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Approximate Controllability of Non-Instantaneous Impulsive Stochastic Evolution Systems Driven by Fractional Brownian Motion with Hurst Parameter $H \in (0, \frac{1}{2})$

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Abstract: This paper initiates a study on the existence and approximate controllability for a type of non-instantaneous impulsive stochastic evolution equation (ISEE) excited by fractional Brownian motion (fBm) with Hurst index $0 < H < 1/2$. First, to overcome the irregular or singular properties of fBm with Hurst parameter $0 < H < 1/2$, we define a new type of control function. Then, by virtue of the stochastic analysis theory, inequality technique, the semigroup approach, Krasnoselskii's fixed-point theorem and Schaefer's fixed-point theorem, we derive two new sets of sufficient conditions for the existence and approximate controllability of the concerned system. In the end, a concrete example is worked out to demonstrate the applicability of our obtained results.

Keywords: approximate controllability; non-instantaneous impulses; fractional Brownian motion; stochastic evolution equations

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1. Introduction

As a significant property of dynamical control systems, controllability implies that it is possible to steer the state of the system from an arbitrary initial state to a target state by choosing a suitable control from the set of admissible controls. The fundamental concept of controllability was introduced by Kalman [1] in 1960. Afterward, extensive studies of controllability for linear and nonlinear systems in finite and infinite dimensional spaces emerged, one after another [2–5]. Moreover, taking into account the reality and inevitability of stochastic effects, many authors investigated the controllability problems of stochastic differential equations (SDEs) with different kinds of noises: for instance, see [6–9] and the references therein.

On the other side, impulsive dynamical systems arise in the description of mathematical modeling of real-world systems which are affected by instantaneous perturbations or non-instantaneous impulses (see, for example, [10–16] and the references therein). Naturally, the controllability of impulsive stochastic differential equations (ISDEs) have been discussed heatedly; for example, see [17–22] and the references therein. In the case of non-instantaneous impulses, for instance, the approximate controllability of a class of multi-valued impulsive fractional stochastic partial integro-differential equation (FISPIDE) with infinite delay was explored by Yan and Lu [23]; recently, Yan and Han [24] derived the approximate controllability result of a type of neutral FISPIDE with noncompact operators. Note that most of the noises they considered in the aforementioned researches are uncorrelated. Based on this problem, the controllability of various types of ISDEs excited by fBm with Hurst index $1/2 < H < 1$ have been researched by many authors; see, for example, [25–27] and their cited references. Here it is worth mentioning that Dhayal et al. [28] obtained the approximate controllability results of a kind of fractional non-instantaneous ISDEs driven by fBm with Hurst parameter $H \in (1/2, 1)$ in Hilbert space. Since the properties of the fBm with $0 < H < 1/2$ are more irregular and singular, this makes it especially difficult to study the approximate controllability of impulsive stochastic systems driven by fBm with $H \in (0, 1/2)$. Fortunately, Li and Yan [29] showed some new estimations on the stochastic integral of fBm with Hurst index H lesser than $1/2$. Very recently, Li, Jing and Xu [30] ran a study on the exact controllability of a type of neutral SEEs with fBm ($0 < H < 1/2$) by the aid of the above-mentioned established estimates and the Banach fixed-point theorem. However, to date, there is no research on the approximate controllability for ISDEs driven by fBm with $H \in (0, 1/2)$, not to mention the case of non-instantaneous impulses. As a weak concept of controllability, approximate controllability is more useful than exact controllability in practice [31]. Accordingly, we urgently need to make up for this deficiency.

In this article, we consider the existence and approximate controllability problem for the non-instantaneous ISEEs excited by fBm with Hurst index $H \in (0, 1/2)$ of the following form:

$$\begin{cases} dx(t) = [Ax(t) + Bu(t) + b(t, x(t))]dt + g(t)dB_Q^H(t), t \in \cup_{k=0}^N (s_k, t_{k+1}], \\ x(t) = I_k(t, x(t)),\ t \in \cup_{k=1}^N (t_k, s_k], \\ x(0) = x_0, \end{cases} \tag{1}$$

where $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}, B : \mathcal{U} \to \mathbb{H}$ is a bounded linear operator, the control function $u(\cdot)$ takes value in $L^2_{\mathcal{F}}([0, T], \mathcal{U})$ and $B_Q^H(t)$ symbolizes a fBm with Hurst index $H \in (0, 1/2)$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{K}$. Let $0 = t_0 = s_0 < t_1 < s_1 < \cdots < t_N < s_N < t_{N+1} = T, b, g$ and I_k be satisfying suitable conditions to be specified later. Moreover, the initial datum x_0 is an $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable independent of $B_Q^H(t)$.

As stated above, this work is devoted to deriving the existence and approximate controllability results of a class of non-instantaneous ISEEs driven by fBm with Hurst index $0 < H < 1/2$. We employ the inequality technique, the estimated results of Li and Yan [29], some technical transformations, Krasnoselskii's fixed-point theorem and Schaefer's fixed-point theorem to overcome difficulties brought by the introduction of fBm with Hurst index $0 < H < 1/2$ and the non-instantaneous impulses. Also worth noting is that we have to define a new control function, which differs from the existing studies on the approximate controllability of ISDEs excited by fBm with $1/2 < H < 1$.

The organization of the rest work is as follows: Section 2 introduces the needed notations, hypotheses, definitions and lemmas. Section 3 formulates and proves two different sets of sufficient conditions for the existence and approximate controllability of system (1). Finally, an example to illustrate our results is given in Section 4.

2. Preliminaries

$(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space endowed with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. $\mathbb{H}, \mathbb{K}$ denotes two real, separable Hilbert spaces. Let $L(\mathbb{K}, \mathbb{H})$ be the space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$. For the sake of simplicity, throughout this paper, the same notation $\|\cdot\|$ is used to denote the norms in different spaces. $\mathcal{C}_T = \mathcal{PC}\big([0, T], L^2(\Omega, \mathbb{H})\big)$ expresses the family of all $\mathcal{F}_t$-adapted, $\mathbb{H}$-valued processes $\{x(t), t \in [0, T]\}$, where $x(t)$ is continuous at $t \neq t_k, k = 1, 2, \cdots, N$, and there exist $x(t_k^+)$ and $x(t_k^-)$ with $x(t_k^-) = x(t_k)$ and $\sup_{t \in [0,T]} \mathbb{E}\|x(t)\|^2 < \infty$, equipped with the norm: $\left(\sup_{t \in [0,T]} \mathbb{E}\|x(t)\|^2\right)^{1/2}$.

Let $J = [0, T]$, $\{B^H(t)\}_{t \in J}$ is a one-dimensional fBm, where the Hurst index $H \in (0, 1)$. When $0 < H < 1/2$, introduce the kernel operator

$$K_H(t, s) = c_H \left[\left(\frac{t(t - s)}{s} \right)^{H-1/2} - (H - 1/2)s^{1/2-H} \int_s^t (u - s)^{H-1/2} u^{H-3/2} du \right], \tag{2}$$

where

$$c_H = \left(\frac{H}{(1-2H)\mathrm{B}(1-2H, H+1/2)} \right)^{1/2},$$

for $t > s$, $\mathrm{B}(\cdot,\cdot)$ is the Beta function. When $t \leq s$, we set $K_H(t,s) = 0$. It follows from (2) that

$$|K_H(t,s)| \leq 2C_H \left(\frac{(t-s)^H}{\sqrt{t-s}} + \frac{s^H}{\sqrt{s}} \right).$$

Additionally, the following inequality holds:

$$\left| \frac{\partial K_H}{\partial t}(t,s) \right| \leq c_H \left(\frac{1}{2} - H \right) (t-s)^{H-\frac{3}{2}}.$$

