\left(x^2 - 9x + 8\right) + \left(-2x^2 + 33x - 31\right)c^2. \tag{63}$$

As $-2x^2 + 33x - 31 \leq 0$, it follows that

$$U(c, x) \geq 8\left(x^2 - 9x + 8\right) \geq 0, \quad x \in \left[0, \frac{1}{2}\right). \tag{64}$$

This implies that $\chi_3(c, x) \geq \chi_4(c, x)$. In virtue of $\chi_2(c, x) \geq 0$, we obtain that $\chi_2(c, x) + \chi_3(c, x) - \chi_4(c, x) \geq 0$ for $(c, x) \in [0,2) \times \left(\frac{1}{2}, 1\right)$. It yields to

$$\Gamma(c, x, 0) \leq \Gamma(c, x, 1), \quad (c, x) \in (1,2] \times \left[0, \frac{1}{2}\right). \tag{65}$$

Therefore, we only need to find the global maximum value of Γ on the face of $y = 1$ if $(c, x) \in [0,2) \times \left[0, \frac{1}{2}\right)$. As it has been proved that $\Gamma(c, x, y) \leq m_0 = \Gamma(0,0,1)$ for $(c, x) \in [0,2) \times \left[\frac{1}{2}, 1\right)$, it is enough to consider the points of (c, x, y) on the face of $y = 1$ to find the global maxima of Γ in Δ.

From the above discussion, we find that we can restrict on the face $y = 1$ to find the maximum value of $\Gamma(c, x, y)$, which is $\Gamma(c, x, y) \leq \max\Gamma(c, x, 1)$.

On $y = 1$, we see that

$$\begin{aligned}\Gamma(c,x,1) \quad = \quad & 15{,}625c^6 + 450\left(4-c^2\right)\left[800c^3 + \left(125c^4 + 4320c^3 + 4320c^2\right)x\right. \\ & \left(1480c^4 - 800c^3 + 4320c^2\right)x^2 + 1080\left(c^2 - 4c - 4\right)c^2x^3\Big] \\ & +450\left(4-c^2\right)^2\Big[4608 + 288cx + \left(1530c^2 + 576c - 4032\right)x^2 \\ & +\left(900c^2 - 288c + 2160\right)x^3 + 144\left(c^2 - 4c - 4\right)x^4\Big] := Q(c,x).\end{aligned}$$

First, we suppose that $x \leq \frac{2}{3}$. As it is observed that $c^2 - 4c - 4 \leq 0$ with $c \in [0, 2]$, we deduce that

$$\begin{aligned}Q(c,x) \quad \leq \quad & 15{,}625c^6 + 450\left(4-c^2\right)\Big[800c^3 + \left(125c^4 + 4320c^3 + 4320c^2\right)x \\ & +\left(1480c^4 - 800c^3 + 4320c^2\right)x^2\Big] + 450\left(4-c^2\right)^2[4608 \\ & +288cx + \left(1530c^2 + 576c - 4032\right)x^2\Big] \\ & +\left(900c^2 - 288c + 2160\right)x^3 := S(c,x),\end{aligned}$$

In virtue of $900c^2 - 288c + 2160 \geq 0$ and $x \leq \frac{2}{3}$, we have

$$\left(900c^2 - 288c + 2160\right)x^3 \leq \left(600c^2 - 96c + 1440\right)x^2. \tag{66}$$

Hence, we know

$$\begin{aligned}S(c,x) \quad \leq \quad & 15{,}625c^6 + 450\left(4-c^2\right)\Big[800c^3 + \left(125c^4 + 4320c^3 + 4320c^2\right)x \\ & +\left(1480c^4 - 800c^3 + 4320c^2\right)x^2\Big] + 450\left(4-c^2\right)^2[4608 + 288cx \\ & +\left(2130c^2 + 480c - 2592\right)x^2\Big] := T(c,x).\end{aligned}$$

