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Solving Two-Sided Fractional Super-Diffusive Partial Differential Equations with Variable Coefficients in a Class of New Reproducing Kernel Spaces

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Abstract: Fractional-order calculus has become a useful mathematical framework to describe the complex super-diffusive process; however, numerical solutions of the two-sided space-fractional super-diffusive model with variable coefficients are difficult to obtain, and almost no method can obtain an analytical solution. In this paper, a class of new fractional dimensional reproducing kernel spaces (RKS) based on Caputo fractional derivatives is given, and we give analytical and numerical solutions of the two-sided space-fractional super-diffusive model based on the class of new RKS. The analytical solution is represented in the form of series in the reproducing kernel space. Numerical experiments indicate that the piecewise reproducing kernel method is more accurate than the traditional reproducing kernel method (RKM), and these new fractional reproducing kernel spaces are efficient for the two-sided space-fractional super-diffusive model.

Keywords: two-sided fractional super-diffusive model; variable coefficient; reproducing kernel space; analytical and numerical solution; reproducing kernel method

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1. Introduction

In the process of solute transport in groundwater, the two-sided space-fractional partial differential equations can be used to describe the abnormal diffusion phenomenon in the process of solute transport in an aquifer. As for the numerical solution of this kind of problem, some scholars [1] have conducted in-depth research on this kind of problem. In this paper, we will consider the following two-sided space-fractional partial differential Equations [2]:

$$u_t(x,t) = C_+(x,t)\mathcal{D}^{\alpha,\rho}_{x,a^+}u(x,t) + C_-(x,t)\mathcal{D}^{\alpha,\rho}_{x,b^-}u(x,t) + g(x,t). \tag{1}$$

with the initial condition $u(x,0) = 0$, $a \le x \le b$, and boundary condition $u(a,t) = 0$, $u(b,t) = 0, 0 \le t \le T$. $g(x,t)$ is a known function, $u(x,t)$ is a unknown function, $C_+(x,t)$ and $C_-(x,t)$ are two known coefficients, $\mathcal{D}^{\alpha,\rho}_{x,a^+}u(x,t)$ and $\mathcal{D}^{\alpha,\rho}_{x,b^-}u(x,t)$ are defined as the left and right Caputo-type fractional derivative $\mathcal{D}^{\alpha,\rho}_{a^+}u(x,t)$ and $\mathcal{D}^{\alpha,\rho}_{b^-}u(x,t)$ of x, $\mathcal{D}^{\alpha,\rho}_{a^+}u(x,t)$ and $\mathcal{D}^{\alpha,\rho}_{b^-}u(x,t)$; see Definition 6. When $1 \le \beta \le 2$, it is a super-diffusive process, when $\beta = 2$, and if $C_+(x,t) = C_-(x,t)$ and $C(x,t) = C_+(x,t) = C_-(x,t)$, then, Equation (1) becomes the standard diffusion Equation

$$\frac{\partial u(x,t)}{\partial t} = C(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t), a \le x \le b, 0 \le t \le T.$$

Numerical solutions of the two-sided space-fractional super-diffusive model are difficult to obtain, and almost no method can obtain the analytical solution. In this paper, we give analytical and numerical solutions of the two-sided space-fractional super-diffusive model based on a class of the new reproducing kernel method.

Reproducing kernel Hilbert spaces arise in a number of areas, including approximation theory, statistics, machine-learning theory, group representation theory and various areas of complex analysis. In 1950, N. Arronszajn [3] published the theory of reproducing kernels, which formed a systematic theory of a reproducing kernel space. In 1986, Cui [4] constructed a reproducing kernel space and gave its reproducing kernel expression for the first time. At the same time, Cui gave the reproducing kernel method of a linear operator equation. Since then, the reproducing kernel method has been widely used to solve various Equations [5–12].

In [5], the authors used the spline reproducing kernel function approximation to develop Filon and Levin methods for highly oscillatory integrals. In [6], the authors found an identification approach for differential equation models by using a reproducing kernel. In [7], the authors presented an optimal reproducing kernel method for nonlocal boundary value problems by combining the piecewise polynomial kernel with polynomial kernel. In [8], the authors presented a numerical technique to obtain the approximation solution for linear Volterra integral equations of the second kind based on the reproducing kernel theory.

In [9], the authors solved the time variable fractional order advection–reaction-diffusion equations based on the piecewise reproducing kernel method. In [11], the authors gave a fitted fractional reproducing kernel algorithm for the numerical solutions of ABC-Fractional Volterra integro-differential equations. In [12], the authors discussed techniques for constructing univariate spline interpolations by employing a class of reproducing kernel functions.

A key of the reproducing kernel method is the reproducing kernel space. Until now, authors [5–12] all used integer-dimensional reproducing kernel space to solve fractional order differential Equations [10–12]. In fact, for fractional order differential equations, the integer dimensional reproducing kernel space is relatively larg. In this paper, a class of new fractional reproducing kernel spaces based on Caputo fractional derivatives are given.

2. Main Notations

Throughout this paper, $\mathbb{N}_0$ will denote the set of non-negative integers. We will use the notation $\lfloor x \rfloor$ to design the integer part of a real number x—that is, the greatest integer less than or equal to x. We also define $\lceil x \rceil = \lfloor x \rfloor + 1$ if $x \notin \mathbb{N}_0$ and $\lceil x \rceil = \lfloor x \rfloor$ if $x \in \mathbb{N}_0$.

Definition 1. *For $c \in \mathbb{R}, p \in \mathbb{N}_0$, the space $L_c^p[a,b]$ and $L_c^\infty[a,b]$ are defined as*

$$\begin{array}{l} L_c^p[a,b] = \{f | f \in L[a,b], \|f\|_{L_c^p(a,b)} := \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{1/p} < \infty, 1 \le p < \infty, \} \\ L_c^\infty[a,b] = \{f | f \in L[a,b], \|f\|_{L_c^\infty(a,b)} := ess \sup\limits_{x \in [a,b]} [x^c |f(x)|] < \infty\}. \end{array} \tag{2}$$

The convention that $L_1^1[a,b] = L[a,b]$ is used.