In addition, notice that $B^{1/2}$ is standard Bm, and B^H has the Wiener integral in the following form:

$$B^H(t) = \int_0^t K_H(t,s) dW(s). \tag{3}$$

Let Λ be the space of step functions on J of the following form:

$$\phi(t) = \sum_{j=1}^{m-1} x_j \chi_{[t_j, t_{j+1})}(t),$$

where $x_j \in \mathbb{R}, 0 = t_1 < t_2 < \cdots < t_m = T$. Denote $\mathfrak{H}$ as the Hilbert space of the closure of Λ with scalar product $\left\langle \chi_{[0,t]}, \chi_{[0,s]} \right\rangle_{\mathfrak{H}} = R_H(t,s)$. Then, the mapping

$$\phi = \sum_{j=1}^{m-1} x_j \chi_{[t_j, t_{j+1})}(t) \mapsto \int_0^T \phi(s) dB^H(s)$$

becomes an isometry between Λ and $span\{B^H, t \in J\}$, and it can be expanded to an isometry between $\mathfrak{H}$ and $\overline{span}^{L^2(\Omega)}\{B^H, t \in J\}$. For any $s < T$, we consider the following linear operator $K_{H,T}^* : \mathfrak{H} \to L^2(J)$,

$$(K_{H,T}^* \phi)(t) = K_H(T,t)\phi(t) + \int_t^T (\phi(s) - \phi(t)) \frac{\partial K_H}{\partial s}(s,t) ds.$$

It is known that $K_{H,T}^*$ turns into an isometry between $\mathfrak{H}$ and $L^2(J)$. In this way, for every $\phi \in \mathfrak{H}$, the following relationship

$$\int_0^T \phi(s) dB^H(s) := B^H(\varphi) = \int_0^T \left(K_{H,T}^* \phi \right)(s) dW(s)$$

holds if and only if $K_{H,T}^* \in L^2(J)$, where the integrals $\int \cdot dB^H$, $\int \cdot dW$ should be interpreted as the Wiener integrals with regard to fBm and the Wiener process W, respectively.

Let $Q \in L(\mathbb{K}, \mathbb{K})$ indicate a non-negative self-adjoint operator, $L_Q^0(\mathbb{K}, \mathbb{H})$ denote the space of all $\xi \in L(\mathbb{K}, \mathbb{H})$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert–Schmidt operator endowed with the norm $\|\xi\|_{L_Q^0(\mathbb{K},\mathbb{H})}^2 = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n} \xi(s) e_n \right\|^2 = tr(\xi Q \xi^*)$. Let $\left\{ B_j^H(t) \right\}_{j \in \mathbb{N}}$ be a sequence of two-sided one-dimensional standard fBm mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$; if we assume further that Q is nuclear, then the infinite-dimensional fBm on $\mathbb{K}$ is defined by

$$B_Q^H(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} e_j B_j^H(t), \quad t \geq 0.$$

Definition 1. *For any $\psi : J \to L_Q^0(\mathbb{K}, \mathbb{H})$ satisfing the condition $\sum_{j=1}^{\infty} \lambda_j \|K_H^*(\varphi e_n)\| < \infty$, the Wiener integral for ψ of the fBm B_Q^H is well defined by*

$$\int_0^t \psi(s) dB_Q^H(s) := \sum_{j=1}^{\infty} \int_0^t \sqrt{\lambda_j} \psi(s) e_j dB_j^H(s)$$
$$= \sum_{j=1}^{\infty} \int_0^t \sqrt{\lambda_j} (K_H^*(\psi e_j))(s) dW_j(s),$$

where W_j is the standard Bm, same as that in (3).

One can refer to [29,30,32,33] for more particulars about $B_Q^H(t)$ and the stochastic integral with regard to $B_Q^H(t)$.

Before proceeding any further, we introduce some needed results on $(-A)^\alpha$ and the analytic semigroup $T(t)$ generated by A (Ref. [34], Theorem 6.13, p. 74).

Lemma 1. *Let A be the infinitesimal generator of an analytic semigroup $\{T(t)\}$. If $0 \in \rho(A)$, then*
(a) $T(t) : \mathbb{H} \to \mathcal{D}(-A)^\alpha$ for every $t > 0$ and $\alpha \geq 0$.
(b) For every $x \in \mathcal{D}(-A)^\alpha$, we have $T(t)(-A)^\alpha x = (-A)^\alpha T(t)x$.
(c) The operator $(-A)^\alpha$ is bounded and

$$\|(-A)^\alpha T(t)\| \leq M_\alpha t^{-\alpha} e^{-\gamma t}, \ t > 0.$$

(d) For $0 < \alpha \leq 1$ and $x \in \mathcal{D}(-A)^\alpha$, there exists $C_\alpha > 0$ such that

$$\left\|(T(t) - \mathbb{I})(-A)^{-\alpha}\right\| \leq C_\alpha t^\alpha.$$

Following Ref. [28], the definition of a mild solution to system (1) is introduced.

Definition 2. *A $\mathbb{H}$-valued stochastic process $x(t)$ is said to be a mild solution of the system (1), if*
(a) $t \in [0, T]$, $x(t)$, is $\mathcal{F}_t$-adapted and has càdlàg paths a.s.
(b) $x(t) = I_k(t, x(t))$ for all $t \in (t_k, s_k], k = 1, 2, \cdots, N$ and $x(t)$ satisfies the following integral equations

$$\begin{aligned} x(t) =& T(t)x_0 + \int_0^t T(t-s)[Bu(s) + b(s, x(s))]ds + \int_0^t T(t-s)g(s)dB_Q^H(s), \\ & \forall t \in [0, t_1], \\ x(t) =& T(t - s_k)I_k(s_k, x(s_k^-)) + \int_{s_k}^t T(t-s)[Bu(s) + b(s, x(s))]ds \\ &+ \int_{s_k}^t T(t-s)g(s)dB_Q^H(s), \ \forall t \in [s_k, t_{k+1}], \ k = 1, 2, \cdots, N. \end{aligned}$$

Definition 3. *The system (1) is said to be approximately controllable on the interval $[0, T]$, if $\overline{R(T, x_0)} = L^2(\Omega, \mathbb{H})$, where*

$$R(T, x_0) = \left\{x(T; x_0, u) : u \in L^2([0, T]; \mathcal{U})\right\}$$

is the reachable set of (1) at terminal time T.

Note that we have the following lemma [35] about the Q-Wiener process $\{W(t)\}_{t \geq 0}$.

Lemma 2. *For any $x_{t_{k+1}} \in L^2(\Omega, \mathcal{F}_{t_{k+1}}, \mathbb{H})$, there exists $\psi_k \in L^2_{\mathcal{F}}\big([s_k, t_{k+1}], L_2^0(\mathbb{K}, \mathbb{H})\big)$ such that*

$$x_{t_{k+1}} = \mathbb{E}x_{t_{k+1}} + \int_{s_k}^{t_{k+1}} \psi_k(s)dW(s).$$

Remark 1. *In the existing literature, they use the similar property of fBm with the Hurst index $1/2 < H < 1$ to define the control function (see, for example, Refs. [25,27,28]). However, since the more irregular or singular properties of fBm with Hurst parameter are $0 < H < 1/2$, we do not have a similar formula for fBm with $0 < H < 1/2$ as in Lemma 2. Hence, here we need to construct a different type of control function.*

