Review

Oscillation Results for Solutions of Fractional-Order Differential Equations

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Abstract: This survey paper is devoted to succinctly reviewing the recent progress in the field of oscillation theory for linear and nonlinear fractional differential equations. The paper provides a fundamental background for all interested researchers who would like to contribute to this topic.

1. Introduction

For many years, integer-order differential equations have been used to describe natural or real-life occurrences. However, the factors at play in these situations are extremely complex and diverse. Therefore, it has been realized that integer-order differential equations cannot incorporate all of such features. One can cover up this gap by using fractional-order differential equations which provide better description and interpretation to construct these models. The origin of fractional calculus is the same as that of classical calculus. However, the growth of fractional calculus has stagnated due to insufficient geometrical and unsuitable physical interpretations of fractional derivatives. Researchers came to appreciate the importance of these derivatives with the advent of high-speed computers and precise computational techniques by creating and applying a specific fractional differential operator to a real-life situation. Fractional calculus has become a popular topic in practically every branch of science and engineering. Indeed, it has been expanded rapidly due to the nonlocal character of fractional operators. As a result, fractional calculus and its many applications have piqued the interest of many researchers [1,2].

For specific reasons, most of the real-life phenomena in the world are non-linear. Therefore, it is possible to understand the nonlinear phenomenon of the actual model through the nonlinear equation. Unlike linear equations, it is not always possible to calculate analytical solutions for nonlinear equations. However, one can obtain an approximate solution to the nonlinear equation to better understand how the equation works. The qualitative properties of nonlinear equations such as existence, uniqueness, stability, oscillation, controllability, bifurcation, chaos, etc., can be easily discussed without solving them. Commenting on solutions to equations without solving them can help scientists tackle many research problems.

Nonlinearity is a qualitative property of equations that can be utilized to create or remove oscillation. Torsion oscillations, cardiac oscillations, sinusoidal oscillations, and harmonic oscillations are all examples of practical applications of the theory of oscillation of differential equations. Many academics have developed a systematic examination of the oscillation and non-oscillation of solutions of integer order differential equations. Because

of the remarkable interest in the theory of fractional calculus, oscillation of solutions for fractional differential equations has been investigated for the past twenty years. By studying the oscillation of nonlinear fractional differential equations, Grace and other scholars initiated and pioneered this topic. The line has continued to progress, and some notable outcomes have been established and elaborated; the reader can consult the papers cited herein.

This study intends to bring together the recent advances in the field of oscillation theory of linear and nonlinear fractional differential equations, as well as provide researchers with insight into future needs in the field of oscillation of fractional differential equations. The results of this paper will be presented based on different fractional operators.

We use the following notations, definitions and known results of fractional calculus throughout the article. Denote by $\mathbb{R}$ the set of all real numbers, and $\mathbb{R}^+$ the set of all positive real numbers.

Definition 1 ([1,2]). *The Euler gamma function is defined by*

$$\Gamma(z) = \int_0^\infty \zeta^{z-1} e^{-\zeta} d\zeta, \quad \Re(z) > 0.$$

Using its reduction formula, the Euler gamma function can also be extended to the half-plane $\Re(z) \le 0$ except for $z \in \{\ldots, -2, -1, 0\}$.

2. Oscillation Results via Riemann–Liouville and Caputo Operators

Definition 2 ([1,2]). *Let $[a, b]$, $(-\infty < a < b < \infty)$, be a finite interval on the real axis $\mathbb{R}$. The (left-sided) Riemann–Liouville fractional integral $\mathcal{I}_a^\kappa$ of order $\kappa \in \mathbb{C}$, $\Re(\kappa) > 0$, is defined by*

$$\mathcal{I}_a^\kappa \mathfrak{f}(\zeta) = \frac{1}{\Gamma(\kappa)} \int_a^\zeta (\zeta - s)^{\kappa-1} \mathfrak{f}(s) ds, \quad \zeta > a.$$

The (left-sided) Riemann–Liouville fractional derivative $\mathcal{D}_a^\kappa$ of order $\kappa \in \mathbb{C}$, $\Re(\kappa) \ge 0$, is defined by

$$\mathcal{D}_a^\kappa \mathfrak{f}(\zeta) = \left(\frac{d}{d\zeta} \right)^n \mathcal{I}_a^{n-\kappa} \mathfrak{f}(\zeta), \quad n = [\Re(\kappa)] + 1, \quad \zeta > a.$$

The (left-sided) Caputo fractional derivative $^C\mathcal{D}_a^\kappa$ of order $\kappa \in \mathbb{C}$, $\Re(\kappa) \ge 0$, is defined via the above (left-sided) Riemann–Liouville fractional derivative by

$$^C\mathcal{D}_a^\kappa \mathfrak{f}(\zeta) = \mathcal{D}_a^\kappa \left[\mathfrak{f}(\zeta) - \sum_{k=0}^{n-1} \frac{\mathfrak{f}^{(k)}(a)}{k!} (\zeta - a)^k \right], \quad n = [\Re(\kappa)] + 1, \quad \zeta > a.$$

Grace et al. initiated the study of oscillation theory for fractional differential equations. Grace et al. obtained oscillation criteria for a class of nonlinear fractional differential equations of the form

$$\begin{cases} \mathcal{D}_a^\kappa u + \mathfrak{f}_1(\zeta, u) = v(\zeta) + \mathfrak{f}_2(\zeta, u), & \zeta > a \ge 0, \\ \mathcal{D}_a^{\kappa-k} u(a) = b_k, & k = 1, 2, \cdots, m-1; \quad \lim_{\zeta \to a^+} \mathcal{I}_a^{m-\kappa} u(\zeta) = b_m, \end{cases} \tag{1}$$

and

$$\begin{cases} ^C\mathcal{D}_a^\kappa u + \mathfrak{f}_1(\zeta, u) = v(\zeta) + \mathfrak{f}_2(\zeta, u), & \zeta > a \ge 0, \\ u^{(k)}(a) = b_k, & k = 0, 1, 2, \cdots, m-1, \end{cases} \tag{2}$$

where $m - 1 < \kappa \le m$, $m \ge 1$ is an integer; $\mathfrak{f}_1, \mathfrak{f}_2 \in C([a, \infty) \times \mathbb{R}, \mathbb{R})$, and $v \in C([a, \infty), \mathbb{R})$.

The authors considered those solutions of (1) (or (2)) which are continuous and continuable to (a, ∞), and are not identically zero on any half-line (b, ∞) for some $b \ge a$.

A solution of (1) (or (2)) is said to be oscillatory if it has arbitrarily large zeros on (a, ∞); otherwise, it is called non-oscillatory. An equation is said to be oscillatory if all of its solutions are oscillatory.

Let p_1, $p_2 \in C([a, \infty), \mathbb{R}^+)$ and β, γ are positive real numbers. The authors made the following assumptions:

(H1) $u\mathfrak{f}_i(\zeta, u) > 0$, $i = 1, 2$, $u \neq 0$, $\zeta \geq a$.

(H2) $|\mathfrak{f}_1(\zeta, u)| \geq p_1(\zeta)|u|^\beta$ and $|\mathfrak{f}_2(\zeta, u)| \leq p_2(\zeta)|u|^\gamma$, $u \neq 0$, $\zeta \geq a$.

(H3) $|\mathfrak{f}_1(\zeta, u)| \leq p_1(\zeta)|u|^\beta$ and $|\mathfrak{f}_2(\zeta, u)| \geq p_2(\zeta)|u|^\gamma$, $u \neq 0$, $\zeta \geq a$.

We find the following popular results of Grace et al. in Reference 20 of [3].

Theorem 1. *Let $\mathfrak{f}_2 = 0$ and condition (H 1) holds. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} v(s) ds = -\infty, \tag{3}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} v(s) ds = \infty, \tag{4}$$

then (1) is oscillatory.

Theorem 2. *Let conditions (H 1) and (H 2) hold with $\beta > 1$ and $\gamma = 1$. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[v(s) + H_\beta(s) \right] ds = -\infty, \tag{5}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[v(s) - H_\beta(s) \right] ds = \infty, \tag{6}$$

where

$$H_\beta(s) = (\beta - 1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s),$$

then (1) is oscillatory.

Theorem 3. *Let conditions (H 1) and (H 2) hold with $\beta = 1$ and $\gamma < 1$. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} [v(s) + H_\gamma(s)] ds = -\infty, \tag{7}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} [v(s) - H_\gamma(s)] ds = \infty, \tag{8}$$

where

$$H_\gamma(s) = (1 - \gamma)\gamma^{\frac{\gamma}{\gamma-1}} p_1^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s),$$

then (1) is oscillatory.

Theorem 4. *Let conditions (H 1) and (H 2) hold with $\beta > 1$ and $\gamma < 1$. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[v(s) + H_{\beta,\gamma}(s) \right] ds = -\infty, \tag{9}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[v(s) - H_{\beta,\gamma}(s) \right] ds = \infty, \tag{10}$$

where

$$H_{\beta,\gamma}(s) = (\beta - 1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}(s)\xi^{\frac{\beta}{\beta-1}}(s) + (1 - \gamma)\gamma^{\frac{\gamma}{1-\gamma}}\xi^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s),$$

with $\xi \in C([a, \infty), \mathbb{R}^+)$, *then* (1) *is oscillatory.*

Theorem 5. *Let* $\mathfrak{f}_2 = 0$ *and condition* (H 1) *holds. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} v(s) ds = -\infty, \tag{11}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} v(s) ds = \infty, \tag{12}$$

then (2) *is oscillatory.*

Theorem 6. *Let conditions* (H 1) *and* (H 2) *hold with* $\beta > 1$ *and* $\gamma = 1$. *If*

$$\liminf_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} \big[v(s) + H_\beta(s)\big] ds = -\infty, \tag{13}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} \big[v(s) - H_\beta(s)\big] ds = \infty, \tag{14}$$

where H_β *is defined as in Theorem* 2, *then* (2) *is oscillatory.*

Theorem 7. *Let conditions* (H 1) *and* (H 2) *hold with* $\beta = 1$ *and* $\gamma < 1$. *If*

$$\liminf_{\zeta \to \infty} t^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} [v(s) + H_\gamma(s)] ds = -\infty, \tag{15}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} [v(s) - H_\gamma(s)] ds = \infty, \tag{16}$$

where H_γ *is defined as in Theorem* 3, *then* (2) *is oscillatory.*

Theorem 8. *Let conditions* (H 1) *and* (H 2) *hold with* $\beta > 1$ *and* $\gamma < 1$. *If*

$$\liminf_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} \big[v(s) + H_{\beta,\gamma}(s)\big] ds = -\infty, \tag{17}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} \big[v(s) - H_{\beta,\gamma}(s)\big] ds = \infty, \tag{18}$$

where $H_{\beta,\gamma}$ *is defined as in Theorem* 4, *then* (2) *is oscillatory.*

In continuation to the above work, Chen et al. [3] established several oscillation theorems for (1) and (2). The authors in [3] observed that the cases $\beta > \gamma > 1$ and $1 > \beta > \gamma > 0$ were not considered for (1) and (2) in Reference 20 of [3]. The purpose of the paper [3] was to cover this gap and establish new oscillation criteria that improve the results in Reference 20 of [3].

Theorem 9 ([3]). *Assume* (H 1) *and* (H 2) *hold with* $\beta > \gamma$. *If*

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta (\zeta - s)^{\kappa-1} [v(s) + G(s)] ds = -\infty, \tag{19}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^{\zeta} (\zeta - s)^{\kappa - 1} [v(s) - G(s)] ds = \infty, \tag{20}$$

for every sufficiently large T, where

$$G(s) = \left(\frac{\beta}{\gamma} - 1\right) \left[\frac{\gamma p_2(s)}{\beta}\right]^{\frac{\beta}{\beta - \gamma}} p_1^{\frac{\gamma}{\gamma - \beta}}(s),$$

then (1) *is oscillatory.*

Theorem 10 ([3]). *Let $\kappa \geq 1$. Assume (H 1) and (H 3) hold with $\beta < \gamma$. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^{\zeta} (\zeta - s)^{\kappa - 1} [v(s) - G(s)] ds = -\infty, \tag{21}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^{\zeta} (\zeta - s)^{\kappa - 1} [v(s) + G(s)] ds = \infty, \tag{22}$$

for every sufficiently large T, where G is defined as in Theorem 9, then every bounded solution of (1) *is oscillatory.*

Theorem 11 ([3]). *Assume (H 1) and (H 2) hold with $\beta > \gamma$. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-m} \int_T^{\zeta} (\zeta - s)^{\kappa - 1} [v(s) + G(s)] ds = -\infty, \tag{23}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-m} \int_T^{\zeta} (\zeta - s)^{\kappa - 1} [v(s) - G(s)] ds = \infty, \tag{24}$$

for every sufficiently large T, where G is defined as in Theorem 9, then (2) *is oscillatory.*

Theorem 12 ([3]). *Let $\kappa \geq 1$. Assume (H 1) and (H 3) hold with $\beta < \gamma$. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-m} \int_T^{\zeta} (\zeta - s)^{\kappa - 1} [v(s) - G(s)] ds = -\infty, \tag{25}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-m} \int_T^{\zeta} (\zeta - s)^{\kappa - 1} [v(s) + G(s)] ds = \infty, \tag{26}$$

for every sufficiently large T, where G is defined as in Theorem 9, then every bounded solution of (2) *is oscillatory.*

Shao et al. [4] considered the oscillation theory for a fractional differential equation with mixed nonlinearities of the type

$$\begin{cases} \mathcal{D}_a^{\kappa} u - p(\zeta) u(\zeta) + \sum_{i=1}^n q_i(\zeta) |u(\zeta)|^{\lambda_i - 1} = r(\zeta), & \zeta > a, \\ \mathcal{D}_a^{\kappa - k} u(a) = b_k, & k = 1, 2, \cdots, m - 1; \quad \lim_{\zeta \to a^+} \mathcal{I}_a^{m - \kappa} u(\zeta) = b_m, \end{cases} \tag{27}$$

and

$$\begin{cases} {}^C\mathcal{D}_a^{\kappa} u - p(\zeta) u(\zeta) + \sum_{i=1}^n q_i(\zeta) |u(\zeta)|^{\lambda_i - 1} = r(\zeta), & \zeta > a, \\ u^{(k)}(a) = b_k, & k = 0, 1, 2, \cdots, m - 1, \end{cases} \tag{28}$$

where $m - 1 < \kappa \le m$, $m \ge 1$ is an integer, $p, r, q_i \in C([a, \infty), \mathbb{R})$ $(1 \le i \le n)$, λ_i $(1 \le i \le n)$ are ratios of odd positive integers with $\lambda_1 > \cdots > \lambda_l > 1 > \lambda_{l+1} > \cdots > \lambda_n$.

Theorem 13 ([4]). *Let*

$$p(\zeta) > 0 \text{ and } q_i(\zeta) \begin{cases} \ge 0, & 1 \le i \le l; \\ \le 0, & l+1 \le i \le n. \end{cases} \tag{29}$$

If for some constant $K > 0$, we have

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[r(s) + K \sum_{i=1}^n p^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] ds = -\infty, \tag{30}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[r(s) + K \sum_{i=1}^n p^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] ds = \infty, \tag{31}$$

then (27) *is oscillatory.*

Corollary 1 ([4]). *Let $l = n$ in* (27)*, then $\lambda_1 > \lambda_2 > \cdots \lambda_n > 1$. Suppose $p(\zeta) > 0$, $q_i(\zeta) \ge 0$, $i = 1, 2, \cdots n$. If* (30) *and* (31) *hold for some constant $K_1 > 0$, then* (27) *is oscillatory.*

Corollary 2 ([4]). *Let $l = 0$ in* (27)*, then $1 > \lambda_1 > \lambda_2 > \cdots \lambda_n$. Suppose $p(\zeta) < 0$, $q_i(\zeta) \le 0$, $i = 1, 2, \cdots n$. If* (30) *and* (31) *hold for some constant $K_2 > 0$, then* (27) *is oscillatory.*

Corollary 3 ([4]). *Let*

$$p(\zeta) \equiv 0 \text{ and } q_i(\zeta) \begin{cases} \ge 0, & 1 \le i \le l; \\ \le 0, & l+1 \le i \le n. \end{cases} \tag{32}$$

If there exists a positive function v_1 on $[a, \infty)$ such that for some constant $K_3 > 0$, we have

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[r(s) + K_3 \sum_{i=1}^n v_1^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] ds = -\infty, \tag{33}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[r(s) + K_3 \sum_{i=1}^n v_1^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] ds = \infty, \tag{34}$$

then (27) *is oscillatory.*

Theorem 14 ([4]). *Assume that condition* (29) *holds. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[r(s) + K \sum_{i=1}^n p^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] ds = -\infty, \tag{35}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[r(s) + K \sum_{i=1}^n p^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] ds = \infty, \tag{36}$$

for some constant $K > 0$, then (28) *is oscillatory.*

Corollary 4 ([4]). *Suppose $p(\zeta) > 0$, $q_i(\zeta) \ge 0$, $i = 1, 2, \cdots n$. If* (35) *and* (36) *hold for some constant $K_1 > 0$, then* (28) *is oscillatory.*

Corollary 5 ([4]). *Suppose $p(\zeta) > 0$, $q_i(\zeta) \leq 0$, $i = 1, 2, \cdots n$. If (35) and (36) hold for some constant $K_2 > 0$, then (28) is oscillatory.*

Corollary 6 ([4]). *Let (32) hold. If there exists a positive function v_1 on $[a, \infty)$ such that for some constant $K_3 > 0$, we have*

$$\liminf_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[r(s) + K_3 \sum_{i=1}^n v_1^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] ds = -\infty, \tag{37}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-m} \int_a^\zeta (\zeta - s)^{\kappa-1} \left[r(s) + K_3 \sum_{i=1}^n v_1^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] ds = \infty, \tag{38}$$

then (28) is oscillatory.

In this line, Wang et al. [5] discussed the oscillations of the fractional order differential equation

$$\mathcal{D}_a^\kappa u + q(\zeta) \mathfrak{f}_3(u) = 0, \quad \zeta > a > 0, \tag{39}$$

where $0 < \kappa \leq 1$, q is a positive real-valued function and $\mathfrak{f}_3 : [0, \infty) \to [0, \infty)$ is a continuous functional satisfying

$$\frac{\mathfrak{f}_3(u)}{\mathcal{I}_a^{2-\kappa} u} \geq K_4 > 0.$$

The Riccati transformation technique is used to obtain some sufficient conditions which guarantee that every solution of the equation is oscillatory or the limit inferior converges to zero.

