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Fractional Sturm–Liouville Eigenvalue Problems, II

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Abstract: We continue the study of a non-self-adjoint fractional three-term Sturm–Liouville boundary value problem (with a potential term) formed by the composition of a left Caputo and left Riemann–Liouville fractional integral under *Dirichlet type* boundary conditions. We study the existence and asymptotic behavior of the real eigenvalues and show that for certain values of the fractional differentiation parameter α , $0 < \alpha < 1$, there is a finite set of real eigenvalues and that, for α near $1/2$, there may be none at all. As $\alpha \rightarrow 1^-$ we show that their number becomes infinite and that the problem then approaches a standard Dirichlet Sturm–Liouville problem with the composition of the operators becoming the operator of second order differentiation.

Keywords: Fractional Sturm–Liouville; fractional calculus; Laplace transform; Mittag-Leffler function; eigenvalues; asymptotics

1. Introduction

This is a continuation of [1] where the results therein are extended to three-term Fractional Sturm–Liouville operators (with a potential term) formed by the composition of a left Caputo and left-Riemann–Liouville fractional integral. Similar kinds of spectral problems have been considered in [2–11]. Specifically, the boundary value problem considered here is of the form,

$$-{}^c D_{0+}^\alpha \circ D_{0+}^\alpha y(t) + q(t)y(t) = \lambda y(t), \quad 1/2 < \alpha < 1, \quad 0 \leq t \leq 1, \quad (1)$$

with boundary conditions

$$\mathcal{I}_{0+}^{1-\alpha} y(t)|_{t=0} = c_1, \quad \text{and} \quad \mathcal{I}_{0+}^{1-\alpha} y(t)|_{t=1} = c_2, \quad (2)$$

where c_1, c_2 are real constants and the real valued unspecified potential function, $q \in L^\infty[0, 1]$. We note that these are not self-adjoint problems and so there may be a non-real spectrum, in general. A well-known property of the Riemann–Liouville integral gives that if the solutions are continuous on $[0, 1]$ then the boundary conditions (2) reduce to the usual fixed-end boundary conditions, $y(0) = y(1) = 0$, as $\alpha \rightarrow 1$.

For the analogue of the Dirichlet problem described above we study the existence and asymptotic behaviour of the real eigenvalues and show that for each α , $0 < \alpha < 1$, there is a finite set of real eigenvalues and that, for α near $1/2$, there may be none at all. As $\alpha \rightarrow 1^-$ we show that their number becomes infinite and that the problem then approaches a standard Dirichlet Sturm–Liouville problem with the composition of the operators becoming the operator of second order differentiation acting on a suitable function space.

Our approach is different from most in this area. Specifically, we start with the existence and uniqueness of solutions of the Equation (1) along with the initial conditions (2), then we formulate the boundary value problem as an integral equation, after which we show that the solution of this integral equation as a function of λ is an entire function of λ of order of at most $1/2\alpha$. Since α is between $1/2$ and 1 , this entire function is of fractional order and therefore must have an infinite number of zeros, some of which may be complex.



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These zeros are the eigenvalues of our problem and therefore we get their existence. Using asymptotic methods and hard analysis, we proved our bounds on the real eigenvalues of which there must be a finite number for each α . Finally, we show that as α tends to 1, the number of real eigenvalues becomes infinite and the original problem approaches the standard Sturm–Liouville eigenvalue problem.

2. Preliminaries

We recall some definitions from fractional calculus and refer the reader to our previous paper [1] for further details.

Definition 1. The left and the right Riemann–Liouville fractional integrals $\mathcal{I}_{a^+}^\alpha$ and $\mathcal{I}_{b^-}^\alpha$ of order $\alpha \in \mathbb{R}^+$ are defined by

$$\mathcal{I}_{a^+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t \in (a, b], \tag{3}$$

and

$$\mathcal{I}_{b^-}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad t \in [a, b), \tag{4}$$

respectively. Here $\Gamma(\alpha)$ denotes Euler’s Gamma function. The following property is easily verified.

Property 1. For a constant C , we have $\mathcal{I}_{a^+}^\alpha C = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \cdot C$.

The proof is by direct calculation.

Definition 2. The left and the right Caputo fractional derivatives ${}^c\mathcal{D}_{a^+}^\alpha$ and ${}^c\mathcal{D}_{b^-}^\alpha$ are defined by

$${}^c\mathcal{D}_{a^+}^\alpha f(t) := \mathcal{I}_{a^+}^{n-\alpha} \circ \mathcal{D}^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \quad t > a, \tag{5}$$

and

$${}^c\mathcal{D}_{b^-}^\alpha f(t) := (-1)^n \mathcal{I}_{b^-}^{n-\alpha} \circ \mathcal{D}^n f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(s)}{(s-t)^{\alpha-n+1}} ds, \quad t < b, \tag{6}$$

respectively, where f is sufficiently differentiable and $n - 1 \leq \alpha < n$.

Definition 3. Similarly, the left and the right Riemann–Liouville fractional derivatives $\mathcal{D}_{a^+}^\alpha$ and $\mathcal{D}_{b^-}^\alpha$ are defined by

$$\mathcal{D}_{a^+}^\alpha f(t) := \mathcal{D}^n \circ \mathcal{I}_{a^+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \quad t > a, \tag{7}$$

and

$$\mathcal{D}_{b^-}^\alpha f(t) := (-1)^n \mathcal{D}^n \circ \mathcal{I}_{b^-}^{n-\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b \frac{f(s)}{(s-t)^{\alpha-n+1}} ds, \quad t < b, \tag{8}$$

respectively, where f is sufficiently differentiable and $n - 1 \leq \alpha < n$.

Property 2. For $\Re(\nu) > -1, 0 < \alpha < 1$, and $t > 0$, we have

$$D_{0^+}^\alpha (t^\nu) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} t^{\nu-\alpha}$$

Property 3. For $\Re(\nu) > 0$, $0 < \alpha < 1$, and $t > 0$, we have

$${}^c D_{0^+}^\alpha (t^\nu) = \frac{\Gamma(1 + \nu)}{\Gamma(1 + \nu - \alpha)} t^{\nu - \alpha}$$

Property 4. If $y(t) \in L^1(a, b)$ and $\mathcal{I}_{a^+}^{1-\alpha} y, \mathcal{I}_{b^-}^{1-\alpha} y \in AC[a, b]$, then

$$\begin{aligned} \mathcal{I}_{a^+}^\alpha \mathcal{D}_{a^+}^\alpha y(t) &= y(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \mathcal{I}_{a^+}^{1-\alpha} y(a), \\ \mathcal{I}_{b^-}^\alpha \mathcal{D}_{b^-}^\alpha y(t) &= y(t) - \frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)} \mathcal{I}_{b^-}^{1-\alpha} y(b). \end{aligned}$$

Property 5. If $y(t) \in AC[a, b]$ and $0 < \alpha \leq 1$, then

$$\begin{aligned} \mathcal{I}_{a^+}^\alpha {}^c \mathcal{D}_{a^+}^\alpha y(t) &= y(t) - y(a), \\ \mathcal{I}_{b^-}^\alpha {}^c \mathcal{D}_{b^-}^\alpha y(t) &= y(t) - y(b). \end{aligned}$$

Property 6. For $0 < \alpha < 1$ we have

$$\mathcal{D}_{a^+}^\alpha f(t) = \frac{f(a)}{\Gamma(1-\alpha)} (t-a)^{-\alpha} + {}^c \mathcal{D}_{a^+}^\alpha f(t).$$

The Mittag-Leffler Function

The function $E_\delta(z)$ defined by

$$E_\delta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k + 1)}, \quad (z \in \mathbb{C}, \Re(\delta) > 0), \quad (9)$$

was introduced by Mittag-Leffler [12]. In particular, when $\delta = 1$ and $\delta = 2$, we have

$$E_1(z) = e^z, \quad E_2(z) = \cosh(\sqrt{z}). \quad (10)$$

The generalized Mittag-Leffler function $E_{\delta,\theta}(z)$ is defined by

$$E_{\delta,\theta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k + \theta)}, \quad (11)$$

where $z, \theta \in \mathbb{C}$ and $\Re(\delta) > 0$. When $\theta = 1$, $E_{\delta,\theta}(z)$ coincides with the Mittag-Leffler function (9):

$$E_{\delta,1}(z) = E_\delta(z). \quad (12)$$

Two other particular cases of (11) are as follows:

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}. \quad (13)$$

Property 7. For any δ with $\Re(\delta) > 0$ and for any $z \neq 0$ we have

$$E_{\delta,\delta}(z) = \frac{1}{z} E_{\delta,0}(z)$$

Further properties of this special function may be found in [13].