Now for any $z > 0$ and $x_{t_{k+1}} \in L^2(\Omega, \mathcal{F}_{t_{k+1}}, \mathbb{H})$, combining the technique shown in Ref. [9], we define the control function:

$$\begin{aligned} u^z(t,x) =& B^*T^*(t_{k+1}-t)\Big[\big(zI + \Pi_{s_k}^{t_{k+1}}\big)^{-1}\big(\mathbb{E}x_{t_{k+1}} - T(t_{k+1}-s_k)I_k(s_k, x(s_k^-))\big)\Big] \\ &-B^*T^*(t_{k+1}-t)\int_{s_k}^{t_{k+1}} \big(zI + \Pi_{s_k}^{t_{k+1}}\big)^{-1} T(t_{k+1}-s)b(s,x(s))ds \\ &-B^*T^*(t_{k+1}-t)\int_{s_k}^{t_{k+1}} \big(zI + \Pi_{s_k}^{t_{k+1}}\big)^{-1} T(t_{k+1}-s)g(s)dB_Q^H(s) \\ &+B^*T^*(t_{k+1}-t)\int_{s_k}^{t_{k+1}} \big(zI + \Pi_{s_k}^{t_{k+1}}\big)^{-1} \psi_k(s)dW(s), \end{aligned} \tag{4}$$

where $x_{t_{k+1}} = \mathbb{E}x_{t_{k+1}} + \int_{s_k}^{t_{k+1}} \psi_k(s)dW(s)$ with Lemma 2 and $k = 0, 1, \cdots, N$, $I_0(0, \cdot) = x_0$, $x_{t_{N+1}} = x_T$.

We end this section by stating Krasnoselskii's fixed-point theorem [36] and Schaefer's fixed-point theorem [37], which are key tools in proving the existence of mild solutions to system (1).

Theorem 1. *Let $\mathfrak{B}$ be a bounded, closed and convex subset of a Banach space $\mathcal{Z}$, and let Φ_1, Φ_2 be maps from $\mathfrak{B}$ to $\mathcal{Z}$ such that $\Phi_1 x + \Phi_2 y \in \mathfrak{B}$ whenever $x, y \in \mathfrak{B}$. If Φ_1 is a contraction mapping and Φ_2 is compact and continuous, then there exists $x \in \mathfrak{B}$ such that $x = \Phi_1 x + \Phi_2 x$.*

Theorem 2. *Let $\mathcal{X}$ be a Banach space and $\Phi : \mathcal{X} \to \mathcal{X}$ be a completely continuous operator. If the set $\mathrm{S}(\Phi) = \{x \in \mathcal{X} : x = \lambda\Phi(x), for\ some\ \lambda \in (0,1)\}$ is bounded, then Φ has a fixed point on $\mathcal{X}$.*

3. Main Results

In this section, our goal is to obtain the results on existence and approximate controllability of system (1). We divide the process into two steps: Step 1, we show the existence of mild solutions to the non-instantaneous ISEEs driven by fBm with Hurst parameter $H \in (0, 1/2)$. Step 2, under given assumptions, we prove that the stochastic control system (1) is approximately controllable on $[0, T]$.

In the first part of this section, we discuss this problem with the following hypotheses.

$(A1)$ $A : \mathcal{D}(A) \to \mathbb{H}$, is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ on $\mathbb{H}$ and for any $t > 0$, $T(t)$ is compact. In this case, there exist two constants $M \geq 1, \lambda > 0$ such that

$$\|T(t)\| \leq Me^{-\lambda t}$$

for all $t \geq 0$.

$(A2)$ The function $b: T_0 \times \mathbb{H} \to \mathbb{H}, T_0 = \cup_{k=0}^{N}(s_k, t_{k+1}]$ satisfies the global Lipschitz condition and the linear growth condition, that is, for all $t \in T_0, x, y \in \mathbb{H}$, there exists two positive constants C_1, C_2 such that

$$\|b(t,x) - b(t,y)\|^2 \leq C_1\|x-y\|^2,\ \|b(t,x)\|^2 \leq C_2\Big(1 + \|x\|^2\Big).$$

$(A3)$ The mapping $g: T_0 \to L_Q^0(\mathbb{K}, \mathbb{H})$ satisfies the Hölder continuous condition, i.e., for any $t, s \in T_0$, there exists a positive constant C_g such that

$$\|g(t) - g(s)\|_{L_Q^0} \leq C_g|t-s|^{\beta},$$

with $\beta > \frac{1}{2} - H$.

$(A4)$ The functions $I_k: T_k \times \mathbb{H} \to \mathbb{H}, T_k = (t_k, s_k], k = 1, 2, \cdots, N$ are continuous and there exist positive constants c_k, d_k such that for $\forall t \in T_k, x, y \in \mathbb{H}$,

$$\|I_k(t,x) - I_k(t,y)\|^2 \leq c_k\|x-y\|^2,\ \|I_k(t,x)\|^2 \leq d_k\Big(1 + \|x\|^2\Big).$$

with $c_k, d_k < 1$ and set $c_0 = d_0 = 0$.

$(A5)$ The operators $z\Big(zI + \Pi_{s_k}^{t_{k+1}}\Big)^{-1} \to 0$ in the strong operator topology as $z \to 0^+$, where

$$\Pi_{s_k}^{t_{k+1}} = \int_{s_k}^{t_{k+1}} T(t_{k+1}-s)BB^*T^*(t_{k+1}-s)ds,$$

i.e., the linear deterministic control system corresponding to system (1) is approximately controllable on $[0, T]$.

$(A6)$ Let $n_k = e^{-2\lambda(t_{k+1}-s_k)}$, and the following inequality holds:

$$\max_{0\leq k\leq N}\left\{P_k, 2M^2c_k + \frac{M^2M_B^2(1-n_k)^2}{4\lambda^2 n_k}M_k\right\} < 1.$$

Remark 2. *Assumption (A6) is a contraction condition to guarantee the existence of a mild solution to system (1), where M_k is defined in Lemma 5 and P_k is defined in Theorem 3.*

For the subsequent work, we state two useful lemmas which can be found in Ref. [29].

Lemma 3. *Let $g: J \to L_Q^0(\mathbb{K}, \mathbb{H})$ meet the condition $(A3)$, then there exist $C_3, C_4 > 0$ depending on M, λ, β and H such that*

$$\mathbb{E}\left\|\int_0^t T(t-s)g(s)dB_Q^H(s)\right\|^2 \leq C_3 + C_4 t^{2H+\gamma-1},\ \gamma \in (1-H, 1).$$

Lemma 4. *Supposed that $g: J \to L_Q^0(\mathbb{K}, \mathbb{H})$ satisfies the assumption $(A3)$. Then, we have*

$$\mathbb{E}\left\|\int_0^{t+\delta} T(t+\delta-s)g(s)dB_Q^H(s) - \int_0^t T(t-s)g(s)dB_Q^H(s)\right\|^2 \leq C\delta^{2\alpha},$$

for each $0 \leq t < t+\delta \leq T, \delta \in (0,1), 0 < \alpha < H$. In particular, we have

$$\mathbb{E}\left\|\int_t^{t+\delta} T(t+\delta-s)g(s)dB_Q^H(s)\right\|^2 \leq C\delta^{2\alpha},$$

where C is a positive constant.

To prove the main results, we also need to show the following lemma.

