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Symmetry Analysis and PT-Symmetric Extension of the Fifth-Order Korteweg-de Vries-Like Equation

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Abstract: In the present paper, PT-symmetric extension of the fifth-order Korteweg-de Vries-like equation are investigated. Several special equations with PT symmetry are obtained by choosing different values, for which their symmetries are obtained simultaneously. In particular, for the particular equation, its conservation laws are obtained, including conservation of momentum and conservation of energy. Reciprocal Bäcklund transformations of conservation laws of momentum and energy are presented for the first time. The important thing is that for the special case of $\epsilon = 3$, the corresponding time fractional case are studied by Lie group method. And what is interesting is that the symmetry of the time fractional equation is obtained, and based on the symmetry, this equation is reduced to a fractional ordinary differential equation. Finally, for the general case, the symmetry of this equation is obtained, and based on the symmetry, the reduced equation is presented. Through the results obtained in this paper, it can be found that the Lie group method is a very effective method, which can be used to deal with many models in natural phenomena.

Keywords: PT-symmetric; fifth-order Korteweg-de vries-like equation; symmetry analysis; conservation laws

1. Introduction

The authors [1] considered the complex PT-symmetric extension of the classical Korteweg-de Vries (KdV) equation

$$u_t - iu(iu_x)^\epsilon + u_{xxx} = 0, \tag{1}$$

where i is the imaginary unit, they discussed the features of these equations for $\epsilon = 0, 1, 3, 2n + 1$. Indeed, the classical KdV equation is PT symmetric, however is not symmetric under P or T. PT symmetric quantum mechanics is related to many integrable models [1–3]. If $\epsilon = 1$, this situation is the classical KdV equation, which has been studied in a large amount of papers. For more description on the classical KdV equation, see [1] and references therein.

Based on the results of [1], the following fifth-order KdV-like equation will be considered in the present paper

$$u_t - iu(iu_x)^\epsilon + \alpha u_{xxx} + \beta u_{xxxxx} = 0, \tag{2}$$

it is clear that this equation is also PT symmetric. This equation includes fifth order nonlinear dispersion term. If $\beta = 0$, it reduced to Equation (3) [1]. While $\alpha = 0$, this equation becomes the fifth-order KdV equation. In general, objectively speaking, higher order equations are more difficult to handle than lower order equations. This is because we know that the higher the order of the equation, the more difficult it will be to calculate and the longer it will take to process. Indeed, there are many nonlinear natural phenomena that might be more reasonably described using higher order nonlinear evolution equations (NLEEs).

Because of the importance of NLEEs, there are many approaches there to deal with them, some of which include but are not limited to, for example, the Hirota bilinear method [4], the inverse scattering transformation method [5], Darboux transformations [6], the structure-preserving method [7–9], the Lie symmetry method [10–18], and so on.

If $\epsilon = 1$ for Equation (2), it is the general Kawahara equation. There have been many papers have investigated Kawahara type equations, including exact solutions, symmetry, etc. Kawahara [19] derived this equation. The author [20] studied solitary wave solution for the generalized Kawahara equation. New solitons solutions and periodic solutions are derived in [21]. Nonlinear self-adjointness of a generalized fifth-order KdV equation are studied in [22]. The author [23] considered symmetry analysis and exact solutions to the fifth-order KdV types of equations. Homotopy analysis method is used to study the Kawahara equation [24]. New analytical cnoidal and solitary wave solutions of the Extended Kawahara equation are presented in [25].

From the known literature, for the PT-symmetric extension of the higher-order fifth-order KdV equation, so far, there is no corresponding references to study this equation. In view of this, this paper uses the symmetry method to systematically study this equation. For different parameters of ϵ, the symmetry of these equations are investigated separately, and especially for $\epsilon = 1$, the conservation law of this equation are derived. The interesting thing is that the reciprocal Bäcklund transformations of the conservation of momentum and energy are presented for this equation.