Property 8 (See [14], p. 43). If $0 < \delta < 2$ and $\mu \in (\frac{\delta\pi}{2}, \min(\pi, \delta\pi))$, then the function $E_{\delta,\theta}(z)$ has the following exponential expansion as $|z| \rightarrow \infty$

$$E_{\delta,\theta}(z) = \begin{cases} \frac{1}{\delta} z^{\frac{1-\theta}{\delta}} \exp(z^{\frac{1}{\delta}}) - \sum_{k=1}^N \frac{1}{\Gamma(\theta-\delta k)} \frac{1}{z^k} + O(\frac{1}{z^{N+1}}), & |\arg(z)| \leq \mu, \\ - \sum_{k=1}^N \frac{1}{\Gamma(\theta-\delta k)} \frac{1}{z^k} + O(\frac{1}{z^{N+1}}), & \mu \leq |\arg(z)| \leq \pi. \end{cases} \tag{14}$$

3. Existence and Uniqueness of the Solution of SLPs

In this section we convert (1) and (2) to an integral equation and prove that it has a solution that satisfies the relevant equations and initial conditions. First, we proceed formally. Separating terms in (1), we get

$${}^c\mathcal{D}_{0^+}^\alpha \circ \mathcal{D}_{0^+}^\alpha y(t) = (q(t) - \lambda)y(t), \quad 1/2 < \alpha < 1, \quad 0 \leq t \leq 1.$$

Taking the left Riemann–Liouville fractional integrals $\mathcal{I}_{a^+}^\alpha$ on both sides of the above equation and using Property 5, we have

$$\mathcal{D}_{0^+}^\alpha y(t) - \mathcal{D}_{0^+}^\alpha y(t)|_{t=0} = \mathcal{I}_{0^+}^\alpha ((q(t) - \lambda)y(t)).$$

Taking the left Riemann–Liouville fractional integrals $\mathcal{I}_{a^+}^\alpha$ from both sides of the above equation once again and using Property 4, we get

$$y(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \mathcal{I}_{0^+}^{1-\alpha} y(t)|_{t=0} - \mathcal{I}_{0^+}^\alpha (\mathcal{D}_{0^+}^\alpha y(t)|_{t=0}) = \mathcal{I}_{0^+}^\alpha (\mathcal{I}_{0^+}^\alpha ((q(t) - \lambda)y(t)))$$

Using Property 1, we can write

$$y(t) = c_1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + c_2 \frac{t^\alpha}{\Gamma(\alpha+1)} + \mathcal{I}_{0^+}^\alpha (\mathcal{I}_{0^+}^\alpha ((q(t) - \lambda)y(t)))$$

in which

$$c_1 = \mathcal{I}_{0^+}^{1-\alpha} y(t)|_{t=0}, \quad c_2 = \mathcal{D}_{0^+}^\alpha y(t)|_{t=0}.$$

We obtain, through the double fractional integral in the above equation, the following:

$$y(t) = c_1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + c_2 \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma^2(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s \frac{(q(r) - \lambda)y(r)}{(s-r)^{1-\alpha}} dr \right) ds.$$

By changing the order of integrals in the above equation we get

$$y(t) = c_1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + c_2 \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma^2(\alpha)} \int_0^t (q(r) - \lambda)y(r) \left(\int_r^t (t-s)^{\alpha-1} (s-r)^{\alpha-1} ds \right) dr$$

Solving the inner integral gives us

$$y(t, \lambda) = c_1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + c_2 \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(2\alpha)} \int_0^t (q(s) - \lambda)y(s, \lambda) (t-s)^{2\alpha-1} ds. \tag{15}$$

We will now show that (15) has a solution that exists in a neighbourhood of $t = 0$ and is unique there. Working backwards will then provide us with a unique solution to (1) and (2). Although this result already appears in [15], we give a shorter proof part of which will be required later.

To this end, let $t > 0$. Define

$$y_n(t, \lambda) = y_0(t, \lambda) + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} (q(s) - \lambda)y_{n-1}(s, \lambda) ds, \tag{16}$$

where

$$y_0(t, \lambda) = c_1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + c_2 \frac{t^\alpha}{\Gamma(\alpha + 1)}. \tag{17}$$

Let $\lambda \in \mathbf{C}$, $|\lambda| < \Lambda$, where $\Lambda > 0$ is arbitrary but fixed. Then,

$$\begin{aligned} |y_1(t, \lambda) - y_0(t, \lambda)| &\leq \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |q(s) - \lambda| |y_0(s, \lambda)| ds \\ &\leq \frac{\|q\|_\infty + \Lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |y_0(s, \lambda)| ds, \end{aligned} \tag{18}$$

in which $\|q\|_\infty = \sup_{t \in [0,1]} |q(t)|$. Substituting (17) in (18) and using the fact that,

$$\int_a^t (t-s)^{\alpha-1} (s-a)^{\beta-1} ds = \frac{(t-a)^{\alpha+\beta-1} \Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)},$$

we have

$$|y_1(t, \lambda) - y_0(t, \lambda)| \leq (\|q\|_\infty + \Lambda) \left(\frac{c_1}{\Gamma(3\alpha)} t^{3\alpha-1} + c_2 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \right),$$

Now, for $n = 2$ in (16) we get

$$\begin{aligned} |y_2(t, \lambda) - y_1(t, \lambda)| &\leq \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |q(s) - \lambda| |y_1(s, \lambda) - y_0(s, \lambda)| ds \\ &\leq \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |q(s) - \lambda| \left((\|q\|_\infty + \Lambda) \left(\frac{c_1}{\Gamma(3\alpha)} s^{3\alpha-1} + c_2 \frac{s^{3\alpha}}{\Gamma(3\alpha + 1)} \right) \right) ds \\ &\leq (\|q\|_\infty + \Lambda)^2 \left(\frac{c_1}{\Gamma(5\alpha)} t^{5\alpha-1} + \frac{c_2}{\Gamma(5\alpha + 1)} t^{5\alpha} \right). \end{aligned} \tag{19}$$