Theorem 15 ([5]). *If there exists a positive function $\sigma \in C'(0, \infty)$ and a sufficiently large $\zeta_2 \geq a$ such that*

$$\limsup_{\zeta \to \infty} \int_{\zeta_2}^\zeta \left[K_4 \sigma(s) q(s) - \frac{(\sigma'_+(s))^2}{4\sigma(s)} \right] ds = \infty, \tag{40}$$

where $\sigma'_+(s) = \max\{\sigma'(s), 0\}$, then either (39) is oscillatory or

$$\liminf_{\zeta \to \infty} u(\zeta) = 0.$$

Corollary 7 ([5]). *If there exists a sufficiently large ζ_2 such that*

$$\limsup_{\zeta \to \infty} \int_{\zeta_2}^\zeta \left[K_4 s q(s) - \frac{1}{4s} \right] ds = \infty, \tag{41}$$

then either (39) is oscillatory or

$$\liminf_{\zeta \to \infty} u(\zeta) = 0.$$

Corollary 8 ([5]). *If there exists a sufficiently large ζ_2 such that*

$$\limsup_{\zeta \to \infty} \int_{\zeta_2}^\zeta q(s) ds = \infty, \tag{42}$$

then either (39) is oscillatory or

$$\liminf_{\zeta \to \infty} u(\zeta) = 0.$$

Theorem 16 ([5]). *Assume that there exist functions $H \in C(D, \mathbb{R}^+)$, $\sigma \in C'(0, \infty)$ such that*

$$H(\zeta, \zeta) = 0, \quad H(\zeta, s) > 0 \text{ for } \zeta > s \geq a,$$

where $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq a\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ on D with respect to the second variable. Also assume there exists a nonnegative continuous function h defined on D and a differentiable positive function σ satisfying for all $\zeta \geq a$

$$\frac{\sigma'_+(s)}{\sigma(s)} H(\zeta, s) + H'_s(\zeta, s) = \frac{1}{\sigma(s)} h(\zeta, s) H^{\frac{1}{2}}(\zeta, s), \tag{43}$$

where $\sigma'_+(s) = \max\{\sigma'(s), 0\}$. If these assumptions hold and

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_1)} \int_{\zeta_1}^{\zeta} \left[K_4 \sigma(s) q(s) H(\zeta, s) - \frac{h^2(\zeta, s)}{4\sigma(s)} \right] ds = \infty, \tag{44}$$

then either (39) is oscillatory or

$$\liminf_{\zeta \to \infty} u(\zeta) = 0.$$

Theorem 17 ([5]). *Assume there is a positive function σ such that σ' is continuous on $(0, \infty)$ and a sufficiently large ζ_1 satisfies*

$$\limsup_{\zeta \to \infty} \frac{1}{\zeta^m} \int_{\zeta_1}^{\zeta} (\zeta - s)^m \left[K_4 \sigma(s) q(s) - \frac{(\sigma'_+(s))^2}{4\sigma(s)} \right] ds = \infty, \tag{45}$$

where $m > 1$, then either (39) is oscillatory or

$$\liminf_{\zeta \to \infty} u(\zeta) = 0.$$

In this line, Grace established some new criteria for the oscillation of fractional differential equations with the Caputo derivative of the form

$${}^C\mathcal{D}_a^\kappa u = e(\zeta) + \mathfrak{f}_4(\zeta, u), \quad a > 1, \quad \kappa \in (1, 2). \tag{46}$$

Moreover, the conditions under which all solutions of this equation are asymptotic to $a\zeta + b$ as $\zeta \to \infty$ for some real numbers a and b, are presented. We find the following results in Reference 10 of [6].

Theorem 18. *Suppose that $p > 1$, $\gamma > 0$, $p(\kappa - 2) + 1 > 0$, $p(\gamma - 1) + 1 > 0$, $q = \frac{p}{p-1}$, and the function $e : \mathbb{R} \to \mathbb{R}$ is continuous such that*

$$\frac{1}{\zeta} \int_a^{\zeta} (\zeta - s)^{\kappa - 1} |e(s)| ds \text{ is bounded for all } t \geq a, \tag{47}$$

and the function $\mathfrak{f}_4(\zeta, u)$ satisfies the following conditions:

1. $\mathfrak{f}_4(\zeta, u)$ *is continuous in $D = \{(\zeta, u) : \zeta \geq 0, u \in \mathbb{R}\}$;*
2. *There are continuous nonnegative functions $g, h : [0, \infty) \to [0, \infty)$, g is nondecreasing and let $0 < \gamma \leq 3 - \alpha - \frac{1}{p}$ such that*

$$|\mathfrak{f}_4(\zeta, u)| \leq \zeta^{\gamma - 1} h(\zeta) g\left(\frac{|u|}{\zeta}\right), \quad \zeta > 0, \quad (\zeta, u) \in D, \tag{48}$$

and

$$\int_a^{\infty} s^{\theta q / p} h^q(s) ds < \infty, \tag{49}$$

where $\theta = p(\kappa + \gamma - 3) + 1 \leq 0$;

3.

$$\int_a^{\infty} \frac{\tau^{q-1}}{g^q(\tau^q)} d\tau = \infty. \tag{50}$$

If u is a solution of (46), *then* $|u(\zeta)| = O(\zeta)$ *as* $\zeta \to \infty$, *that is,*

$$\limsup_{\zeta \to \infty} \frac{|u(\zeta)|}{\zeta} < \infty. \tag{51}$$

Note that the Theorem 18 is remains valid if $g(z) = z$.

Theorem 19. *Let the constants* κ, p, q, γ *and* θ *be defined as is in Theorem* 18, *conditions* (47)–(50) *hold. If for every constant* c, $0 < c < 1$,

$$\liminf_{\zeta \to \infty} \left[c\zeta + \int_a^\zeta (\zeta - s)^{\kappa-1} e(s) ds \right] = -\infty, \tag{52}$$

and

$$\limsup_{\zeta \to \infty} \left[c\zeta + \int_a^\zeta (\zeta - s)^{\kappa-1} e(s) ds \right] = \infty, \tag{53}$$

then (46) *oscillatory.*

Theorem 20. *Let the constants* κ, p, q, γ *and* θ *be defined as is in Theorem* 18. *Assume that* $\mathfrak{f}_4 : [a, \infty) \times \mathbb{R} \to (0, \infty)$ *is continuous and there exists a continuous function* $h : [a, \infty) \to (0, \infty)$ *and a real number* λ *with* $0 < \lambda < 1$ *such that*

$$0 \le u \mathfrak{f}_4(\zeta, u) \le \zeta^{\gamma-1} h(\zeta) |u|^{\lambda+1}, \quad u \neq 0, \quad \zeta \ge a. \tag{54}$$

Denote by

$$g_\pm(\zeta) = \frac{1}{\Gamma(\kappa)} \int_{\zeta_1}^\zeta (\zeta - s)^{\kappa-1} \left[e(s) \pm (1-\lambda) \lambda^{\frac{\lambda}{1-\lambda}} s^{\gamma-1} m_1^{\frac{\lambda}{\lambda-1}}(s) h^{\frac{1}{1-\lambda}}(s) \right] ds. \tag{55}$$

Here $\zeta \ge \zeta_1$ *for some* $\zeta_1 \ge a$, *and* $m_1 : [a, \infty) \to (0, \infty)$ *is a given continuous function. Suppose*

$$\liminf_{\zeta \to \infty} \left(\frac{g_+(\zeta)}{\zeta} \right) > -\infty, \quad \limsup_{\zeta \to \infty} \left(\frac{g_-(\zeta)}{\zeta} \right) < \infty, \tag{56}$$

and

$$\int_a^\infty s^{\theta q/p} m_1^q(s) ds < \infty. \tag{57}$$

If u is a non-oscillatory solution of (46), *then*

$$\limsup_{\zeta \to \infty} \frac{|u(\zeta)|}{\zeta} < \infty.$$

Theorem 21. *Let* $0 < \lambda < 1$ *and condition* (56) *of Theorem* 20 *be replaced by*

$$\int_{\zeta_1}^\zeta (\zeta - s)^{\kappa-1} \left[\int_{\zeta_1}^\tau s^{\gamma-1} m_1^{\frac{\lambda}{\lambda-1}}(s) h^{\frac{1}{1-\lambda}}(s) ds \right] d\tau < \infty, \tag{58}$$

and

$$\liminf_{\zeta \to \infty} \frac{1}{\zeta} \int_a^\zeta (\zeta - s)^{\kappa-1} e(s) ds > -\infty, \quad \limsup_{\zeta \to \infty} \frac{1}{\zeta} \int_a^\zeta (\zeta - s)^{\kappa-1} e(s) ds < \infty, \tag{59}$$

then the conclusion of Theorem (20) *holds.*

Theorem 22. *Let* $0 < \lambda < 1$, *the constants* κ, p, q, γ *and* θ *be defined as is in Theorem* 18, *and conditions* (54)–(57) *hold. If for every constant* M, $0 < M < 1$,

$$\liminf_{\zeta \to \infty} [M\zeta + g_+(\zeta)] = -\infty, \tag{60}$$

and

$$\limsup_{\zeta\to\infty}[M\zeta + g_-(\zeta)] = \infty, \tag{61}$$

then (46) *oscillatory.*

Theorem 23. *Let* $\lambda = 1$*, the constants* κ*,* p*,* q*,* γ *and* θ *be defined as is in Theorem* 18*, and conditions* (57) *and* (59) *hold with* $h(\zeta) = m_1(\zeta)$*. Then every non-oscillatory solution of* (46) *satisfies* (51)*.*

Theorem 24. *Let* $\lambda = 1$*, the constants* κ*,* p*,* q*,* γ *and* θ *be defined as is in Theorem* 18*, and conditions* (57) *and* (59) *hold with* $h(\zeta) = m_1(\zeta)$*. If for every constant* M*,* $0 < M < 1$*,*

$$\liminf_{\zeta\to\infty}\left[M\zeta + \int_a^\zeta (\zeta - s)^{\kappa-1}e(s)ds\right] = -\infty, \tag{62}$$

and

$$\limsup_{\zeta\to\infty}\left[M\zeta + \int_a^\zeta (\zeta - s)^{\kappa-1}e(s)ds\right] = \infty, \tag{63}$$

then (46) *oscillatory.*

Yang et al. [7] studied forced oscillatory properties of solutions to nonlinear fractional differential equations with damping,

$$\begin{cases} \mathcal{D}_0^{\kappa+1}u + p_1(\zeta)\mathcal{D}_0^\kappa u + q_1(\zeta)\mathfrak{f}_5(u) = g_1(\zeta), & \zeta > 0, \\ \lim_{\zeta\to 0^+}\mathcal{I}_0^{1-\kappa}u(\zeta) = b \in \mathbb{R}, \end{cases} \tag{64}$$

where $0 < \kappa < 1$, $p_1 \in C(\mathbb{R}^+, \mathbb{R})$, $q_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\mathfrak{f}_5 \in C(\mathbb{R}, \mathbb{R})$, $g_1 \in C(\mathbb{R}^+, \mathbb{R})$, and

$$\frac{\mathfrak{f}_5(u)}{u} > 0, \quad u \neq 0.$$

Theorem 25 ([7])**.** *Suppose that*

$$\liminf_{\zeta\to\infty}\int_0^\zeta \frac{(\zeta - s)^{\kappa-1}}{V(s)}\left[M + \int_{\zeta_0}^s g_1(\xi)V(\xi)d\xi\right]ds < 0, \tag{65}$$

and

$$\limsup_{\zeta\to\infty}\int_0^\zeta \frac{(\zeta - s)^{\kappa-1}}{V(s)}\left[M + \int_{\zeta_0}^s g_1(\xi)V(\xi)d\xi\right]ds > 0, \tag{66}$$

where M is a constant and

$$V(\zeta) = \exp\left(\int_{\zeta_0}^\zeta p(s)ds\right).$$

Then (64) *is oscillatory.*

Using Riccati type transformations, Tunč et al. [8] established some new oscillation criteria for the fractional differential equation

$$\mathcal{D}_0^{\kappa+1}u + p_2(\zeta)\mathcal{D}_0^\kappa u + q_2(\zeta)\mathfrak{f}_6(G_1(\zeta)) = 0, \quad \zeta \geq t_0 > 0, \tag{67}$$

where $0 < \kappa < 1$, $p_2 \in C([\zeta_0, \infty), \mathbb{R})$ with $p_2(\zeta) < 0$, $q_2 \in C([\zeta_0, \infty), \mathbb{R}^+)$ with $q_2(\zeta) \geq 0$, $\mathfrak{f}_6 \in C(\mathbb{R}, \mathbb{R})$ with

$$u\mathfrak{f}_6(u) > 0, \quad u \neq 0,$$

and there exists a constant $K_5 > 0$ such that

$$\frac{\mathfrak{f}_6(u)}{u} \geq K_5, \quad u \neq 0,$$

and

$$G_1(\zeta) = \int_0^\zeta (\zeta - s)^{-\kappa} u(s) ds.$$

Theorem 26 ([8]). *If*

$$\lim_{\zeta \to \infty} \left[\frac{1}{4\Gamma(1-\kappa)} \int_{\zeta_0}^\zeta \left(4\Gamma(1-\kappa)K_5 q_2(s) - p_2^2(s) \right) ds \right] = \infty, \tag{68}$$

then (67) *oscillatory.*

Theorem 27 ([8]). *Assume that there exists a positive function $g_2 \in C'[\zeta_0, \infty)$ such that*

$$\lim_{\zeta \to \infty} \int_{\zeta_0}^\zeta \frac{\Gamma(1-\kappa)}{g_2(s)} ds = \infty, \tag{69}$$

and

$$\lim_{\zeta \to \infty} \left[-\frac{1}{\sqrt{4\Gamma(1-\kappa)}} \int_{\zeta_0}^\zeta \Psi(s) ds + \frac{g_2'(s)}{\sqrt{4\Gamma(1-\kappa)}} \right] = \infty, \tag{70}$$

where

$$\Psi(s) = p_2^2(s)g_2(s) + \frac{[g_2'(s)]^2}{g_2(s)} - 2p_2(s)g_2'(s) - 4K\Gamma(1-\kappa)g_2(s)q_2(s), \tag{71}$$

then (67) *oscillatory.*

Grace dealt with the asymptotic behavior of non-oscillatory solutions of fractional differential equations of the form

$$^C\mathcal{D}_a^\kappa v = e(\zeta) + \mathfrak{f}_4(\zeta, u), \quad \zeta \geq a \geq 0, \quad \kappa \in (0,1). \tag{72}$$

The following particular cases are considered:

$$v(\zeta) = \left(r_1(\zeta) |u'|^{\delta-1} u' \right)', \quad \delta \geq 1, \tag{73}$$

$$v(\zeta) = u', \tag{74}$$

$$v(\zeta) = u, \tag{75}$$

where $r_1 : [a, \infty) \to (0, \infty)$ and $\mathfrak{f}_4 : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ satisfies $u\mathfrak{f}_4(\zeta, u) > 0$ for $u \neq 0$ and $\zeta \geq a$. We find the following results in Reference 8 of [9].

Theorem 28. *Consider* (72) *with* (73). *Assume that the function $\mathfrak{f}_4$ satisfies*

$$u\mathfrak{f}_4(\zeta, u) \leq t^{\gamma-1} h_1(\zeta) |u|^{\beta+1}, \quad u \neq 0, \quad t \geq a,$$

for some function $h_1 : (a, \infty) \to \mathbb{R}^+$ and real numbers $\gamma > 0$ and $0 < \beta < \delta$. For the sake of simplification, define

$$R(\zeta) = \int_a^\zeta r_1^{-1/\delta}(s) ds,$$

and

$$g_2(\zeta) = \int_a^\zeta (\zeta - s)^{\kappa-1} s^{\gamma-1} m_2^{\beta/(\beta-\delta)}(s) h_1^{\delta/(\delta-\beta)}(s) ds,$$

where $m_2 : (a, \infty) \to \mathbb{R}^+$ is continuous function. Let q be a conjugate number of $p > 1$, $p(\kappa - 1) + 1 > 0$, and $\gamma = 2 - \kappa - \frac{1}{p}$. Suppose that for any $T \geq \max\{1, a\}$, we have

$$\int_{\zeta}^{\infty} s^q R^{q\delta}(s) m_2^q(s) ds < \infty, \tag{76}$$

$$\limsup_{\zeta \to \infty} \frac{1}{\zeta} \int_{\zeta}^{\zeta} g_2(s) ds < \infty, \tag{77}$$

$$\liminf_{\zeta \to \infty} \frac{1}{\zeta} \int_{\zeta}^{\zeta} \int_{a}^{\tau} (\tau - s)^{\kappa-1} e(s) ds d\tau > -\infty, \tag{78}$$

$$\limsup_{\zeta \to \infty} \frac{1}{\zeta} \int_{\zeta}^{\zeta} \int_{a}^{\tau} (\tau - s)^{\kappa-1} e(s) ds d\tau < \infty. \tag{79}$$

Then every non-oscillatory solution u of (72) satisfies

$$|u(\zeta)| = O\left(\zeta^{1/\delta} R(\zeta)\right), \quad \zeta \to \infty.$$

Theorem 29. *Consider* (72) *with* (74). *Assume that the function* $\mathfrak{f}_4$ *satisfies*

$$u\mathfrak{f}_4(\zeta, u) \le t^{\gamma-1} h_1(\zeta) |u|^{\lambda+1}, \quad u \ne 0,$$

for some function $h_1 : [a, \infty) \to \mathbb{R}^+$ *and real numbers* $\gamma > 0$ *and* $0 < \lambda < 1$. *For the sake of simplification, define*

$$g_3(\zeta) = \int_{a}^{\zeta} (\zeta - s)^{\kappa-1} s^{\gamma-1} m_3^{\lambda/(\lambda-1)}(s) h_1^{1/(1-\lambda)}(s) ds,$$

where $m_3 : (a, \infty) \to \mathbb{R}^+$ *is continuous function. Let* q *be a conjugate number of* $p > 1$, $p(\kappa - 1) + 1 > 0$, *and* $\gamma = 2 - \kappa - \frac{1}{p}$. *Suppose that for any* $T \ge \max\{1, a\}$, *we have*

$$\int_{\zeta}^{\infty} s^q m_3^q(s) ds < \infty, \tag{80}$$

$$\limsup_{\zeta \to \infty} g_3(\zeta) < \infty, \tag{81}$$

$$\liminf_{\zeta \to \infty} \int_{\zeta}^{\zeta} (\zeta - s)^{\kappa-1} e(s) ds > -\infty, \tag{82}$$

$$\limsup_{\zeta \to \infty} \int_{\zeta}^{\zeta} (\zeta - s)^{\kappa-1} e(s) ds < \infty. \tag{83}$$

Then every non-oscillatory solution u of (72) satisfies

$$|u(\zeta)| = O(\zeta), \quad \zeta \to \infty.$$

Theorem 30. *Consider* (72) *with* (75). *Let* q *be a conjugate number of* $p > 1$, $p(\kappa - 1) + 1 > 0$, *and* $\gamma = 2 - \kappa - \frac{1}{p}$. *Suppose that for any* $T \ge \max\{1, a\}$, *we have*

$$\int_{\zeta}^{\infty} m_3^q(s) ds < \infty, \tag{84}$$

$$\limsup_{\zeta \to \infty} g_3(\zeta) < \infty, \tag{85}$$

$$\liminf_{\zeta \to \infty} \int_{\zeta}^{\zeta} (\zeta - s)^{\kappa-1} e(s) ds > -\infty, \tag{86}$$

$$\limsup_{\zeta \to \infty} \int_{\zeta}^{\zeta} (\zeta - s)^{\kappa-1} e(s) ds < \infty. \tag{87}$$

Then every non-oscillatory solution u of (72) is bounded.

Grace et al. [10] established some new criteria for the oscillation of fractional differential equations with the Caputo derivative of the form

$$^{C}\mathcal{D}_a^{\kappa}u = e(\zeta) + \mathfrak{f}_4(\zeta, u), \quad a > 1, \quad \zeta > 0, \tag{88}$$

where $\kappa = \alpha + n - 1$, $\alpha \in (0,1)$, and $n \geq 1$ is a natural number. Assume that $\mathfrak{f}_4 : [a, \infty) \times \mathbb{R} \to (0, \infty)$ is continuous and there exists a continuous function $h : [a, \infty) \to (0, \infty)$ and a real number λ with $0 < \lambda \leq 1$ such that (54) holds. Denote by

$$g_4(\zeta) = \frac{1}{\Gamma(\kappa)} \int_{\zeta_1}^{\zeta} (\zeta - s)^{\kappa-1} (1-\lambda) \lambda^{\frac{\lambda}{1-\lambda}} s^{\gamma-1} m_4^{\frac{\lambda}{\lambda-1}}(s) h^{\frac{1}{1-\lambda}}(s) ds. \tag{89}$$

Here, $0 < \lambda < 1$, $t \geq \zeta_1$ for some $\zeta_1 \geq a$, and $m_4 : [a, \infty) \to (0, \infty)$ is a given continuous function.