For $c \leq \frac{7}{10}$, it is clear that $1480c^4 - 800c^3 + 4320c^2 \geq 0$ and $2130c^2 + 480c - 2592 \geq 0$. Using $x \leq \frac{2}{3}$, $x^2 \leq \frac{4}{9}$ and $x^2 \leq \frac{2}{3}x$, we have

$$\begin{aligned}T(c,x) \quad \leq \quad & 15{,}625c^6 + 500\left(4-c^2\right)\left(667c^4 + 2992c^3 + 4320c^2\right) \\ & +450\left(4-c^2\right)^2\Big[4608 + \left(1420c^2 + 1248c - 1728\right)x\Big] := L(c,x).\end{aligned}$$

In virtue of $1420c^2 + 1248c - 1728 \leq 0$ for $c \in \left[0, \frac{7}{10}\right]$, we know

$$\begin{aligned}L(c,x) \quad \leq \quad & 15{,}625c^6 + 500\left(4-c^2\right)\left(667c^4 + 2992c^3 + 4320c^2\right) \\ & +2{,}073{,}600\left(4-c^2\right)^2 := \vartheta(c).\end{aligned}$$

It is a simple excise to be verified that $\vartheta(c)$ attains its maximum value $m_0 = 3.31776 \times 10^7$ at $c = 0$ for $c \in \left[0, \frac{7}{10}\right]$.

For $c > \frac{7}{10}$, it is easy to find that $\frac{\partial T}{\partial x} \geq 0$ with $x \in \left[0, \frac{2}{3}\right)$. Thus, obtain

$$T(c,x) \leq T\left(c, \frac{2}{3}\right) \leq 3.205172 \times 10^7 < m_0. \tag{67}$$

Hence, we conclude that

$$Q(c,x) \le m_0, \quad (c,x) \in [0,1) \times \left[0, \frac{2}{3}\right]. \tag{68}$$

Second, we assume that $x \in \left(\frac{2}{3}, 1\right)$. In this case, we have $x^4 \le \frac{2}{3}x^3$. Since $c^2 - 4c - 4 \le 0$ for $c \in [0,2]$, we obtain that

$$114\left(c^2 - 4c - 4\right)x^4 \le 96\left(c^2 - 4c - 4\right)x^3. \tag{69}$$

It follows that

$$\begin{aligned} Q(c,x) \le\ & 15{,}625c^6 + 450\left(4 - c^2\right)\left[800c^3 + \left(125c^4 + 4320c^3 + 4320c^2\right)x\right. \\ & \left.\left(1480c^4 - 800c^3 + 4320c^2\right)x^2 + 1080\left(c^2 - 4c - 4\right)c^2x^3\right] \\ & +450\left(4 - c^2\right)^2\left[4608 + 288cx + \left(1530c^2 + 576c - 4032\right)x^2\right. \\ & \left. + \left(996c^2 - 672c + 1776\right)x^3\right] := K(c,x). \end{aligned}$$

By observing that $c^2 - 4c - 4 \le 0$ and $996c^2 - 672c + 1776 \ge 0$ with $c \in [0,2]$, it is clear that

$$1080\left(c^2 - 4c - 4\right)c^2x^3 \le 720\left(c^2 - 4c - 4\right)c^2x^2 \tag{70}$$

and

$$\left(996c^2 - 672c + 1776\right)x^3 \le \left(996c^2 - 672c + 1776\right)x^2. \tag{71}$$

This yields to

$$\begin{aligned} K(c,x) \le\ & 15{,}625c^6 + 450\left(4 - c^2\right)\left[800c^3 + \left(125c^4 + 4320c^3 + 4320c^2\right)x\right. \\ & \left.\left(2200c^4 - 3680c^3 + 1440c^2\right)x^2\right] + 450\left(4 - c^2\right)^2[4608 + 288cx \\ & \left. + \left(2526c^2 - 96c - 2256\right)x^2\right] := \Pi(c,x). \end{aligned}$$