Continuing in this way we get that the series

$$y(t, \lambda) := y_0(t, \lambda) + \sum_{n=1}^{\infty} (y_n(t, \lambda) - y_{n-1}(t, \lambda)) \tag{20}$$

where

$$\sum_{n=1}^{\infty} |y_n(t, \lambda) - y_{n-1}(t, \lambda)| \leq c_1 t^{-1} \sum_{n=1}^{\infty} \frac{(\|q\|_\infty + \Lambda)^n}{\Gamma(2n\alpha + \alpha)} t^{2n\alpha + \alpha} + c_2 \sum_{n=1}^{\infty} \frac{(\|q\|_\infty + \Lambda)^n}{\Gamma(2n\alpha + \alpha + 1)} t^{2n\alpha + \alpha} \tag{21}$$

converges uniformly on compact subsets of $(0, 1]$. Denote the sum of the infinite series in (20) by $y(t, \lambda)$. So, by virtue of (17) and (21), (20) gives us,

$$\begin{aligned} |y(t, \lambda)| &\leq |y_0(t, \lambda)| + \sum_{n=1}^{\infty} |y_n(t, \lambda) - y_{n-1}(t, \lambda)| \\ &\leq \frac{c_1}{\Gamma(\alpha)} t^{\alpha-1} + \frac{c_2}{\Gamma(\alpha + 1)} t^\alpha + c_1 t^{-1} \sum_{n=1}^{\infty} \frac{(\|q\|_\infty + \Lambda)^n}{\Gamma(2n\alpha + \alpha)} t^{2n\alpha + \alpha} + c_2 \sum_{n=1}^{\infty} \frac{(\|q\|_\infty + \Lambda)^n}{\Gamma(2n\alpha + \alpha + 1)} t^{2n\alpha + \alpha} \\ &= c_1 t^{-1} \sum_{n=0}^{\infty} \frac{(\|q\|_\infty + \Lambda)^n}{\Gamma(2n\alpha + \alpha)} t^{2n\alpha + \alpha} + c_2 \sum_{n=0}^{\infty} \frac{(\|q\|_\infty + \Lambda)^n}{\Gamma(2n\alpha + \alpha + 1)} t^{2n\alpha + \alpha} \\ &= c_1 t^{\alpha-1} E_{2\alpha, \alpha}((\|q\|_\infty + \Lambda) t^{2\alpha}) + c_2 t^\alpha E_{2\alpha, \alpha+1}((\|q\|_\infty + \Lambda) t^{2\alpha}). \end{aligned}$$

Note that for a solution $y(t, \lambda)$ of (15) to be $C([0, 1])$, it is necessary and sufficient that $c_1 = 0$, i.e., $\mathcal{I}_{0+}^{1-\alpha} y(t)|_{t=0} = 0$. This then proves the global existence of a solution of (15) on $[\delta, 1]$, $\delta > 0$, since $q \in L^\infty[0, 1]$ for given c_1 and c_2 , as defined in (2).

From the proof comes the following a priori estimate when $c_1 = 0$, that is,

$$|y(t, \lambda)| \leq c_2 t^\alpha \left(\frac{1}{\Gamma(\alpha + 1)} + \left| E_{2\alpha, \alpha+1}((\|q\|_\infty + \Lambda)t^{2\alpha}) \right| \right) \leq c_2 \left(\frac{1}{\Gamma(\alpha + 1)} + \left| E_{2\alpha, \alpha+1}((\|q\|_\infty + \Lambda)t^{2\alpha}) \right| \right)$$

valid for each $t \in [0, 1]$ and all $|\lambda| < \Lambda$.

The previous bound can be made into an absolute constant by taking the sup over all t and $|\lambda| < \Lambda$. Of course, the bound goes to infinity as $|\lambda| \rightarrow \infty$ over non-real values, as it must. Thus,

$$|y(t, \lambda)| \leq c_2 \left(\frac{1}{\Gamma(\alpha + 1)} + \sup_{|\lambda| < \Lambda, t \in [0, 1]} \left| E_{2\alpha, \alpha+1}((\|q\|_\infty + \Lambda)t^{2\alpha}) \right| \right) = c_2 \left(\frac{1}{\Gamma(\alpha + 1)} + |E_{2\alpha, \alpha+1}((\|q\|_\infty + \Lambda))| \right) := c_3. \tag{22}$$

for all $|\lambda| < \Lambda, t \in [0, 1]$. Uniqueness follows easily by means of Gronwall’s inequality, as usual. Let $\varepsilon > 0$. Assume that (15) has two solutions $y(t, \lambda), z(t, \lambda)$. Since $q \in L^\infty[0, 1]$ and $|\lambda| < \Lambda$ we can derive that,

$$|y(t, \lambda) - z(t, \lambda)| \leq \varepsilon e^{\frac{1}{\Gamma(2\alpha)}(\|q\|_\infty + \Lambda)t^{\frac{2\alpha}{2\alpha}}}.$$

and since $t \in [0, 1]$, we get

$$|y(t, \lambda) - z(t, \lambda)| \leq O(\varepsilon)$$

where the O -term can be made independent of both t, λ . Letting $\varepsilon \rightarrow 0$ yields uniqueness for $t \in [0, 1]$ and $|\lambda| < \Lambda$.

4. Another Integral Equation

In the previous section we showed that (15) has a solution that, for each $\lambda \in \mathbf{C}$, exists on $[0, 1]$, is unique, and is continuous there if and only if $c_1 = 0$. On the other hand, if $c_1 \neq 0$ then the solution is merely continuous on all compact subsets of $(0, 1]$. In this section we find another expression for the integral equation which is equivalent to both (15) and the problem (1) with boundary conditions (2).

Lemma 1. For $0 < \alpha < 1$ and $0 < t < 1$, we have

$$-{}^c\mathcal{D}_{0+}^\alpha \mathcal{D}_{0+}^\alpha \left(t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha}) \right) = \lambda t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha})$$

Proof. Using properties of the Mittag-Leffler function we can write

$$\begin{aligned} \mathcal{D}_{0+}^\alpha \left(t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha}) \right) &= \mathcal{D}_{0+}^\alpha \left(\sum_{k=0}^\infty \frac{(-\lambda)^k t^{2\alpha k + \alpha - 1}}{\Gamma(2\alpha k + \alpha)} \right) \\ &= \sum_{k=0}^\infty \frac{(-\lambda)^k \mathcal{D}_{0+}^\alpha \left(t^{2\alpha k + \alpha - 1} \right)}{\Gamma(2\alpha k + \alpha)} \\ &= \sum_{k=0}^\infty \frac{(-\lambda)^k t^{2\alpha k - 1}}{\Gamma(2\alpha k)} \\ &= t^{-1} E_{2\alpha, 0}(\lambda t^{2\alpha}) \\ &= -\lambda t^{2\alpha-1} E_{2\alpha, 2\alpha}(-\lambda t^{2\alpha}), \end{aligned} \tag{23}$$

in which the third and the last equalities come from Property 2 and Property 7, respectively. Now, taking the left Caputo fractional derivative of both sides of (23) we get

$$\begin{aligned}
 -{}^c\mathcal{D}_{0+}^\alpha \mathcal{D}_{0+}^\alpha \left(t^{\alpha-1} E_{2\alpha,\alpha}(-\lambda t^{2\alpha}) \right) &= {}^c\mathcal{D}_{0+}^\alpha \left(\lambda t^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda t^{2\alpha}) \right) \\
 &= \lambda {}^c\mathcal{D}_{0+}^\alpha \left(\sum_{k=0}^\infty \frac{(-\lambda)^k t^{2\alpha k+2\alpha-1}}{\Gamma(2\alpha k+2\alpha)} \right) \\
 &= \lambda \left(\sum_{k=0}^\infty \frac{(-\lambda)^k {}^c\mathcal{D}_{0+}^\alpha \left(t^{2\alpha k+2\alpha-1} \right)}{\Gamma(2\alpha k+2\alpha)} \right) \\
 &= \lambda t^{\alpha-1} \sum_{k=0}^\infty \frac{(-\lambda)^k t^{2\alpha k}}{\Gamma(2\alpha k+\alpha)} \\
 &= \lambda t^{\alpha-1} E_{2\alpha,\alpha}(-\lambda t^{2\alpha})
 \end{aligned}$$

as required. \square

Lemma 2. For $0 < \alpha < 1$ and $0 < t < 1$, we have

$$-{}^c\mathcal{D}_{0+}^\alpha \mathcal{D}_{0+}^\alpha \left(t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) \right) = \lambda t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha})$$