Theorem 31 ([10]). *Let $0 < \lambda < 1$. Suppose that $p > 1$, $p(\kappa - 1) + 1 > 0$, $p(\gamma - 1) + 1 > 0$, $q = \frac{p}{p-1}$, $\gamma = (n - \kappa) + \frac{1}{q}$, and the function $e : \mathbb{R} \to \mathbb{R}$ is continuous such that*

$$\frac{g_4(\zeta)}{\zeta^{n-1}} \text{ is bounded for all } \zeta \geq a, \tag{90}$$

$$\frac{1}{\zeta^{n-1}} \int_a^{\zeta} (\zeta - s)^{\kappa-1} |e(s)| ds \text{ is bounded for all } \zeta \geq a, \tag{91}$$

and

$$\int^{\infty} s^{(n-1)q} m_4^q(s) ds < \infty. \tag{92}$$

If u is any non-oscillatory solution of (88), then

$$\limsup_{\zeta \to \infty} \frac{|u(\zeta)|}{\zeta^{n-1}} < \infty. \tag{93}$$

Theorem 32 ([10]). *Let $\lambda = 1$, the constants κ, p, q, γ and θ be defined as is in Theorem 31, and conditions (90)–(92) hold with $h(\zeta) = m_4(\zeta)$. Then every non-oscillatory solution of (88) satisfies (93).*

Theorem 33 ([10]). *Let $0 < \lambda < 1$, the constants κ, p, q, γ and θ be defined as is in Theorem 31, and conditions (90)–(92) hold. If for every constant $M > 0$,*

$$\liminf_{\zeta \to \infty} \left[M\zeta^{n-1} + \int_{\zeta_1}^{\zeta} (\zeta - s)^{\kappa-1} e(s) ds \right] = -\infty, \tag{94}$$

and

$$\limsup_{\zeta \to \infty} \left[M\zeta^{n-1} + \int_{\zeta_1}^{\zeta} (\zeta - s)^{\kappa-1} e(s) ds \right] = \infty, \tag{95}$$

then (88) oscillatory.

Theorem 34 ([10]). *Let $\lambda = 1$ and let the hypotheses of Theorem 33 hold with $h(\zeta) = m_4(\zeta)$. Then the conclusion of Theorem 33 holds.*

Grace [11] presented the conditions under which every non-oscillatory solution of the forced fractional differential equation

$$^{C}\mathcal{D}_a^{\kappa}v = e(\zeta) + \mathfrak{f}_4(\zeta, u), \quad a > 1, \quad \kappa \in (0,1), \tag{96}$$

where $\mathfrak{f}_4 : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and assume that there exists a continuous function $h : [a, \infty) \to (0, \infty)$ and a real number λ with $0 < \lambda \leq 1$ such that

$$0 \leq u \mathfrak{f}_4(\zeta, u) \leq h(\zeta)|u|^{\lambda+1}, \quad u \neq 0, \quad \zeta \geq a,$$

holds. Assume, for $\zeta \geq a \geq 1$,

$$R_1(\zeta) = \int_a^\zeta r_1^{-1/\delta}(s)ds \to \infty \text{ as } \zeta \to \infty. \tag{97}$$

Denote by

$$g_5(\zeta) = \left(\frac{\delta-\lambda}{\lambda}\right)\left(\frac{\lambda}{\delta}\right)^{\frac{\delta}{\delta-\lambda}} \int_{\zeta_1}^\zeta (\zeta-s)^{\kappa-1} m_5^{\frac{\lambda}{\lambda-\delta}}(s) h^{\frac{\delta}{\delta-\lambda}}(s)ds. \tag{98}$$

Here, $0 < \lambda < 1$, $\zeta \geq \zeta_1$ for some $\zeta_1 \geq a$, and $m_5 : [a,\infty) \to (0,\infty)$ is a given continuous function.

Theorem 35 ([11]). *Consider* (96) *with the particular case*

$$v(\zeta) = \left(r_1(\zeta)\left(u'(\zeta)\right)^\delta\right)', \quad c_0 = y(a), \quad \delta \geq 1,$$

where $r_1 : [a,\infty) \to (0,\infty)$ is a continuous function. Let $0 < \lambda < 1$. Suppose that $p > 1$, $p(\kappa-1)+1 > 0$, $q = \frac{p}{p-1}$,

$$\lim_{\zeta\to\infty} g_5(\zeta) < \infty, \tag{99}$$

$$\lim_{\zeta\to\infty} \int_a^\zeta (\zeta-s)^{\kappa-1}|e(s)|ds < \infty, \tag{100}$$

and

$$\int_{\zeta_1}^\infty s^q m_5^q(s) R_1^{\delta q}(s)ds < \infty. \tag{101}$$

If u is any non-oscillatory solution of (96)*, then*

$$\limsup_{\zeta\to\infty} \frac{e^{-t}|u(\zeta)|}{\zeta^{1/\delta}R_1(\zeta)} < \infty. \tag{102}$$

Theorem 36 ([11]). *Consider* (96) *with the particular case*

$$v(\zeta) = u'(\zeta), \quad c_0 = y(a), \quad \delta \geq 1.$$

Let $0 < \lambda < 1$. Suppose that $p > 1$, $p(\kappa-1)+1 > 0$, $q = \frac{p}{p-1}$,

$$\lim_{\zeta\to\infty} g_5(\zeta) < \infty \text{ for all } \zeta \geq \zeta_1 \geq a, \tag{103}$$

$$\lim_{\zeta\to\infty} \int_a^\zeta (\zeta-s)^{\kappa-1}|e(s)|ds < \infty, \tag{104}$$

and

$$\int_{\zeta_1}^\infty s^q m_5^q(s)ds < \infty. \tag{105}$$

If u is any non-oscillatory solution of (96)*, then*

$$\limsup_{\zeta\to\infty} \frac{e^{-t}|u(\zeta)|}{\zeta} < \infty. \tag{106}$$

Theorem 37 ([11]). *Consider* (96) *where $u(a) = c_0$ and c_0 is a real constant. Let $0 < \lambda < 1$. Suppose that $p > 1$, $p(\kappa-1)+1 > 0$, $q = \frac{p}{p-1}$,*

$$\lim_{\zeta \to} g_5(\zeta) < \infty \text{ for all } \zeta \geq \zeta_1 \geq a, \tag{107}$$

$$\lim_{\zeta \to} \int_a^\zeta (\zeta - s)^{\kappa-1}|e(s)|ds < \infty, \tag{108}$$

and

$$\int_{\zeta_1}^\infty m_5^q(s)ds < \infty. \tag{109}$$

If u is any non-oscillatory solution of (96)*, then* $e^{-\zeta}|u(\zeta)|$ *is bounded.*

Theorem 38 ([11]). *Let* $\lambda = 1$ *and the hypotheses of Theorems* 35–37 *hold with* $m_5(\zeta) = h(\zeta)$. *Then the conclusion of Theorems* 35–37 *holds.*

Grace et al. dealt with the boundedness of non-oscillatory solutions of the forced fractional differential equation with positive and negative terms

$$^C\mathcal{D}_a^\kappa v + \mathfrak{f}_6(\zeta, u) = e_1(\zeta) + k_1(\zeta)u + h_2(\zeta, u), \quad a > 1, \quad \kappa \in (0,1), \quad \zeta \geq a, \tag{110}$$

with the particular cases

$$v(\zeta) = (r_2(\zeta)u')', \tag{111}$$
$$v(\zeta) = r_2(\zeta)u'. \tag{112}$$

The following conditions are always assumed to hold:

1. $r_2, k_1 : [a, \infty) \to (0, \infty)$ and $e_1 : [a, \infty) \to \mathbb{R}$ are continuous functions;
2. $\mathfrak{f}_6, h_2 : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exist continuous functions $m_6, m_7 : [a, \infty) \to (0, \infty)$ and positive real numbers λ_1 and γ_1 with $\lambda_1 > \gamma_1$ such that for $u \neq 0$ and $\zeta \geq a$,

$$u\mathfrak{f}_6(\zeta, u) \geq m_6(\zeta)|u|^{\lambda_1+1}, \quad 0 \leq uh_2(\zeta, u) \leq m_7(\zeta)|u|^{\gamma_1+1}.$$

We find the following results in Reference 11 of [9].

Theorem 39. *Assume there exist real number* $p > 1$ *such that* $p(\kappa - 1) + 1 > 0$, *and there are real numbers* $S > 0$ *and* $\sigma_1 > 1$ *such that*

$$\left(\frac{\zeta}{r_2(\zeta)}\right) \leq Se^{-\sigma_1\zeta}. \tag{113}$$

If

$$\int_a^\infty e^{-qs}k_1^q(s)ds < \infty, \quad q = \frac{p}{p-1}, \tag{114}$$

$$\lim_{\zeta \to \infty} \int_a^\zeta (\zeta - s)^{\kappa-1}|e_1(s)|ds < \infty, \tag{115}$$

$$\lim_{\zeta \to \infty} \int_a^\zeta (\zeta - s)^{\kappa-1}g_6(s)ds < \infty, \tag{116}$$

where

$$g_6(\zeta) = \left(\frac{\lambda_1 - \gamma_1}{\gamma_1}\right)\left[\frac{\gamma_1}{\lambda_1}m_7(\zeta)\right]^{\frac{\lambda_1}{\lambda_1-\gamma_1}}[m_7(\zeta)]^{\frac{\gamma_1}{\gamma_1-\lambda_1}}, \tag{117}$$

then every non-oscillatory solution of (110)*,* (111) *is bounded.*

Theorem 40. *Assume there exist real number* $p > 1$ *such that* $p(\kappa - 1) + 1 > 0$, *and there are real numbers* $S > 0$ *and* $\sigma_1 > 1$ *such that*

$$\left(\frac{1}{r_2(\zeta)}\right) \leq Se^{-\sigma_1\zeta}. \tag{118}$$

If

$$\int_a^\infty e^{-qs}k_1^q(s)ds < \infty, \quad q = \frac{p}{p-1}, \tag{119}$$

$$\lim_{\zeta\to\infty}\int_a^\zeta (\zeta - s)^{\kappa-1}|e_1(s)|ds < \infty, \tag{120}$$

$$\lim_{\zeta\to\infty}\int_a^\zeta (\zeta - s)^{\kappa-1}g_6(s)ds < \infty, \tag{121}$$

where

$$g_6(\zeta) = \left(\frac{\lambda_1 - \gamma_1}{\gamma_1}\right)\left[\frac{\gamma_1}{\lambda_1}m_7(\zeta)\right]^{\frac{\lambda_1}{\lambda_1-\gamma_1}}[m_7(\zeta)]^{\frac{\gamma_1}{\gamma_1-\lambda_1}}, \tag{122}$$

then every non-oscillatory solution of (110), (112) *is bounded.*

Seemab et al. [6] established the oscillation criteria and asymptotic behavior of solutions for a class of fractional differential equations by considering equations of the form

$$\begin{cases} \mathcal{D}_0^\kappa u + \lambda_2 u = \mathfrak{f}_7(\zeta, u), & \zeta > 0, \\ \mathcal{D}_0^{\kappa-1}u(0) = u_0, & \lim_{\zeta\to 0^+}\mathcal{I}_a^{2-\kappa}u(0) = u_1, \end{cases} \tag{123}$$

where $1 < \kappa \leq 2$, $\lambda_2 \in [1, \infty)$; $u_0, u_1 \in [0, \infty)$ and $\mathfrak{f}_7 : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function.

Theorem 41 ([6]). *Let $1 < \kappa < 2$, $u_1 = 0$, $\mathfrak{f}_7 : [0, \infty) \times \mathbb{R} \to [0, \infty)$ be a continuous function, and there exists a constant $M_1 > 0$ such that*

$$|\mathfrak{f}_7(\zeta, u)| \leq \frac{M_1}{\Gamma(1-\kappa)(\zeta-a)^\kappa}, \text{ for some } \zeta > a > 0. \tag{124}$$

Then all unbounded solutions of (123) *are oscillatory.*

Theorem 42 ([6]). *Let $u_1 = 0$ and $\mathfrak{f}_7 : [0, \infty) \times \mathbb{R} \to [0, \infty)$ satisfy $\mathfrak{f}_7(\zeta, -u) = -\mathfrak{f}_7(\zeta, u)$ and $u_2 \leq u_3$ implies $\mathfrak{f}_7(\zeta, u_2) \geq \mathfrak{f}_7(\zeta, u_3)$ for each fixed ζ. Let*

$$\lim_{\zeta\to\infty}\int_\rho^\zeta \mathfrak{f}_7(s, L)ds = \infty, \text{ for some } \zeta \geq \rho > 0, \ L > 0. \tag{125}$$

Then all bounded solutions of (123) *are oscillatory.*

Theorem 43 ([6]). *Let $\mathfrak{f}_7 : [0, \infty) \times \mathbb{R} \to [0, \infty)$ satisfy $\mathfrak{f}_7(\zeta, -u) = -\mathfrak{f}_7(\zeta, u)$ and $\mathfrak{f}_7(\zeta, u)$ be monotonically increasing in u for each fixed t. Let*

$$\liminf_{\zeta\to\infty}\int_0^\zeta \mathfrak{f}_7(s, L)ds = -\infty, \text{ for some } L > 0. \tag{126}$$

Then all bounded solutions of (123) *are non-oscillatory.*

Theorem 44 ([6]). *Let $\mathfrak{f}_7 : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be monotonically increasing in u for each fixed t and it satisfy $\mathfrak{f}_7(\zeta, -u) = -\mathfrak{f}_7(\zeta, u)$ and $u\mathfrak{f}_7(\zeta, u) < 0$ if $u \neq 0$. Let*

$$\lim_{\zeta\to\infty}\int_\rho^\zeta \mathfrak{f}_7(s, L)ds = \infty, \text{ for some } \zeta \geq \rho > 0, \ L > 0. \tag{127}$$

If $u(\zeta)$ are oscillatory solutions of (123) *such that* $\lim_{\zeta \to \infty} u(\zeta)$ *exists, then*

$$\lim_{\zeta \to \infty} u(\zeta) = 0.$$

Theorem 45 ([6]). *Let $u_1 = 0$, $\mathfrak{f}_7 : [0, \infty) \times \mathbb{R} \to [0, \infty)$ be monotonically decreasing in u for each fixed ζ and it satisfy $\mathfrak{f}_7(\zeta, -u) = -\mathfrak{f}_7(\zeta, u)$. Let*

$$\lim_{\zeta \to \infty} \int_0^\zeta \mathfrak{f}_7(s, L)ds = \infty, \text{ for some } L > 0, \tag{128}$$

then all bounded solutions of (123) *are eventually negative.*

Theorem 46 ([6]). *Let $u_1 = 0$, $\mathfrak{f}_7 : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ that satisfy $\mathfrak{f}_7(\zeta, -u) = -\mathfrak{f}_7(\zeta, u)$ and $u\mathfrak{f}_7(\zeta, u) < 0$ if $u \neq 0$. Moreover, $u_2 \leq u_3$ implies $\mathfrak{f}_7(\zeta, u_2) \leq \mathfrak{f}_7(\zeta, u_3)$ for each fixed t. Let*

$$\lim_{\zeta \to \infty} \int_\rho^\zeta \mathfrak{f}_7(s, L)ds = \infty, \text{ for some } \zeta \geq \rho > 0, \ L > 0, \tag{129}$$

then no non-oscillatory solution of (123) *is bounded away from zero as $\zeta \to \infty$.*

Theorem 47 ([6]). *Let $u_1 = 0$, $\mathfrak{f}_7 : [0, \infty) \times \mathbb{R} \to [0, \infty)$ that satisfying $\mathfrak{f}_7(\zeta, -u) = -\mathfrak{f}_7(\zeta, u)$ and, $u_2 \leq u_3$ implies $\mathfrak{f}_7(\zeta, u_2) \geq \mathfrak{f}_7(\zeta, u_3)$ for each fixed ζ. Let*

$$\lim_{\zeta \to \infty} \int_\rho^\zeta \mathfrak{f}_7(s, L)ds = \infty, \text{ for some } \zeta \geq \rho > 0, \ L > 0, \tag{130}$$

then no non-oscillatory solution of (123) *goes to zero as $\zeta \to \infty$.*

Theorem 48 ([6]). *Let $u_1 = 0$, $\mathfrak{f}_7 : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ that satisfy $\mathfrak{f}_7(\zeta, -u) = -\mathfrak{f}_7(\zeta, u)$ and $u\mathfrak{f}_7(\zeta, u) < 0$ if $u \neq 0$. Moreover, $u_2 \leq u_3$ implies $\mathfrak{f}_7(\zeta, u_2) \leq \mathfrak{f}_7(\zeta, u_3)$ for each fixed ζ. Let*

$$\lim_{\zeta \to \infty} \int_\rho^\zeta \mathfrak{f}_7(s, L)ds = \infty, \text{ for some } \zeta \geq \rho > 0, \ L > 0. \tag{131}$$

If $u(\zeta)$ is a non-oscillatory solution of (123) *such that $\lim_{\zeta \to \infty} u(\zeta)$ exists, then $\lim_{\zeta \to \infty} u(\zeta) = 0$.*

Graef et al. [9] dealt with the boundedness of non-oscillatory solutions of the forced fractional differential equation with positive and negative terms

$$^C\mathcal{D}_a^\kappa v + \mathfrak{f}_6(\zeta, u) = e_1(\zeta) + k_1(\zeta)u^\eta + h_2(\zeta, u), \quad \kappa \in (0, 1), \quad \zeta \geq a \geq 1, \tag{132}$$

with the particular cases

$$v(\zeta) = \left(r_2(\zeta) \left(u' \right)^\eta \right)', \tag{133}$$

$$v(\zeta) = r_2(\zeta) \left(u' \right)^\eta. \tag{134}$$

Here, $\eta \geq 1$ is the ratio of positive odd integers. The following conditions are always assumed to hold:
1. $r_2, k_1 : [a, \infty) \to (0, \infty)$ and $e_1 : [a, \infty) \to \mathbb{R}$ are continuous functions;
2. $\mathfrak{f}_6, h_2 : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exist continuous functions $m_6, m_7 : [a, \infty) \to (0, \infty)$ and positive real numbers λ_1 and γ_1 with $\lambda_1 > \gamma_1$ such that for $u \neq 0$ and $t \geq a$,

$$u\mathfrak{f}_6(\zeta, u) \geq m_6(\zeta)|u|^{\lambda_1 + 1}, \quad 0 \leq uh_2(\zeta, u) \leq m_7(\zeta)|u|^{\gamma_1 + 1}.$$

Theorem 49 ([9])*. Assume there exist real number $p > 1$ such that $p(\kappa - 1) + 1 > 0$. If*

$$\int_a^\infty k_1^q(s)s^q R_2^{\eta q}(s)ds < \infty, \quad q = \frac{p}{p-1}, \tag{135}$$

$$\lim_{\zeta \to \infty} \int_a^\zeta (\zeta - s)^{\kappa - 1}|e_1(s)|ds < \infty, \tag{136}$$

$$\lim_{\zeta \to \infty} \int_a^\zeta (\zeta - s)^{\kappa - 1}g_6(s)ds < \infty, \tag{137}$$

where

$$g_6(\zeta) = \left(\frac{\lambda_1 - \gamma_1}{\gamma_1}\right)\left[\frac{\gamma_1}{\lambda_1}m_7(\zeta)\right]^{\frac{\lambda_1}{\lambda_1 - \gamma_1}}[m_6(\zeta)]^{\frac{\gamma_1}{\gamma_1 - \lambda_1}}, \tag{138}$$

and

$$R_2(\zeta) = \int_a^\zeta r_2^{-1/\eta}(s) \to \infty \text{ as } \zeta \to \infty. \tag{139}$$

then every non-oscillatory solution u of (132)*,* (133) *satisfies*

$$\limsup_{\zeta \to \infty} \frac{|u(\zeta)|}{\zeta^{1/\eta}e^{\zeta/\eta}R_2(\zeta)} < \infty. \tag{140}$$

Theorem 50 ([9])*. Assume there exist real number $p > 1$ such that $p(\kappa - 1) + 1 > 0$. If*

$$\int_a^\infty k_1^q(s)R_2^{\eta q}(s)ds < \infty, \quad q = \frac{p}{p-1}, \tag{141}$$

$$\lim_{\zeta \to \infty} \int_a^\zeta (\zeta - s)^{\kappa - 1}|e_1(s)|ds < \infty, \tag{142}$$

$$\lim_{\zeta \to \infty} \int_a^\zeta (\zeta - s)^{\kappa - 1}g_6(s)ds < \infty, \tag{143}$$

where

$$g_6(\zeta) = \left(\frac{\lambda_1 - \gamma_1}{\gamma_1}\right)\left[\frac{\gamma_1}{\lambda_1}m_7(\zeta)\right]^{\frac{\lambda_1}{\lambda_1 - \gamma_1}}[m_6(\zeta)]^{\frac{\gamma_1}{\gamma_1 - \lambda_1}}, \tag{144}$$

and

$$R_2(\zeta) = \int_a^\zeta r_2^{-1/\eta}(s) \to \infty \text{ as } \zeta \to \infty. \tag{145}$$

then every non-oscillatory solution u of (132)*,* (133) *satisfies*

$$\limsup_{\zeta \to \infty} \frac{|u(\zeta)|}{e^{\zeta/\eta}R_2(\zeta)} < \infty. \tag{146}$$

In this line, Grace dealt with the asymptotic behavior of positive solutions of certain forced fractional differential equations of the form (96) with the particular cases

$$v(\zeta) = \left(r_1(\zeta)u'(\zeta)\right)', \quad a_0 = v(a), \quad a_0 \in \mathbb{R}, \tag{147}$$

$$v(\zeta) = u'(\zeta), \quad a_0 = v(a), \quad a_0 \in \mathbb{R}, \tag{148}$$

$$a_0 = u(a), \quad a_0 \in \mathbb{R}, \tag{149}$$

where $r_1 : [a, \infty) \to (0, \infty)$ is a continuous function, $\mathfrak{f}_4 : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and assume that there exists a continuous function $h : [a, \infty) \to (0, \infty)$ and a real number λ with $0 < \lambda \leq 1$ such that (54) holds. Denote by

$$g_7(\zeta) = (1 - \lambda)\lambda^{\frac{\lambda}{1-\lambda}}\frac{1}{\Gamma(\kappa)}\int_{\zeta_1}^\zeta (\zeta - s)^{\kappa - 1}s^{\gamma_2 - 1}m_5^{\frac{\lambda}{\lambda - 1}}(s)h^{\frac{1}{1-\lambda}}(s)ds. \tag{150}$$

Here, $0 < \lambda < 1$, $\zeta \geq \zeta_1$ for some $t_1 \geq a$, and $m_5 : [a, \infty) \to (0, \infty)$ is a given continuous function. We find the following results in Reference 11 of [11].