Definition 2. *For $f(x) \in L_c^p[a,b]$, α, ρ are two positive real numbers. The left generalized fractional integral $\mathcal{I}_{a^+}^{\alpha,\rho}$ of order α is defined for any real number $x > a$ is defined as*

$$\mathcal{I}_{a^+}^{\alpha,\rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho-1} f(\tau)}{(x^\rho - \tau^\rho)^{1-\alpha}} \, d\tau. \tag{3}$$

This definition is a left fractional fractional integral of the n-fold of the form

$$\mathcal{I}_{a^+}^{\alpha,\rho} f(x) = \int_a^x t_1^{\rho-1} \, dt_1 \int_a^{t_1} t_2^{\rho-1} \, dt_2 \cdots \int_a^{t_{n-1}} t_n^{\rho-1} f(t_n) \, dt_n. \tag{4}$$

Similarly, the right generalized fractional integral $\mathcal{I}_{b^-}^{\alpha,\rho} f$ of order α is defined for any real number. $x < b$ is defined as

$$\mathcal{I}_{b^-}^{\alpha,\rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\rho-1} f(\tau)}{(\tau^\rho - x^\rho)^{1-\alpha}} \, d\tau. \tag{5}$$

This definition is a right fractional fractional integral of n-fold of the form

$$\mathcal{I}_{a^+}^{\alpha,\rho} f(x) = \int_x^b t_1^{\rho-1} \, dt_1 \int_{t_1}^b t_2^{\rho-1} \, dt_2 \cdots \int_{t_{n-1}}^b t_n^{\rho-1} f(t_n) \, dt_n. \tag{6}$$

Definition 3. *For $0 \le \varepsilon < 1$, ρ is a non-negative real number, and the space $C_\varepsilon^\rho[a,b]$ is defined as*

$$C_\varepsilon^\rho[a,b] = \{f| \; if \; \rho \neq 0, \; (x^\rho - a^\rho)^\varepsilon f(x) \in C[a,b]\}, \;\; if \; \rho = 0, \; (\log x - \log a)^\varepsilon f(x) \in C[a,b]\}. \tag{7}$$

The norm is defined as

$$\|f\|_{C_\varepsilon^\rho} = \begin{cases} \int_a^b |(x^\rho - a^\rho)^\varepsilon f(x)| dx, & \rho \neq 0, \\ \int_a^b |(\log x - \log a)^\varepsilon f(x)| dx, & \rho = 0. \end{cases} \tag{8}$$

The convention that $C_{0,\rho}[a,b] = C[a,b]$ is used. ρ is weighted.

Definition 4. *For $0 \le \varepsilon < 1, n \in \mathbb{N}_0$, ρ is a non-negative real number, and space $C_{\gamma,\varepsilon}^n[a,b]$ is defined as*

$$C_{\gamma,\varepsilon}^n[a,b] = \{f|\gamma^{n-1} f \in C[a,b], \gamma^n f \in C_\varepsilon^\rho[a,b], \gamma = x^{1-\rho} \frac{d}{dx}\},$$

The norm is defined as

$$\|f\|_{C_{\gamma,\varepsilon}^n} = \sum_{k=0}^{n-1} \|\gamma^k f\|_C + \|\gamma^n f\|_{C_{\varepsilon,\rho}}.$$

The convention $C_{\gamma,0}^n[a,b] = C_\gamma^n[a,b]$ endowed with the norm $\|f\|_{C_\gamma^n} = \sum_{k=0}^n \|\gamma^k f\|_C$ is used.

Definition 5. *For $n \in \mathbb{N}_0$, ρ is a non-negative real number, and space $AC_\rho^n[a,b]$ is defined as*

$$AC_\rho^n[a,b] = \{f|\gamma^{n-1} f \in AC[a,b], \gamma = x^{1-\rho} \frac{d}{dx}\}, \quad AC_\rho^1[a,b] = AC[a,b],$$

with $\gamma := x^{1-\rho} \frac{d}{dx}$ and $AC_\rho^1 = AC[a,b]$.

It has been shown in [11] that the space $AC_\rho^n[a,b]$ consists of those and only those functions f that are represented in the form:

$$f(x) = \sum_{k=0}^{n-1} c_k (x^\rho - a^\rho)^k + \int_a^x (x^\rho - t^\rho)^{n-1} g(t) \, dt$$

with $g \in L^1[a,b]$ *and* $c_k \in \mathbb{R}$.

Definition 6. *If $f(x) \in AC_\rho^n[a,b]$, then the left and right corresponding generalized Caputo fractional derivatives of order α of f to these generalized integrals (3) and (5) are defined by [13,14]:*

$$\mathcal{D}_{a^+}^{\alpha,\rho} f(x) = \mathcal{I}_{a^+}^{n-\alpha,\rho} (\gamma^n f)(x) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^x \frac{\tau^{\rho-1} (\gamma^n f)(\tau)}{(x^\rho - \tau^\rho)^{\alpha-n+1}} \, d\tau, \alpha \notin \mathbb{N}_0. \tag{9}$$

$$\mathcal{D}_{b^-}^{\alpha,\rho}f(x)=\mathcal{I}_{b^-}^{n-\alpha,\rho}(\gamma^n f)(x)=\frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_x^b\frac{\tau^{\rho-1}(\gamma^n f)(\tau)}{(\tau^\rho-x^\rho)^{\alpha-n+1}}d\tau,\alpha\notin\mathbb{N}_0. \tag{10}$$

where $\alpha\geq 0$ *and* $n=\lfloor\alpha\rceil$.