Proof. Once again, using the properties of the Mittag-Leffler function we can write

$$\begin{aligned}
 \mathcal{D}_{0+}^\alpha \left(t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) \right) &= \mathcal{D}_{0+}^\alpha \left(\sum_{k=0}^\infty \frac{(-\lambda)^k t^{2\alpha k+\alpha}}{\Gamma(2\alpha k+\alpha+1)} \right) \\
 &= \sum_{k=0}^\infty \frac{(-\lambda)^k \mathcal{D}_{0+}^\alpha \left(t^{2\alpha k+\alpha} \right)}{\Gamma(2\alpha k+\alpha+1)} \tag{24} \\
 &= \sum_{k=0}^\infty \frac{(-\lambda)^k t^{2\alpha k}}{\Gamma(2\alpha k+1)} \\
 &= E_{2\alpha,1}(-\lambda t^{2\alpha}).
 \end{aligned}$$

in which the third equality comes from Property 3. Now, taking the left Caputo fractional derivative of both sides of (24) we get

$$\begin{aligned}
 {}^c\mathcal{D}_{0+}^\alpha \mathcal{D}_{0+}^\alpha \left(t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) \right) &= {}^c\mathcal{D}_{0+}^\alpha \left(E_{2\alpha,1}(\lambda t^{2\alpha}) \right) \\
 &= {}^c\mathcal{D}_{0+}^\alpha \left(\sum_{k=0}^\infty \frac{(-\lambda)^k t^{2\alpha k}}{\Gamma(2\alpha k+1)} \right) \\
 &= \sum_{k=0}^\infty \frac{(-\lambda)^k {}^c\mathcal{D}_{0+}^\alpha \left(t^{2\alpha k} \right)}{\Gamma(2\alpha k+1)} \\
 &= \sum_{k=1}^\infty \frac{(-\lambda)^k t^{\alpha(2k-1)}}{\Gamma(1+\alpha(2k-1))} \\
 &= \sum_{k=0}^\infty \frac{(-\lambda)^{k+1} t^{\alpha(2k+1)}}{\Gamma(1+\alpha(2k+1))} \\
 &= -\lambda t^\alpha \sum_{k=0}^\infty \frac{(-\lambda)^k t^{2\alpha k}}{\Gamma(2\alpha k+\alpha+1)} \\
 &= -\lambda t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha})
 \end{aligned}$$

as desired. \square

Lemma 3. For $0 < \alpha < 1$ and $0 < t < 1$, we have

$${}^c\mathcal{D}_{0+}^\alpha \mathcal{D}_{0+}^\alpha \left(\int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda(t-s)^{2\alpha}) q(s)y(s) ds \right) = -q(t)y(t) + \lambda \int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda(t-s)^{2\alpha}) q(s)y(s) ds$$

Proof. Let $c_4 = 1/\Gamma(1-\alpha)$. Observe that,

$$\begin{aligned} \mathcal{I}_{0+}^{1-\alpha} \int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda(t-s)^{2\alpha}) q(s)y(s) ds &= c_4 \int_0^t \frac{\int_0^r (r-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda(r-s)^{2\alpha}) q(s)y(s) ds}{(t-r)^\alpha} dr \\ &= c_4 \int_0^t q(s)y(s) \left(\int_s^t \frac{(r-s)^{2\alpha-1}}{(t-r)^\alpha} E_{2\alpha,2\alpha}(-\lambda(r-s)^{2\alpha}) dr \right) ds \\ &= c_4 \int_0^t q(s)y(s) \left(\sum_{k=0}^\infty \frac{(-\lambda)^k}{\Gamma(2\alpha k + 2\alpha)} \int_s^t \frac{(r-s)^{2\alpha-1+2\alpha k}}{(t-r)^\alpha} dr \right) ds \quad (25) \\ &= \int_0^t q(s)y(s) \left(\sum_{k=0}^\infty \frac{(-\lambda)^k (t-s)^{2\alpha k + \alpha}}{\Gamma(2\alpha k + \alpha + 1)} \right) ds \\ &= \int_0^t q(s)y(s) (t-s)^\alpha E_{2\alpha,\alpha+1}(-\lambda(t-s)^{2\alpha}) ds. \end{aligned}$$

Next, differentiating both sides of (25) with respect to t and noting that $\mathcal{D}_{0+}^\alpha = D(\mathcal{I}_{0+}^{1-\alpha})$ we find,

$$\mathcal{D}_{0+}^\alpha \left(\int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda(t-s)^{2\alpha}) q(s)y(s) ds \right) = \int_0^t (t-s)^{\alpha-1} E_{2\alpha,\alpha}(-\lambda(t-s)^{2\alpha}) q(s)y(s) ds. \quad (26)$$

as $\frac{d}{dt}(t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha})) = t^{\alpha-1} E_{2\alpha,\alpha}(-\lambda t^{2\alpha})$. Next, we are going to take the left Caputo fractional derivative of both sides of (26). However, since the right hand side of (26) as a function of t is zero at $t = 0$, we can use Property 6 and replace the Caputo fractional derivative ${}^c\mathcal{D}_{0+}^\alpha$ by the Riemann–Liouville one \mathcal{D}_{0+}^α . In order to do so, first we need to apply $\mathcal{I}_{0+}^{1-\alpha}$ followed by the classical derivative of the right hand side of (26) as follows,

$$\begin{aligned} \mathcal{I}_{0+}^{1-\alpha} \int_0^t (t-s)^{\alpha-1} E_{2\alpha,\alpha}(-\lambda(t-s)^{2\alpha}) q(s)y(s) ds &= c_4 \int_0^t \frac{\int_0^r (r-s)^{\alpha-1} E_{2\alpha,\alpha}(-\lambda(r-s)^{2\alpha}) q(s)y(s) ds}{(t-r)^\alpha} dr \\ &= c_4 \int_0^t q(s)y(s) \left(\int_s^t \frac{(r-s)^{\alpha-1}}{(t-r)^\alpha} E_{2\alpha,\alpha}(-\lambda(r-s)^{2\alpha}) dr \right) ds \\ &= c_4 \int_0^t q(s)y(s) \left(\sum_{k=0}^\infty \frac{(-\lambda)^k}{\Gamma(2\alpha k + \alpha)} \int_s^t \frac{(r-s)^{\alpha-1+2\alpha k}}{(t-r)^\alpha} dr \right) ds \\ &= \int_0^t q(s)y(s) \left(\sum_{k=0}^\infty \frac{(-\lambda)^k (t-s)^{2\alpha k}}{\Gamma(2\alpha k + 1)} \right) ds \\ &= \int_0^t q(s)y(s) E_{2\alpha,1}(-\lambda(t-s)^{2\alpha}) ds. \end{aligned}$$

Taking the derivative of the previous equation and using the fact stated in the previous paragraph, we get

$$\begin{aligned} {}^c\mathcal{D}_{0+}^\alpha \int_0^t (t-s)^{\alpha-1} E_{2\alpha,\alpha}(-\lambda(t-s)^{2\alpha}) q(s)y(s) ds &= q(t)y(t) + \int_0^t q(s)y(s) (t-s)^{-1} E_{2\alpha,0}(-\lambda(t-s)^{2\alpha}) ds \\ &= q(t)y(t) - \lambda \int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda(t-s)^{2\alpha}) q(s)y(s) ds. \quad (27) \end{aligned}$$

where we used Property 7 to arrive at the second equality above. Combining (26) and (27) completes the proof. \square

Theorem 1. For $1/2 < \alpha < 1$, the integral equation

$$y(t, \lambda) = c_1 t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha}) + c_2 t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha}) + \int_0^t (t-s)^{2\alpha-1} E_{2\alpha, 2\alpha}(-\lambda(t-s)^{2\alpha}) q(s) y(s, \lambda) ds \quad (28)$$

satisfies (1) with initial conditions $\mathcal{I}_{0+}^{1-\alpha} y(t)|_{t=0} = c_1$ and $\mathcal{D}_{0+}^\alpha y(t)|_{t=0} = c_2$ in which c_1 and c_2 are given constants, and that this solution is unique.