In Section 2, symmetry analysis and conservation laws of this Equation (2) for $\epsilon = 1$ are presented. In Section 3, symmetry analysis and travelling wave solutions for $\epsilon = 0$ are displayed. Symmetry analysis for $\epsilon = 3$ are derived, and the time fractional form of this equation is studied in Section 4. Symmetry analysis and reductions for $\epsilon = 2n + 1$ are given in Section 5. In the last Section 6, the conclusion of this paper is obtained.

2. Symmetry Analysis and Conservation Laws for $\epsilon = 1$

2.1. Symmetry Analysis

If $\epsilon = 1$, one can get

$$u_t + u(u_x) + \alpha u_{xxx} + \beta u_{xxxxx} = 0, \tag{3}$$

this is the general Kawahara equation [22,23,26–29], the Lie algebra is spanned by the following vector fields

$$V_1 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, V_2 = \frac{\partial}{\partial t}, V_3 = \frac{\partial}{\partial x}. \tag{4}$$

Additionally, one can get the high order Lie-Bäcklund symmetries as follows

$$\eta_u = c_2 t u_x + c_3 u u_x + c_1 u_x + c_3 u_{xxx} + c_3 u_{xxxxx} - c_2. \tag{5}$$

2.2. Conservation Laws

For the one-dimensional case, the conservation law can be written in the following form

$$T^t + T^x = 0, \tag{6}$$

using the method proposed in [11], the following multiplier can be obtained

$$\Lambda = c_1 u_t - c_1 x + c_2 u_{xxxx} + c_2 \frac{\alpha}{\beta} u_{xx} + c_2 \frac{u^2}{2\beta} + c_3 u + c_4, \tag{7}$$

for this multiplier, one can get the following conservation laws:

Conservation of momentum P:

$$\partial_t(u) + \partial_x\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right) = 0, \tag{8}$$

thus,

$$\frac{d}{dt}P = 0, P = \int_{-\infty}^{\infty} u\,dx. \tag{9}$$

Conservation of Energy E:

$$\partial_t\left(\frac{1}{2}u^2\right) + \partial_x\left(\frac{1}{3}u^3 + \alpha u u_{xx} + \beta u u_{xxxx} - \frac{1}{2}\alpha u_x^2 + \frac{1}{2}\beta u_{xx}^2 - \beta u_x u_{xxx}\right) = 0, \tag{10}$$

thus,

$$\frac{d}{dt}E = 0, E = \int_{-\infty}^{\infty} u^2\,dx. \tag{11}$$

In addition, the other two conservation laws are

$$T^t = \frac{1}{2}u^2 t - ux,$$
$$T^x = \frac{1}{3}u^3 t - \frac{1}{2}u^2 x + \alpha u t u_{xx} + \beta u t u_{xxxx} - \frac{1}{2}\alpha t u_x^2 - \beta t u_x u_{xxx} \tag{12}$$
$$+ \frac{1}{2}\beta t u_{xx}^2 + \alpha u_x - \alpha x u_{xx} + \beta u_{xxx} - x\beta u_{xxxx},$$

and

$$T^t = \frac{1}{6\beta}u\left(3\alpha u_{xx} + 3\beta u_{xxxx} + u^2\right),$$
$$T^x = \frac{1}{8\beta}\left(4\alpha^2 u_{xx}^2 + 8\alpha\beta u_{xx}u_{xxxx} + 4\alpha u^2 u_{xx} + 4\beta^2 u_{xxxx}^2 + 4\beta u^2 u_{xxxx} + u^4 - 4\alpha u u_{tx}\right. \tag{13}$$
$$\left. + 4\alpha u_t u_x - 4\beta u u_{txxx} + 4\beta u_t u_{xxx} - 4\beta u_{tx}u_{xx} + 4\beta u_{txx}u_x\right).$$

2.3. Reciprocal Bäcklund Transformations to Conservation of Momentum and Energy

In order to get reciprocal Bäcklund transformations to conservation of Momentum and Energy, we consider the following results [30]

$$(T^t)'_{t'} + (T^x)'_{x'} = 0,$$
$$\frac{\partial}{\partial_{t'}} = \frac{F}{T}\frac{\partial}{\partial_x} + \frac{\partial}{\partial_t}, \frac{\partial}{\partial_{x'}} = \frac{1}{T}\frac{\partial}{\partial_x}. \tag{14}$$