Let $\Pi(c,x) = d_1(c) + d_2(c)x + d_3(c)x^2$. For $c < \frac{9}{10}$, it is noted that $d_2(c) \ge 0$ and $d_3(c) \le 0$. Then, we have

$$\Pi(c,x) \le d_1(c) + d_2(c) + \frac{4}{9}d_3(c) := \zeta(c). \tag{72}$$

A basic calculation shows that $\zeta(c)$ achieves its maximum value 3.214558×10^7 at $c = \frac{9}{10}$. If $c > \frac{9}{10}$, it is not hard to be checked that $\frac{\partial K}{\partial x} \ge 0$ for all $x \in \left(\frac{2}{3}, 1\right)$. Therefore, we obtain that

$$K(c,x) \le K(c,1) \le 3.215749 \times 10^7 < m_0. \tag{73}$$

Hence, we have

$$Q(c,x) < m_0, \quad (c,x) \in [0,2) \times \left(\frac{2}{3}, 1\right). \tag{74}$$

Combining (68) and (74), we have

$$\Gamma(c,x,1) = Q(c,x) \le m_0, \quad (c,x) \in [0,2) \times [0,1). \tag{75}$$

Since it is proved that the global maximum value of Γ is sure to be attained on the face $y = 1$ of Δ, we have

$$\Gamma(c,x,y) \leq m_0, \quad (c,x) \in [0,2) \times [0,1) \times [0,1]. \tag{76}$$

In addition, it is shown that, for all the points on the faces of $c = 2$ and $x = 1$, Γ have a maxima less than m_0. Then, we can conclude that

$$\Gamma(c,x,y) \leq m_0, \quad (c,x) \in [0,2] \times [0,1] \times [0,1]. \tag{77}$$

Using (36), we obtain that

$$\left|a_3a_5 - a_4^2\right| \leq \frac{m_0}{M_0} = \frac{25}{324}. \tag{78}$$

The equality is achieved by the function given by

$$f_3(z) = z\exp\left(\int_0^z \left(\frac{5}{6}t^2 + \frac{1}{6}t^{14}\right)dt\right) = z + \frac{5}{18}z^4 + \cdots. \tag{79}$$

The proof of Theorem 5 is thus completed. □

Now we will determine the bounds of $\mathcal{H}_{3,1}(f)$ for $f \in \mathcal{S}^*_{4\mathcal{L}}$.

Theorem 6. *Let f be the form of (1) and $f \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$|\mathcal{H}_{3,1}(f)| \leq \frac{25}{324} \approx 0.0772. \tag{80}$$

This result is sharp.

Proof. By using (22)–(25) for $n = 5$, along with $c_1 = c$, we have

$$\begin{aligned} \mathcal{H}_{3,1}(f) &= 2a_2a_3a_4 - a_2^2a_5 - a_3^3 + a_3a_5 - a_4^2 && (81)\\ &= \frac{1}{M_0}\Big(B_1c^6 + B_2c^4c_2 + B_3c^3c_3 + B_4c^2c_2^2 + B_5c_2c_4 \\ &\quad + B_6c^2c_4 + B_7cc_2c_3 + B_8c_2^3 + B_9c_3^2\Big), && (82)\end{aligned}$$

where B_1 =51,425, $B_2 = -801{,}900$, B_3 =2,534,400, $B_4 = -97{,}200$, B_5 =9,331,200, $B_6 = -8{,}553{,}600$, B_7 =12,441,600, B_8 =6,609,600 and $B_9 = -8{,}294{,}400$. By using Lemma 1, we deduce that

$$\mathcal{H}_{3,1}(f) = \frac{1}{M_0}\Big(v_1(c,x) + v_3(c,x)\delta^2 + v_2(c,x)\delta + \Phi(c,x,\delta)\rho\Big), \tag{83}$$