Proof. We apply ${}^{-c}\mathcal{D}_{0+}^\alpha \mathcal{D}_{0+}^\alpha$ on both sides of (28) to find,

$$\begin{aligned} {}^{-c}\mathcal{D}_{0+}^\alpha \mathcal{D}_{0+}^\alpha (y(t, \lambda)) &= {}^{-c}\mathcal{D}_{0+}^\alpha \mathcal{D}_{0+}^\alpha \left(c_1 t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha}) + c_2 t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha}) \right) + \\ &\quad {}^{-c}\mathcal{D}_{0+}^\alpha \mathcal{D}_{0+}^\alpha \left(\int_0^t (t-s)^{2\alpha-1} E_{2\alpha, 2\alpha}(-\lambda(t-s)^{2\alpha}) q(s) y(s, \lambda) ds \right) \\ &= \lambda c_1 t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha}) + \lambda c_2 t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha}) - q(t)y(t) + \\ &\quad \lambda \int_0^t (t-s)^{2\alpha-1} E_{2\alpha, 2\alpha}(-\lambda(t-s)^{2\alpha}) q(s) y(s, \lambda) ds \\ &= -q(t)y(t) + \lambda \left(c_1 t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha}) + c_2 t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha}) \right) + \\ &\quad \lambda \left(\int_0^t (t-s)^{2\alpha-1} E_{2\alpha, 2\alpha}(-\lambda(t-s)^{2\alpha}) q(s) y(s, \lambda) ds \right) \\ &= -q(t)y(t) + \lambda(y(t, \lambda)), \end{aligned} \quad (29)$$

in which second equality come from Lemmas 1–3. We verify the initial conditions. Taking $\mathcal{I}_{0+}^{1-\alpha}$ of both sides (28), we get,

$$\begin{aligned} \mathcal{I}_{0+}^{1-\alpha} (y(t, \lambda)) &= \mathcal{I}_{0+}^{1-\alpha} \left(c_1 t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha}) + c_2 t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha}) \right) + \\ &\quad \mathcal{I}_{0+}^{1-\alpha} \left(\int_0^t (t-s)^{2\alpha-1} E_{2\alpha, 2\alpha}(-\lambda(t-s)^{2\alpha}) q(s) y(s, \lambda) ds \right) \\ &= c_1 E_{2\alpha, 1}(-\lambda t^{2\alpha}) + c_2 t E_{2\alpha, 2}(-\lambda t^{2\alpha}) + \int_0^t (t-s)^\alpha E_{2\alpha, 2\alpha+1}(-\lambda(t-s)^{2\alpha}) q(s) y(s, \lambda) ds, \end{aligned} \quad (30)$$

where the third term of the second equality comes from (25). Since $E_{2\alpha, 1}(-\lambda t^{2\alpha})|_{t=0} = 1$ and the other two terms of the above equality vanish when $t = 0$, we have verified the first initial condition. Again Taking \mathcal{D}_{0+}^α on both sides (28), we can find,

$$\begin{aligned} \mathcal{D}_{0+}^\alpha (y(t, \lambda)) &= \mathcal{D}_{0+}^\alpha \left(c_1 t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha}) + c_2 t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha}) \right) + \\ &\quad \mathcal{D}_{0+}^\alpha \left(\int_0^t (t-s)^{2\alpha-1} E_{2\alpha, 2\alpha}(-\lambda(t-s)^{2\alpha}) q(s) y(s, \lambda) ds \right) \\ &= -c_1 \lambda t^{2\alpha-1} E_{2\alpha, 2\alpha}(-\lambda t^{2\alpha}) + c_2 E_{2\alpha, 1}(-\lambda t^{2\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{2\alpha, \alpha}(-\lambda(t-s)^{2\alpha}) q(s) y(s, \lambda) ds, \end{aligned} \quad (31)$$

where the second equality above comes from (23), (24) and (26). The second initial condition can readily be obtained by substituting $t = 0$ in (31). \square

5. Analyticity of Solutions with Respect to the Parameter λ

In this section we show that the solutions (15) or (28) are, generally speaking, entire functions of the parameter λ for each t under consideration and $\lambda \in \mathbf{C}$. First, we show continuity with respect to said parameter. Consider the case where $c_1 = 0$, i.e., $y \in C[0, 1]$.

Lemma 4. Let $y \in C[0, 1]$, $\lambda \in \mathbf{C}$. Then, for each fixed $t \in [0, 1]$, $y(t, \lambda)$ is continuous with respect to λ .

Proof. Let $\Lambda > 0$ be arbitrary but fixed, and let $|\lambda|, |\lambda_0| < \Lambda$. Using (28),

$$\begin{aligned}
 y(t, \lambda) - y(t, \lambda_0) &= \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} ((q(s) - \lambda)y(s, \lambda) - (q(s) - \lambda_0)y(s, \lambda_0)) ds \\
 &= \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} ((\lambda_0 - \lambda)y(s, \lambda) + (q(s) - \lambda_0)(y(s, \lambda) - y(s, \lambda_0))) ds.
 \end{aligned}$$

So,

$$y(t, \lambda) - y(t, \lambda_0) = -(\lambda - \lambda_0) \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} y(s, \lambda) ds + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} (q(s) - \lambda_0)(y(s, \lambda) - y(s, \lambda_0)) ds. \tag{32}$$

Now, let $\epsilon > 0$ and $|\lambda - \lambda_0| < \delta$ where $\delta > 0$ is to be chosen later. Then,

$$|y(t, \lambda) - y(t, \lambda_0)| \leq \delta \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |y(s, \lambda)| ds + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |q(s) - \lambda_0| |y(s, \lambda) - y(s, \lambda_0)| ds.$$

Using (22) and Gronwall’s inequality, we get

$$\begin{aligned}
 |y(t, \lambda) - y(t, \lambda_0)| &\leq \frac{\delta c_3 t^{2\alpha}}{2\alpha \Gamma(2\alpha)} + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |q(s) - \lambda_0| |y(s, \lambda) - y(s, \lambda_0)| ds \\
 &\leq \frac{\delta c_3}{\Gamma(2\alpha + 1)} e^{\frac{1}{\Gamma(2\alpha)} \int_0^1 (t-s)^{2\alpha-1} |q(s) - \lambda_0| ds} \\
 &\leq \frac{\delta c_3}{\Gamma(2\alpha + 1)} e^{\frac{1}{\Gamma(2\alpha)} \int_0^1 (1-s)^{2\alpha-1} |q(s) - \lambda_0| ds} := C\delta
 \end{aligned}$$

where

$$C = \frac{c_3}{\Gamma(2\alpha + 1)} e^{\frac{1}{\Gamma(2\alpha)} \int_0^1 (1-s)^{2\alpha-1} |q(s) - \lambda_0| ds}$$

is a function of α and λ_0 only as $q \in L^\infty(0, 1)$. Thus, for any $t \in [0, 1]$, the continuity of $y(t, \lambda)$ follows by choosing $\delta < \frac{\epsilon}{C}$. It also follows from this that,

$$\sup_{t \in [0,1]} |y(t, \lambda) - y(t, \lambda_0)| < \epsilon, \quad |\lambda - \lambda_0| < \delta. \tag{33}$$

□

Next, we consider the differentiability of $y(t, \lambda)$ with respect to λ .