Lemma 5. *For any $x, y \in \mathcal{C}_T$, there exist positive constants M_k and R_k such that*

$$\mathbb{E}\|u^z(t,x) - u^z(t,y)\|^2 \leq M_k \|x-y\|^2_{\mathcal{PC}},\ \mathbb{E}\|u^z(t,x)\|^2 \leq R_k,$$

where

$$\begin{aligned} M_k =& \frac{M_B^2 M^2 e^{-2\lambda(t_{k+1}-t)}}{z^2}\left(2M^2 e^{-2\lambda(t_{k+1}-s_k)} c_k + \frac{M^2 C_1 (t_{k+1}-s_k)}{\lambda}\right), \\ R_k =& \frac{4M_B^2 M^2 e^{-2\lambda(t_{k+1}-t)}}{z^2}\bigg[M^2 e^{-2\lambda(t_{k+1}-s_k)} d_k (1+K) + \mathbb{E}\big\|x_{t_{k+1}}\big\|^2 \\ &+ \frac{M^2 C_2 (1+K)(t_{k+1}-s_k)}{2\lambda} + C_3 + C_4 (t_{k+1}-s_k)^{2H+\gamma-1}\bigg], \end{aligned}$$

and $K \geq \max\limits_{1\leq k\leq N}\left[\frac{Q_0}{1-P_0}, \frac{d_k}{1-d_k}, \frac{Q_k}{1-P_k}\right]$, Q_k are defined in Theorem 3.

Proof. For any $x, y \in \mathcal{C}_T$, it follows from Equation (4) that

$$\begin{aligned} &\mathbb{E}\|u^z(t,x) - u^z(t,y)\|^2 \\ \leq& \frac{M_B^2 M^2 e^{-2\lambda(t_{k+1}-t)}}{z^2}\bigg(2M^2 e^{-2\lambda(t_{k+1}-s_k)} \mathbb{E}\big\|I_k(s_k, x(s_k^-)) - I_k(s_k, y(s_k^-))\big\|^2 \\ &+ 2M^2 \int_{s_k}^{t_{k+1}} e^{-2\lambda(t_{k+1}-s)} ds \int_{s_k}^{t_{k+1}} \mathbb{E}\|b(s,x(s)) - b(s,y(s))\|^2 ds\bigg), \end{aligned}$$

then, the hypotheses $(A2)$ and $(A4)$ lead to

$$\begin{aligned} &\mathbb{E}\|u^z(t,x) - u^z(t,y)\|^2 \\ \leq& \frac{M_B^2 M^2 e^{-2\lambda(t_{k+1}-t)}}{z^2}\left(2M^2 e^{-2\lambda(t_{k+1}-s_k)} c_k + \frac{M^2 C_1 (t_{k+1}-s_k)}{\lambda}\right)\|x-y\|^2_{\mathcal{PC}}. \end{aligned}$$

To continue, for Equation (4), the elementary inequality and Lemma 3 yield that

$$\begin{aligned} &\mathbb{E}\|u^z(t,x)\|^2 \\ \leq& \frac{M_B^2 M^2 e^{-2\lambda(t_{k+1}-t)}}{z^2}\bigg(4M^2 e^{-2\lambda(t_{k+1}-s_k)} \mathbb{E}\big\|I_k(s_k, x(s_k^-))\big\|^2 + 4\mathbb{E}\big\|x_{t_{k+1}}\big\|^2 \\ &+ 4M^2 \int_{s_k}^{t_{k+1}} e^{-2\lambda(t_{k+1}-s)} ds \int_{s_k}^{t_{k+1}} \mathbb{E}\|b(s,x(s))\|^2 ds \\ &+ 4C_3 + 4C_4 (t_{k+1}-s_k)^{2H+\gamma-1}\bigg), \end{aligned}$$

that is,

$$\begin{aligned} \mathbb{E}\|u^z(t,x)\|^2 \leq& \frac{4M_B^2 M^2 e^{-2\lambda(t_{k+1}-t)}}{z^2}\bigg[M^2 e^{-2\lambda(t_{k+1}-s_k)} d_k (1+K) + \mathbb{E}\big\|x_{t_{k+1}}\big\|^2 \\ &+ \frac{M^2 C_2 (1+K)(t_{k+1}-s_k)}{2\lambda} + C_3 + C_4 (t_{k+1}-s_k)^{2H+\gamma-1}\bigg]. \end{aligned}$$

Hence, the statements of Lemma 5 are proved. □

Theorem 3. *Assume that the hypotheses $(A1)$–$(A6)$ are satisfied. Then the non-instantaneous impulsive stochastic control system (1) has at least one mild solution on $[0, T]$.*

Proof. We transform the existence problem of (1) into a fixed-point one. Consider the following two operators Φ_1 and Φ_2 on

$$\mathcal{S}_K = \left\{x \in C_T, \|x\|^2_{\mathcal{PC}} \leq K\right\} \subseteq \mathcal{C}_T$$

of the form

$$(\Phi_1 x)(t) = \begin{cases} T(t)x_0 + \int_0^t T(t-s)Bu^z(s,x)ds, & t \in [0,t_1], \\ I_k(t,x(t)), & t \in (t_k, s_k], \\ T(t-s_k)I_k(s_k, x(s_k^-)) + \int_{s_k}^t T(t-s)Bu^z(s,x)ds, & t \in (s_k, t_{k+1}], \end{cases}$$

and

$$(\Phi_2 x)(t) = \begin{cases} \int_0^t T(t-s)b(s,x(s))ds + \int_0^t T(t-s)g(s)dB_Q^H(s), & t \in [0,t_1], \\ 0, & t \in (t_k, s_k], \\ \int_{s_k}^t T(t-s)b(s,x(s))ds + \int_{s_k}^t T(t-s)g(s)dB_Q^H(s), & t \in (s_k, t_{k+1}]. \end{cases}$$

Next, we divide our proof into three steps. In step 1, we show that $\Phi_1 x + \Phi_2 y \in \mathcal{S}_K$ for any $x, y \in \mathcal{S}_K$. In Step 2, we demonstrate Φ_1 is a contraction. Then we prove that Φ_2 is continuous and compact in Step 3. As a result, we combine steps 1 through 3 to complete the proof based on Theorem 1.
Step 1. For any $t \in [0, t_1]$ and $x, y \in \mathcal{S}_K$, the elementary inequality yields that

$$\begin{aligned} \mathbb{E}\|(\Phi_1 x)(t) + (\Phi_2 y)(t)\|^2 \leq & 4\mathbb{E}\|T(t)x_0\|^2 + 4\mathbb{E}\left\|\int_0^t T(t-s)Bu^z(s,x)ds\right\|^2 \\ & +4\mathbb{E}\left\|\int_0^t T(t-s)b(s,y(s))ds\right\|^2 \\ & +4\mathbb{E}\left\|\int_0^t T(t-s)g(s)dB_Q^H(s)\right\|^2. \end{aligned}$$

With the aid of hypotheses $(A1)$–$(A4)$, Lemma 3, we have

$$\begin{aligned} &\mathbb{E}\|(\Phi_1 x)(t) + (\Phi_2 y)(t)\|^2 \\ &\leq 4M^2\mathbb{E}\|x_0\|^2 + \frac{2M^2M_B^2}{\lambda}\int_0^{t_1}\mathbb{E}\|u^z(s,x)\|^2 ds + \frac{2M^2}{\lambda}\int_0^{t_1} C_2(1+\mathbb{E}\|y(s)\|^2)ds \\ &+4(C_3 + C_4 t_1^{2H+\alpha-1}) \\ &\leq 4M^2\mathbb{E}\|x_0\|^2 + \frac{6M^4M_B^4}{z^2\lambda^2}\left[\mathbb{E}\|x_{t_1}\|^2 + \frac{M^2C_2(1+K)t_1}{2\lambda} + C_3 + C_4 t_1^{2H+\gamma-1}\right] \\ &+\frac{2M^2C_2(1+K)t_1}{\lambda} + 4(C_3 + C_4 t_1^{2H+\gamma-1}) \\ &\leq Q_0 + P_0 K \leq K, \end{aligned} \tag{5}$$

where

$$\begin{aligned} Q_0 = \; & 4M^2\mathbb{E}\|x_0\|^2 + \frac{6M^4M_B^4}{z^2\lambda^2}\mathbb{E}\|x_{t_1}\|^2 + \left(\frac{6M^4M_B^4}{z^2\lambda^2} + 1\right)\frac{2M^2C_2t_1}{\lambda} \\ & + \left(\frac{6M^4M_B^4}{z^2\lambda^2} + 4\right)\left(C_3 + C_4 t_1^{2H+\gamma-1}\right) \end{aligned}$$

and

$$P_0 = \left(\frac{3M^4M_B^4}{2z^2\lambda^2} + 1\right)\frac{2M^2C_2t_1}{\lambda}.$$