3. A Class of Fractional Reproducing Kernel Space

Theorem 1. *If* $f\in AC_\rho^n[a,b]$ *and* $\mathcal{D}_{a^+}^{(n+1)\alpha,\rho}f\in C[a,b]$. *Then, the generalized Taylor expansion of* f *with the generalized left Caputo-type fractional derivatives could also be written with the remainder in integral form.*

$$f(x)=\sum_{j=0}^{n}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{j\alpha}\frac{\mathcal{D}_{a^+}^{j\alpha,\rho}f(a)}{\Gamma(j\alpha+1)}+\frac{1}{\Gamma((n+1)\alpha)}\int_a^x(\frac{x^\rho-t^\rho}{\rho})^{(n+1)\alpha-1}\frac{\mathcal{D}_{a^+}^{(n+1)\alpha,\rho}f(t)}{t^{1-\rho}}dt. \tag{11}$$

Similarly, if $f\in AC_\rho^n[a,b]$ *and* $\mathcal{D}_{b^-}^{(n+1)\alpha,\rho}f\in C[a,b]$. *Then, the generalized Taylor expansion of* f *with the generalized right Caputo-type fractional derivatives could also be written with the remainder in integral form.*

$$f(x)=\sum_{j=0}^{n}\left(\frac{b^\rho-x^\rho}{\rho}\right)^{j\alpha}\frac{\mathcal{D}_{b^-}^{j\alpha,\rho}f(b)}{\Gamma(j\alpha+1)}+\frac{\rho^{1-(n+1)\alpha}}{\Gamma((n+1)\alpha)}\int_x^b\frac{\tau^{\rho-1}\mathcal{D}_{b^-}^{(n+1)\alpha,\rho}f(\tau)}{(\tau^\rho-x^\rho)^{1-(n+1)\alpha}}d\tau, \tag{12}$$

where $\rho>0$, $\alpha>0$, $n\in\mathbb{N}_0$.

Theorem 2. *If* $\rho\geq 1$, $\alpha>0$, $n\in\mathbb{N}_0$, *space* $W_{a^+}^{\alpha,\rho,m}[a,b]$ *is defined as*

$$W_{a^+}^{\alpha,\rho,m}[a,b]=\{f|f\in AC_\rho^m[a,b],\mathcal{D}_{a^+}^{(n+1)\alpha,\rho}f\in C[a,b]\}. \tag{13}$$

The inner product of space $W_{a^+}^{\alpha,\rho,m}[a,b]$ *is defined with the following form*

$$\langle u(x),v(x)\rangle_{W_{a^+}^{\alpha,\rho,m}[a,b]}=\sum_{j=0}^{m}\mathcal{D}_{a^+}^{j\alpha,\rho}u(a)\mathcal{D}_{a^+}^{j\alpha,\rho}v(a)+\int_a^b\mathcal{D}_{a^+}^{(m+1)\alpha,\rho}u(t)\mathcal{D}_{a^+}^{(m+1)\alpha,\rho}v(t)dt. \tag{14}$$

The norm is

$$\|u(x)\|=\left(\langle u(x),v(x)\rangle_{W_{a^+}^{\alpha,\rho,m}[a,b]}\right)^{\frac{1}{2}}.$$

Then, the space $W_{a^+}^{\alpha,\rho,m}[a,b]$ *is a reproducing kernel space, and its reproducing kernel is*

$$K_{a^+}^{\alpha,\rho,m}(x,y)=\begin{cases}\sum\limits_{j=0}^{m}\left(\frac{1}{\Gamma(j\alpha+1)}\right)^2\left(\frac{x^\rho-a^\rho}{\rho}\right)^{j\alpha}\left(\frac{y^\rho-a^\rho}{\rho}\right)^{j\alpha}+\left(\frac{1}{\Gamma((m+1)\alpha)}\right)^2\int_a^x t^{2\rho-2}\left(\frac{(x^\rho-t^\rho)(y^\rho-t^\rho)}{\rho^2}\right)^{(m+1)\alpha-1}dt, x<y\\ \sum\limits_{j=0}^{m}\left(\frac{1}{\Gamma(j\alpha+1)}\right)^2\left(\frac{x^\rho-a^\rho}{\rho}\right)^{j\alpha}\left(\frac{y^\rho-a^\rho}{\rho}\right)^{j\alpha}+\left(\frac{1}{\Gamma((m+1)\alpha)}\right)^2\int_a^y t^{2\rho-2}\left(\frac{(x^\rho-t^\rho)(y^\rho-t^\rho)}{\rho^2}\right)^{(m+1)\alpha-1}dt, y<x\end{cases} \tag{15}$$

where $Re(\alpha)\geq 0$, $\rho>0$, $\gamma=x^{1-\rho}\frac{d}{dx}$, $AC_\rho^1=AC[a,b]$ *and* $m\in\mathbb{N}_0$.

Proof. Suppose that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $W_{a^+}^{\alpha,\rho,m}[a,b]$—that is,

$$\begin{aligned}&\|f_{n+p}-f_n\|_{W_{a^+}^{\alpha,\gamma,m}[a,b]}\\&=\sum_{j=0}^{m}\left(\mathcal{D}_{a^+}^{j\alpha,\rho}f_{n+p}(a)-\mathcal{D}_{a^+}^{j\alpha,\rho}f_n(a)\right)^2+\int_a^b\left(\mathcal{D}_{a^+}^{(m+1)\alpha,\rho}f_{n+p}(x)-\mathcal{D}_{a^+}^{(m+1)\alpha,\rho}f_n(x)\right)^2dx\to 0,(n\to\infty).\end{aligned} \tag{16}$$

We know that $\{\mathcal{D}_{a^+}^{j\alpha,\rho} f_n(a)\}_{n=1}^{\infty}$, $(j = 0, 1, \ldots, m)$ are all real number Cauchy sequences, and $\{\mathcal{D}_{a^+}^{(m+1)\alpha,\rho} f_n(x)\}_{n=1}^{\infty}$ is a Cauchy function sequence in $L^2[a,b]$. Therefore, there exist a real number r_i, $(i = 0, 1, 2, \ldots, m-1)$ and a real function $f \in C[a,b]$, such that

$$\lim_{n\to\infty} \mathcal{D}_{a^+}^{j\alpha,\rho} f_n(a) = r_j, \quad \int_a^b (\mathcal{D}_{a^+}^{(m+1)\alpha,\rho} f_n(x) - f(x))^2 dx \to 0, (n \to \infty). \tag{17}$$

Let

$$g(x) = \sum_{j=0}^{m} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{j\alpha} \frac{r_j}{\Gamma(j\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha)} \int_a^x (\frac{x^\rho - t^\rho}{\rho})^{(m+1)\alpha-1} \frac{\mathcal{D}_{a^+}^{(m+1)\alpha,\rho} f(t)}{t^{1-\rho}} dt. \tag{18}$$

It was shown in [13] that the space $AC_\rho^n[a,b]$ consists of those and only those functions $g(x)$ that are represented in the form

$$g(x) = \sum_{j=0}^{m-1} c_j (x^\rho - a^\rho)^j + \int_a^x (x^\rho - t^\rho)^{m-1} f(t)\, dt$$

with $g \in L^1[a,b]$ *and* $c_j \in \mathbb{R}$.