Theorem 51. *Consider* (96) *and* (147). *Let* $0 < \lambda < 1$. *Suppose that* $p > 1$, $p(\kappa - 1) + 1 > 0$, $q = \frac{p}{p-1}$, $\gamma_2 = 2 - \kappa - \frac{1}{p}$, $p(\gamma_2 - 1) + 1 > 0$, *and* $\frac{\zeta}{r_1(\zeta)}$ *is bounded on* $[a, \infty)$,

$$\int_{\zeta_1}^{\infty} \frac{s}{r_1(s)} ds < \infty, \tag{151}$$

$$\int_{\zeta_1}^{\infty} \left(s^2 m_5(s)\right)^q ds < \infty \tag{152}$$

$$\limsup_{\zeta \to \infty} \frac{1}{\zeta^2} \int_a^{\zeta} \frac{1}{r_1(x)} \int_{\zeta_1}^{x} \int_a^{y} (y - s)^{\kappa - 1} e(s) ds dy dx < \infty, \tag{153}$$

$$\liminf_{\zeta \to \infty} \frac{1}{\zeta^2} \int_a^{\zeta} \frac{1}{r_1(x)} \int_{\zeta_1}^{x} \int_a^{y} (y - s)^{\kappa - 1} e(s) ds dy dx > -\infty, \tag{154}$$

and

$$\lim_{\zeta \to \infty} \frac{1}{\zeta^2} \int_{\zeta_1}^{\zeta} \frac{1}{r_1(x)} \int_{\zeta_1}^{x} g_7(s) ds dx < \infty. \tag{155}$$

If u is a positive solution of (96), *then*

$$\limsup_{\zeta \to \infty} \frac{u(\zeta)}{\zeta^2} < \infty. \tag{156}$$

Remark 1. *Conditions* (153) *and* (154) *can be replaced by*

$$\lim_{\zeta \to \infty} \frac{1}{\zeta^2} \int_a^{\zeta} \frac{1}{r_1(x)} \int_{\zeta_1}^{x} \int_a^{y} (y - s)^{\kappa - 1} |e(s)| ds dy dx < \infty,$$

and the result remains valid.

Theorem 52. *Consider* (96) *and* (148). *Let* $0 < \lambda < 1$. *Suppose that* $p > 1$, $p(\kappa - 1) + 1 > 0$, $q = \frac{p}{p-1}$, $\gamma_2 = 2 - \kappa - \frac{1}{p}$, $p(\gamma_2 - 1) + 1 > 0$,

$$\int_{\zeta_1}^{\infty} (sm_5(s))^q ds < \infty \tag{157}$$

$$\limsup_{\zeta \to \infty} \frac{1}{\zeta} \int_{\zeta_1}^{\zeta} \int_a^{x} (x - s)^{\kappa - 1} e(s) ds dx < \infty, \tag{158}$$

$$\liminf_{\zeta \to \infty} \frac{1}{\zeta} \int_{\zeta_1}^{\zeta} \int_a^{x} (x - s)^{\kappa - 1} e(s) ds dx > -\infty, \tag{159}$$

and

$$\lim_{\zeta \to \infty} \frac{1}{\zeta} \int_{\zeta_1}^{\zeta} g_7(s) ds < \infty. \tag{160}$$

If u is a positive solution of (96), *then*

$$\limsup_{\zeta \to \infty} \frac{u(\zeta)}{\zeta} < \infty. \tag{161}$$

Theorem 53. *Consider* (96) *and* (147). *Let* $0 < \lambda < 1$. *Suppose that* $p > 1$, $p(\kappa - 1) + 1 > 0$, $q = \frac{p}{p-1}$, $\gamma_2 = 2 - \kappa - \frac{1}{p}$, $p(\gamma_2 - 1) + 1 > 0$,

$$\int_{\zeta_1}^{\infty} (m_5(s))^q ds < \infty \tag{162}$$

$$\limsup_{\zeta\to\infty}\int_{\zeta_1}^{\zeta}(\zeta-s)^{\kappa-1}e(s)ds<\infty, \tag{163}$$

$$\liminf_{\zeta\to\infty}\int_{\zeta_1}^{\zeta}(\zeta-s)^{\kappa-1}e(s)ds>-\infty, \tag{164}$$

and

$$\lim_{\zeta\to\infty}g_7(\zeta)<\infty. \tag{165}$$

If u is a positive solution of (96), *then* $u(\zeta)$ *is bounded.*

Theorem 54. *Let* $\lambda=1$ *and the hypotheses of Theorems 51–53 hold with* $m_5(\zeta)=h(\zeta)$. *Then the conclusions of Theorems 51–53 hold.*

In [12], Grace concerned with the asymptotic behavior of non-oscillatory solutions of forced fractional differential equations of the form (96) with the particular case

$$v(\zeta)=u'''(\zeta), \tag{166}$$

where $r_1:[a,\infty)\to(0,\infty)$ is a continuous function, $\mathfrak{f}_4:[a,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and assume that there exists a continuous function $h:[a,\infty)\to(0,\infty)$ and a real number λ with $0<\lambda\le1$ such that (54) holds. Let $m_5:[a,\infty)\to(0,\infty)$ is a given continuous function.

Theorem 55 ([12]). *Consider* (96) *and* (166). *Let* $0<\lambda<1$. *Suppose that* $p>1$, $p(\kappa-1)+1>0$, $q=\frac{p}{p-1}$, $\gamma_2=2-\kappa-\frac{1}{p}$, $p(\gamma_2-1)+1>0$,

$$\int_a^{\infty}\left(s^3 m_5(s)\right)^q ds<\infty \tag{167}$$

$$\limsup_{\zeta\to\infty}\frac{1}{\zeta^3}\int_a^{\zeta}(\zeta-s)^{3+(\kappa-1)}e(s)ds<\infty, \tag{168}$$

$$\liminf_{\zeta\to\infty}\frac{1}{\zeta^3}\int_a^{\zeta}(\zeta-s)^{3+(\kappa-1)}e(s)ds>-\infty, \tag{169}$$

and

$$\lim_{\zeta\to\infty}\frac{1}{\zeta^3}\int_a^{\zeta}(\zeta-s)^{3+(\kappa-1)}\left(s^{\gamma_2-1}m_5^{\frac{\lambda}{\lambda-1}}(s)h^{\frac{1}{1-\lambda}}(s)\right)ds<\infty. \tag{170}$$

If u is a non-oscillatory solution of (96), *then*

$$\limsup_{\zeta\to\infty}\frac{|u(\zeta)|}{\zeta^3}<\infty. \tag{171}$$

Remark 2 ([12]). *Conditions* (168) *and* (169) *can be replaced by*

$$\lim_{\zeta\to\infty}\frac{1}{\zeta^3}\int_a^{\zeta}(\zeta-s)^{3+(\kappa-1)}|e(s)|ds<\infty,$$

and the result remains valid.

Theorem 56. *Let* $\lambda=1$ *and the hypotheses of Theorem 55 holds with* $m_5(\zeta)=h(\zeta)$. *Then the conclusion of Theorem 55 holds.*

Theorem 57 ([12]). *Consider* (96) *and* (166). *Let* $0<\lambda<1$. *Suppose that* $p>1$, $p(\kappa-1)+1>0$, $q=\frac{p}{p-1}$, $\gamma_2=2-\kappa-\frac{1}{p}$, $p(\gamma_2-1)+1>0$,

$$\int_a^{\infty}(s^n m_5(s))^q ds<\infty \tag{172}$$

$$\limsup_{\zeta \to \infty} \frac{1}{\zeta^n} \int_a^\zeta (\zeta - s)^{n+(\kappa-1)} e(s) ds < \infty, \tag{173}$$

$$\liminf_{\zeta \to \infty} \frac{1}{\zeta^n} \int_a^\zeta (\zeta - s)^{n+(\kappa-1)} e(s) ds > -\infty, \tag{174}$$

and

$$\lim_{\zeta \to \infty} \frac{1}{\zeta^n} \int_a^\zeta (\zeta - s)^{n+(\kappa-1)} \left(s^{\gamma_2-1} m_5^{\frac{\lambda}{\lambda-1}}(s) h^{\frac{1}{1-\lambda}}(s) \right) ds < \infty. \tag{175}$$

If u is a non-oscillatory solution of (96) *with* $v(\zeta) = u^{(n)}(\zeta)$, *then*

$$\limsup_{\zeta \to \infty} \frac{|u(\zeta)|}{\zeta^n} < \infty. \tag{176}$$

3. Oscillation Results via Liouville Operators

Definition 3 ([1,2]). *The (right-sided) Liouville fractional integral is defined by*

$$\mathcal{I}_-^\kappa \mathfrak{f}(\zeta) = \frac{1}{\Gamma(\kappa)} \int_\zeta^\infty (s - \zeta)^{\kappa-1} \mathfrak{f}(s) ds, \quad \zeta > 0, \quad \Re(\kappa) > 0.$$

The (right-sided) Liouville fractional derivative is defined by

$$\mathcal{D}_-^\kappa \mathfrak{f}(\zeta) = \left(-\frac{d}{d\zeta} \right)^n \mathcal{I}_-^{n-\kappa} \mathfrak{f}(\zeta), \quad n = [\Re(\kappa)] + 1, \quad \Re(\kappa) \geq 0, \quad \zeta > 0.$$

Chen [13] obtained several oscillation theorems for the fractional differential equation

$$\left[r(\zeta)(\mathcal{D}_-^\kappa u(\zeta))^\eta \right]' - q(\zeta) f \left(\int_\zeta^\infty (s - \zeta)^{-\kappa} u(s) ds \right) = 0, \quad \zeta > 0, \tag{177}$$

where $0 < \kappa < 1$, $\eta > 0$ is a quotient of odd positive integers, r and q are positive continuous functions on $[\zeta_0, \infty)$ for a certain $\zeta_0 > 0$ and $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\mathfrak{f}(u)/(u^\eta) \geq K$ for a certain constant $K > 0$ and for all $u \neq 0$.

Theorem 58 ([13]). *Suppose that*

$$\int_{\zeta_0}^\infty r^{-\frac{1}{\eta}}(\zeta) d\zeta = \infty, \tag{178}$$

holds. Furthermore, assume that there exists a positive function $b \in C^1[t_0, \infty)$ *such that*

$$\limsup_{\zeta \to \infty} \int_{\zeta_0}^\zeta \left[Kb(s)q(s) - \frac{r(s)[b'_+(s)]^{\eta+1}}{(\eta+1)^{\eta+1}[\Gamma(1-\kappa)b(s)]^\eta} \right] ds = \infty, \tag{179}$$

where $b'_+(s) = \max\{b'(s), 0\}$. *Then* (177) *is oscillatory.*

Theorem 59 ([13]). *Suppose that* (178) *holds. Furthermore, assume that there exists a positive function* $b \in C^1[t_0, \infty)$ *and a function* $H \in C(D, \mathbb{R})$, *where* $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ *such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq t_0\}$ *and H has a nonpositive continuous partial derivative* $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ *on* D_0 *with respect to the second variable and satisfies*

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_0}^{\zeta-1} \left[b(s)q(s)H(\zeta, s) - \frac{b(s)r(s)h_+^\eta(\zeta, s)}{K(\eta+1)^{\eta+1}[\Gamma(1-\kappa)H(\zeta, s)]^\eta} \right] ds = \infty, \tag{180}$$

where

$$h_+(\zeta, s) = \max\left\{0, \frac{b'_+(s)}{b(s)} H(\zeta, s) + H'_s(\zeta, s)\right\}, \quad (\zeta, s) \in D_0, \tag{181}$$

and $b'_+(s) = \max\{b'(s), 0\}$. Then (177) is oscillatory.

Theorem 60 ([13]). *Suppose that*

$$\int_{\zeta_0}^{\infty} r^{-\frac{1}{\eta}}(\zeta) d\zeta < \infty, \tag{182}$$

holds. Assume that there exists a positive function $b \in C^1[t_0, \infty)$ such that (179) holds. Furthermore, assume that for every constant $C \geq t_0$,

$$\int_C^{\infty} \left[\frac{1}{r(\zeta)} \int_C^{\zeta} q(s) ds\right]^{\frac{1}{\eta}} d\zeta = \infty. \tag{183}$$

Then every solution u of (177) is oscillatory or satisfies

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{184}$$

Theorem 61 ([13]). *Suppose that (182) holds. Let $b(\zeta)$ and $H(\zeta, s)$ be defined as in Theorem 59 such that (180) holds. Furthermore, assume that for every constant $C \geq \zeta_0$, (183) holds. Then every solution u of (177) is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{185}$$

Remark 3. *From Theorems 58–61, we can derive many different sufficient conditions for the oscillation of (177) with different choices of the functions b and H.*

In [14], Chen discussed the oscillatory behavior of the fractional differential equation with damping

$$\mathcal{D}_-^{1+\kappa} u(\zeta) - p(\zeta) \mathcal{D}_-^{\kappa} u(\zeta) + q(\zeta) \mathfrak{f}\left(\int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds\right) = 0, \quad \zeta > 0, \tag{186}$$

where $0 < \kappa < 1$, $p \geq 0$ and $q > 0$ are continuous functions on $[\zeta_0, \infty)$ for a certain $\zeta_0 > 0$ and $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\mathfrak{f}(u)/(u) \geq K$ for a certain constant $K > 0$ and for all $u \neq 0$, and

$$\int_{\zeta_0}^{\infty} \exp\left(-\int_{\zeta_0}^{\zeta} p(s) ds\right) d\zeta = \infty.$$

Theorem 62 ([14]). *Suppose that there exists a positive function $b \in C^1[\zeta_0, \infty)$ such that*

$$\limsup_{\zeta \to \infty} \int_{\zeta_0}^{\zeta} \left[K b(s) q(s) V(s) - v b'_+(s)\right] ds = \infty, \tag{187}$$

for any constant $v > 0$, where $b'_+(s) = \max\{b'(s), 0\}$, and

$$V(s) = \exp\left(\int_{\zeta_0}^{s} p(s) ds\right), \quad s \geq \zeta_0. \tag{188}$$

Then (186) is oscillatory.

Theorem 63 ([14]). *Suppose that there exists a positive function $b \in C^1[\zeta_0, \infty)$ such that*

$$\limsup_{\zeta \to \infty} \int_{\zeta_0}^{\zeta} \left[Kb(s)q(s) - \frac{[M_+(s)]^2}{4\Gamma(1-\kappa)b(s)} \right] ds = \infty, \tag{189}$$

where $M_+(s) = \max\{b'_+(s) - b(s)p(s), 0\}$, and b'_+ is defined as in Theorem 62. Then (186) is oscillatory.

Theorem 64 ([14]). *Assume that there exists a positive function $b \in C^1[\zeta_0, \infty)$ and a function $H \in C(D, \mathbb{R})$, where $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ on D_0 with respect to the second variable and that there exists a function $h \in C(D, \mathbb{R})$ such that

$$H'_s(\zeta, s) + \frac{M'_+(s)}{b(s)} H(\zeta, s) = \frac{h(\zeta, s)}{b(s)} \sqrt{H(\zeta, s)}, \quad (\zeta, s) \in D, \tag{190}$$

and

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_0}^{\zeta} \left[b(s)q(s)q(s)H(\zeta, s) - \frac{h_+^2(\zeta, s)}{4K\Gamma(1-\kappa)b(s)} \right] ds = \infty, \tag{191}$$

where M_+ is defined as in Theorem 63 and $h_+(\zeta, s) = \max\{h(\zeta, s), 0\}$. Then (177) is oscillatory.

Remark 4. *From Theorems 62–64, we can derive many different sufficient conditions for the oscillation of (186) with different choices of the functions b and H.*

Take $b(s) = 1$. Then from Theorem 63 we obtain the following result.

Corollary 9. *Assume that the following condition hold:*

$$\int_{\zeta_0}^{\infty} q(s)ds = \infty. \tag{192}$$

Then (186) is oscillatory.

Take $b(s) = 1$. Then from Theorem 62 we obtain the following result.

Corollary 10. *Assume that the following condition hold:*

$$\int_{\zeta_0}^{\infty} \left[q(s) \exp\left(\int_{\zeta_0}^{\tau} p(\tau)d\tau \right) \right] ds = \infty. \tag{193}$$

Then (186) is oscillatory.

Note that, since

$$q(s) \leq q(s) \exp\left(\int_{\zeta_0}^{s} p(\tau)d\tau \right), \quad s \geq \zeta_0,$$

Corollary 9 can also be derived from Corollary 10. Obviously, Corollary 10 is better than Corollary 9.

Take $b(s) = s$. Then from Theorem 63 we obtain the following result.

Corollary 11. *Assume that the following condition hold:*

$$\limsup_{\zeta \to \infty} \int_{\zeta_0}^{\zeta} \left[sq(s) - \frac{[M_+(s)]^2}{4K\Gamma(1-\kappa)s} \right] ds = \infty, \tag{194}$$

where $M_+(s) = \max\{1 - sp(s), 0\}$. Then (186) is oscillatory.

Take $b(s) = 1$ and $H(\zeta, s) = (\zeta - s)^m$, where $m \geq 2$ is a constant. Then Theorem 64 implies the following result.

Corollary 12. *Suppose that there exists a constant $m \geq 2$ such that*

$$\limsup_{\zeta \to \infty} \frac{1}{\zeta^m} \int_{\zeta_0}^{\zeta} q(s)(\zeta - s)^m ds = \infty. \tag{195}$$

Then (186) is oscillatory.

By the generalized Riccati transformation technique, Han et al. [15] obtained oscillation criteria for a class of nonlinear fractional differential equations of the form

$$[r(\zeta)g(\mathcal{D}^\kappa_- u(\zeta))]' - q(\zeta)\mathfrak{f}\left(\int_{\zeta}^{\infty}(s-\zeta)^{-\kappa}u(s)ds\right) = 0, \quad \zeta > 0, \tag{196}$$

where $0 < \kappa < 1$, r and q are positive continuous functions on $[\zeta_0, \infty)$ for a certain $\zeta_0 > 0$; $\mathfrak{f}$, $g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that

$$u\mathfrak{f}(u) > 0, \quad ug(u) > 0, \quad u \neq 0,$$

and there exist positive constants k_1, k_2 such that

$$\frac{\mathfrak{f}(u)}{u} \geq k_1, \quad \frac{g(u)}{u} \geq k_2, \quad u \neq 0.$$

Moreover, $g^{-1} : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$ug^{-1}(u) > 0, \quad u \neq 0,$$

and there exists some positive constant γ_1 such that

$$g^{-1}(uv) \geq \gamma_1 g^{-1}(u)g^{-1}(v), \quad uv \neq 0.$$

Theorem 65 ([15]). *Suppose that*

$$\int_{\zeta_0}^{\infty} g^{-1}\left(\frac{1}{r(\zeta)}\right) d\zeta = \infty, \tag{197}$$

holds. Furthermore, assume that there exists a positive function $b \in C^1[\zeta_0, \infty)$ such that

$$\limsup_{\zeta \to \infty} \int_{\zeta_0}^{\zeta} \left[k_1 b(s)q(s) - \frac{r(s)[b'(s)]^2}{4k_2 \Gamma(1-\kappa)b(s)} \right] ds = \infty. \tag{198}$$

Then (196) is oscillatory.