Lemma 5. *Let $y \in C[0, 1]$, $\lambda \in \mathbf{C}$. Then, for each fixed $t \in [0, 1]$, $y(t, \lambda)$ is differentiable with respect to λ .*

Proof. As before let $|\lambda| < \Lambda$, $t \in [0, 1]$. Equation (32) can be rewritten as

$$\frac{y(t, \lambda) - y(t, \lambda_0)}{\lambda - \lambda_0} = -\frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} y(s, \lambda) ds + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} (q(s) - \lambda_0) \frac{y(s, \lambda) - y(s, \lambda_0)}{\lambda - \lambda_0} ds.$$

As $y(t, \lambda_0)$ is given, we define $h(t, \lambda_0)$ to be the unique solution of the Volterra integral equation of the second kind,

$$h(t, \lambda_0) = -\frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} y(s, \lambda_0) ds + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} (q(s) - \lambda_0) h(s, \lambda_0) ds.$$

So,

$$\left| \frac{y(t, \lambda) - y(t, \lambda_0)}{\lambda - \lambda_0} - h(t, \lambda_0) \right| \leq \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |y(s, \lambda) - y(s, \lambda_0)| ds + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |q(s) - \lambda_0| \left| \frac{y(s, \lambda) - y(s, \lambda_0)}{\lambda - \lambda_0} - h(s, \lambda_0) \right| ds.$$

Let $\varepsilon > 0$ and choose $\delta > 0$ as in (33). Using Gronwall’s inequality and (33) we get, for $t \in [0, 1]$,

$$\left| \frac{y(t, \lambda) - y(t, \lambda_0)}{\lambda - \lambda_0} - h(t, \lambda_0) \right| \leq \frac{\varepsilon}{2\alpha\Gamma(2\alpha)} + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |q(s) - \lambda_0| \left| \frac{y(s, \lambda) - y(s, \lambda_0)}{\lambda - \lambda_0} - h(s, \lambda_0) \right| ds, \tag{34}$$

$$\leq \frac{\varepsilon}{\Gamma(2\alpha + 1)} e^{\frac{1}{\Gamma(2\alpha)} \int_0^1 (t-s)^{2\alpha-1} |q(s) - \lambda_0| ds} = O(\varepsilon).$$

for λ near λ_0 since, for $t \in [0, 1]$, $\int_0^1 (t-s)^{2\alpha-1} |q(s) - \lambda_0| ds = O(1)$. Thus,

$$\frac{\partial y(t, \lambda)}{\partial \lambda} \Big|_{\lambda=\lambda_0} := \lim_{\lambda \rightarrow \lambda_0} \frac{y(t, \lambda) - y(t, \lambda_0)}{\lambda - \lambda_0} = h(t, \lambda_0),$$

exists at λ_0 . Since λ_0 is arbitrary $y_\lambda(t, \lambda)$ exists for all λ with $|\lambda| < \Lambda$, real or complex and the result follows. \square

Theorem 2. For each $t \in [0, 1]$, $y(t, \lambda)$ is an entire function of λ .

Proof. This follows from Lemma 5 since $\lambda \in \mathbf{C}$ and $|\lambda| < \Lambda$ where $\Lambda > 0$ is arbitrary. \square

6. A Dirichlet-Type Problem

Let $y \in C[0, 1]$, $\lambda \in \mathbf{C}$ be fixed. In this case we note that the first of the boundary conditions (2) is equivalent to the usual fixed end (Dirichlet) boundary conditions, that is,

$$y \in C[0, 1] \iff \mathcal{I}_{0+}^{1-\alpha} y(t, \lambda) \Big|_{t=0} = 0 \iff y(0, \lambda) = 0.$$

The continuity assumption implies that there is a number M such that $|y(t, \lambda)| \leq M$, for all $t \in [0, 1]$. Thus,

$$|\mathcal{I}_{0+}^{1-\alpha} y(t, \lambda)| \leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha} ds = \frac{M t^{1-\alpha}}{(1-\alpha)\Gamma(\alpha)},$$

and so $\mathcal{I}_{0+}^{1-\alpha} y(t, \lambda) \Big|_{t=0} = 0$. On the other hand (15) now implies that $c_1 = 0$, i.e., $y(0, \lambda) = 0$, so that $y \in C[0, 1]$. However, the condition $y(1, \lambda) = 0$ is independent of the statement that $\mathcal{I}_{0+}^{1-\alpha} y(t, \lambda) \Big|_{t=1} = 0$.

Since, for any $z \neq 0$, the Mittag-Leffler functions satisfy

$$E_{\delta, \delta}(z) = \frac{1}{z} E_{\delta, 0}(z),$$

we get

$$t^{2\alpha-1} E_{2\alpha, 2\alpha}(-\lambda t^{2\alpha}) = -\frac{1}{\lambda t} E_{2\alpha, 0}(-\lambda t^{2\alpha}). \tag{35}$$

Hence, using (28) and (35) we get

$$y(t, \lambda) = c_1 t^{\alpha-1} E_{2\alpha, \alpha}(-\lambda t^{2\alpha}) + c_2 t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha}) - \int_0^t \frac{E_{2\alpha, 0}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)} q(s) y(s, \lambda) ds. \tag{36}$$

Remark 1. When $\alpha \rightarrow 1$, the integral Equation (36) becomes

$$y(t, \lambda) = y(0, \lambda) \cos(\sqrt{\lambda}t) + y'(0, \lambda) \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} + \int_0^t \frac{\sin(\sqrt{\lambda}(t-s))}{\sqrt{\lambda}} q(s)y(s, \lambda) ds, \quad (37)$$

which is exactly the integral equation equivalent of the classical Sturm–Liouville equation $-y'' + q(t)y = \lambda y$ for $\lambda > 0$.

Remark 2. Observe that, for each α ,

$$\lim_{s \rightarrow t^-} \frac{E_{2\alpha,0}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)} = \begin{cases} 0, & \text{if } \alpha \in (1/2, 1], \\ 1, & \text{if } \alpha = 1/2. \end{cases}$$

and so, for each $1/2 < \alpha < 1$, the kernel appearing in (36) is uniformly bounded on $[0, 1]$. This agrees with the equivalent result for the classical case (37).

7. Existence and Asymptotic Distribution of the Eigenvalues

Without loss of generality we may assume that $c_2 = 1$ in (36) and $y(t, \lambda)$ is the corresponding solution. In the sequel we always assume that $1/2 < \alpha < 1$.

Lemma 6. For each $t \in [0, 1]$, $1/2 < \alpha < 1$, and $|\arg(-\lambda)| \leq \mu$ where $\mu \in (\alpha\pi, \pi)$, we have $|t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha})| \rightarrow 0$ as $|\lambda| \rightarrow \infty$.

Proof. By (14) we can write

$$\begin{aligned} t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) &= t^\alpha \left(\frac{1}{2\alpha} (-\lambda t^{2\alpha})^{\frac{1-(\alpha+1)}{2\alpha}} \right) \exp \left\{ (-\lambda t^{2\alpha})^{\frac{1}{2\alpha}} \right\} + \mathcal{O} \left(\frac{1}{\lambda} \right) \\ &= -\frac{i}{2\alpha\sqrt{\lambda}} \exp \left\{ (-\lambda)^{\frac{1}{2\alpha}} t \right\} + \mathcal{O} \left(\frac{1}{\lambda} \right) \\ &= -\frac{i}{2\alpha\sqrt{\lambda}} \exp \left\{ |\lambda|^{\frac{1}{2\alpha}} \left(\cos\left(\frac{\arg(-\lambda)}{2\alpha}\right) + i \sin\left(\frac{\arg(-\lambda)}{2\alpha}\right) \right) t \right\} + \mathcal{O} \left(\frac{1}{\lambda} \right). \end{aligned}$$

Therefore,

$$\left| t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) \right| = \frac{1}{2\alpha\sqrt{\lambda}} \exp \left\{ |\lambda|^{\frac{1}{2\alpha}} \cos\left(\frac{\arg(-\lambda)}{2\alpha}\right) t \right\}.$$

Regarding the assumption on $\arg(-\lambda)$, we have $\cos\left(\frac{\arg(-\lambda)}{2\alpha}\right) < 0$ and it completes the proof. \square

Lemma 7. For each $t \in [0, 1]$, $s \in [0, t]$, $1/2 < \alpha < 1$, and $|\arg(-\lambda)| \leq \mu$ where $\mu \in (\alpha\pi, \pi)$, we have $\left| \frac{E_{2\alpha,0}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)} \right| \rightarrow 0$ as $|\lambda| \rightarrow \infty$.