We can obtain $g(x) \in W_{a^+}^{\alpha,\rho,n,m}[a,b]$, and $\mathcal{D}_{a^+}^{j\alpha,\rho} g(a) = \mathcal{D}_{a^+}^{j\alpha,\rho} f(a) = r_j, (j = 0, 1, \ldots m)$, $\|\mathcal{D}_{a^+}^{(m+1)\alpha,\rho} f_n(x) - f(x)\|_{W_{a^+}^{\alpha,\rho,m}[a,b]} \to 0, (n \to \infty)$. Thus, $W_{a^+}^{\alpha,\rho,m}[a,b]$ is complete—namely, $W_{a^+}^{\alpha,\rho,m}[a,b]$ is a Hilbert space.

Next, we prove that space $W_{a^+}^{\alpha,\rho,m}[a,b]$ is a reproducing kernel space.

For any $f(x) \in W_{a^+}^{\alpha,\rho,m}[a,b]$ of

$$f(x) = \sum_{j=0}^{m} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{j\alpha} \frac{\mathcal{D}_{a^+}^{j\alpha,\rho} f(a)}{\Gamma(j\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha)} \int_a^x (\frac{x^\rho - t^\rho}{\rho})^{(m+1)\alpha-1} \frac{\mathcal{D}_{a^+}^{(m+1)\alpha,\rho} f(t)}{t^{1-\rho}} dt, \tag{19}$$

we find that

$$\begin{aligned}
|f(x)| &\le \left|\sum_{j=0}^{m} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{j\alpha} \frac{\mathcal{D}_{a^+}^{j\alpha,\rho} f(a)}{\Gamma(j\alpha+1)}\right| + \left|\frac{1}{\Gamma((m+1)\alpha)} \int_a^x (\frac{x^\rho - t^\rho}{\rho})^{(m+1)\alpha-1} \frac{\mathcal{D}_{a^+}^{(m+1)\alpha,\rho} f(t)}{t^{1-\rho}} dt\right| \\
&\le \left(\sum_{j=0}^{m} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{2j\alpha} \left(\frac{1}{\Gamma(j\alpha+1)}\right)^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{m} \left(\mathcal{D}_{a^+}^{j\alpha,\rho} f(a)\right)^2\right)^{\frac{1}{2}} \\
&+ \frac{\left(\int_a^b t^{2\rho-2} (b^\rho - t^\rho)^{2(m+1)\alpha-2} dt\right)^{\frac{1}{2}}}{\rho^{(m+1)\alpha-1} (\Gamma((m+1)\alpha)} \left(\int_a^b \left(\mathcal{D}_{a^+}^{(m+1)\alpha,\rho} f(t)\right)^2 dt\right)^{\frac{1}{2}} \\
&\le \left(\left(\sum_{j=0}^{m} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{2j\alpha} \left(\frac{1}{\Gamma(j\alpha+1)}\right)^2\right)^{\frac{1}{2}} + \frac{\left(\int_a^b t^{2\rho-2} (b^\rho - t^\rho)^{2(m+1)\alpha-2} dt\right)^{\frac{1}{2}}}{\rho^{(m+1)\alpha-1} (\Gamma((m+1)\alpha)}\right) \|f\|_{AC_\gamma^n[a,b]}
\end{aligned} \tag{20}$$

There exist C with

$$|f(x)| \le C\|f\|_{W_{a^+}^{\alpha,\rho,m}[a,b]},$$

where $C = \left(\sum_{j=0}^{m} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{2j\alpha} \left(\frac{1}{\Gamma(j\alpha+1)}\right)^2\right)^{\frac{1}{2}} + \frac{\left(\int_a^b t^{2\rho-2} (b^\rho - t^\rho)^{2(m+1)\alpha-2} dt\right)^{\frac{1}{2}}}{\rho^{(m+1)\alpha-1} (\Gamma((m+1)\alpha)}$.

Thus, $I(f) = f(x)$ is bounded on $W_{a^+}^{\alpha,\rho,m}[a,b]$. The space $W_{a^+}^{\alpha,\rho,n,m}[a,b]$ is a reproducing kernel space.

Next, we verify that $K_{a^+}^{\alpha,\rho,m}(x,y)$ is a reproducing kernel of the space $W_{a^+}^{\alpha,\rho,m}[a,b]$. Note that $K_y(x) = K_{a^+}^{\alpha,\rho,m}(x,y)$. For any $u(y) \in W_{a^+}^{\alpha,\rho,m}[a,b]$, we have

$$\begin{aligned}\langle f(x), K_y(x)\rangle_{W_{a^+}^{\alpha,\rho,n}[a,b]} &= \sum_{j=0}^{m} \mathcal{D}_{a^+}^{j\alpha,\rho} f(a)\mathcal{D}_{a^+}^{j\alpha,\rho} K_y(a) + \int_a^b \mathcal{D}_{a^+}^{(m+1)\alpha,\rho} f(t)\mathcal{D}_{a^+}^{(m+1)\alpha,\rho} K_y(t)dt. \\ &= \sum_{j=0}^{m} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{j\alpha} \frac{\mathcal{D}_{a^+}^{j\alpha,\rho} f(a)}{\Gamma(j\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha)} \int_a^x (\frac{x^\rho - t^\rho}{\rho})^{(m+1)\alpha-1} \frac{\mathcal{D}_{a^+}^{(m+1)\alpha,\rho} f(t)}{t^{1-\rho}} dt, \\ &= f(y).\end{aligned} \tag{21}$$

Thus, $K_y(x) = K_{a^+}^{\alpha,\rho,m}(x,y)$ is the reproducing kernel of the space $W_{a^+}^{\alpha,\rho,m}[a,b]$. □

Similarly, we can obtain the following Theorem 3.