Theorem 66 ([15]). *Assume that there exists a positive function $b \in C^1[\zeta_0, \infty)$ and a function $H \in C(D, \mathbb{R})$, where $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : t > s \geq t_0\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_0}^{\zeta-1} H(\zeta, s) \left[k_1 b(s) q(s) - \frac{r(s)[b'(s)]^2}{4k_2 \Gamma(1-\kappa)b(s)} \right] ds = \infty. \tag{199}$$

Then (196) *is oscillatory.*

Theorem 67 ([15]). *Suppose that*

$$\int_{\zeta_0}^{\infty} g^{-1} \left(\frac{1}{r(\zeta)} \right) d\zeta < \infty, \tag{200}$$

holds and g is an increasing function. Assume that there exists a positive function $b \in C^1[\zeta_0, \infty)$ such that (198) *holds. Furthermore, assume that for every constant $C \ge \zeta_0$,*

$$\int_{C}^{\infty} g^{-1} \left[\frac{1}{r(\zeta)} \int_{C}^{\zeta} q(s) ds \right] dt = \infty. \tag{201}$$

Then every solution u of (196) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{202}$$

Theorem 68 ([15]). *Suppose that* (200) *holds and g is an increasing function. Let $b(\zeta)$ and $H(\zeta, s)$ be defined as in Theorem 66 such that* (199) *holds. Furthermore, assume that for every constant $C \ge \zeta_0$,* (201) *holds. Then every solution u of* (196) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{203}$$

Qi et al. [16] established some new interval oscillation criteria based on the certain Riccati transformation and inequality technique for a class of fractional differential equations with damping term of the form

$$\left(p_1(\zeta)[r_1(\zeta)\mathcal{D}_-^\kappa u(\zeta)]' \right)' + p(\zeta)[r_1(\zeta)\mathcal{D}_-^\kappa u(\zeta)]'$$
$$- q(\zeta) \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0, \quad \zeta > 0, \tag{204}$$

where $0 < \kappa < 1$, $p_1 \in C^1([t_0, \infty), \mathbb{R}^+)$, $r_1 \in C^2([t_0, \infty), \mathbb{R}^+)$, p and q are positive continuous functions on $[\zeta_0, \infty)$ for a certain $\zeta_0 > 0$. Denote by

$$A(\zeta) = \int_{\zeta_0}^{\zeta} \frac{p(s)}{p_1(s)} ds, \tag{205}$$

$$\delta_1(\zeta, a) = \int_{a}^{\zeta} \frac{1}{e^{A(s)} p_1(s)} ds, \tag{206}$$

$$\delta_2(\zeta, a) = \int_{a}^{\zeta} \frac{\delta_1(s, a)}{r_1(s)} ds. \tag{207}$$

Theorem 69 ([16]). *Assume*

$$\int_{\zeta_0}^{\infty} \frac{1}{e^{A(s)} p_1(s)} ds = \infty, \tag{208}$$

$$\int_{\zeta_0}^{\infty} \frac{1}{r_1(s)} ds = \infty, \tag{209}$$

$$\int_{\zeta_0}^{\infty} \frac{1}{r_1(\xi)} \int_{\xi}^{\infty} \frac{1}{e^{A(\tau)} p_1(\tau)} \int_{\tau}^{\infty} e^{A(s)} q(s) ds d\tau d\xi = \infty, \tag{210}$$

hold, and there exist two functions $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ and $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ such that

$$\int_\zeta^\infty \left(\phi(s)q(s)e^{A(s)} - \phi(s)\psi'(s) + \frac{\phi(s)\Gamma(1-\kappa)\delta_1(s,T)\psi^2(s)}{r_1(s)} \right.$$

$$\left. - \left[2\phi(s)\psi(s)\Gamma(1-\kappa)\delta_1(s,T) + r_1(s)\phi'(s) \right]^2 \times \left(4\Gamma(1-\kappa)\phi(s)\delta_1(s,T)r_1(s) \right)^{-1} \right) ds = \infty, \quad (211)$$

for all sufficiently large T. Then every solution of (204) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_\zeta^\infty (s-\zeta)^{-\kappa} u(s)ds = 0. \quad (212)$$

Theorem 70 ([16]). *Assume* (208)–(210) *hold. Furthermore, assume that there exist two functions $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ and $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ and a function $H \in C^1(D, \mathbb{R})$, where $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_0}^\zeta H(\zeta, s)$$

$$\left(\phi(s)q(s)e^{A(s)} - \phi(s)\psi'(s) + \frac{\phi(s)\Gamma(1-\kappa)\delta_1(s,T)\psi^2(s)}{r_1(s)} \right.$$

$$\left. - \left[2\phi(s)\psi(s)\Gamma(1-\kappa)\delta_1(s,T) + r_1(s)\phi'(s) \right]^2 \times \left(4\Gamma(1-\kappa)\phi(s)\delta_1(s,T)r_1(s) \right)^{-1} \right) ds = \infty, \quad (213)$$

for all sufficiently large T. Then every solution of (204) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_\zeta^\infty (s-\zeta)^{-\kappa} u(s)ds = 0. \quad (214)$$

In Theorem 70, if we take $H(\zeta, s)$ for some special functions such as $(\zeta - s)^m$ or $\ln \frac{\zeta}{s}$, then we can obtain some corollaries as follows.

Corollary 13 ([16]). *Assume* (208)–(210) *hold, and*

$$\limsup_{\zeta \to \infty} \frac{1}{(\zeta - \zeta_0)^m} \int_{\zeta_0}^\zeta (\zeta - s)^m$$

$$\left(\phi(s)q(s)e^{A(s)} - \phi(s)\psi'(s) + \frac{\phi(s)\Gamma(1-\kappa)\delta_1(s,T)\psi^2(s)}{r_1(s)} \right.$$

$$\left. - \left[2\phi(s)\psi(s)\Gamma(1-\kappa)\delta_1(s,T) + r_1(s)\phi'(s) \right]^2 \times \left(4\Gamma(1-\kappa)\phi(s)\delta_1(s,T)r_1(s) \right)^{-1} \right) ds = \infty, \quad (215)$$

for all sufficiently large T. Then every solution of (204) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_\zeta^\infty (s-\zeta)^{-\kappa} u(s)ds = 0. \quad (216)$$

Corollary 14 ([16]). *Assume* (208)–(210) *hold, and*

$$\limsup_{\zeta \to \infty} \frac{1}{(\ln \zeta - \ln \zeta_0)} \int_{\zeta_0}^{\zeta} (\ln \zeta - \ln s)$$

$$\left(\phi(s)q(s)e^{A(s)} - \phi(s)\psi'(s) + \frac{\phi(s)\Gamma(1-\kappa)\delta_1(s,T)\psi^2(s)}{r_1(s)} \right.$$

$$\left. - \left[2\phi(s)\psi(s)\Gamma(1-\kappa)\delta_1(s,T) + r_1(s)\phi'(s) \right]^2 \times \left(4\Gamma(1-\kappa)\phi(s)\delta_1(s,T)r_1(s) \right)^{-1} \right) ds = \infty, \quad (217)$$

for all sufficiently large T. Then every solution of (204) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{218}$$

Xu [17] established several oscillation criteria for nonlinear fractional differential equations of the form

$$\left(p_1(\zeta) \left[(r_1(\zeta)\mathcal{D}_-^{\kappa} u(\zeta))' \right]^{\eta} \right)' - F\left(\zeta, \int_{\zeta}^{\infty} (s-\zeta)^{-\kappa} u(s) ds \right) = 0, \quad t \geq t_0 > 0, \tag{219}$$

where $0 < \kappa < 1$, η is a quotient of odd positive integers, $p_1 \in C^1([\zeta_0, \infty), \mathbb{R}^+)$, $r_1 \in C^2([\zeta_0, \infty), \mathbb{R}^+)$,

$$\int_{\zeta_0}^{\infty} \frac{ds}{p_1^{\frac{1}{\eta}}(s)} = \infty,$$

$$\int_{\zeta_0}^{\infty} \frac{ds}{r_1(s)} = \infty,$$

$F(\zeta, G) \in C([\zeta_0, \infty) \times \mathbb{R}, \mathbb{R})$, there exists a function $q_1 \in C([\zeta_0, \infty), \mathbb{R}^+)$ such that

$$\frac{F(\zeta, G)}{G^{\eta}} \geq q_1(\zeta), \quad G \neq 0, \quad u \neq 0, \quad \zeta \geq \zeta_0.$$

Denote by

$$B(\zeta_1, \zeta) = \int_{\zeta_1}^{\zeta} p_1^{-\frac{1}{\eta}}(s) ds. \tag{220}$$

Theorem 71 ([17]). *Assume*

$$\int_{\zeta_0}^{\infty} \frac{1}{r_1(\xi)} \int_{\xi}^{\infty} \left[\frac{1}{p_1(\tau)} \int_{\tau}^{\infty} q(s) ds \right]^{\frac{1}{\eta}} d\tau d\xi = \infty, \tag{221}$$

holds, and there exist a function $b \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ *such that*

$$\int_{\zeta}^{\infty} \left[b(s)q(s) - \frac{1}{(\eta+1)^{\eta+1}} \left(\frac{b'_+(s)}{b(s)} \right)^{\eta+1} \times \frac{b(s)r_1^{\eta}(s)}{[\Gamma(1-\kappa)B(\zeta_2, s)]^{\eta}} \right] ds \quad = \quad \infty, \tag{222}$$

for all sufficiently large constants t_2 *and* T, *where* $b'_+(s) = \max\{b'(s), 0\}$, $B(\zeta_2, t)$ *is defined for* $t \geq T \geq \zeta_2 \geq \zeta_0 > 0$. *Then every solution of* (219) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{223}$$

Corollary 15 ([17]). *Assume (221) holds, and there exist a function $b \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ such that*

$$\int_\zeta^\infty b(s)q(s)ds = \infty, \tag{224}$$

and

$$\int_\zeta^\infty \left(\frac{b'_+(s)}{b(s)}\right)^{\eta+1} \frac{b(s)r_1^\eta(s)}{B^\eta(\zeta_2, s)} ds < \infty, \tag{225}$$

for all sufficiently large constants ζ_2 and T, where $b'_+(s) = \max\{b'(s), 0\}$, $B(\zeta_2, \zeta)$ is defined for $\zeta \geq T \geq \zeta_2 \geq \zeta_0 > 0$. Then every solution of (219) is oscillatory or satisfies

$$\lim_{\zeta \to \infty} \int_\zeta^\infty (s - \zeta)^{-\kappa} u(s)ds = 0. \tag{226}$$

Theorem 72 ([17]). *Assume (221) holds. Furthermore, assume that there exist two functions $b \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ and a function $H \in C^1(D, \mathbb{R})$, where $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_2}^\zeta b(s)$$
$$\times \left(H(\zeta, s)q(s) - \frac{r_1^\eta(s)h_+^{\eta+1}(\zeta, s)}{(\eta+1)^{\eta+1}[\Gamma(1-\kappa)B(\zeta_2, s)H(\zeta, s)]^\eta}\right) ds = \infty, \tag{227}$$

where t_2 is sufficiently large, $\zeta \geq \zeta_2 \geq \zeta_0$,

$$H'_s(\zeta, s) + \frac{b'_+(s)}{b(s)}H(\zeta, s) = h(\zeta, s), \tag{228}$$

and $h_+(\zeta, s) = \max\{h(\zeta, s), 0\}$. Then every solution of (219) is oscillatory or satisfies

$$\lim_{\zeta \to \infty} \int_\zeta^\infty (s - \zeta)^{-\kappa} u(s)ds = 0. \tag{229}$$

Corollary 16 ([17]). *Assume (221) holds. Furthermore, assume that there exist two functions $b \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ and a function $H \in C^1(D, \mathbb{R})$, where $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_2}^\zeta b(s)H(\zeta, s)q(s)ds = \infty, \tag{230}$$

and

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_2}^\zeta \frac{b(s)r_1^\eta(s)h_+^{\eta+1}(\zeta, s)}{[B(\zeta_2, s)H(\zeta, s)]^\eta} ds < \infty, \tag{231}$$

where t_2 is sufficiently large, $\zeta \geq \zeta_2 \geq \zeta_0$. Then every solution of (219) is oscillatory or satisfies

$$\lim_{\zeta \to \infty} \int_\zeta^\infty (s - \zeta)^{-\kappa} u(s)ds = 0. \tag{232}$$

With an appropriate choice of the functions H and b, one can derive from Theorem 72 a number of oscillation criteria for (219). Let $H(\zeta, s) = \ln(\zeta/s)$, $(\zeta, s) \in D$, and $b(\zeta) = \zeta^v$. Then, we obtain the following corollary.

Corollary 17 ([17]). *Assume* (221) *holds. If*

$$\limsup_{\zeta \to \infty} \frac{1}{\ln t} \int_{\zeta_2}^{\zeta} s^v$$

$$\times \left(\ln(\zeta/s)q(s) - \frac{r_1^{\eta}(s)h_+^{\eta+1}(\zeta, s)}{(\eta+1)^{\eta+1}[\Gamma(1-\kappa)B(\zeta_2, s)\ln(\zeta/s)]^{\eta}} \right) ds = \infty, \quad (233)$$

where ζ_2 is sufficiently large, $\zeta \geq \zeta_2 \geq \zeta_0$. Then every solution of (219) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \quad (234)$$

Corollary 18 ([17]). *Assume* (221) *holds. If*

$$\limsup_{\zeta \to \infty} \frac{1}{\ln \zeta} \int_{\zeta_2}^{\zeta} s^v q(s) \ln(\zeta/s) ds = \infty, \quad (235)$$

and

$$\limsup_{\zeta \to \infty} \frac{1}{\ln \zeta} \int_{\zeta_2}^{\zeta} \frac{s^v r_1^{\eta}(s)[(1/s)(v \ln(\zeta/s) - 1)]^{\eta+1}}{[B(\zeta_2, s) \ln(\zeta/s)]^{\eta}} ds < \infty, \quad (236)$$

where ζ_2 is sufficiently large, $\zeta \geq \zeta_2 \geq \zeta_0$. Then every solution of (219) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \quad (237)$$

By the generalized Riccati transformation technique, Zheng et al. [18] obtained oscillation criteria for a class of nonlinear fractional differential equations of the form

$$\left(p_1(\zeta) \left[(r_1(\zeta) \mathcal{D}_-^{\kappa} u(\zeta))' \right]^{\eta} \right)' + p(\zeta) \left[(r_1(\zeta) \mathcal{D}_-^{\kappa} u(\zeta))' \right]^{\eta}$$

$$- q(\zeta) \mathfrak{f} \left(\int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds \right) = 0, \quad \zeta \geq \zeta_0, \quad (238)$$

where $0 < \kappa < 1$, η is a quotient of odd positive integers, $p_1 \in C^1([\zeta_0, \infty), \mathbb{R}^+)$, $r_1 \in C^2([\zeta_0, \infty), \mathbb{R}^+)$, $p, q \in C([\zeta_0, \infty), \mathbb{R}^+)$, and $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $u f(u) > 0$ and $\mathfrak{f}(u)/(u^{\eta}) \geq K$ for a certain constant $K > 0$ and for all $u \neq 0$. Denote by

$$A(\zeta) = \int_{\zeta_0}^{\zeta} \frac{p(s)}{p_1(s)} ds, \quad (239)$$

$$\theta_1(\zeta, a) = \int_a^{\zeta} \frac{1}{\left[e^{A(s)} p_1(s) \right]^{\frac{1}{\eta}}} ds, \quad (240)$$

$$\theta_2(\zeta, a) = \int_a^{\zeta} \frac{\theta_1(s, a)}{r_1(s)} ds. \quad (241)$$

Theorem 73 ([18]). *Assume*

$$\int_{\zeta_0}^{\infty} \frac{1}{\left[e^{A(s)}p_1(s)\right]^{\frac{1}{\eta}}} ds = \infty, \tag{242}$$

$$\int_{\zeta_0}^{\infty} \frac{1}{r_1(s)} ds = \infty, \tag{243}$$

$$\int_{\zeta_0}^{\infty} \frac{1}{r_1(\xi)} \int_{\xi}^{\infty} \left[\frac{1}{e^{A(\tau)}p_1(\tau)} \int_{\tau}^{\infty} e^{A(s)}q(s)ds\right]^{\frac{1}{\eta}} d\tau d\xi = \infty, \tag{244}$$

hold, and there exist two functions $\phi \in C^1([t_0, \infty), \mathbb{R}^+)$ and $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ such that

$$\int_{\zeta}^{\infty} \left(K\phi(s)q(s)e^{A(s)} - \phi(s)\psi'(s) + \frac{\phi(s)\Gamma(1-\kappa)\theta_1(s,T)\psi^{1+\frac{1}{\eta}}(s)}{r_1(s)} \right.$$

$$- \left[(\eta+1)\psi^{\frac{1}{\eta}}(s)\phi(s)\Gamma(1-\kappa)\theta_1(s,T) + r_1(s)\phi'(s)\right]^{\eta+1}$$

$$\times \left. \left((\eta+1)^{\eta+1}[\Gamma(1-\kappa)\phi(s)\theta_1(s,T)]^{\eta}r_1(s)\right)^{-1} \right) ds = \infty, \tag{245}$$

for all sufficiently large T. Then every solution of (238) is oscillatory or satisfies

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s-\zeta)^{-\kappa} u(s)ds = 0. \tag{246}$$

Theorem 74 ([18]). *Assume* (242)–(244) *hold. Furthermore, assume there exist two functions* $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ *and* $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ *such that*

$$\int_{\zeta}^{\infty} \left(K\phi(s)q(s)e^{A(s)} - \phi(s)\psi'(s) + \frac{\eta\phi(s)(\Gamma(1-\kappa))^{\eta}\theta_1(s,T)\theta_2^{\eta-1}(s,T)\psi^2(s)}{r_1(s)} \right.$$

$$- \left[2\eta\psi(s)\phi(s)(\Gamma(1-\kappa))^{\eta}\theta_1(s,T)\theta_2^{\eta-1}(s,T) + r_1(s)\phi'(s)\right]^2$$

$$\times \left. \left(4(\Gamma(1-\kappa))^{\eta}\theta_1(s,T)\theta_2^{\eta-1}(s,T)r_1(s)\phi(s)\right)^{-1} \right) ds = \infty, \tag{247}$$

for all sufficiently large T. Then every solution of (238) is oscillatory or satisfies

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s-\zeta)^{-\kappa} u(s)ds = 0. \tag{248}$$

Theorem 75 ([18]). *Assume* (242)–(244) *hold. Furthermore, assume that there exist two functions* $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ *and* $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ *and a function* $H \in C^1(D, \mathbb{R})$, *where* $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ *such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta,s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$
\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_0}^{\zeta} H(\zeta, s)
$$

$$
\left(K\phi(s)q(s)e^{A(s)} - \phi(s)\psi'(s) + \frac{\phi(s)\Gamma(1-\kappa)\theta_1(s,T)\psi^{1+\frac{1}{\eta}}(s)}{r_1(s)} \right.
$$

$$
- \left[(\eta+1)\psi^{\frac{1}{\eta}}(s)\phi(s)\Gamma(1-\kappa)\theta_1(s,T) + r_1(s)\phi'(s) \right]^{\eta+1}
$$

$$
\left. \times \left((\eta+1)^{\eta+1}[\Gamma(1-\kappa)\phi(s)\theta_1(s,T)]^{\eta} r_1(s) \right)^{-1} \right) ds = \infty, \quad (249)
$$

for all sufficiently large T. Then every solution of (204) is oscillatory or satisfies

$$
\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s-\zeta)^{-\kappa} u(s) ds = 0. \quad (250)
$$

Theorem 76 ([18]). *Assume (242)–(244) hold. Furthermore, assume that there exist two functions $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ and $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ and a function $H \in C^1(D, \mathbb{R})$, where $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ such that*

$$
H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,
$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta,s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$
\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_0}^{\zeta} H(\zeta, s)
$$

$$
\left(K\phi(s)q(s)e^{A(s)} - \phi(s)\psi'(s) + \frac{\eta\phi(s)(\Gamma(1-\kappa))^{\eta}\theta_1(s,T)\theta_2^{\eta-1}(s,T)\psi^2(s)}{r_1(s)} \right.
$$

$$
- \left[2\eta\psi(s)\phi(s)(\Gamma(1-\kappa))^{\eta}\theta_1(s,T)\theta_2^{\eta-1}(s,T) + r_1(s)\phi'(s) \right]^2
$$

$$
\left. \times \left(4(\Gamma(1-\kappa))^{\eta}\theta_1(s,T)\theta_2^{\eta-1}(s,T)r_1(s)\phi(s) \right)^{-1} \right) ds = \infty, \quad (251)
$$

for all sufficiently large T. Then every solution of (204) is oscillatory or satisfies

$$
\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s-\zeta)^{-\kappa} u(s) ds = 0. \quad (252)
$$

In Theorems 75 and 76, if we take $H(\zeta, s)$ for some special functions such as $(\zeta - s)^m$ or $\ln \frac{\zeta}{s}$, then we can obtain some corollaries.