From this transformation, it should be possible to obtain the following statement:

Corollary 1. *Reciprocal Bäcklund transformations to conservation of Momentum*

$$\begin{cases} (T^t)' = \frac{1}{u}, \\ (T^x)' = -\frac{\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)}{u}. \end{cases} \tag{15}$$

Proof. First, one has

$$(T^t)'_{t'} = \frac{\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)}{u}\frac{-u_x}{u^2} + \frac{-u_t}{u^2}$$
$$= \frac{-\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)u_x - u u_t}{u^3},$$
$$(T^x)'_{x'} = \frac{1}{u}\left(-\frac{\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)}{u}\right)_x = \tag{16}$$
$$\frac{-\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)_x u + u_x\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)}{u^3}.$$

Thus,

$$
\begin{aligned}
(T^t)'_{t'} + (T^x)'_{x'} &= \frac{-\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)u_x - uu_t}{u^3} \\
&+ \frac{-\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)_x u + u_x\left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)}{u^3} \\
&= \frac{-uu_t - \left(\frac{1}{2}u^2 + \alpha u_{xx} + \beta u_{xxxx}\right)_x u}{u^3} = 0.
\end{aligned}
\tag{17}
$$

In the same proof process, one can get

Corollary 2. *Reciprocal Bäcklund transformations to conservation of energy:*

$$
\begin{cases}
(T^t)' = \frac{2}{u^2}, \\
(T^x)' = -\frac{2\left(\frac{1}{3}u^3 + \alpha u u_{xx} + \beta u u_{xxxx} - \frac{1}{2}\alpha u_x^2 + \frac{1}{2}\beta u_{xx}^2 - \beta u_x u_{xxx}\right)}{u^2}.
\end{cases}
\tag{18}
$$

□

3. Symmetry Analysis and Travelling Wave Solutions for $\epsilon = 0$

3.1. Symmetry Analysis

While $\epsilon = 0$, one has

$$
u_t - iu + \alpha u_{xxx} + \beta u_{xxxxx} = 0,
\tag{19}
$$

using the transformation

$$
u(x,t) = e^{it}v(x,t),
\tag{20}
$$

one obtains the following linear partial differential equation (PDE):

$$
v_t + \alpha v_{xxx} + \beta v_{xxxxx} = 0,
\tag{21}
$$

as it is a linear equation, it contains an infinite number of conservation laws.

The corresponding vector field can be obtained as follows

$$
V_1 = v\frac{\partial}{\partial v}, \; V_2 = \frac{\partial}{\partial t}, \; V_3 = \frac{\partial}{\partial x}, \; V_4 = F\frac{\partial}{\partial v},
\tag{22}
$$

where F satisfy the following PDE:

$$
F_t + \alpha F_{xxx} + \beta F_{xxxxx} = 0.
\tag{23}
$$

3.2. Travelling Wave Solutions

For the travelling wave transformation $V_2 + \lambda V_3$, the invariant and invariant functions are

$$
\xi = x - \lambda t, \; v = v(\xi),
\tag{24}
$$

substituting Equation (24) into Equation (21), one has

$$
-\lambda v' + \alpha v''' + \beta v^{(5)} = 0,
\tag{25}
$$

solving this equation, one can get

$$
\begin{aligned}
v(\xi) =& c_1 e^{-\frac{1}{2}\frac{\sqrt{-2\beta\left(\alpha + \sqrt{\alpha^2 + 4\beta\lambda}\right)}\xi}{\beta}} + c_2 e^{\frac{1}{2}\frac{\sqrt{-2\beta\left(\alpha + \sqrt{\alpha^2 + 4\beta\lambda}\right)}\xi}{\beta}} \\
&+ c_3 e^{-\frac{1}{2}\frac{\sqrt{2}\sqrt{\beta\left(-\alpha + \sqrt{\alpha^2 + 4\beta\lambda}\right)}\xi}{\beta}} + c_4 e^{\frac{1}{2}\frac{\sqrt{2}\sqrt{\beta\left(-\alpha + \sqrt{\alpha^2 + 4\beta\lambda}\right)}\xi}{\beta}},
\end{aligned}
\tag{26}
$$