where

$$\begin{aligned} v_1(c,x) &= -450\Big(4-c^2\Big)x\Big[18\Big(4-c^2\Big)x\Big(-8x^2c^2 + 50xc^2 - 35c^2 + 120x\Big) \\ &\quad +1080c^4x^2 - 680c^4x - 125c^4 + 4320xc^2\Big] - 15{,}625c^6, \\ v_2(c,x) &= -14{,}400\Big(4-c^2\Big)\Big(1-|x|^2\Big)c\Big[\Big(18x^2 - 90x\Big)\Big(4-c^2\Big) - 135xc^2 - 25c^2\Big], \\ v_3(c,x) &= -129{,}600\Big(4-c^2\Big)\Big(1-|x|^2\Big)\Big[\Big(2|x|^2+16\Big)\Big(4-c^2\Big) - 15\bar{x}c^2\Big], \\ \Phi(c,x,\delta) &= 388{,}800\Big(4-c^2\Big)\Big(1-|x|^2\Big)\Big(1-|\delta|^2\Big)\Big[6\Big(4-c^2\Big)x - 5c^2\Big]. \end{aligned}$$

Now by utilizing $|\delta| = y$, $|x| = x$ and taking $|\rho| \leq 1$, we achieve

$$\begin{aligned} |\mathcal{H}_{3,1}(f)| &\leq \frac{1}{M_0}\Big(|v_1(c,x)| + |v_2(c,x)|y + |v_3(c,x)|y^2 + |\Phi(c,x,y)|\Big), \\ &\leq \frac{1}{M_0}H(c,x,y), && (84)\end{aligned}$$

where

$$H(c,x,y) = h_1(c,x) + h_2(c,x)y + h_3(c,x)y^2 + h_4(c,x)\left(1-y^2\right) \tag{85}$$

and

$$\begin{aligned} h_1(c,x) &= 450\left(4-c^2\right)x\Big[18\left(4-c^2\right)x\left(8x^2c^2+50xc^2+35c^2+120x\right) \\ &\quad +1080c^4x^2+680c^4x+125c^4+4320xc^2\Big]+15{,}625c^6, \\ h_2(c,x) &= 14{,}400\left(4-c^2\right)\left(1-x^2\right)c\Big[\left(18x^2+90x\right)\left(4-c^2\right)+135xc^2+25c^2\Big], \\ h_3(c,x) &= 129{,}600\left(4-c^2\right)\left(1-x^2\right)\Big[\left(2x^2+16\right)\left(4-c^2\right)+15xc^2\Big], \\ h_4(c,x) &= 388{,}800\left(4-c^2\right)\left(1-x^2\right)\Big[6\left(4-c^2\right)x+5c^2\Big]. \end{aligned}$$

By observing that $h_1(c,x) \le \chi_1(c,x)$, $h_2(c,x) = \chi_2(c,x)$, $h_3(c,x) = \chi_3(c,x)$ and $h_4(c,x) = \chi_4(c,x)$, we have

$$H(c,x,y) \le \Gamma(c,x,y) \le m_0. \tag{86}$$

It follows that

$$\mathcal{H}_{3,1}(f) \le \frac{m_0}{M_0} = \frac{25}{324}. \tag{87}$$

If $f \in \mathcal{S}^*_{4\mathcal{L}}$, then the equality is achieved by the function given by

$$f_3(z) = z\exp\left(\int_0^z \left(\frac{5}{6}t^2 + \frac{1}{6}t^{14}\right)dt\right) = z + \frac{5}{18}z^4 + \cdots. \tag{88}$$

Theorem 6 is thus proved as asserted. □

5. Conclusions

In the current article, we consider a subfamily of starlike function $\mathcal{S}^*_{n-1,\mathcal{L}}$ associated with a domain bounded by an epicycloid with $n-1$ cusps. For functions belonging to this class, we obtain some coefficient inequalities and the upper bounds of second and third Hankel determinants. In particular, for a four-leaf shaped domain, we obtain the sharp bounds of the third Hankel determinant. For the general class, we conjecture that the sharp upper bounds of $|\mathcal{H}_{3,1}(f)|$ for $f \in \mathcal{S}^*_{n-1,\mathcal{L}}$ is $\frac{n^2}{9(n+1)^2}$ with equality achieved by the function given in (30).

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