By the generalized Riccati transformation technique, Xiang et al. [19] obtained oscillation criteria for a class of nonlinear fractional differential equations of the form

$$
(p_1(\zeta)[r_2(\zeta) + r_1(\zeta)\mathcal{D}_-^{\kappa} u(\zeta)]^{\eta})' - q(\zeta)\mathfrak{f}\left(\int_{\zeta}^{\infty} (s-\zeta)^{-\kappa} u(s) ds \right) = 0, \quad \zeta \geq \zeta_0 > 0, \quad (253)
$$

where $0 < \kappa < 1$, η is a quotient of odd positive integers, $p_1 \in C([\zeta_0, \infty), \mathbb{R}^+)$, $r_1 \in C([\zeta_0, \infty), \mathbb{R}^+)$, $q \in C([\zeta_0, \infty), \mathbb{R}^+)$, r_2 is a nonnegative continuous function on $[\zeta_0, \infty)$, and $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\mathfrak{f}(u)/(u^\eta) \geq K$ for a certain constant $K > 0$ and for all $u \neq 0$. There exists $N > 0$, $q(\zeta) \leq N$, for $\zeta \in [\zeta_0, \infty)$.

Theorem 77 ([19]). *Assume that*

$$\int_{\zeta_0}^{\infty} \frac{r_2(\zeta)}{r_1(\zeta)} dt < \infty,$$

$$\int_{\zeta_0}^{\infty} \frac{ds}{p_1^{\frac{1}{\eta}}(s)} = \infty,$$

hold and there exist a function $b \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ *such that*

$$\limsup_{\zeta \to \infty} \int_{\zeta_0}^{\zeta} \left[Kb(s)q(s) - \left(\frac{b'_+(s)}{b(s)} \right)^{\eta+1} \times \frac{N^\eta b(s) p_1(s)}{(\eta+1)^{\eta+1}[\Gamma(1-\kappa)]^\eta} \right] ds \quad = \quad \infty, \quad (254)$$

where $b'_+(s) = \max\{b'(s), 0\}$. *Then* (253) *is oscillatory.*

Theorem 78 ([19]). *Assume that*

$$\int_{\zeta_0}^{\infty} \frac{r_2(\zeta)}{r_1(\zeta)} dt < \infty,$$

$$\int_{\zeta_0}^{\infty} \frac{ds}{p_1^{\frac{1}{\eta}}(s)} = \infty,$$

hold and there exist two functions $b \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ *and a function* $H \in C^1(D, \mathbb{R})$, *where* $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ *such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}$ *and* H *has a nonpositive continuous partial derivative* $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ *on* D_0 *with respect to the second variable and satisfies*

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_0}^{\zeta-1}$$

$$\times \left(KH(\zeta, s)b(s)q(s) - \frac{N^\eta b(s)p_1(s)h_+^{\eta+1}(\zeta, s)}{(\eta+1)^{\eta+1}[\Gamma(1-\kappa)H(\zeta, s)]^\eta} \right) ds = \infty, \quad (255)$$

where

$$H'_s(\zeta, s) + \frac{b'_+(s)}{b(s)} H(\zeta, s) = h(\zeta, s), \quad (256)$$

and $h_+(\zeta, s) = \max\{h(\zeta, s), 0\}$. *Then* (253) *is oscillatory.*

Corollary 19. *Assume that*

$$\int_{\zeta_0}^{\infty} \frac{r_2(\zeta)}{r_1(\zeta)} dt < \infty,$$

$$\int_{\zeta_0}^{\infty} \frac{ds}{p_1^{\frac{1}{\eta}}(s)} < \infty, \quad (257)$$

and

$$\left(\frac{r_2(\zeta)}{r_1(\zeta)} \right)' \neq 0, \quad \zeta \in [\zeta_0, \infty). \quad (258)$$

hold, and there exist a function $b \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ such that (254) holds. Furthermore, assume that, for every constant $T \geq \zeta_0$,

$$\int_\zeta^\infty \left[\frac{1}{p_1(\zeta)} \int_\zeta^\zeta q(s)ds \right]^{\frac{1}{\eta}} d\zeta = \infty. \tag{259}$$

Then every solution u of (253) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \frac{d}{d\zeta} \left[\int_\zeta^\infty (s - \zeta)^{-\kappa} u(s)ds \right] = 0, \tag{260}$$

or

$$\lim_{\zeta \to \infty} \int_\zeta^\infty (s - \zeta)^{-\kappa} u(s)ds = 0. \tag{261}$$

Corollary 20. *Assume that*

$$\int_{\zeta_0}^\infty \frac{r_2(\zeta)}{r_1(\zeta)} d\zeta < \infty,$$

(257) *and* (258) *hold. Let* $b(\zeta)$ *and* $H(\zeta, s)$ *be defined as in Theorem* 78 *such that* (255) *holds. Further, assume that, for every constant* $T \geq \zeta_0$, (259) *holds. Then every solution u of* (253) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \frac{d}{d\zeta} \left[\int_\zeta^\infty (s - \zeta)^{-\kappa} u(s)ds \right] = 0, \tag{262}$$

or

$$\lim_{\zeta \to \infty} \int_\zeta^\infty (s - \zeta)^{-\kappa} u(s)ds = 0. \tag{263}$$

With an appropriate choice of the functions H and b, one can derive from Theorem 77, Theorem 78, Corollary 19 and Corollary 20 a number of oscillation criteria for (253).

By the generalized Riccati transformation technique, Pan et al. [20] obtained oscillation criteria for a class of nonlinear fractional differential equations of the form

$$\left(p_1(\zeta) \left[[r_1(\zeta)g(\mathcal{D}_-^\kappa u(\zeta))]' \right]^\eta \right)' - F\left(\zeta, \int_\zeta^\infty (s - \zeta)^{-\kappa} u(s)ds \right) = 0, \quad \zeta > 0, \tag{264}$$

where $0 < \kappa < 1$, η is a quotient of odd positive integers, $p_1 \in C^1([\zeta_0, \infty), \mathbb{R}^+)$, $r_1 \in C^2([\zeta_0, \infty), \mathbb{R}^+)$;

$$\int_{\zeta_0}^\infty \frac{ds}{p_1^{\frac{1}{\eta}}(s)} = \infty;$$

$g \in C^2(\mathbb{R}, \mathbb{R})$; g is an increasing function and there exists positive k such that

$$\frac{u}{g(u)} \geq k > 0, \quad ug(u) \neq 0.$$

Moreover, $g^{-1} : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$ug^{-1}(u) > 0, \quad u \neq 0,$$

and there exists some positive constant γ_1 such that

$$g^{-1}(uv) \geq \gamma_1 g^{-1}(u)g^{-1}(v), \quad uv \neq 0;$$

$F(\zeta, G) \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, there exists a function $q_1 \in C([\zeta_0, \infty), \mathbb{R}^+)$ such that

$$\frac{F(\zeta, G)}{G^\eta} \geq q_1(\zeta), \quad G \neq 0, \quad u \neq 0, \quad \zeta \geq \zeta_0.$$

Denote by

$$A_1(\zeta_1, \zeta) = \int_{\zeta_1}^{\zeta} \frac{1}{[p_1(s)]^{\frac{1}{\eta}}} ds, \tag{265}$$

$$A_2(\zeta_1, \zeta) = \int_{\zeta_1}^{\zeta} \frac{A_1(\zeta_1, s)}{r_1(s)} ds. \tag{266}$$

Theorem 79 ([20]). *Assume*

$$\int_{\zeta_0}^{\infty} g^{-1}\left(\frac{1}{r_1(s)}\right) ds = \infty, \tag{267}$$

$$\int_{\zeta_0}^{\infty} g^{-1}\left(\frac{1}{r_1(\xi)} \int_{\xi}^{\infty} \left[\frac{1}{p_1(\tau)} \int_{\tau}^{\infty} q(s) ds\right]^{\frac{1}{\eta}} d\tau\right) d\xi = \infty, \tag{268}$$

hold, and there exist two functions $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ and $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ such that

$$\int_{\zeta}^{\infty} \left(\phi(s) q(s) - \phi(s) \psi'(s) + \frac{k \phi(s) \Gamma(1-\kappa) A_1(\zeta, s) \psi^{1+\frac{1}{\eta}}(s)}{r_1(s)} \right.$$

$$- \left[(\eta+1) k \psi^{\frac{1}{\eta}}(s) \phi(s) \Gamma(1-\kappa) A_1(\zeta, s) + r_1(s) \phi'(s) \right]^{\eta+1}$$

$$\times \left. \left((\eta+1)^{\eta+1} [k\Gamma(1-\kappa)\phi(s) A_1(\zeta, s)]^{\eta} r_1(s) \right)^{-1} \right) ds = \infty, \quad (269)$$

for all sufficiently large T. Then every solution of (264) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{270}$$

Theorem 80 ([20]). *Assume* (267)–(268) *hold. Furthermore, assume there exist two functions $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ and $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ such that*

$$\int_{\zeta}^{\infty} \left(\phi(s) q(s) - \phi(s) \psi'(s) + \frac{\eta \phi(s) (k\Gamma(1-\kappa))^{\eta} A_1(\zeta, s) A_2^{\eta-1}(\zeta, s) \psi^2(s)}{r_1(s)} \right.$$

$$- \left[2\eta \psi(s) \phi(s) (k\Gamma(1-\kappa))^{\eta} A_1(\zeta, s) A_2^{\eta-1}(\zeta, s) + r_1(s) \phi'(s) \right]^2$$

$$\times \left. \left(4\eta (k\Gamma(1-\kappa))^{\eta} A_1(\zeta, s) A_2^{\eta-1}(\zeta, s) r_1(s) \phi(s) \right)^{-1} \right) ds = \infty, \quad (271)$$

for all sufficiently large T. Then every solution of (264) *is oscillatory or satisfies*

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{272}$$

Theorem 81 ([20]). *Assume* (267)–(268) *hold. Furthermore, assume that there exist two functions $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ and $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ and a function $H \in C^1(D, \mathbb{R})$, where $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_0}^{\zeta} H(\zeta, s)$$

$$\left(\phi(s)q(s) - \phi(s)\psi'(s) + \frac{k\phi(s)\Gamma(1-\kappa)A_1(\zeta, s)\psi^{1+\frac{1}{\eta}}(s)}{r_1(s)} \right.$$

$$- \left[(\eta+1)k\psi^{\frac{1}{\eta}}(s)\phi(s)\Gamma(1-\kappa)A_1(\zeta, s) + r_1(s)\phi'(s) \right]^{\eta+1}$$

$$\left. \times \left((\eta+1)^{\eta+1}[k\Gamma(1-\kappa)\phi(s)A_1(\zeta, s)]^{\eta}r_1(s) \right)^{-1} \right) ds = \infty, \quad (273)$$

for all sufficiently large T. Then every solution of (264) is oscillatory or satisfies

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{274}$$

Theorem 82 ([20]). *Assume (267)–(268) hold. Furthermore, assume that there exist two functions $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ and $\psi \in C^1([\zeta_0, \infty), [0, \infty))$ and a function $H \in C^1(D, \mathbb{R})$, where $D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\}$ such that*

$$H(\zeta, \zeta) = 0 \text{ for } \zeta \geq \zeta_0, \quad H(\zeta, s) > 0 \text{ for } (\zeta, s) \in D_0,$$

where $D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}$ and H has a nonpositive continuous partial derivative $H'_s(\zeta, s) = \frac{\partial H(\zeta, s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$\limsup_{\zeta \to \infty} \frac{1}{H(\zeta, \zeta_0)} \int_{\zeta_0}^{\zeta} H(\zeta, s)$$

$$\left(\phi(s)q(s) - \phi(s)\psi'(s) + \frac{\eta\phi(s)(k\Gamma(1-\kappa))^{\eta}A_1(\zeta, s)A_2^{\eta-1}(\zeta, s)\psi^2(s)}{r_1(s)} \right.$$

$$- \left[2\eta\psi(s)\phi(s)(k\Gamma(1-\kappa))^{\eta}A_1(\zeta, s)A_2^{\eta-1}(\zeta, s) + r_1(s)\phi'(s) \right]^2$$

$$\left. \times \left(4\eta(k\Gamma(1-\kappa))^{\eta}A_1(\zeta, s)A_2^{\eta-1}(\zeta, s)r_1(s)\phi(s) \right)^{-1} \right) ds = \infty, \quad (275)$$

for all sufficiently large T. Then every solution of (264) is oscillatory or satisfies

$$\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s - \zeta)^{-\kappa} u(s) ds = 0. \tag{276}$$

In Theorems 81 and 82, if we take $H(\zeta, s)$ for some special functions such as $\ln \frac{\zeta}{s}$, then we can obtain the following two corollaries.

Corollary 21 ([20]). *Assume* (267)–(268) *hold. Furthermore, assume that there exist two functions* $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ *and* $\psi \in C^1([\zeta_0, \infty), [0, \infty))$. *If*

$$
\limsup_{\zeta \to \infty} \frac{1}{\ln \zeta - \ln \zeta_0} \int_{\zeta_0}^{\zeta} \ln(\zeta/s)
$$
$$
\left(\phi(s)q(s) - \phi(s)\psi'(s) + \frac{k\phi(s)\Gamma(1-\kappa)A_1(\zeta,s)\psi^{1+\frac{1}{\eta}}(s)}{r_1(s)} \right.
$$
$$
- \left[(\eta+1)k\psi^{\frac{1}{\eta}}(s)\phi(s)\Gamma(1-\kappa)A_1(\zeta,s) + r_1(s)\phi'(s) \right]^{\eta+1}
$$
$$
\left. \times \left((\eta+1)^{\eta+1}[k\Gamma(1-\kappa)\phi(s)A_1(\zeta,s)]^\eta r_1(s) \right)^{-1} \right) ds = \infty, \quad (277)
$$

for all sufficiently large T. Then every solution of (264) *is oscillatory or satisfies*

$$
\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s-\zeta)^{-\kappa} u(s) ds = 0. \quad (278)
$$

Corollary 22 ([20]). *Assume* (267)–(268) *hold. Furthermore, assume that there exist two functions* $\phi \in C^1([\zeta_0, \infty), \mathbb{R}^+)$ *and* $\psi \in C^1([\zeta_0, \infty), [0, \infty))$. *If*

$$
\limsup_{\zeta \to \infty} \frac{1}{\ln \zeta - \ln \zeta_0} \int_{\zeta_0}^{\zeta} \ln(\zeta/s)
$$
$$
\left(\phi(s)q(s) - \phi(s)\psi'(s) + \frac{\eta\phi(s)(k\Gamma(1-\kappa))^\eta A_1(\zeta,s)A_2^{\eta-1}(\zeta,s)\psi^2(s)}{r_1(s)} \right.
$$
$$
- \left[2\eta\psi(s)\phi(s)(k\Gamma(1-\kappa))^\eta A_1(\zeta,s)A_2^{\eta-1}(\zeta,s) + r_1(s)\phi'(s) \right]^2
$$
$$
\left. \times \left(4\eta(k\Gamma(1-\kappa))^\eta A_1(\zeta,s)A_2^{\eta-1}(\zeta,s)r_1(s)\phi(s) \right)^{-1} \right) ds = \infty, \quad (279)
$$

for all sufficiently large T. Then every solution of (264) *is oscillatory or satisfies*

$$
\lim_{\zeta \to \infty} \int_{\zeta}^{\infty} (s-\zeta)^{-\kappa} u(s) ds = 0. \quad (280)
$$

4. Oscillation Results via Hadamard Operators

Definition 4 ([1,2]). *Let* (a,b), $(0 \le a < b \le \infty)$, *be a finite or infinite interval of the half-axis* $\mathbb{R}^+$, *and let* $\Re(\kappa) > 0$ *and* $v \in \mathbb{C}$. *The (left-sided) Hadamard fractional integral* $\mathcal{I}_a^\kappa$ *of order* $\kappa \in \mathbb{C}$, $\Re(\kappa) > 0$, *is defined by*

$$
\mathcal{I}_a^\kappa \mathfrak{f}(\zeta) = \frac{1}{\Gamma(\kappa)} \int_a^\zeta \left[\ln\left(\frac{\zeta}{s}\right) \right]^{\kappa-1} \frac{\mathfrak{f}(s)}{s} ds, \quad a < \zeta < b.
$$

The (left-sided) Hadamard fractional derivative $\mathcal{D}_a^\kappa$ *of order* $\kappa \in \mathbb{C}$, $\Re(\kappa) \ge 0$, *is defined by*

$$
\mathcal{D}_a^\kappa \mathfrak{f}(\zeta) = \left(\zeta \frac{d}{dt} \right)^n \mathcal{I}_a^{n-\kappa} \mathfrak{f}(\zeta), \quad n = [\Re(\kappa)] + 1, \quad a < \zeta < b.
$$

The (left-sided) Caputo type Hadamard fractional derivative $^{C}\mathcal{D}_{a}^{\kappa}$ of order $\kappa \in \mathbb{C}$, $\Re(\kappa) \geq 0$, is defined by

$$^{C}\mathcal{D}_{a}^{\kappa}\mathfrak{f}(\zeta) = \mathcal{I}_{a}^{n-\kappa}\left(\zeta \frac{d}{dt}\right)^{n}\mathfrak{f}(\zeta), \quad n = [\Re(\kappa)] + 1, \quad a < \zeta < b.$$

Abdalla et al. [21] established sufficient conditions for the oscillation of solutions of the following fractional differential equations in the frame of left Hadamard fractional derivatives in the Riemann–Liouville and the Caputo settings.

$$\begin{cases} \mathcal{D}_{a}^{\kappa}u + \mathfrak{f}_{1}(\zeta, u) = v(\zeta) + \mathfrak{f}_{2}(\zeta, u), & \zeta > a, \\ \lim_{\zeta \to a^{+}} \mathcal{D}_{a}^{\kappa-j}u(\zeta) = b_{j}, & j = 1, 2, \cdots, n, \end{cases} \tag{281}$$

and

$$\begin{cases} ^{C}\mathcal{D}_{a}^{\kappa}u + \mathfrak{f}_{1}(\zeta, u) = v(\zeta) + \mathfrak{f}_{2}(\zeta, u), & \zeta > a, \\ \left(\zeta \frac{d}{dt}\right)^{k}u(a) = b_{k}, & k = 0, 1, 2, \cdots, n-1, \end{cases} \tag{282}$$

where $n = \lceil \kappa \rceil$, $\Re(\kappa) \geq 0$, $b_{0}, b_{1}, \cdots, b_{n} \in \mathbb{R}$; $\mathfrak{f}_{1}, \mathfrak{f}_{2} \in C([a, \infty) \times \mathbb{R}, \mathbb{R})$, and $v \in C([a, \infty), \mathbb{R})$.