Theorem 3. *If space $W_{b^-}^{\alpha,\rho,m}[a,b]$ is defined as*

$$W_{a^+}^{\alpha,\rho,m}[a,b] = \{f|f \in AC_\gamma^m[a,b], \mathcal{D}_{b^-}^{(m+1)\alpha,\rho} f \in C[a,b]\}. \tag{22}$$

The inner product of space $W_{b^-}^{\alpha,\rho,m}[a,b]$ is the following form

$$\langle u(x), v(x)\rangle_{W_{b^-}^{\alpha,\rho,m}[a,b]} = \sum_{j=0}^{m} \mathcal{D}_{b^-}^{j\alpha,\rho} u(b)\mathcal{D}_{b^-}^{j\alpha,\rho} v(b) + \int_a^b \mathcal{D}_{b^-}^{(m+1)\alpha,\rho} u(t)\mathcal{D}_{b^-}^{(m+1)\alpha,\rho} v(t)dt. \tag{23}$$

The norm is $\|u(x)\| = \left(\langle u(x), v(x)\rangle_{W_{b^-}^{\alpha,\rho,m}[a,b]}\right)^{\frac{1}{2}}$. Then, the space $W_{b^-}^{\alpha,\rho,m}[a,b]$ is a reproducing kernel space, and its reproducing kernel is

$$R_{b^-}^{\alpha,\rho,m}(x,y) = \begin{cases} \sum_{j=0}^{m} \left(\frac{1}{\Gamma(j\alpha+1)}\right)^2 \left(\frac{b^\rho - x^\rho}{\rho}\right)^{j\alpha} \left(\frac{b^\rho - y^\rho}{\rho}\right)^{j\alpha} \\ + \left(\frac{1}{\Gamma((m+1)\alpha)}\right)^2 \int_x^b t^{2\rho-2} \left(\frac{(t^\rho - x^\rho)(t^\rho - y^\rho)}{\rho^2}\right)^{(m+1)\alpha-1} dt, y < x, \\ \sum_{j=0}^{m} \left(\frac{1}{\Gamma(j\alpha+1)}\right)^2 \left(\frac{b^\rho - x^\rho}{\rho}\right)^{j\alpha} \left(\frac{b^\rho - y^\rho}{\rho}\right)^{j\alpha} \\ + \left(\frac{1}{\Gamma((m+1)\alpha)}\right)^2 \int_y^b t^{2\rho-2} \left(\frac{(t^\rho - x^\rho)(t^\rho - y^\rho)}{\rho^2}\right)^{(m+1)\alpha-1} dt, x < y, \end{cases} \tag{24}$$

where $Re(\alpha) \geq 0$, $\rho > 0$, $\gamma = x^{1-\rho}\frac{d}{dx}$, $AC_\rho^1 = AC[a,b]$ and $n, m \in \mathbb{N}_0$.

Thus, we can obtain some reproducing kernels by choosing different parameters α, ρ and n in (24). These reproducing kernels are shown in Table 1.

Table 1. Some new reproducing kernel functions with $a = 0$ and $x > y$.

α	ρ	m	$K_{a^+}^{\alpha,\rho,m}(x,y)$
$\alpha = \frac{1}{2}$	$\rho = 1$	1	$\frac{4xy}{\pi} + y + 1$
$\alpha = 2$	$\rho = 1$	1	$\frac{35x^3y^4 - 21x^2y^5 + 7x(y^5+180)y - y^7 + 5040}{5040}$
$\alpha = \frac{1}{2}$	$\rho = \frac{4}{3}$	1	$\frac{9x^{4/3}y^{4/3}}{4\pi} + \frac{3y^{5/3}}{5} + 1$
$\alpha = \frac{1}{2}$	$\rho = \frac{6}{5}$	1	$\frac{25x^{6/5}y^{6/5}}{9\pi} + \frac{5y^{7/5}}{7} + 1$