When $t \in (t_k, s_k]$, the hypothesis $(A4)$ leads to

$$\mathbb{E}\|(\Phi_1 x)(t) + (\Phi_2 y)(t)\|^2 = \mathbb{E}\|I_k(t, x(t))\|^2 \le d_k(1+K) \le K. \tag{6}$$

For $t \in (s_k, t_{k+1}]$, estimating as above, we obtain

$$\begin{aligned}
&\mathbb{E}\|(\Phi_1 x)(t) + (\Phi_2 y)(t)\|^2 \\
\le& 4M^2 d_k(1 + \mathbb{E}\|x(s_k^-)\|^2) + \frac{2M^2 M_B^2}{\lambda}\int_{s_k}^{t_{k+1}} \mathbb{E}\|u^z(s,x)\|^2 ds \\
&+ \frac{2M^2}{\lambda}\int_{s_k}^{t_{k+1}} C_2(1+\mathbb{E}\|y(s)\|^2)ds + 4(C_3 + C_4(t_{k+1}-s_k)^{2H+\gamma-1}) \\
\le& 4M^2 d_k(1+K) + \frac{8M^4 M_B^4}{z^2\lambda^2}\Big[M^2 e^{-2\lambda(t_{k+1}-s_k)} d_k(1+K) + \mathbb{E}\|x_{t_{k+1}}\|^2 \\
&+ \frac{M^2 C_2(1+K)(t_{k+1}-s_k)}{2\lambda} + C_3 + C_4(t_{k+1}-s_k)^{2H+\gamma-1}\Big] \\
&+ \frac{2M^2 C_2(1+K)(t_{k+1}-s_k)}{\lambda} + 4(C_3 + C_4(t_{k+1}-s_k)^{2H+\gamma-1}) \\
\le& Q_k + P_k K \le K,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
Q_k =& 4M^2 d_k + \frac{8M^6 M_B^4}{z^2\lambda^2}\left(n_k d_k + \frac{C_2(t_{k+1}-s_k)}{2\lambda}\right) + \frac{2M^2 C_2(t_{k+1}-s_k)}{\lambda} \\
&+ \frac{8M^4 M_B^4}{z^2\lambda^2}\mathbb{E}\|x_{t_{k+1}}\|^2 + \left(\frac{8M^4 M_B^4}{z^2\lambda^2} + 4\right)(C_3 + C_4(t_{k+1}-s_k)^{2H+\gamma-1})
\end{aligned}$$

and

$$P_k = 4M^2 d_k + \frac{8M^6 M_B^4}{z^2\lambda^2}\left(n_k d_k + \frac{C_2(t_{k+1}-s_k)}{2\lambda}\right) + \frac{2M^2 C_2(t_{k+1}-s_k)}{\lambda}.$$

The above arguments imply that $\Phi_1 x + \Phi_2 y \in \mathcal{S}_K$ whenever $x, y \in \mathcal{S}_K$.

Step 2. For any $t \in [0, t_1]$ and $x, y \in \mathcal{S}_K$, by Lemma 5, one can easily obtain

$$\begin{aligned}
\mathbb{E}\|(\Phi_1 x)(t) - (\Phi_1 y)(t)\|^2 \le& \mathbb{E}\left\|\int_0^t T(t-s)B(u^z(s,x) - u^z(s,y))ds\right\|^2 \\
\le& \frac{M^6 M_B^4 C_1 t_1}{4z^2\lambda^3}\|x-y\|_{\mathcal{PC}}^2 \\
\le& J_0\|x-y\|_{\mathcal{PC}}^2,
\end{aligned} \tag{8}$$

where $J_0 = \frac{M^6 M_B^4 C_1 t_1}{4z^2\lambda^3}$. In turn, for $t \in (t_k, s_k], k = 1, 2, \cdots, N$, we have

$$\mathbb{E}\|(\Phi_1 x)(t) - (\Phi_1 y)(t)\|^2 \le \mathbb{E}\|I_k(t, x(t)) - I_k(t, y(t))\|^2 \le c_k\|x-y\|_{\mathcal{PC}}^2. \tag{9}$$

When $t \in (s_k, t_{k+1}], k = 1, 2, \cdots, N$, a similar computation as before yields

$$\begin{aligned}
&\mathbb{E}\|(\Phi_1 x)(t) - (\Phi_1 y)(t)\|^2 \\
\le& 2M^2 c_k\|x-y\|_{\mathcal{PC}}^2 \\
&+ \frac{M^6 M_B^4}{4\lambda^2 z^2}(1-n_k)^2\left(2n_k c_k + \frac{C_1(t_{k+1}-s_k)}{\lambda}\right)\|x-y\|_{\mathcal{PC}}^2 \\
\le& J_k\|x-y\|_{\mathcal{PC}}^2,
\end{aligned} \tag{10}$$

where $J_k = 2M^2\left[c_k + \frac{M^4M_B^4}{8\lambda^2z^2}(1-n_k)^2\left(2n_kc_k + \frac{C_1(t_{k+1}-s_k)}{\lambda}\right)\right]$.

The above inequalities (8)–(10) together with the assumption $(A6)$ imply that Φ_1 is a contraction.

Step 3. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence such that $y_n \to y$ in $\mathcal{S}_K$. For $t \in (s_k, t_{k+1}], k = 0, 1, \cdots, N$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi_2 y_n)(t) - (\Phi_2 y)(t)\|^2 \leq & \frac{M^2C_1}{2\lambda}\int_{s_k}^{t}\mathbb{E}\|y_n(s) - y(s)\|^2 ds \\ \leq & \frac{M^2C_1(t_{k+1}-s_k)}{2\lambda}\|y_n - y\|^2_{\mathcal{PC}},\end{aligned}$$

then, $\mathbb{E}\|(\Phi_2 x_n)(t) - (\Phi_2 x)(t)\|^2 \to 0$ as $n \to \infty$, that is, Φ_2 is continuous on $\mathcal{S}_K$.

It is time to prove that Φ_2 is compact. Our first goal is to show that $\{(\Phi_2 y)(t) : y \in \mathcal{S}_K\}$ is equicontinuous. Set $\tau_1, \tau_2 \in (s_k, t_{k+1}], k = 0, 1, \cdots, N, \tau_1 < \tau_2, \tau_2 = \tau_1 + h (h > 0)$. In virtue of the elementary inequality, hypotheses $(A2)$, $(A3)$ and Lemma 4, we arrive at

$$\begin{aligned}&\mathbb{E}\|(\Phi_2 y)(\tau_2) - (\Phi_2 y)(\tau_1)\|^2 \\ \leq & 4C_2(1+K)(t_{k+1}-s_k)\int_{s_k}^{\tau_1}\|T(\tau_2 - s) - T(\tau_1 - s)\|^2 ds \\ &+ \frac{2M^2}{\lambda}C_2(1+K)(\tau_2 - \tau_1) \\ &+ 2\mathbb{E}\left\|\int_{s_k}^{\tau_1+h} T(\tau_1 + h - s)g(s)dB_Q^H(s) - \int_{s_k}^{\tau_1} T(\tau_1 - s)g(s)dB_Q^H(s)\right\|^2 \\ \leq & 4C_2(1+K)(t_{k+1}-s_k)\|T(h) - \mathbb{I}\|^2\int_{s_k}^{\tau_1} e^{-2\lambda(\tau_1 - s)}ds \\ &+ \frac{2M^2}{\lambda}C_2(1+K)(\tau_2 - \tau_1) \\ &+ 2Ch^{2\alpha}.\end{aligned}$$

Then, $\mathbb{E}\|(\Phi_2 y)(\tau_2) - (\Phi_2 y)(\tau_1)\|^2 \to 0$ as $h \to 0$(i.e., $\tau_2 \to \tau_1$). Together with the compactness of $T(t)$ for $t > 0$, the equicontinuity of $\{(\Phi_2 y)(t) : y \in \mathcal{S}_K\}$ is proved.