Theorem 83 ([21]). *Let $\mathfrak{f}_{2} = 0$ and condition (H 1) holds. If*

$$\liminf_{\zeta \to \infty}(\ln \zeta)^{1-\kappa} \int_{T}^{\zeta}\left(\ln \frac{\zeta}{s}\right)^{\kappa-1}\frac{v(s)}{s}ds = -\infty, \tag{283}$$

and

$$\limsup_{\zeta \to \infty}(\ln \zeta)^{1-\kappa} \int_{T}^{\zeta}\left(\ln \frac{\zeta}{s}\right)^{\kappa-1}\frac{v(s)}{s}ds = \infty, \tag{284}$$

for every sufficiently large T, then (281) is oscillatory.

Theorem 84 ([22]). *Let the assumptions (H 1) and (H 2) hold with $\beta > \gamma$. If*

$$\liminf_{\zeta \to \infty}(\ln \zeta)^{1-\kappa} \int_{T}^{\zeta}\left(\ln \frac{\zeta}{s}\right)^{\kappa-1}\frac{[v(s) + H_{\beta,\gamma}(s)]}{s}ds = -\infty, \tag{285}$$

and

$$\limsup_{\zeta \to \infty}(\ln \zeta)^{1-\kappa} \int_{T}^{\zeta}\left(\ln \frac{\zeta}{s}\right)^{\kappa-1}\frac{[v(s) - H_{\beta,\gamma}(s)]}{s}ds = \infty, \tag{286}$$

for every sufficiently large T, where

$$H_{\beta,\gamma}(s) = \frac{\beta - \gamma}{\gamma}[p_{1}(s)]^{\frac{\gamma}{\gamma-\beta}}\left[\frac{\gamma p_{2}(s)}{\beta}\right]^{\frac{\beta}{\beta-\gamma}}, \tag{287}$$

then (281) is oscillatory.

Theorem 85 ([21]). *Let $\kappa \geq 1$ and suppose that the assumptions (H 1) and (H 3) hold with $\beta < \gamma$. If*

$$\liminf_{\zeta \to \infty}(\ln \zeta)^{1-\kappa} \int_{T}^{\zeta}\left(\ln \frac{\zeta}{s}\right)^{\kappa-1}\frac{[v(s) - H_{\beta,\gamma}(s)]}{s}ds = -\infty, \tag{288}$$

and

$$\limsup_{\zeta \to \infty}(\ln \zeta)^{1-\kappa} \int_{T}^{\zeta}\left(\ln \frac{\zeta}{s}\right)^{\kappa-1}\frac{[v(s) + H_{\beta,\gamma}(s)]}{s}ds = \infty, \tag{289}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), then every bounded solution of the problem (281) is oscillatory.

Theorem 86 ([21]). *Let* $\mathfrak{f}_2 = 0$ *and condition* (H 1) *holds. If*

$$\liminf_{\zeta \to \infty} (\ln \zeta)^{1-n} \int_T^\zeta \left(\ln \frac{\zeta}{s}\right)^{\kappa-1} \frac{v(s)}{s} ds = -\infty, \tag{290}$$

and

$$\limsup_{\zeta \to \infty} (\ln \zeta)^{1-n} \int_T^\zeta \left(\ln \frac{\zeta}{s}\right)^{\kappa-1} \frac{v(s)}{s} ds = \infty, \tag{291}$$

for every sufficiently large T, then (282) *is oscillatory.*

Theorem 87 ([22]). *Let the assumptions* (H 1) *and* (H 2) *hold with* $\beta > \gamma$. *If*

$$\liminf_{\zeta \to \infty} (\ln \zeta)^{1-n} \int_T^\zeta \left(\ln \frac{\zeta}{s}\right)^{\kappa-1} \frac{[v(s) + H_{\beta,\gamma}(s)]}{s} ds = -\infty, \tag{292}$$

and

$$\limsup_{\zeta \to \infty} (\ln \zeta)^{1-n} \int_T^\zeta \left(\ln \frac{\zeta}{s}\right)^{\kappa-1} \frac{[v(s) - H_{\beta,\gamma}(s)]}{s} ds = \infty, \tag{293}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ *is defined by* (287), *then* (282) *is oscillatory.*

Theorem 88 ([21]). *Let* $\kappa \geq 1$ *and suppose that the assumptions* (H 1) *and* (H 3) *hold with* $\beta < \gamma$. *If*

$$\liminf_{\zeta \to \infty} (\ln \zeta)^{1-n} \int_T^\zeta \left(\ln \frac{\zeta}{s}\right)^{\kappa-1} \frac{[v(s) - H_{\beta,\gamma}(s)]}{s} ds = -\infty, \tag{294}$$

and

$$\limsup_{\zeta \to \infty} (\ln \zeta)^{1-n} \int_T^\zeta \left(\ln \frac{\zeta}{s}\right)^{\kappa-1} \frac{[v(s) + H_{\beta,\gamma}(s)]}{s} ds = \infty, \tag{295}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ *is defined by* (287), *then every bounded solution of the problem* (282) *is oscillatory.*

5. Oscillation Results via Conformable Operators

Definition 5 ([23–25]). *The left conformable derivative starting from a of a function* $\mathfrak{f} : [a, \infty) \to \mathbb{R}$ *of order* $0 < \rho \leq 1$ *is defined by*

$$\mathcal{D}_a^\rho \mathfrak{f}(\zeta) = \lim_{\varepsilon \to 0} \frac{\mathfrak{f}(\zeta + \varepsilon(\zeta - a)^{1-\rho}) - \mathfrak{f}(\zeta)}{\varepsilon}. \tag{296}$$

If $\mathcal{D}_a^\rho \mathfrak{f}(\zeta)$ *exists on* (a, b), *then*

$$\mathcal{D}_a^\rho \mathfrak{f}(a) = \lim_{\zeta \to a^+} \mathcal{D}_a^\rho \mathfrak{f}(\zeta).$$

If f *is differentiable, then*

$$\mathcal{D}_a^\rho \mathfrak{f}(\zeta) = (\zeta - a)^{1-\rho} \mathfrak{f}'(\zeta).$$

The corresponding left conformable integral is defined as

$$\mathcal{I}_a^\rho \mathfrak{f}(\zeta) = \int_a^\zeta \mathfrak{f}(s)(s - a)^{\rho-1} ds, \quad 0 < \rho \leq 1. \tag{297}$$

Definition 6 ([23–25]). *The left conformable integral operator is defined by*

$$\mathcal{I}_a^{\kappa,\rho} \mathfrak{f}(\zeta) = \frac{1}{\Gamma(\kappa)} \int_a^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho}\right)^{\kappa-1} \mathfrak{f}(s)(s - a)^{\rho-1} ds, \tag{298}$$

where $\kappa \in \mathbb{C}$, $\Re(\kappa) \geq 0$.

Definition 7 ([23–25]). *The left fractional conformable derivative of order $\kappa \in \mathbb{C}$, $\Re(\kappa) \geq 0$, in the Riemann–Liouville setting is defined by*

$$\mathcal{D}_a^{\kappa,\rho} \mathfrak{f}(\zeta) = \mathcal{D}_a^{n,\rho} \mathcal{I}_a^{n-\kappa,\rho} \mathfrak{f}(\zeta), \tag{299}$$

where $n = \lceil \Re(\kappa) \rceil$, $\mathcal{D}_a^{n,\rho} = \mathcal{D}_a^{\rho} \mathcal{D}_a^{\rho} \cdots \mathcal{D}_a^{\rho}$ (n times), $\mathcal{D}_a^{\rho}$ is the left conformable differential operator presented in Definition 5.

Definition 8 ([23–25]). *The left fractional conformable derivative of order $\kappa \in \mathbb{C}$, $\Re(\kappa) \geq 0$, in the Caputo setting is defined by*

$$\mathcal{D}_a^{\kappa,\rho} \mathfrak{f}(\zeta) = \mathcal{I}_a^{n-\kappa,\rho} \mathcal{D}_a^{n,\rho} \mathfrak{f}(\zeta), \tag{300}$$

where $n = \lceil \Re(\kappa) \rceil$, $\mathcal{D}_a^{n,\rho} = \mathcal{D}_a^{\rho} \mathcal{D}_a^{\rho} \cdots \mathcal{D}_a^{\rho}$(n times), $\mathcal{D}_a^{\rho}$ is the left conformable differential operator presented in Definition 5.

Abdalla et al. [26] established sufficient conditions for the oscillation of solutions of the following fractional differential equations in the frame of left Hadamard fractional derivatives in the Riemann–Liouville and the Caputo settings.

$$\begin{cases} \mathcal{D}_a^{\kappa,\rho} u + \mathfrak{f}_1(\zeta, u) = v(\zeta) + \mathfrak{f}_2(\zeta, u), & \zeta > a \geq 0, \\ \lim_{\zeta \to a^+} \mathcal{I}_a^{j-\kappa,\rho} u(\zeta) = b_j, & j = 1, 2, \cdots, n, \end{cases} \tag{301}$$

and

$$\begin{cases} {}^C\mathcal{D}_a^{\kappa,\rho} u + \mathfrak{f}_1(\zeta, u) = v(\zeta) + \mathfrak{f}_2(\zeta, u), & \zeta > a \geq 0, \\ \mathcal{D}^{\rho,k} u(a) = b_k, & k = 0, 1, 2, \cdots, n-1, \end{cases} \tag{302}$$

where $n = \lceil \kappa \rceil$, $\Re(\kappa) \geq 0$, $0 < \rho \leq 1$; $b_0, b_1, \cdots, b_n \in \mathbb{R}$; $\mathfrak{f}_1, \mathfrak{f}_2 \in C([a,\infty) \times \mathbb{R}, \mathbb{R})$, and $v \in C([a,\infty), \mathbb{R})$.

Theorem 89 ([26]). *Let $\mathfrak{f}_2 = 0$ and condition (H 1) holds. If*

$$\liminf_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-\kappa} \int_T^\zeta \left(\frac{(\zeta-a)^\rho - (s-a)^\rho}{\rho} \right)^{\kappa-1} \frac{v(s)}{(s-a)^{1-\rho}} ds = -\infty, \tag{303}$$

and

$$\limsup_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-\kappa} \int_T^\zeta \left(\frac{(\zeta-a)^\rho - (s-a)^\rho}{\rho} \right)^{\kappa-1} \frac{v(s)}{(s-a)^{1-\rho}} ds = \infty, \tag{304}$$

for every sufficiently large T, then (301) is oscillatory.

Theorem 90 ([26]). *Let the assumptions (H 1) and (H 2) hold with $\beta > \gamma$. If*

$$\liminf_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-\kappa} \int_T^\zeta \left(\frac{(\zeta-a)^\rho - (s-a)^\rho}{\rho} \right)^{\kappa-1} \frac{[v(s) + H_{\beta,\gamma}(s)]}{(s-a)^{1-\rho}} ds = -\infty, \tag{305}$$

and

$$\limsup_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-\kappa} \int_T^\zeta \left(\frac{(\zeta-a)^\rho - (s-a)^\rho}{\rho} \right)^{\kappa-1} \frac{[v(s) - H_{\beta,\gamma}(s)]}{(s-a)^{1-\rho}} ds = \infty, \tag{306}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), then (301) is oscillatory.

Theorem 91 ([26]). *Let $\kappa \geq 1$ and suppose that the assumptions (H 1) and (H 3) hold with $\beta < \gamma$. If*

$$\liminf_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-\kappa} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho} \right)^{\kappa - 1} \frac{[v(s) - H_{\beta,\gamma}(s)]}{(s - a)^{1-\rho}} ds = -\infty, \tag{307}$$

and

$$\limsup_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-\kappa} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho} \right)^{\kappa - 1} \frac{[v(s) + H_{\beta,\gamma}(s)]}{(s - a)^{1-\rho}} ds = \infty, \tag{308}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), then every bounded solution of the problem (301) is oscillatory.

Theorem 92 ([26]). *Let $\mathfrak{f}_2 = 0$ and condition (H 1) holds. If*

$$\liminf_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-n} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho} \right)^{\kappa - 1} \frac{v(s)}{(s - a)^{1-\rho}} ds = -\infty, \tag{309}$$

and

$$\limsup_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-n} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho} \right)^{\kappa - 1} \frac{v(s)}{(s - a)^{1-\rho}} ds = \infty, \tag{310}$$

for every sufficiently large T, then (302) is oscillatory.

Theorem 93 ([26]). *Let the assumptions (H 1) and (H 2) hold with $\beta > \gamma$. If*

$$\liminf_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-n} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho} \right)^{\kappa - 1} \frac{[v(s) + H_{\beta,\gamma}(s)]}{(s - a)^{1-\rho}} ds = -\infty, \tag{311}$$

and

$$\limsup_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-n} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho} \right)^{\kappa - 1} \frac{[v(s) - H_{\beta,\gamma}(s)]}{(s - a)^{1-\rho}} ds = \infty, \tag{312}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), then (302) is oscillatory.

Theorem 94 ([26]). *Let $\kappa \geq 1$ and suppose that the assumptions (H 1) and (H 3) hold with $\beta < \gamma$. If*

$$\liminf_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-n} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho} \right)^{\kappa - 1} \frac{[v(s) - H_{\beta,\gamma}(s)]}{(s - a)^{1-\rho}} ds = -\infty, \tag{313}$$

and

$$\limsup_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho} \right)^{1-n} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho} \right)^{\kappa - 1} \frac{[v(s) + H_{\beta,\gamma}(s)]}{(s - a)^{1-\rho}} ds = \infty, \tag{314}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), then every bounded solution of the problem (302) is oscillatory.

Motivated by the works in Reference 20 of [3,7,26], Aphithana et al. [27] studied forced oscillatory properties of solutions to the conformable initial value problem with damping in the Riemann–Liouville and the Caputo settings as follows:

$$\begin{cases} \mathcal{D}_a^{1+\kappa,\rho} u + p(\zeta) \mathcal{D}_a^{\kappa,\rho} u + q(\zeta) \mathfrak{f}(u) = g(\zeta), & \zeta > a \geq 0, \\ \lim_{\zeta \to a^+} \mathcal{I}_a^{j-\kappa,\rho} u(\zeta) = b_j, & j = 1, 2, \cdots, n, \end{cases} \tag{315}$$

and

$$\begin{cases} {}^{C}\mathcal{D}_a^{1+\kappa,\rho}u + p(\zeta)\,{}^{C}\mathcal{D}_a^{\kappa,\rho}u + q(\zeta)\mathfrak{f}(u) = g(\zeta), & \zeta > a \geq 0, \\ \mathcal{D}^{\rho,k}u(a) = b_k, & k = 0, 1, 2, \cdots, n-1, \end{cases} \qquad (316)$$

where $n = \lceil \kappa \rceil$, $\Re(\kappa) \geq 0$, $0 < \rho \leq 1$; $b_0, b_1, \cdots, b_n \in \mathbb{R}$; $p, g \in C(\mathbb{R}^+, \mathbb{R})$, $q \in C(\mathbb{R}^+, \mathbb{R}^+)$, and $\mathfrak{f} \in C(\mathbb{R}, \mathbb{R})$ such that

$$\frac{\mathfrak{f}(u)}{u} > 0, \quad u \neq 0.$$

Theorem 95 ([27]). *If*

$$\liminf_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho}\right)^{1-\kappa} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho}\right)^{\kappa-1}$$

$$\times \left[\frac{M + \mathcal{I}_{\zeta_1}^{1,\rho}(g(s)V(s))}{V(s)}\right] \frac{ds}{(s-a)^{1-\rho}} = -\infty, \quad (317)$$

and

$$\limsup_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho}\right)^{1-\kappa} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho}\right)^{\kappa-1}$$

$$\times \left[\frac{M + \mathcal{I}_{\zeta_1}^{1,\rho}(g(s)V(s))}{V(s)}\right] \frac{ds}{(s-a)^{1-\rho}} = \infty, \quad (318)$$

for every sufficiently large T, where

$$V(\zeta) = \exp\left[\int_{\zeta_1}^\zeta (s-a)^{\rho-1} ds\right], \quad \zeta_1 > a, \qquad (319)$$

and

$$M = \mathcal{D}_a^{\kappa,\rho} u(\zeta_1) V(\zeta_1) \qquad (320)$$

then (315) *is oscillatory.*

Theorem 96 ([27]). *If*

$$\liminf_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho}\right)^{1-n} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho}\right)^{\kappa-1}$$

$$\times \left[\frac{M^* + \mathcal{I}_{\zeta_1}^{1,\rho}(g(s)V(s))}{V(s)}\right] \frac{ds}{(s-a)^{1-\rho}} = -\infty, \quad (321)$$

and

$$\limsup_{\zeta \to \infty} \left(\frac{\zeta^\rho}{\rho}\right)^{1-n} \int_T^\zeta \left(\frac{(\zeta - a)^\rho - (s - a)^\rho}{\rho}\right)^{\kappa-1}$$

$$\times \left[\frac{M^* + \mathcal{I}_{\zeta_1}^{1,\rho}(g(s)V(s))}{V(s)}\right] \frac{ds}{(s-a)^{1-\rho}} = \infty, \quad (322)$$

for every sufficiently large T, where V is defined as in (319) *and*

$$M^* = {}^{C}\mathcal{D}_a^{\kappa,\rho} u(\zeta_1) V(\zeta_1) \qquad (323)$$

then (316) *is oscillatory.*

6. Oscillation Results via Generalized Proportional Operators

Definition 9 ([28]). *For $\rho \in (0,1]$, $\kappa \in \mathbb{C}$, $\Re(\kappa) \geq 0$, the generalized proportional fractional integral of $\mathfrak{f}$ of order κ is*

$$\mathcal{I}_a^{\kappa,\rho}\mathfrak{f}(\zeta) = \frac{1}{\rho^\kappa \Gamma(\kappa)} \int_a^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}\mathfrak{f}(s)(\zeta-s)^{\kappa-1}f(s)ds. \tag{324}$$

Definition 10 ([28]). *For $\rho \in (0,1]$, $\kappa \in \mathbb{C}$, $\Re(\kappa) \geq 0$, $n = [\Re(\kappa)] + 1$, the generalized proportional fractional derivative of Riemann–Liouville type of f of order κ is*

$$\mathcal{D}_a^{\kappa,\rho}\mathfrak{f}(\zeta) = \mathcal{D}^{n,\rho}\mathcal{I}_a^{n-\kappa,\rho}\mathfrak{f}(\zeta), \tag{325}$$

where $\mathcal{D}^{n,\rho} = \mathcal{D}^\rho \mathcal{D}^\rho \cdots \mathcal{D}^\rho (n \text{ times})$, $\mathcal{D}^\rho$ is the proportional derivative defined in [29].

Definition 11 ([28]). *For $\rho \in (0,1]$, $\kappa \in \mathbb{C}$, $\Re(\kappa) \geq 0$, $n = [\Re(\kappa)] + 1$, the generalized proportional fractional derivative of Caputo type of f of order κ is*

$$\mathcal{D}_a^{\kappa,\rho}\mathfrak{f}(\zeta) = \mathcal{D}^{n,\rho}\mathcal{I}_a^{n-\kappa,\rho}\mathfrak{f}(\zeta), \tag{326}$$

where $\mathcal{D}^{n,\rho} = \mathcal{D}^\rho \mathcal{D}^\rho \cdots \mathcal{D}^\rho (n \text{ times})$, $\mathcal{D}^\rho$ is the proportional derivative defined in [29].

Sudsutad et al. [22] established several oscillation criteria of solutions for the generalized proportional fractional differential equation with initial conditions of the form

$$\begin{cases} \mathcal{D}_a^{\kappa,\rho}u + \mathfrak{f}_1(\zeta,u) = v(\zeta) + \mathfrak{f}_2(\zeta,u), & \zeta > a \geq 0, \\ \lim_{\zeta \to a^+} \mathcal{I}_a^{j-\kappa,\rho}u(\zeta) = b_j, & j = 1,2,\cdots,n, \end{cases} \tag{327}$$

and

$$\begin{cases} {}^C\mathcal{D}_a^{\kappa,\rho}u + \mathfrak{f}_1(\zeta,u) = v(\zeta) + \mathfrak{f}_2(\zeta,u), & \zeta > a \geq 0, \\ \mathcal{D}^{k,\rho}u(a) = b_k, & k = 0,1,2,\cdots,n-1, \end{cases} \tag{328}$$

where $n = \lceil \kappa \rceil$, $\Re(\kappa) \geq 0$, $0 < \rho \leq 1$; $b_0, b_1, \cdots, b_n \in \mathbb{R}$; $\mathfrak{f}_1, \mathfrak{f}_2 \in C([a,\infty) \times \mathbb{R}, \mathbb{R})$, and $v \in C([a,\infty), \mathbb{R})$.