where c_1, c_2, c_3, c_4 are constants. Putting (26) into (20), one can get

$$u(x,t) = e^{it}\left(c_1 e^{-\frac{1}{2}\frac{\sqrt{-2\beta\left(\alpha+\sqrt{\alpha^2+4\beta\lambda}\right)}(x-\lambda t)}{\beta}} + c_2 e^{\frac{1}{2}\frac{\sqrt{-2\beta\left(\alpha+\sqrt{\alpha^2+4\beta\lambda}\right)}(x-\lambda t)}{\beta}} \right. $$

$$\left. + c_3 e^{-\frac{1}{2}\frac{\sqrt{2}\sqrt{\beta\left(-\alpha+\sqrt{\alpha^2+4\beta\lambda}\right)}(x-\lambda t)}{\beta}} + c_4 e^{\frac{1}{2}\frac{\sqrt{2}\sqrt{\beta\left(-\alpha+\sqrt{\alpha^2+4\beta\lambda}\right)}(x-\lambda t)}{\beta}} \right). \tag{27}$$

4. Symmetry Analysis for $\epsilon = 3$

When $\epsilon = 3$, from Equation (2), one should obtain the following PDE

$$u_t - u(u_x)^3 + \alpha u_{xxx} + \beta u_{xxxxx} = 0, \tag{28}$$

unfortunately, we cannot write this equation in the form of a conservation law. However, it is still possible to study this equation using the symmetry method.

If $\alpha \neq 0, \beta = 0$, this equation reduces approximately to Equation (10) in [1]

$$u_t - u(u_x)^3 + \alpha u_{xxx} = 0. \tag{29}$$

After tedious calculations, one can obtain

$$V_1 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t}, V_2 = \frac{\partial}{\partial t}, V_3 = \frac{\partial}{\partial x}. \tag{30}$$

When $\alpha = 0, \beta \neq 0$, the following vector fields are derived

$$V_1 = 3x\frac{\partial}{\partial x} + 15t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}, V_2 = \frac{\partial}{\partial t}, V_3 = \frac{\partial}{\partial x}. \tag{31}$$

While $\alpha \neq 0, \beta \neq 0$, one gets the vector fields are as follows:

$$V_1 = \frac{\partial}{\partial t}, V_2 = \frac{\partial}{\partial x}. \tag{32}$$

Symmetry Analysis for Time Fractional form of Equation (28)

For this case, one can have

$$u_t^\gamma - u(u_x)^3 + \alpha u_{xxx} + \beta u_{xxxxx} = 0, \tag{33}$$

where $0 < \gamma \leq 1$, it is clear that this equation is a new PDE. If $\gamma = 1$, in the discussion above, this equation has PT symmetry. To study the more general case, we again use Lie symmetry to study this equation. Generally speaking, because this is a fractional differential equation, due to the nature of fractional differential equations, if the Lie symmetry method is used to study it, it is slightly different from the ordinary Lie symmetry method.

Firstly, considering the following one parameter Lie group of point transformations [31,32]

$$\begin{aligned}
t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\
x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\
u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \\
\frac{\partial^\gamma \bar{u}}{\partial \bar{t}^\gamma} &= \frac{\partial^\gamma u}{\partial t^\gamma} + \epsilon\eta_\gamma^0(x,t,u) + O(\epsilon^2), \\
\frac{\partial^3 \bar{u}}{\partial \bar{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \epsilon\eta^{xxx}(x,t,u) + O(\epsilon^2), \\
\frac{\partial^5 \bar{u}}{\partial \bar{x}^5} &= \frac{\partial^5 u}{\partial x^5} + \epsilon\eta^{xxxxx}(x,t,u) + O(\epsilon^2),
\end{aligned} \tag{34}$$