Table 1. *Cont.*

α	ρ	m	$K_{a^+}^{\alpha,\rho,m}(x,y)$
$\alpha=1$	$\rho=\frac{3}{2}$	1	$\frac{2}{63}x^{3/2}(3y^2+14)y^{3/2}-\frac{4y^5}{105}+1$
$\alpha=\frac{3}{2}$	$\rho=\frac{5}{4}$	1	$\frac{256x^{5/4}y^{5/4}}{225\pi}+\frac{16y^4\left(-52x^{5/4}y^{5/4}+91x^{5/2}+11y^{5/2}\right)}{75075}+1$
$\alpha=2$	$\rho=\frac{3}{2}$	1	$\frac{1}{9}x^{3/2}y^{3/2}+\frac{2\left(231x^{3/2}y^{19/2}+836x^{9/2}y^{13/2}-627x^3y^8-35y^{11}\right)}{7702695}+1$
$\alpha=\frac{1}{3}$	$\rho=1$	2	$\frac{x^2y^2}{\Gamma\left(\frac{5}{3}\right)^2}+\frac{xy}{\Gamma\left(\frac{4}{3}\right)^2}+y+1$
$\alpha=\frac{1}{3}$	$\rho=\frac{5}{4}$	2	$\frac{256x^{5/2}y^{5/2}}{625\Gamma\left(\frac{5}{3}\right)^2}+\frac{16x^{5/4}y^{5/4}}{25\Gamma\left(\frac{4}{3}\right)^2}+\frac{2y^{3/2}}{3}+1$
$\alpha=\frac{1}{2}$	$\rho=1$	2	$x^2y^2+\frac{4xy}{\pi}+\frac{2\sqrt{x}\sqrt{y}(x+y)-(x-y)^2\left(2\log\left(\sqrt{x}+\sqrt{y}\right)-\log(x-y)\right)}{2\pi}+1$
$\alpha=1$	$\rho=\frac{3}{2}$	2	$-\frac{4x^{3/2}\left(4y^5-455\right)y^{3/2}}{4095}+\frac{2x^3\left(9y^2+70\right)y^3}{2835}+\frac{y^8}{1170}+1$
$\alpha=1$	$\rho=\frac{3}{2}$	3	$\frac{2x^{3/2}\left(y^8+7410\right)y^{3/2}}{33345}+\frac{8x^{9/2}\left(81y^2+910\right)y^{9/2}}{2985255}+x^3\left(\frac{4y^3}{81}-\frac{2y^8}{12285}\right)-\frac{2y^{11}}{220077}+1$
$\alpha=\frac{1}{2}$	$\rho=1$	3	$x^2y^2+\frac{4xy\left(4x^2y^2+9\right)}{9\pi}+\frac{xy^2}{2}-\frac{y^3}{6}+1$
$\alpha=\frac{1}{2}$	$\rho=1$	4	$\frac{x^4y^4}{4}+\frac{16x^3y^3}{9\pi}+x^2y^2+\frac{3(x-y)^4\left(2\log\left(\sqrt{x}+\sqrt{y}\right)-\log(x-y)\right)-2\sqrt{x}\sqrt{y}(x+y)\left(3x^2-14xy+3y^2\right)}{72\pi}+\frac{4xy}{\pi}+1$
$\alpha=\frac{1}{2}$	$\rho=1$	5	$\frac{32xy\left(16x^4y^4+100x^2y^2+225\right)+15\pi\left(30x^4y^4+10x^2(y+12)y^2-5xy^4+y^5+120\right)}{1800\pi}$
$\alpha=\frac{1}{2}$	$\rho=\frac{4}{3}$	5	$-\frac{27x^{4/3}y^{17/3}}{3536}+\frac{27x^{8/3}y^{8/3}\left(8y^{5/3}+195\right)}{16640}+\frac{6561x^{16/3}y^{16/3}}{262144}+\frac{9}{4\pi}x^{4/3}y^{4/3}+\frac{6561x^{20/3}y^{20/3}}{409600\pi}+\frac{81x^4y^4}{256\pi}+\frac{81y^7}{49504}+1$

4. Representation of Solutions

Next, we give analytical and numerical solutions of the two-sided space-fractional super-diffusive model based on this class of RKS. Let

$$\mathbb{L}_{(x,t)}u(x,t)=u_t(x,t)-C_+(x,t)\mathcal{D}_{x,a^+}^{\alpha,\rho}u(x,t)-C_-(x,t)\mathcal{D}_{x,b^-}^{\alpha,\rho}u(x,t). \tag{25}$$

Equation (1) is converted to the following form:

$$\begin{cases}\mathbb{L}_{(x,t)}u(x,t)=g(x,t),(x,t)\in\Omega=[a,b]\times[0,T],\\ u(x,0)=0,0\leq x\leq T,\\ u(a,t)=0,u(b,t)=0,0\leq t\leq T.\end{cases} \tag{26}$$

First, we calculate the reproducing kernel for Equation (26).

Using method of [15], we can obtain the following reproducing kernel $K(x,y,s,t)$ for Equation (26).

$$K(x,y,s,t)=\tilde{\tilde{K}}_{a^+}^{\alpha,\rho,m}(x,y)\times\overline{K}_{0^+}^{1,1,2}(s,t),$$

where

$$\tilde{K}_{a^+}^{\alpha,\rho,m}(x,y)=K_{a^+}^{\alpha,\rho,m}(x,y)-\frac{K_{a^+}^{\alpha,\rho,m}(a,y)\times K_{a^+}^{\alpha,\rho,m}(x,a)}{K_{a^+}^{\alpha,\rho,m}(a,a)}.$$

$$\tilde{\tilde{K}}_{a^+}^{\alpha,\rho,m}(x,y)=\tilde{K}_{a^+}^{\alpha,\rho,m}(x,y)-\frac{\tilde{K}_{a^+}^{\alpha,\rho,m}(b,y)\times\tilde{K}_{a^+}^{\alpha,\rho,m}(x,b)}{\tilde{K}_{a^+}^{\alpha,\rho,m}(b,b)}.$$

$$\overline{K}_{0^+}^{1,1,2}(s,t)=K_{0^+}^{1,1,2}(s,t)-\frac{K_{0^+}^{1,1,2}(0,t)\times K_{0^+}^{1,1,2}(s,0)}{K_{0^+}^{1,1,2}(0,0)}.$$

Secondly, we find an analytical solution for Equation (26).

Theorem 4. *If $\mathbb{L}^{-1}$ is existing and $\{x_i,t_i\}_{i=1}^{\infty}$ is denumerable dense points in Ω, let*

$$\begin{aligned}\psi_i(x,t)&=\mathbb{L}_{(y,s)}K(x,y,s,t)|_{(y,s)=(x_i,t_i)},\\ &=\left(\frac{\partial}{\partial s}K(x,y,s,t)-C_+(x,t)\mathcal{D}_{y,a^+}^{\alpha,\rho}K(x,y,s,t)-C_-(x,t)\mathcal{D}_{y,b^-}^{\alpha,\rho}K(x,y,s,t)\right)|_{(y,s)=(x_i,t_i)},\end{aligned} \tag{27}$$

$$\overline{\psi}_i(x,t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x,t), (\beta_{ii} > 0, i = 1,2,\cdots,\infty),$$

where the β_{ik} *are the Gram–Schmidt orthogonalization coefficients. Then,*

$$u(x,t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k,t_k)\overline{\psi}_i(x,t) \tag{28}$$

is an analytical solution of Equation (26).