To proceed, we show $\mathcal{R}(t) = \{(\Phi_2 y)(t) : y \in \mathcal{S}_K\}$ is relatively compact in $\mathbb{H}$. Firstly, $\mathcal{R}(0)$ is compact. Then, for $t \in (s_k, t_{k+1}], k = 0, 1, \cdots, N$, let ε be a fixed number with $0 < \varepsilon < t$, and we define

$$(\Phi_2^{\varepsilon} y)(t) = \begin{cases} \int_0^{t-\varepsilon} T(t-s)b(s, y(s))ds + \int_0^{t-\varepsilon} T(t-s)g(s)dB_Q^H(s), & t \in [0, t_1], \\ 0, & t \in (t_k, s_k], \\ \int_{s_k}^{t-\varepsilon} T(t-s)b(s, x(s))ds + \int_{s_k}^{t-\varepsilon} T(t-s)g(s)dB_Q^H(s), & t \in (s_k, t_{k+1}]. \end{cases}$$

Since $T(t)(t > 0)$ is compact, then for every $\varepsilon, \mathcal{R}^{\varepsilon}(t) = \left\{(\Phi_2^{\varepsilon} y)(t) : y \in \mathcal{S}_K\right\}$ is the relatively compact set in the space $\mathbb{H}$. In addition,

$$\begin{aligned}&\mathbb{E}\|(\Phi_2 y)(t) - (\Phi_2^{\varepsilon} y)(t)\|^2 \\ \leq & \mathbb{E}\left\|\int_{t-\varepsilon}^{t} T(t-s)f(s, x(s))ds\right\|^2 + \mathbb{E}\left\|\int_{t-\varepsilon}^{t} T(t-s)g(s)dB_Q^H(s)\right\|^2 \\ \leq & \frac{M^2C_2(1+K)}{\lambda}\varepsilon + 2C\varepsilon^{2\alpha},\end{aligned}$$

that is to say, $\mathbb{E}\left\|(\Phi_2 y)(t) - (\Phi_2^{\varepsilon} y)(t)\right\|^2 \to 0$ as ε tends to 0. It means that the set $\mathcal{R}(t)$ and its relatively compact set $\mathcal{R}^{\varepsilon}(t)$ are arbitrarily close. Based on the above analysis, it can be concluded from the Arzelà–Ascoli theorem that Φ_2 is compact.

Consequently, by virtue of Krasnoselskii's fixed-point theorem, the non-instantaneous impulsive stochastic control system (1) admits at least one mild solution on $[0, T]$. □

Theorem 4. *Let the hypotheses of Theorem 3 hold, and assume further that the function b is uniformly bounded. Then the non-instantaneous impulsive stochastic control system (1) is approximately controllable on* $[0, T]$.

Proof. Let x^z be a fixed point of $\Phi_1 + \Phi_2$. By employing the stochastic Fubini theorem, we see that

$$\begin{aligned} x^z(t_{k+1}) =& x_{t_{k+1}} - z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}\left[\mathbb{E}x_{t_{k+1}} - T(t_{k+1} - s_k)I_k(s_k, x(s_k^-))\right] \\ &- \int_{s_k}^{t_{k+1}} z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}\psi_k(s)dW(s) \\ &+ \int_{s_k}^{t_{k+1}} z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}T(t_{k+1} - s)b(s, x^z(s))ds \\ &+ \int_{s_k}^{t_{k+1}} z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}T(t_{k+1} - s)g(s)dB_Q^H(s). \end{aligned} \tag{11}$$

The uniform boundedness of b guarantees that there exists a constant $\widehat{C} > 0$ such that $\|b(s, x^z(s))\|^2 \leq \widehat{C}$, and there is a sub-sequence denoted by $\{b(s, x^z(s))\}$ which weakly converges to say $b(s)$ in $\mathbb{H}$. The compactness of $T(t)$ ensures that $T(t_{k+1} - s)b(s, x^z(s)) \to T(t_{k+1} - s)b(s)$. It derives from (11) that

$$\begin{aligned} &\mathbb{E}\|x^z(t_{k+1}) - x_{t_{k+1}}\|^2 \\ \leq& 6\mathbb{E}\left\|z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}\mathbb{E}\|x_{t_{k+1}}\|\right\|^2 + 6\mathbb{E}\left\|\int_{s_k}^{t_{k+1}} z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}\psi_k(s)dW(s)\right\|^2 \\ &+6\mathbb{E}\left\|z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}T(t_{k+1} - s_k)I_k(s_k, x^z(s_k^-))\right\|^2 \\ &+6\mathbb{E}\left\|\int_{s_k}^{t_{k+1}} z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}T(t_{k+1} - s)[b(s, x^z(s)) - b(s)]ds\right\|^2 \\ &+6\mathbb{E}\left\|\int_{s_k}^{t_{k+1}} z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}T(t_{k+1} - s)b(s)ds\right\|^2 \\ &+6\mathbb{E}\left\|\int_{s_k}^{t_{k+1}} z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}T(t_{k+1} - s)g(s)dB_Q^H(s)\right\|^2. \end{aligned} \tag{12}$$

By the hypothesis $(A5)$, the operators $z\left(zI + \Pi_{s_k}^{t_{k+1}}\right)^{-1}$ tend to 0 strongly when $z \to 0$, furthermore, $\left\|z(zI + \Pi_{s_k}^{t_{k+1}})^{-1}\right\| \leq 1$, then, with the aid of the Lebesgue-dominated convergence theorem and Lemma 3, we deduce that

$$\mathbb{E}\|x^z(t_{k+1}) - x_{t_{k+1}}\|^2 \to 0 \tag{13}$$

as $z \to 0$. This leads to the approximate controllability of the non-instantaneous impulsive stochastic control system (1) on $[0, T]$. □

In the second part, we will prove the existence and approximate controllability of system (1) under another new set of conditions.

$(A7)$ For all $t \in T_0$, the function $b(t, x)$ is continuous in x, for all $x \in \mathbb{H}$, $b(t, x)$ is $\mathcal{F}_t$-measurable. For any positive integer, there exists $h_q : J \to L^1(J)$ such that

$$\|b(t, x)\|^2 \leq h_q(t) \textit{ for all } \|x\|^2 \leq q \textit{ and for almost all } t \in T_0.$$

$(A8)$ For all $t \in T_0, x \in \mathbb{H}$, there exist two functions $\psi : [0,\infty) \to [0,\infty)$ and $p \in L^1(J;\mathbb{R}^+)$ such that

$$\|b(t,x)\|^2 \le p(t)\psi(\|x\|^2),$$

where ψ is a continuous non-decreasing function, and

$$\frac{2M^2}{\lambda}\int_0^{t_1} p(s)ds \le \int_{c_1}^{\infty}\frac{ds}{\psi(s)},\ c_1 = 4M^2\mathbb{E}\|x_0\|^2,$$

$$\frac{2M^2}{\lambda}\int_{s_k}^{t_{k+1}} p(s)ds \le \int_{c_2}^{\infty}\frac{ds}{\psi(s)},\ c_2 = 4M^2 m_k\ for\ all\ 1 \le k \le N.$$

$(A9)$ The functions $I_k : T_k \times \mathbb{H} \to \mathbb{H}, T_k = (t_k, s_k], k = 1,2,\cdots,N$ are continuous and there exist positive constants m_k such that for $\forall t \in T_k, x, y \in \mathbb{H}$,

$$\|I_k(t,x)\|^2 \le m_k.$$

Theorem 5. *Assume that the hypotheses $(A1)$, $(A3)$, $(A7)$–$(A9)$ are satisfied. Then the non-instantaneous impulsive stochastic control system (1) has at least one mild solution on $[0,T]$.*

To prove Theorem 5, we need the following lemma.