where

$$
\begin{aligned}
\eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau), \\
\eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi), \\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi), \\
\eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt} D_x(\tau) - u_{xxxx} D_x(\xi), \\
\eta^{xxxxx} &= D_x(\eta^{xxxx}) - u_{xxxxt} D_x(\tau) - u_{xxxxx} D_x(\xi),
\end{aligned}
\tag{35}
$$

where D_x is given by the following results

$$
D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xxx} \frac{\partial}{\partial u_{xx}} + u_{xxxx} \frac{\partial}{\partial u_{xxx}} + u_{xxxxx} \frac{\partial}{\partial u_{xxxx}} + \cdots ,
\tag{36}
$$

the infinitesimal generator is given by

$$
V = \tau(x,t,u) \frac{\partial}{\partial t} + \xi(x,t,u) \frac{\partial}{\partial x} + \eta(x,t,u) \frac{\partial}{\partial u}.
\tag{37}
$$

From [31,32], one has

$$
\eta_\gamma^0 = D_t^\gamma(\eta) - \gamma D_t(\tau) \frac{\partial^\gamma u}{\partial t^\gamma} - \sum_{n=1}^{\infty} \binom{\gamma}{n} D_t^n(\xi) D_t^{\gamma-n}(u_x) - \sum_{n=1}^{\infty} \binom{\gamma}{n+1} D_t^{n+1}(\tau) D_t^{\gamma-n}(u),
\tag{38}
$$

and

$$
D_t^\gamma(\eta) = \frac{\partial^\gamma \eta}{\partial t^\gamma} + \eta_u \frac{\partial^\gamma u}{\partial t^\gamma} - u \frac{\partial^\gamma \eta_u}{\partial t^\gamma} + \sum_{n=1}^{\infty} \binom{\gamma}{n} \frac{\partial^n \eta_u}{\partial t^n} D_t^{\gamma-n}(u) + \mu,
\tag{39}
$$

where

$$
\mu = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{\gamma}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\gamma}}{\Gamma(n+1-\gamma)} [-u]^r \frac{\partial^m}{\partial t^m} [u^{k-r}] \frac{\partial^{n-m+k} \eta}{\partial t^{n-m} \partial u^k},
\tag{40}
$$

and

$$
\begin{aligned}
\eta_\gamma^0 &= \frac{\partial^\gamma \eta}{\partial t^\gamma} + (\eta_u - \gamma D_t(\tau)) \frac{\partial^\gamma u}{\partial t^\gamma} - u \frac{\partial^\gamma \eta_u}{\partial t^\gamma} + \mu \\
&\quad + \sum_{n=1}^{\infty} \left[\binom{\gamma}{n} \frac{\partial^\gamma \eta_u}{\partial t^\gamma} - \binom{\gamma}{n+1} D_t^{n+1}(\tau) \right] D_t^{\gamma-n}(u) \\
&\quad - \sum_{n=1}^{\infty} \binom{\gamma}{n} D_t^n(\xi) D_t^{\gamma-n}(u_x).
\end{aligned}
\tag{41}
$$

From the above analysis, it can be seen that, if $\alpha \neq 0, \beta \neq 0$, for this general case, the vector fields are shown by:

$$
V_1 = \frac{\partial}{\partial x}.
\tag{42}
$$

When $\alpha \neq 0, \beta = 0$, one can obtain the following equation

$$
u_t^\gamma - u(u_x)^3 + \alpha u_{xxx} = 0,
\tag{43}
$$

if $\gamma = 1$, it is can be found in paper [1]. Based on the above analysis, one can get

$$
V_1 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}.
\tag{44}
$$

While $\alpha = 0, \beta \neq 0$, one has

$$
u_t^\gamma - u(u_x)^3 + \beta u_{xxxxx} = 0,
\tag{45}
$$

vector fields are presented as follows

$$V_1 = 3x\gamma \frac{\partial}{\partial x} + 15t \frac{\partial}{\partial t} - 2\gamma u \frac{\partial}{\partial u}, V_2 = \frac{\partial}{\partial x}. \tag{46}$$