Proof. $u(x,t)$ can be expanded to a Fourier series in terms of the normal orthogonal basis $\{\bar{\psi}_i(x,t)\}_{i=1}^{\infty}$,

$$\begin{aligned}
u(x,t) &= \sum_{i=1}^{\infty}\langle u(x,t), \bar{\psi}_i(x,t)\rangle \bar{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle u(x,t), \psi_k(x,t)\rangle \bar{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle u(x,t), \mathbb{L}_{(y,s)}K(x,y,s,t)|_{(y,s)=(x_k,t_k)}\rangle \bar{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(\mathbb{L}_{(y,s)}\langle u(x,t), K(x,y,s,t)\rangle)|_{(y,s)=(x_k,t_k)} \bar{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(\mathbb{L}_{(y,s)}u(y,s))|_{(y,s)=(x_k,t_k)} \bar{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k,t_k)\bar{\psi}_i(x,t).
\end{aligned}$$

□

Finally, we find a numerical solution for Equation (26).

Deriving from the form of (28), we obtain an N-term numerical solution of Equation (26) as

$$v_N(x,t) = \sum_{i=1}^{N}\sum_{k=1}^{i} \beta_{ik} m(x_k,t_k)\overline{\psi}_i(x,t). \tag{29}$$

However, the direct application of (29) could possibly not have a good numerical simulation effect for Equation (26). The focus of this paper is to fill this gap; thus, we combine the piecewise method with RKM.

Dividing $t \in [0,1]$ into n, the equal subintervals $[t_i, t_{i+1}]$. Let $h = \frac{1}{n}$, $t_i = ih$, $i = 0,1,2,\ldots,n-1$. On the subregion $\Omega_1 = [0,1] \times [t_i, t_{i+1}]$, stretching $[t_i, t_{i+1}]$ to $[0,1]$, (26) is turned into

$$\begin{cases} \mathbb{L}_{(x,t_i)}v(x,t_i) = g(x,t_1), (x,t_i) \in \Omega = [0,1] \times [0,1], \\ v(x,0) = 0, 0 \le x \le 1, \\ v(0,t_i) = 0, v(1,t_i) = 0, 0 \le t_1 \le 1. \end{cases} \tag{30}$$

Solving (30) by the RKM, we can obtain the $v_{i,N}(x,t)$ of (26) on $[0,1] \times [t_i, t_{i+1}]$. Clearly, $v_{i,N}(x,t)$ can provide an approximate solution of $C(x,t_i)$. Combining all these solutions, we can obtain the approximate solution $C_N(x,t)$ of (1).

Regarding the convergence analysis and error estimation, those detailed proofs can be seen in [8,10,11].

5. Numerical Experiment

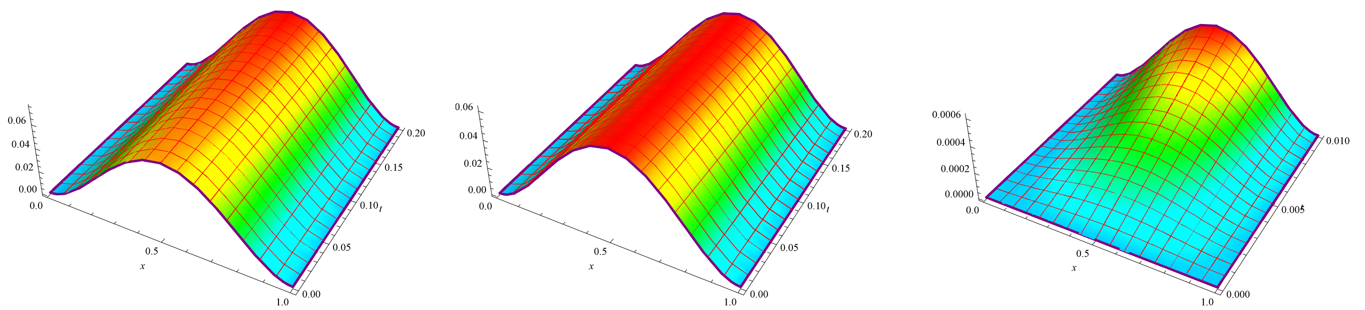
In order to prove the accuracy and effectiveness of this method, the results of numerical simulations are presented.

Experiment 1. *Consider the model (1),* $C_+(x,t) = \Gamma(1.7)x^{1.3}$, $C_-(x,t) = \Gamma(1.7)(1-x)^{1.3}$, *and then model (1) is the following form:*

$$\begin{cases} u_t(x,t) = \Gamma(1.7)x^{1.3}\mathcal{D}^{\alpha,\rho}_{x,0^+}u(x,t) + \Gamma(1.7)(1-x)^{1.3}\mathcal{D}^{\alpha,\rho}_{x,1^-}u(x,t) + g(x,t), \\ u(x,0) = 4x^2(1-x)^2, 0 \le x \le 1, \\ u(0,t) = 0, u(1,t) = 0, 0 \le t \le 1. \end{cases} \tag{31}$$

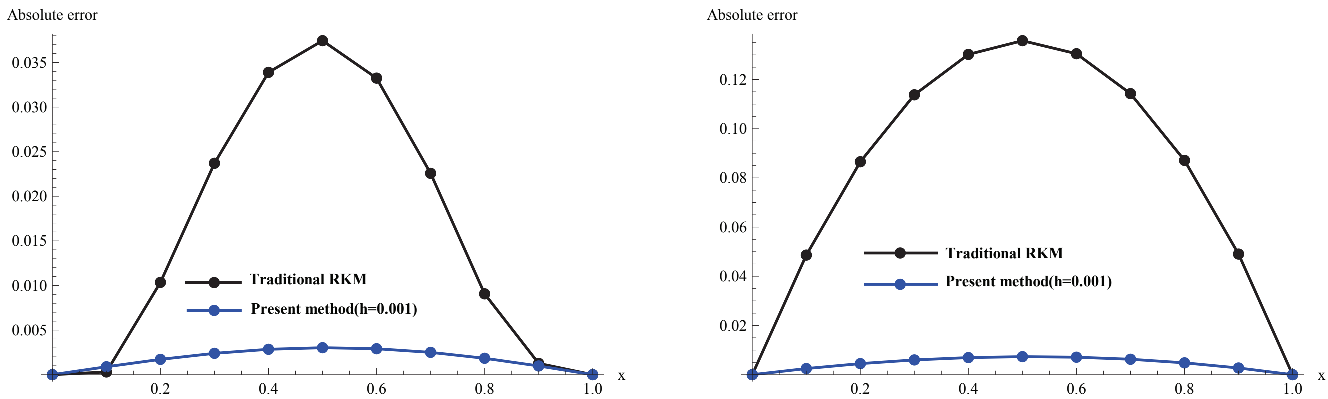
The exact solution is $ue(x,t) = 4x^2(1-x)^2(t+1)$, where $g(x,t) = 4x^2(1-x)^2(t+1) - \Gamma(1.7)x^{1.3}\mathcal{D}^{\alpha,\rho}_{x,0^+}(4x^2(1-x)^2(t+1)) - \Gamma(1.7)(1-x)^{1.3}\mathcal{D}^{\alpha,\rho}_{x,2^-}(4x^2(1-x)^2(t+1))$.