Lemma 6. *If the hypotheses $(A1)$, $(A3)$, $(A7)$, and $(A9)$ are satisfied, then for any $x, y \in \mathcal{C}_T$, there exist positive constants R_k such that*

$$\mathbb{E}\|u^z(t,x)\|^2 \le \frac{R_k}{z^2}\left(1+\int_{s_k}^{t_{k+1}} h_q(s)ds\right).$$

Proof. For Equation (4), by applying the elementary inequality, the hypothesis $(A1)$ and Lemma 3, we have

$$\begin{aligned}
&\mathbb{E}\|u^z(t,x)\|^2\\
\le&\frac{M_B^2 M^2 e^{-2\lambda(t_{k+1}-t)}}{z^2}\Bigg(4M^2 e^{-2\lambda(t_{k+1}-s_k)}\mathbb{E}\big\|I_k(s_k, x(s_k^-))\big\|^2 + 4\mathbb{E}\big\|x_{t_{k+1}}\big\|^2\\
&+4M^2\int_{s_k}^{t_{k+1}} e^{-2\lambda(t_{k+1}-s)}ds\mathbb{E}\int_{s_k}^{t_{k+1}}\|b(s,x(s))\|^2 ds\\
&+4C_3 + 4C_4(t_{k+1}-s_k)^{2H+\gamma-1}\Bigg),
\end{aligned}$$

then, the hypotheses $(A7)$, $(A9)$ lead to

$$\begin{aligned}
\mathbb{E}\|u^z(t,x)\|^2 \le&\frac{P_k}{z^2} + \frac{Q_k}{z^2}\int_{s_k}^{t_{k+1}} h_q(s)ds\\
\le&\frac{R_k}{z^2}\left(1+\int_{s_k}^{t_{k+1}} h_q(s)ds\right),
\end{aligned}$$

where, P_k, Q_k are positive constants, $R_k = max\{P_k, Q_k\}$. Hence, the statement of Lemma 6 is proved. □

We are now turning to the proof of Theorem 5.

Proof. Consider the following operator $\Phi : \mathcal{C}_T \to \mathcal{C}_T$:

$$(\Phi x)(t) = \begin{cases} T(t)x_0 + \int_0^t T(t-s)Bu^z(s,x)ds + \int_0^t T(t-s)b(s,x(s))ds \\ + \int_0^t T(t-s)g(s)dB_Q^H(s), t \in [0,t_1], \\ I_k(t,x(t)), t \in (t_k,s_k], \\ T(t-t_k)I_k(s_k,x(s_k^-)) + \int_{s_k}^t T(t-s)Bu^z(s,x)ds \\ + \int_{s_k}^t T(t-s)b(s,x(s))ds + \int_{s_k}^t T(t-s)g(s)dB_Q^H(s), t \in (s_k,t_{k+1}]. \end{cases} \tag{14}$$

Following the proof of Theorem 3.3 in Ref. [37], it is not difficult to examine Φ is completely continuous. Rather than giving a proof, we outline it. Step 1. We first show that operator Φ maps uniformly bounded set into an equicontinuous family; Step 2. We proceed to demonstrate Φ maps uniformly bounded set into a precompact set; Step 3. It remains to show that Φ is continuous. Finally, combining the Arzelà–Ascoli theorem and Steps 1–3, we see that Φ is a completely continuous operator. Now, we only need to prove that the set

$$\mathrm{S}(\Phi) = \{x \in C_T : x = \lambda\Phi(x),\ for\ some\, \lambda \in (0,1)\}$$

is bounded.

Let $x(t) \in \mathrm{S}(\Phi)$, then for some $\lambda \in (0,1)$, $x(t) = \lambda(\Phi x)(t)$. Thus, for any $t \in [0,t_1]$, we have

$$\begin{aligned} \mathbb{E}\|x(t)\|^2 \leq & 4M^2e^{-2\lambda t}\mathbb{E}\|x_0\|^2 + \frac{2M^2M_B^2}{\lambda}\mathbb{E}\int_0^t \|u^z(s,x)\|^2 ds \\ & + \frac{2M^2}{\lambda}\mathbb{E}\int_0^t \|b(s,x(s))\|^2 ds + 4\mathbb{E}\left\|\int_0^t T(t-s)g(s)dB_Q^H(s)\right\|^2. \end{aligned}$$

Denote by $\mu(t) = \sup\limits_{0\leq s\leq t} \mathbb{E}\|x(s)\|^2, 0 \leq t \leq T$. Then, for all $t \in [0,t_1]$, we have

$$\begin{aligned} \mu(t) \leq & 4M^2\mathbb{E}\|x_0\|^2 + \frac{2M^2M_B^2}{\lambda}\int_0^{t_1} \mathbb{E}\|u^z(s,x)\|^2 ds \\ & + \frac{2M^2}{\lambda}\int_0^t \|p(s)\psi(\mu(s))\|^2 ds + 4\mathbb{E}\left\|\int_0^{t_1} T(t-s)g(s)dB_Q^H(s)\right\|^2. \end{aligned} \tag{15}$$

The right side of inequality (15) is denoted by $v(t)$. Then, we get $v(0) = 4M^2\mathbb{E}\|x_0\|^2$, $\mu(t) \leq v(t)$. In addition,

$$v'(t) = \frac{2M^2}{\lambda}p(t)\psi(\mu(t)) \leq \frac{2M^2}{\lambda}p(t)\psi(v(t)).$$

Using the condition $(A8)$, we derive that

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \frac{2M^2}{\lambda}\int_0^t p(s)ds \leq \frac{2M^2}{\lambda}\int_0^{t_1} p(s)ds \leq \int_{c_1}^{\infty} \frac{ds}{\psi(s)}.$$

Hence, there exists a positive constant K such that $v(t) \leq K$, that is $\mu(t) \leq v(t) \leq K$, $t \in [0,t_1]$.

For all $t \in (t_k,s_k]$, the hypothesis $(A9)$ gives that $\mathbb{E}\|x(t)\|^2 \leq m_k$.

For all $t \in (s_k,t_{k+1}]$, reproducing the above estimating method, it follows from $(A8)$ that $\mu(t) \leq v(t) \leq K, t \in (s_k,t_{k+1}]$. It implies that $\mathrm{S}(\Phi)$ is bounded.

Consequently, according to Schaefer's fixed-point theorem, Φ admits a fixed point, which is a mild solution of system (1). This completes the proof of Theorem 5. □

Theorem 6. *Let the hypotheses (A1), (A3), and (A6)–(A9) hold, and assume further that the function b is uniformly bounded. Then the non-instantaneous impulsive stochastic control system (1) is approximately controllable on* $[0, T]$.

Theorem 6 can be proved similarly as Theorem 4, so the proof will not be stated here.