Theorem 97 ([22]). *Let $\mathfrak{f}_2 = 0$ and condition (H 1) holds. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}v(s)ds = -\infty, \tag{329}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}v(s)ds = \infty, \tag{330}$$

for every sufficiently large T, then (327) is oscillatory.

Theorem 98 ([22]). *Let the assumptions (H 1) and (H 2) hold with $\beta > \gamma$. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}\left[v(s) + H_{\beta,\gamma}(s)\right]ds = -\infty, \tag{331}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}\left[v(s) - H_{\beta,\gamma}(s)\right]ds = \infty, \tag{332}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), then (327) is oscillatory.

Theorem 99 ([22]). *Let $\kappa \geq 1$ and suppose that the assumptions (H 1) and (H 3) hold with $\beta < \gamma$.
If*

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}\big[v(s) + H_{\beta,\gamma}(s)\big]ds = -\infty, \tag{333}$$

and

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}\big[v(s) - H_{\beta,\gamma}(s)\big]ds = \infty, \tag{334}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), then every bounded solution of the problem (327) is oscillatory.

Theorem 100 ([22]). *Let $\mathfrak{f}_2 = 0$ and condition (H 1) holds. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-n} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}v(s)ds = -\infty, \tag{335}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-n} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}v(s)ds = \infty, \tag{336}$$

for every sufficiently large T, then (328) is oscillatory.

Theorem 101 ([22]). *Let the assumptions (H 1) and (H 2) hold with $\beta > \gamma$. If*

$$\liminf_{\zeta \to \infty} \zeta^{1-n} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}\big[v(s) + H_{\beta,\gamma}(s)\big]ds = -\infty, \tag{337}$$

and

$$\limsup_{\zeta \to \infty} t^{1-n} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}\big[v(s) - H_{\beta,\gamma}(s)\big]ds = \infty, \tag{338}$$

for every sufficiently large T, where where $H_{\beta,\gamma}$ is defined by (287), then (328) is oscillatory.

Theorem 102 ([22]). *Let $\kappa \geq 1$ and suppose that the assumptions (H 1) and (H 3) hold with $\beta < \gamma$. If*

$$\limsup_{\zeta \to \infty} \zeta^{1-n} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}\big[v(s) + H_{\beta,\gamma}(s)\big]ds = -\infty, \tag{339}$$

and

$$\liminf_{\zeta \to \infty} \zeta^{1-n} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)}(\zeta-s)^{\kappa-1}\big[v(s) - H_{\beta,\gamma}(s)\big]ds = \infty, \tag{340}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), then every bounded solution of the problem (328) is oscillatory.

In continuation to the above work, Alzabut et al. [30] established some sufficient conditions for forced oscillation criteria of all solutions of the generalized proportional fractional initial value problem with damping term of the form:

$$\begin{cases} \mathcal{D}_a^{1+\kappa,\rho}u + p(\zeta)\mathcal{D}_a^{\kappa,\rho}u + q(\zeta)\mathfrak{f}(u) = g(\zeta), & \zeta > a \geq 0, \\ \lim_{\zeta \to a^+} \mathcal{I}_a^{j-\kappa,\rho}u(\zeta) = b_j, & j = 1, 2, \cdots, n, \end{cases} \tag{341}$$

and

$$\begin{cases} {}^C\mathcal{D}_a^{1+\kappa,\rho}u + p(\zeta)\,{}^C\mathcal{D}_a^{\kappa,\rho}u + q(\zeta)\mathfrak{f}(u) = g(\zeta), & \zeta > a \geq 0, \\ \mathcal{D}^{k,\rho}u(a) = b_k, & k = 0, 1, 2, \cdots, n-1, \end{cases} \tag{342}$$

where $n = \lceil \kappa \rceil$, $\Re(\kappa) \geq 0$, $0 < \rho \leq 1$; $b_0, b_1, \cdots, b_n \in \mathbb{R}$; $p, g \in C(\mathbb{R}^+, \mathbb{R})$, $q \in C(\mathbb{R}^+, \mathbb{R}^+)$, and $\mathfrak{f} \in C(\mathbb{R}, \mathbb{R})$ such that

$$\frac{\mathfrak{f}(u)}{u} > 0, \quad u \neq 0.$$

The following results improve and generalize the oscillation results in [27].

Theorem 103 ([30]). *If*

$$\liminf_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)} (\zeta-s)^{\kappa-1} \left[\frac{e^{\frac{\rho-1}{\rho}(s-t_1)} M + \mathcal{I}_{\zeta_1}^{1,\rho}(\rho g(s) V(s))}{V(s)} \right] ds = -\infty, \quad (343)$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)} (\zeta-s)^{\kappa-1} \left[\frac{e^{\frac{\rho-1}{\rho}(s-\zeta_1)} M + \mathcal{I}_{\zeta_1}^{1,\rho}(\rho g(s) V(s))}{V(s)} \right] ds = \infty, \quad (344)$$

for every sufficiently large T, where

$$V(\zeta) = \exp\left[\int_{\zeta_1}^\zeta \frac{\rho p(s) - (1-\rho)}{\rho} ds \right], \quad (345)$$

and

$$M = \mathcal{D}_{\zeta_1}^{\kappa,\rho} u(\zeta_1) V(\zeta_1) \quad (346)$$

then (341) *is oscillatory.*

Theorem 104 ([30]). *If*

$$\liminf_{\zeta \to \infty} \zeta^{1-n} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)} (\zeta-s)^{\kappa-1} \left[\frac{e^{\frac{\rho-1}{\rho}(s-\zeta_1)} M^* + \mathcal{I}_{\zeta_1}^{1,\rho}(\rho g(s) V(s))}{V(s)} \right] ds = -\infty, \quad (347)$$

and

$$\limsup_{\zeta \to \infty} \zeta^{1-\kappa} \int_T^\zeta e^{\frac{\rho-1}{\rho}(\zeta-s)} (\zeta-s)^{\kappa-1} \left[\frac{e^{\frac{\rho-1}{\rho}(s-\zeta_1)} M * + \mathcal{I}_{\zeta_1}^{1,\rho}(\rho g(s) V(s))}{V(s)} \right] ds = \infty, \quad (348)$$

for every sufficiently large T, where V is defined as in (345) *and*

$$M^* = {}^C \mathcal{D}_a^{\kappa,\rho} u(a) V(a) \quad (349)$$

then (342) *is oscillatory.*

Remark 5. *If we put $\rho = 1$ in Theorem 103 and Theorem 104, then they reduce to Theorem 95 and Theorem 96, respectively.*

7. Oscillation Results via Fractional Operators Involving Mittag–Leffler Kernel

Definition 12 ([31]). *Let $f \in H^1(a, b)$, $a < b$, and $0 < \kappa < 1$, then the left fractional integral with Mittag–Leffler nonsingular kernel is defined by*

$$^{AB}\mathcal{I}_a^\kappa \mathfrak{f}(\zeta) = \frac{1-\kappa}{B(\kappa)} f(\zeta) + \frac{\kappa}{B(\kappa)} \mathcal{I}_a^\kappa \mathfrak{f}(\zeta), \quad (350)$$

where $B(\kappa) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$ and $\mathcal{I}_a^\kappa$ is the κ-th order left-sided Riemann–Liouville fractional integral.

Definition 13 ([31]). *The left Caputo fractional derivative with Mittag–Leffler nonsingular kernel is defined by*

$$^{ABC}\mathcal{D}_a^\kappa \mathfrak{f}(\zeta) = \frac{B(\kappa)}{1 - \kappa} \int_a^\zeta \mathfrak{f}'(s) E_\kappa \left[-\kappa \frac{(\zeta - s)^\kappa}{1 - \kappa} \right] ds, \tag{351}$$

where $E_\kappa(w)$ is the Mittag–Leffler function with one parameter defined by

$$E_\kappa(w) = \sum_{n=0}^\infty \frac{w^n}{\Gamma(\kappa n + 1)}, \quad w \in \mathbb{C}, \quad \Re(\kappa) > 0.$$

Abdalla et al. [31] derived sufficient conditions to prove the oscillation for solutions of Caputo fractional differential equations with Mittag–Leffler nonsingular kernel of the form

$$\begin{cases} ^{ABC}\mathcal{D}_a^\kappa u + \mathfrak{f}_1(\zeta, u) = v(\zeta) + \mathfrak{f}_2(\zeta, u), & \zeta > a, \\ u^{(k)}(a) = b_k, & k = 0, 1, 2, \cdots, n-1, \end{cases} \tag{352}$$

where $n < \kappa \le n + 1, b_0, b_1, \cdots, b_{n-1} \in \mathbb{R}; \mathfrak{f}_1, \mathfrak{f}_2 \in C([a, \infty) \times \mathbb{R}, \mathbb{R})$, and $v \in C([a, \infty), \mathbb{R})$.

Theorem 105 ([31]). *Let $\mathfrak{f}_2 = 0$ and condition* (H 1) *holds. If*

$$\liminf_{\zeta \to \infty} \zeta^{-n} \left[\Gamma(\kappa) \frac{1 - \kappa + n}{B(\kappa - n)} \mathcal{I}_a^n v(\zeta) + \frac{\kappa - n}{B(\kappa - n)} \int_T^\zeta (\zeta - s)^{\kappa - 1} v(s) ds \right] = -\infty, \tag{353}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{-n} \left[\Gamma(\kappa) \frac{1 - \kappa + n}{B(\kappa - n)} \mathcal{I}_a^n v(\zeta) + \frac{\kappa - n}{B(\kappa - n)} \int_T^\zeta (\zeta - s)^{\kappa - 1} v(s) ds \right] = \infty, \tag{354}$$

for every sufficiently large T, then (352) *is oscillatory.*

Theorem 106 ([31]). *Let the assumptions* (H 1) *and* (H 2) *hold with $\beta > \gamma$. If*

$$\liminf_{\zeta \to \infty} \zeta^{-n} \left[\Gamma(\kappa) \frac{1 - \kappa + n}{B(\kappa - n)} \mathcal{I}_a^n [v(\zeta) + H_{\beta,\gamma}(\zeta)] \right.$$
$$\left. + \frac{\kappa - n}{B(\kappa - n)} \int_T^\zeta (\zeta - s)^{\kappa - 1} [v(s) + H_{\beta,\gamma}(s)] ds \right] = -\infty, \tag{355}$$

and

$$\limsup_{\zeta \to \infty} \zeta^{-n} \left[\Gamma(\kappa) \frac{1 - \kappa + n}{B(\kappa - n)} \mathcal{I}_a^n [v(\zeta) + H_{\beta,\gamma}(\zeta)] \right.$$
$$\left. + \frac{\kappa - n}{B(\kappa - n)} \int_T^\zeta (\zeta - s)^{\kappa - 1} [v(s) - H_{\beta,\gamma}(s)] ds \right] = \infty, \tag{356}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), *then* (352) *is oscillatory.*

Theorem 107 ([22]). *Let $\kappa \geq$ and suppose that the assumptions ($A1$) and ($A3$) hold with $\beta < \gamma$.*
If

$$\liminf_{\zeta \to \infty} \zeta^{-n} \left[\Gamma(\kappa) \frac{1 - \kappa + n}{B(\kappa - n)} \mathcal{I}_a^n [v(\zeta) + H_{\beta,\gamma}(\zeta)] \right.$$

$$\left. + \frac{\kappa - n}{B(\kappa - n)} \int_T^\zeta (\zeta - s)^{\kappa-1} [v(s) - H_{\beta,\gamma}(s)] ds \right] = -\infty, \quad (357)$$

and

$$\limsup_{\zeta \to \infty} \zeta^{-n} \left[\Gamma(\kappa) \frac{1 - \kappa + n}{B(\kappa - n)} \mathcal{I}_a^n [v(\zeta) + H_{\beta,\gamma}(\zeta)] \right.$$

$$\left. + \frac{\kappa - n}{B(\kappa - n)} \int_T^\zeta (\zeta - s)^{\kappa-1} [v(s) + H_{\beta,\gamma}(s)] ds \right] = \infty, \quad (358)$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), then every bounded solution of the
problem (352) is oscillatory.

8. Oscillation Results via General Riemann–Liouville and Caputo Operators

Definition 14 ([32]). *Let $\kappa > 0$, $I = [a, b]$ be a finite or infinite interval, $\mathfrak{f}$ an integrable function*
defined on I and $\psi \in C^1(I)$ an increasing function such that $\psi'(s) \neq 0$ for all $s \in I$. Fractional
integrals and fractional derivatives of a function f with respect to another function ψ are defined

$$\mathcal{I}_a^{\kappa,\psi} \mathfrak{f}(\zeta) = \frac{1}{\Gamma(\kappa)} \int_a^\zeta \psi'(s)(\psi(\zeta) - \psi(s))^{\kappa-1} \mathfrak{f}(s) ds, \quad \zeta > a,$$

and

$$\mathcal{D}_a^{\kappa,\psi} \mathfrak{f}(\zeta) = \left(\frac{1}{\psi'(\zeta)} \frac{d}{dt} \right)^n \mathcal{I}_a^{n-\kappa,\psi} \mathfrak{f}(\zeta), \quad n = \lceil \kappa \rceil, \quad \zeta > a.$$

Definition 15 ([32]). *Let $\kappa > 0$, $n \in \mathbb{N}$, I is the interval $-\infty \leq a < b \leq \infty$, $f, \psi \in C^n(I)$ such*
that ψ is an increasing function such that $\psi'(s) \neq 0$ for all $s \in I$. The left ψ-Caputo fractional
derivative of f of order κ is defined by

$${}^C\mathcal{D}_a^{\kappa,\psi} \mathfrak{f}(\zeta) = \mathcal{I}_a^{n-\kappa,\psi} \left(\frac{1}{\psi'(\zeta)} \frac{d}{dt} \right)^n \mathfrak{f}(\zeta).$$

Abdalla et al. [32] studied the oscillation of general fractional differential equations of
the form

$$\begin{cases} \mathcal{D}_a^{\kappa,\psi} u + \mathfrak{f}_1(\zeta, u) = v(\zeta) + \mathfrak{f}_2(\zeta, u), & \zeta > a \geq 0, \\ \lim_{\zeta \to a^+} \mathcal{D}_a^{\kappa-j,\psi} u(\zeta) = b_j, & j = 1, 2, \cdots, n, \end{cases} \quad (359)$$

and

$$\begin{cases} {}^C\mathcal{D}_a^{\kappa,\psi} u + \mathfrak{f}_1(\zeta, u) = v(\zeta) + \mathfrak{f}_2(\zeta, u), & \zeta > a \geq 0, \\ \mathcal{D}^{k,\psi} u(a) = b_k, & k = 0, 1, 2, \cdots, n - 1, \end{cases} \quad (360)$$

where $n = \lceil \kappa \rceil$, $\Re(\kappa) \geq 0$, $b_0, b_1, \cdots, b_n \in \mathbb{R}$; $\mathfrak{f}_1, \mathfrak{f}_2 \in C([a, \infty) \times \mathbb{R}, \mathbb{R})$, and $v \in C([a, \infty), \mathbb{R})$.

Theorem 108 ([32]). *Let $\mathfrak{f}_2 = 0$ and condition (H 1) holds. If*

$$\liminf_{\zeta \to \infty} (\psi(\zeta))^{1-\kappa} \int_\zeta^\zeta \psi'(s)(\psi(\zeta) - \psi(s))^{\kappa-1} v(s) ds = -\infty, \quad (361)$$

and

$$\limsup_{\zeta\to\infty}(\psi(\zeta))^{1-\kappa}\int_\zeta^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}v(s)ds=\infty, \tag{362}$$

for every sufficiently large T, then (359) *is oscillatory.*

Theorem 109 ([22]). *Let the assumptions* (H 1) *and* (H 2) *hold with* $\beta>\gamma$. *If*

$$\liminf_{\zeta\to\infty}(\psi(\zeta))^{1-\kappa}\int_\zeta^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}\big[v(s)+H_{\beta,\gamma}(s)\big]ds=-\infty, \tag{363}$$

and

$$\limsup_{\zeta\to\infty}(\psi(\zeta))^{1-\kappa}\int_\zeta^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}\big[v(s)-H_{\beta,\gamma}(s)\big]ds=\infty, \tag{364}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ *is defined by* (287), *then* (359) *is oscillatory.*

Theorem 110 ([22]). *Let* $\kappa\geq 1$ *and suppose that the assumptions* (H 1) *and* (H 3) *hold with* $\beta<\gamma$. *If*

$$\limsup_{\zeta\to\infty}(\psi(\zeta))^{1-\kappa}\int_\zeta^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}\big[v(s)+H_{\beta,\gamma}(s)\big]ds=\infty, \tag{365}$$

and

$$\liminf_{\zeta\to\infty}(\psi(\zeta))^{1-\kappa}\int_\zeta^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}\big[v(s)-H_{\beta,\gamma}(s)\big]ds=-\infty, \tag{366}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ *is defined by* (287), *then every bounded solution of the problem* (359) *is oscillatory.*

Remark 6. *If we let* $\psi(\zeta)=\zeta$, $\psi(\zeta)=\ln\zeta$ *and* $\psi(\zeta)=\frac{(\zeta-a)^\rho}{\rho}$ *then we recover the Riemann–Liouville and Hadamard fractional oscillation results in Reference 20 of* [3,21,26], *respectively.*

Theorem 111 ([32]). *Let* $\mathfrak{f}_2=0$ *and condition* (H 1) *holds. If*

$$\liminf_{\zeta\to\infty}(\psi(\zeta))^{1-n}\int_\zeta^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}v(s)ds=-\infty, \tag{367}$$

and

$$\limsup_{\zeta\to\infty}(\psi(\zeta))^{1-n}\int_\zeta^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}v(s)ds=\infty, \tag{368}$$

for every sufficiently large T, then (360) *is oscillatory.*

Theorem 112 ([22]). *Let the assumptions* (H 1) *and* (H 2) *hold with* $\beta>\gamma$. *If*

$$\liminf_{\zeta\to\infty}(\psi(\zeta))^{1-n}\int_\zeta^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}\big[v(s)+H_{\beta,\gamma}(s)\big]ds=-\infty, \tag{369}$$

and

$$\limsup_{\zeta\to\infty}(\psi(\zeta))^{1-n}\int_\zeta^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}\big[v(s)-H_{\beta,\gamma}(s)\big]ds=\infty, \tag{370}$$

for every sufficiently large T, where $H_{\beta,\gamma}$ *is defined by* (287), *then* (360) *is oscillatory.*

Theorem 113 ([22]). *Let $\kappa \geq 1$ and suppose that the assumptions* (H 1) *and* (H 3) *hold with* $\beta < \gamma$. *If*

$$\limsup_{\zeta \to \infty}(\psi(\zeta))^{1-n}\int_{\zeta}^{\zeta}\psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}\big[v(s)+H_{\beta,\gamma}(s)\big]ds = \infty, \qquad (371)$$

and

$$\liminf_{\zeta \to \infty}(\psi(\zeta))^{1-n}\int_{\zeta}^{\zeta}\psi'(s)(\psi(\zeta)-\psi(s))^{\kappa-1}\big[v(s)-H_{\beta,\gamma}(s)\big]ds = -\infty, \qquad (372)$$

for every sufficiently large T, where $H_{\beta,\gamma}$ is defined by (287), *then every bounded solution of the problem* (360) *is oscillatory.*

Remark 7. *If we let $\psi(\zeta) = \zeta$, $\psi(\zeta) = \ln\zeta$ and $\psi(\zeta) = \frac{(\zeta-a)^{\rho}}{\rho}$ then we recover the Riemann–Liouville and Hadamard fractional oscillation results in the frame of Caputo in Reference 20 of [3,21,26], respectively.*

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