Proof. By (14) we can write

$$\begin{aligned} \frac{E_{2\alpha,0}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)} &= \frac{\left(\frac{1}{2\alpha} (-\lambda(t-s)^{2\alpha})^{\frac{1}{2\alpha}} \right) \exp \left\{ (-\lambda(t-s)^{2\alpha})^{\frac{1}{2\alpha}} \right\} + \mathcal{O} \left(\frac{1}{\lambda} \right)}{\lambda(t-s)} \\ &= \frac{1}{2\alpha} \frac{(-\lambda)^{\frac{1}{2\alpha}}}{\lambda} \exp \left\{ (-\lambda)^{\frac{1}{2\alpha}} (t-s) \right\} + \mathcal{O} \left(\frac{1}{\lambda^2} \right). \end{aligned}$$

Then,

$$\left| \frac{E_{2\alpha,0}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)} \right| = \frac{1}{2\alpha|\lambda|^{(2\alpha-1)/2\alpha}} \exp \left\{ (t-s)|\lambda|^{1/2\alpha} \cos\left(\frac{\arg(-\lambda)}{2\alpha}\right) \right\} + \mathcal{O} \left(\frac{1}{|\lambda|^2} \right).$$

Arguing as in the previous lemma we reach the desired conclusion. \square

Lemma 8. For each $t \in [0, 1], s \in [0, t], 1/2 < \alpha < 1$, and $|\arg(-\lambda)| \leq \mu$ where $\mu \in (\alpha\pi, \pi)$, we have $\left| \frac{E_{2\alpha, 1-\alpha}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)^\alpha} \right| \rightarrow 0$ as $|\lambda| \rightarrow \infty$

Proof. By (14) we can write

$$\begin{aligned} \frac{E_{2\alpha, 1-\alpha}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)^\alpha} &= \frac{\left(\frac{1}{2\alpha}(-\lambda(t-s)^{2\alpha})^{\frac{1-(1-\alpha)}{2\alpha}}\right) \exp\left\{(-\lambda(t-s)^{2\alpha})^{\frac{1}{2\alpha}}\right\} + \mathcal{O}\left(\frac{1}{\lambda}\right)}{\lambda(t-s)^\alpha} \\ &= \frac{i}{\sqrt{\lambda 2\alpha}} \exp\left\{(-\lambda)^{\frac{1}{2\alpha}}(t-s)\right\} + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \end{aligned}$$

Then,

$$\left| \frac{E_{2\alpha, 1-\alpha}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)^\alpha} \right| = \frac{1}{2\alpha\sqrt{|\lambda|}} \exp\left\{(t-s)|\lambda|^{1/2\alpha} \cos\left(\frac{\arg(-\lambda)}{2\alpha}\right)\right\}. \tag{38}$$

The result follows since the exponential term is uniformly bounded. \square

Lemma 9. For each $t \in [0, 1]$, and $1/2 < \alpha < 1$, the solution $y(t, \lambda)$ is an entire function of λ of order at most $1/2\alpha$.

Proof. Let $\lambda \in \mathbb{C}$. Define f by

$$y(t, \lambda) = \exp\left\{t|\lambda|^{1/2\alpha} \cos\left(\frac{\arg(-\lambda)}{2\alpha}\right)\right\} f(t). \tag{39}$$

Then, using (36),

$$\begin{aligned} f(t) &= t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha}) \exp\left\{-t|\lambda|^{1/2\alpha} \cos\left(\frac{\arg(-\lambda)}{2\alpha}\right)\right\} \\ &\quad - \int_0^t \frac{E_{2\alpha, 0}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)} \exp\left\{-(t-s)|\lambda|^{1/2\alpha} \cos\left(\frac{\arg(-\lambda)}{2\alpha}\right)\right\} q(s) f(s) ds \end{aligned}$$

Applying Lemma 6 there exists $\Lambda \in \mathbb{R}^+$ such that for all $|\lambda| > \Lambda$ we have

$$|f(t)| \leq 1 + \frac{1}{2\alpha|\lambda|^{(2\alpha-1)/2\alpha}} \int_0^t |q(s)| |f(s)| ds$$

which, on account of Gronwall’s inequality, gives us

$$|f(t)| \leq \exp\left\{\frac{1}{2\alpha|\lambda|^{(2\alpha-1)/2\alpha}} \int_0^1 |q(s)| ds\right\} \tag{40}$$

for all sufficiently large $|\lambda|$. Thus, $f \in L^\infty[0, 1]$ so that (39) yields, for some M ,

$$|y(t, \lambda)| \leq M \exp\left(|\lambda|^{1/2\alpha}\right)$$

and the order claim is verified. \square

Lemma 10. For each $t \in [0, 1], \mathcal{I}_0^{1-\alpha} y(t, \lambda)$ is an entire function of λ of order at most 2α .

Proof. This is clear from the definition, the possible values of α , and since $y(t, \lambda)$ is itself entire and of order at most $1/2\alpha$, from Lemma 9. \square

Lemma 11. *The boundary value problem (1) and (2) has infinitely many complex eigenvalues (real eigenvalues are not to be excluded here).*

Proof. By Lemma 10, we know that $\mathcal{I}_{0+}^{1-\alpha}y(t, \lambda)$ is entire for each $t \in [0, 1]$, and $1/2 < \alpha < 1$ as well. So, the eigenvalues of our problem are given by the zeros of $\mathcal{I}_{0+}^{1-\alpha}y(1, \lambda)$, which must be countably infinite in number since the latter function is of fractional order $1/2\alpha$ (on account of the restriction on α). This gives us the existence of infinitely many eigenvalues, generally in \mathbf{C} . \square

Next, we give the asymptotic distribution of these eigenvalues when α is either very close to $1/2$ from the right or very close to 1 from the left. Recall (36) with $c_2 = 1$, so that

$$y(t, \lambda) = t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha}) - \int_0^t \frac{E_{2\alpha, 0}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)} q(s)y(s, \lambda) ds. \quad (41)$$

An iterative method for solving for approximate solutions of (41) maybe found in [16]. Keeping in mind the boundary condition (2) at $t = 1$, we calculate $\mathcal{I}_{0+}^{1-\alpha}y(t, \lambda)$ and then evaluate this at $t = 1$ in order to find the dispersion relation for the eigenvalues. However, our derivation is theoretical in nature. A straightforward though lengthy calculation using (41) and the definition of the Mittag-Leffler functions show that

$$\begin{aligned} \mathcal{I}_{0+}^{1-\alpha}y(t, \lambda) &= \mathcal{I}_{0+}^{1-\alpha}\{t^\alpha E_{2\alpha, \alpha+1}(-\lambda t^{2\alpha})\} + \mathcal{I}_{0+}^{1-\alpha}\left(\int_0^t \frac{E_{2\alpha, 0}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)} q(s)y(s, \lambda) ds\right) \\ &= tE_{2\alpha, 2}(-\lambda t^{2\alpha}) + \frac{1}{\lambda} \int_0^t \frac{E_{2\alpha, 1-\alpha}(-\lambda(t-s)^{2\alpha})}{(t-s)^\alpha} q(s)y(s, \lambda) ds \end{aligned} \quad (42)$$

so that the eigenvalues of (1) and (2) are given by those $\lambda \in \mathbf{C}$ such that

$$E_{2\alpha, 2}(-\lambda) + \frac{1}{\lambda} \int_0^1 \frac{E_{2\alpha, 1-\alpha}(-\lambda(1-s)^{2\alpha})}{(1-s)^\alpha} q(s)y(s, \lambda) ds = 0. \quad (43)$$

Let us consider first the case where $\lambda \in \mathbf{R}$. Lemma 8 implies that the right side of (38) tends to 0 as $\lambda \rightarrow \infty$. Indeed this, combined with (39), implies that

$$\left| \frac{E_{2\alpha, 1-\alpha}(-\lambda(t-s)^{2\alpha})}{\lambda(t-s)^\alpha} y(s, \lambda) \right| = O\left(\frac{1}{\sqrt{|\lambda|}}\right)$$

for all sufficiently large λ .