Remark 3. *It should be pointed out that the assumptions of Theorem 3–6 are the sufficient conditions but not the necessary conditions for the existence and approximate controllability results of system (1).*

Remark 4. *In fact, by the means of the fractional power of the operator* $-A$ *(A is the infinitesimal generator of an analytic semigroup), our techniques, after little modification, can be extended to study the approximate controllability of the non-instantaneous impulsive neutral SEEs excited by fBm with Hurst index* $0 < H < 1/2$ *in the following form:*

$$\begin{cases} d[x(t) + h(t, x(t))] = [Ax(t) + Bu(t) + b(t, x(t))]dt \\ \qquad\qquad\qquad\qquad + g(t)dB_Q^H(t),\ t \in \cup_{k=0}^{N}(s_k, t_{k+1}], \\ x(t) = I_k(t, x(t)),\ \ t \in \cup_{k=1}^{N}(t_k, s_k], \\ x(0) = x_0. \end{cases} \tag{16}$$

Here we give the definition of mild solution to system (16).

Definition 4. *A* $\mathbb{H}$*-valued stochastic process* $x(t)$ *is said to be a mild solution of the system* (16)*, if*
(a) $x(t)$*, is* $\mathcal{F}_t$*-adapted and has càdlàg paths on* $t \in [0, T]$ *a.s.*
(b) $x(t) = I_k(t, x(t))$ *for all* $t \in (t_k, s_k], k = 1, 2, \cdots, N$ *and* $x(t)$ *satisfies the following integral equations*

$$\begin{aligned} x(t) =& T(t)[x_0 + h(0, x_0)] - h(t, x(t)) - \int_0^t AT(t-s)h(s, x(s))ds \\ &+ \int_0^t T(t-s)[Bu(s) + b(s, x(s))]ds + \int_0^t T(t-s)g(s)dB_Q^H(s), \\ &\forall t \in [0, t_1], \\ x(t) =& T(t-s_k)\left[I_k(s_k, x(s_k^-)) + h(s_k, x(s_k^-))\right] - h(t, x(t)) \\ &- \int_{s_k}^t AT(t-s)h(s, x(s))ds + \int_{s_k}^t T(t-s)[Bu(s) + b(s, x(s))]ds \\ &+ \int_{s_k}^t T(t-s)g(s)dB_Q^H(s),\ \forall t \in [s_k, t_{k+1}],\ k = 1, 2, \cdots, N. \end{aligned}$$

4. Example

Example 1. *In this section, we provide an example to illustrate the proposed theory. Consider the following non-instantaneous impulsive SPDE excited by fBm with Hurst index* $0 < H < 1/2$.

$$\begin{cases} dx(t, \zeta) = \left[\frac{\partial^2}{\partial \zeta^2}x(t, \zeta) + Bu(t)(\zeta) + 0.5x(t, \zeta)\right]dt + t^{\frac{1}{3}}dB_Q^H(t), \\ \qquad\qquad t \in (0, 0.3] \cup (0.6, 1],\ \zeta \in [0, \pi], \\ x(t, 0) = 0 = x(t, \pi), \\ x(t, \zeta) = \frac{1}{6}(\sin t)x(t, \zeta), t \in (0.3, 0.6], \\ x(0, \zeta) = x_0(\zeta), \zeta \in [0, \pi], \end{cases} \tag{17}$$

where $0 = t_0 = s_0 < t_1 < s_1 < t_2 = 1$ *with* $t_1 = 0.3$ *and* $s_1 = 0.6$. *Let* $\mathbb{H} = L^2[0,\pi]$, *and* $A = \frac{\partial^2}{\partial \xi^2}$ *with the domain* $\mathcal{D}(A) := \mathbb{H}_0^1(0,\pi) \cap \mathbb{H}^2(0,\pi)$. *Then*

$$Aw = -\sum_{n=1}^{\infty} n^2 \langle w, e_n(\xi) \rangle e_n(\xi),$$

for any $w \in \mathcal{D}(A)$, *where* $e_n(\xi) = \sqrt{\frac{2}{\pi}} sin(n\xi), 0 \le \xi \le \pi, n \in \mathbb{N}$.

It is well known [34] that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \ge 0}$ *and it is given by*

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} \langle w, e_n(\xi) \rangle e_n(\xi), w \in \mathbb{H} \quad \text{and} \quad \|T(t)\| \le e^{-t},$$

It implies that $\{T(t)\}_{t \ge 0}$ *is compact. Now, we define the bounded linear operator B from*

$$\mathcal{U} = \left\{ u = \sum_{n=2}^{\infty} u_n e_n : \|u\|_{\mathcal{U}}^2 := \sum_{n=2}^{\infty} u_n^2 < \infty \right\}$$

to $\mathbb{H}$:

$$Bu = 2u_2 e_1 + \sum_{n=2}^{\infty} u_n e_n.$$

Putting $x(t)(\xi) = x(t,\xi)$, *we can rewrite the system (17) to the abstract form (1), and the functions* f, g, I_k *are*

$$f(t, x(t)) = 0.5x(t),\ g(t) = t^{\frac{1}{3}},\ I_1(t, x(t)) = \frac{1}{6}(\sin t)x(t).$$

Then, we have

$$M = 1,\ \lambda = 1,\ C_1 = C_2 = 0.25,\ t^{\frac{1}{3}} - s^{\frac{1}{3}} < (t-s)^{\frac{1}{3}} (s < t),\ c_1 = d_1 = \frac{1}{36}.$$

Since B is a bounded linear operator, we choose $M_B = 1, z = 1$. *Thus, one can obtain*

$$P_0 = 0.375, \quad 2M^2 c_0 + \frac{M^2 M_B^2 (1-n_0)^2}{4\lambda^2 n_0} M_0 = \frac{(1-n_0)^2 M_0}{4n_0} \approx 0.007,$$
$$P_1 \approx 0.81, \quad 2M^2 c_1 + \frac{M^2 M_B^2 (1-n_1)^2}{4\lambda^2 n_1} M_1 = \frac{1}{18} + \frac{(1-n_1)^2 M_1}{4n_1} \approx 0.077,$$

where $n_0 = e^{-0.6} \approx 0.55, n_1 = e^{-0.8} \approx 0.45, M_0 \approx 0.075$ *and* $M_1 \approx 0.125$, *that is,*

$$\max_{k=0,1} \{P_k, 2M^2 c_k + \frac{M^2 M_B^2 (1-n_k)^2}{4\lambda^2 n_k} M_k\} < 1.$$

In that case, all the conditions are verified. As a result, it follows from Theorem 4 that the system (17) is approximately controllable on $[0,1]$.

5. Conclusions

In infinite dimensional spaces, the concept of exact controllability is usually too strict [31]. So, this paper considered the approximate controllability for a class of non-instantaneous ISEEs excited by fBm with Hurst index $H \in (0, 1/2)$. Since the properties of the fBm with $0 < H < 1/2$ are more irregular and singular, we cannot define the control function like the case with fBm with Hurst parameter $H \in (1/2, 1)$. Hence, a different type of control function was defined. Then we used two different fixed-point theorems to overcome the difficulties brought by the introduction of non-instantaneous impulses, and obtained two new sets of sufficient conditions to ensure the existence and approximate controllability of the system. In our future work, we will consider the following three issues. Firstly, we will discuss the approximate controllability of instantaneous and non-instantaneous impulsive systems [38] driven by fBm with Hurst index $H \in (0, 1/2)$. Secondly, we will explore the optimal control for non-instantaneous ISEEs excited by fBm with Hurst parameter $H \in (0, 1/2)$. Thirdly, based on our method and recent studies on the controllability of deterministic non-instantaneous impulsive differential equations with non-local conditions [39,40], we will investigate the approximate controllability of non-instantaneous impulsive stochastic differential systems driven by fBm with non-local conditions in detail.

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