Now for the operator V_1, one has the corresponding characteristic equations as follows

$$\frac{dx}{3x} = \frac{\gamma dt}{15t} = \frac{-du}{2u}, \tag{47}$$

solving this equation generates the following similarity variable and functions

$$\xi = xt^{\frac{-\gamma}{5}}, \ u = t^{\frac{-2\gamma}{15}} f(\xi), \tag{48}$$

using the Erdelyi-Kober fractional differential operator $P_\beta^{\tau,\alpha}$ of order [31,32]

$$(P_\beta^{\tau,\alpha} g) := \prod_{j=0}^{n-1} \left(\tau + j - \frac{1}{\beta} \xi \frac{d}{d\xi} \right) (K_\beta^{\tau+\alpha,n-\alpha} g)(\xi), \tag{49}$$

$$n = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases} \tag{50}$$

and the Erd*é*lyi-Kober fractional integral operator [31,32]

$$(K_\beta^{\tau,\alpha} g)(\xi) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^\infty (u-1)^{\alpha-1} u^{-(\tau+\alpha)} g(\xi u^{\frac{1}{\beta}}) du, & \alpha > 0, \\ g(\xi), & \alpha = 0 \end{cases} \tag{51}$$

one can reduce Equation (2) into an ordinary differential equation of fractional order as follows

$$\left(P_{\frac{5}{\gamma}}^{1-\frac{2\gamma}{15}-\gamma,\gamma} f \right)(\xi) = uu_\xi^3 - \beta f_{\xi\xi\xi\xi\xi}. \tag{52}$$

5. Symmetry Analysis and Reductions for $\epsilon = 2n + 1$

5.1. Symmetry Analysis

When $\epsilon = 2n + 1$ is an odd integer, for this case, one has

$$u_t + (-1)^n u(u_x)^{2n+1} + \alpha u_{xxx} + \beta u_{xxxxx} = 0, \tag{53}$$

for the general case, one can derive the following vector fields

$$V_1 = \frac{\partial}{\partial t}, V_2 = \frac{\partial}{\partial x}. \tag{54}$$

5.2. Reductions

Case 1: V_2

For this case, invariant and invariant functions are

$$\xi = x, u = u(\xi), \tag{55}$$

substituting Equation (55) into Equation (53), one can get

$$(-1)^n u(u_\xi)^{2n+1} + \alpha u_{\xi\xi\xi} + \beta u_{\xi\xi\xi\xi\xi} = 0. \tag{56}$$

Case 2: V_1

In this case, one has invariant and invariant functions

$$\tau = t, u = u(\tau), \tag{57}$$

putting Equation (57) into Equation (53), one obtains

$$u_\tau = 0, \tag{58}$$

from Equation (58) only a trivial solution can be obtained.

Case 3: $V_2 + \lambda V_3$

It is clear that this is travelling wave transformation, one can get the invariant and invariant functions are

$$\xi = x - \lambda t, u = u(\xi), \tag{59}$$

substituting Equation (59) into Equation (53), one has

$$-\lambda u_\xi + (-1)^n u \left(u_\xi \right)^{2n+1} + \alpha u_{\xi\xi\xi} + \beta u_{\xi\xi\xi\xi\xi} = 0. \tag{60}$$

6. Conclusions

In this paper, symmetries and PT-symmetric extension of the fifth-order KdV-like equation are considered. Taking different values for ϵ, several different equations with PT symmetry properties are obtained. And using the symmetry method, the symmetries of these equations are obtained. In particular, for ϵ equal to 1, this equation was systematically studied and its symmetry as well as conservation laws are obtained. It should be emphasized that the reciprocal Bäcklund transformations of conservation laws of momentum and energy are derived. For the special case of $\epsilon = 3$, the corresponding integer order and fractional order symmetry are discussed, and for the time fractional order form, the equation is simplified into a fractional order ordinary differential equation on the basis of symmetry. Finally, the general case is considered, for which two symmetries are obtained.

In conclusion, this paper has shown the following two results, the first one is to preserve the PT symmetry, and the second one is how to extend symmetry analysis to fifth-order KdV-like equations. However for other cases such as variable coefficients, they will be investigated in future work.

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