Thus, the real eigenvalues of the problem (1) and (2) become the zeros of a transcendental equation of the form,

$$E_{2\alpha, 2}(-\lambda) + O\left(\frac{1}{\sqrt{\lambda}}\right) = 0.$$

We are concerned with the asymptotic behaviour of these real zeros. Recall the distribution of the real zeros of $E_{2\alpha, 2}(-\lambda)$ in [1]. There we showed that, for each $n = 0, 1, 2, \dots, N^* - 1$, where N^* depends on α , the interval

$$I_n(\alpha) := \left(\left(\frac{(2n + \frac{1}{2} + \frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})} \right)^{2\alpha}, \left(\frac{(2n + \frac{3}{2} + \frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})} \right)^{2\alpha} \right), \quad (44)$$

always contains at least two real zeros of $E_{2\alpha,2}(-\lambda)$. For $\alpha \rightarrow 1$, these intervals approach the intervals

$$\left((2n+1)^2 \pi^2, (2n+2)^2 \pi^2 \right),$$

whose end-points are each eigenvalues of the Dirichlet problem for the classical equation $-y'' = \lambda y$ on $[0, 1]$. Since each interval I_n contains two zeros we can denote the first of these two zeros by $\lambda_{2n}(\alpha)$. Equation (44) now gives the *a-priori* estimate

$$\left(\frac{(2n + \frac{1}{2} + \frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})} \right)^{2\alpha} \leq \lambda_{2n}(\alpha) \leq \left(\frac{(2n + \frac{3}{2} + \frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})} \right)^{2\alpha}. \quad (45)$$

For each $\alpha < 1$, and close to 1, and for large λ , the real zeros of the preceding equation approach those of $E_{2\alpha,2}(-\lambda)$ and spread out towards the end-points of intervals of the form (44). For α close to $1/2$ there are no zeros, the first two zeros appearing only when $\alpha \approx 0.7325$. For α larger than this critical value, the zeros appear in pairs and in intervals of the form (44).

Next, recall that for $\alpha < 1$ there are only *finitely many* such real zeros, (see [1]) their number growing without bound as $\alpha \rightarrow 1$. It also follows from Lemma 11 that, for each α , the remaining infinitely many eigenvalues must be non-real. As $\alpha \rightarrow 1^-$ these non-real eigenvalues tend to the real axis thereby forming more and more real eigenvalues until the spectrum is totally real when $\alpha = 1$ and the problem then reduces to a (classical) regular Sturm–Liouville problem.

Finally, for α close to 1, (45) leads to the approximation,

$$\lambda_{2n}(\alpha) \approx \left(\frac{(2n+2)\pi}{\sin(\frac{\pi}{2\alpha})} \right)^{2\alpha},$$

from which this, in conjunction with (44) and $\alpha \rightarrow 1$, we can derive the classical eigenvalue asymptotics, $\lambda_n \sim n^2 \pi^2$ as $n \rightarrow \infty$.

8. Conclusions

We consider the fractional eigenvalue problem,

$$-{}^c\mathcal{D}_{0+}^\alpha \circ \mathcal{D}_{0+}^\alpha y(t) + q(t)y(t) = \lambda y(t), \quad 0 \leq t \leq 1,$$

where α is a real parameter, $1/2 < \alpha < 1$, λ is a generally unspecified complex parameter, with mixed Caputo and Riemann–Liouville derivatives and q an essentially bounded function, subject to the following boundary conditions involving the Riemann–Liouville integrals,

$$\mathcal{I}_{0+}^{1-\alpha} y(t)|_{t=0} = 0, \quad \text{and} \quad \mathcal{I}_{0+}^{1-\alpha} y(t)|_{t=1} = 0.$$

We show that this problem admits, for each α under consideration, and for eigenfunctions that are in $C[0, 1]$, a finite number of real eigenvalues and an infinite number of non-real eigenvalues. The real eigenvalues, though finite in number for each α , are approximated by (44) and (45), which as $\alpha \rightarrow 1$ gives the classical asymptotic relation $\lambda_n \sim n^2 \pi^2$ as $n \rightarrow \infty$.

As $\alpha \rightarrow 1^-$ we observe that the spectrum obtained approaches the Sturm–Liouville spectrum of the classical problem

$$-y'' + q(t)y = \lambda y, \quad y(0) = y(1) = 0.$$

The same results hold if the eigenfunctions are merely $C(0, 1]$ (i.e., $c_1 \neq 0$) except that now the latter have an infinite discontinuity at $t = 0$ for each α . The proofs are identical and are therefore omitted.

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References

1. Dehghan, M.; Mingarelli, A.B. Fractional Sturm-Liouville eigenvalue problems, I. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **2020**, *114*, 46.
2. Aleroev, T.S.; Aleroeva, H.T. Problems of Sturm-Liouville type for differential equations with fractional derivatives. In *Volume 2 Fractional Differential Equations*; Kochubei, A., Luchko, Y., Eds.; De Gruyter: Berlin, Germany; Boston, MA, USA, 2019; pp. 21–46. [[CrossRef](#)]
3. Al-Mdallal, Q.M. An efficient method for solving fractional Sturm-Liouville problems. *Chaos Solitons Fractals* **2009**, *40*, 183–189.
4. Blaszczyk, T.; Ciesielski, M. Numerical solution of fractional Sturm-Liouville equation in integral form. *Fract. Calculus Appl. Anal.* **2014**, *17*, 307–320.
5. Heydarpour, Z.; Izadi J.; George, R.; Ghaderi, M.; Rezapour, S. On a Partial Fractional Hybrid Version of Generalized Sturm-Liouville-Langevin Equation. *Fractal Fract.* **2022**, *6*, 269.
6. Klimek, M.; Agrawal, O.P. On a regular fractional Sturm-Liouville problem with derivatives of order $(0, 1)$. In Proceedings of the 13th International Carpathian Control Conference, High Tatras, Slovakia, 28–31 May 2012. [[CrossRef](#)]
7. Klimek, M.; Agrawal, O.P. Fractional Sturm-Liouville problem. *Comput. Math. Appl.* **2013**, *66*, 795–812.
8. Klimek, M.; Odziejewicz, T.; Malinowska, A.B. Variational methods for the fractional Sturm-Liouville problem. *J. Math. Anal. Appl.* **2014**, *416*, 402–426. [[CrossRef](#)]
9. Klimek, M.; Ciesielski, M.; Blaszczyk, T. Exact and Numerical Solution of the Fractional Sturm-Liouville Problem with Neumann Boundary Conditions. *Entropy* **2022**, *24*, 143. [[CrossRef](#)] [[PubMed](#)]
10. Min, D.; Chen, F. Variational methods to the p-Laplacian type nonlinear fractional order impulsive differential equations with Sturm-Liouville boundary-value problem. *Fract. Calc. Appl. Anal.* **2021**, *24*, 1069–1093. [[CrossRef](#)]
11. Zayernouri, M.; Karniadakis, G.E. Fractional Sturm-Liouville eigen-problems: theory and numerical approximation. *J. Comput. Phys.* **2013**, *252*, 495–517.
12. Mittag-Leffler, G. Sur la nouvelle fonction $E_\alpha(x)$. *C. R. Acad. Sci. Paris* **1903**, *137*, 554–558.
13. Erdélyi, A.; Bateman, H. *Higher Transcendental Functions*; McGraw-Hill Book Company, Inc.: New York, NY, USA, 1953; Volume 3.
14. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Application of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
15. Maralani, E.M.; Saei, F.D.; Akbarfam, A.J.; Ghanbari, K. Eigenvalues of fractional Sturm-Liouville problems by successive method. *Comput. Methods Differ. Equ.* **2021**, *9*, 1163–1175.
16. Ma, H.J.; Wang, Y. Full Information Control of Borel-Measurable Markov Jump. *Mathematics* **2022**, *10*, 37. [[CrossRef](#)]