The numerical results are shown in Figures 1–3 and Table 2. Table 2 shows the absolute errors in two different reproducing kernel spaces. The traditional reproducing kernel method refers to the reproducing kernel method of [15], and the present method refers to the piecewise reproducing kernel method. It can be seen that, for Experiment 1, the numerical results obtained by two reproducing kernel spaces are feasible and effective. From Figures 1–3 and Table 2, it can be seen more intuitively that the smaller the value of h is, the smaller the error.



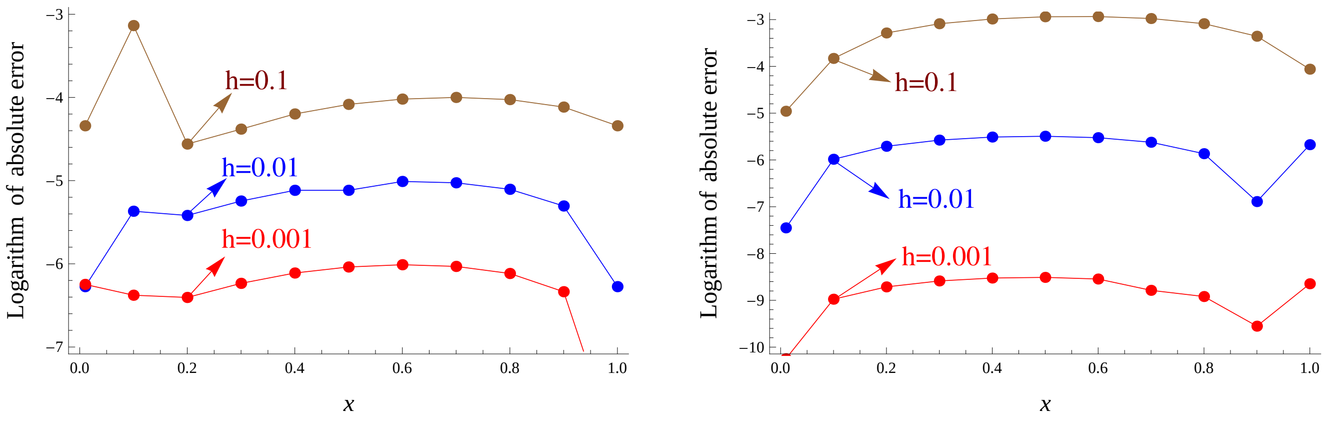
(**a**) The exact solution. (**b**) The numerical solution. (**c**) The absolute error.

Figure 1. The numerical results at $\alpha = 1.3, \rho = 1, m = 2$. The exact solution is $ue(x,t)$, the numerical solution is $u_4(x,t)$, and the absolute error is $|ue(x,t) - u_4(x,t)|$.



(**a**) (**b**)

Figure 2. Comparison of the absolute errors for two reproducing kernel methods at $t = 0.01$ and different reproducing kernel spaces. (**a**) The absolute errors in reproducing kernel space 1 ($\alpha = 1$, $\rho = 1, andm = 2$). (**b**) The absolute errors in reproducing kernel space 2 ($\alpha = \frac{1}{3}, \rho = \frac{4}{5}, andm = 2$).



(**a**) (**b**)

Figure 3. The logarithm of the absolute error at different reproducing kernel spaces $t = 0.01$. (**a**) The logarithm of the absolute error in reproducing kernel space 1 ($\alpha = 1, \rho = 1, m = 2$). (**b**) The logarithm of the absolute error in reproducing kernel space 2 ($\alpha = \frac{1}{3}, \rho = \frac{4}{5}, m = 2$).

Table 2. Comparison of the absolute errors for Experiment 1 at $t = 0.1$ (Two forms of the reproducing kernel spaces).

(x, t)	Reproducing Kernel 1 ($\alpha = 1, \rho = 1, m = 2$)			Reproducing Kernel 2 ($\alpha = \frac{1}{3}, \rho = \frac{4}{5}, m = 2$)		
	$h = 0.1$	$h = 0.01$	$h = 0.001$	$h = 0.1$	$h = 0.01$	$h = 0.001$
(0.1, 0.1)	1.4440×10^{-2}	2.8809×10^{-3}	3.0098×10^{-4}	1.4464×10^{-3}	1.0353×10^{-3}	1.0435×10^{-4}
(0.3, 0.3)	2.0026×10^{-1}	2.2921×10^{-2}	2.3158×10^{-3}	3.6627×10^{-2}	1.1251×10^{-3}	1.1362×10^{-4}
(0.5, 0.5)	4.8845×10^{-1}	5.0010×10^{-2}	5.0002×10^{-3}	1.1624×10^{-2}	3.1989×10^{-3}	3.0835×10^{-4}
(0.6, 0.6)	6.1882×10^{-1}	6.0286×10^{-2}	5.9944×10^{-3}	1.5634×10^{-2}	4.0257×10^{-3}	3.8357×10^{-4}
(0.7, 0.7)	7.0611×10^{-1}	6.4478×10^{-2}	6.3604×10^{-3}	1.8338×10^{-2}	4.1408×10^{-3}	3.8142×10^{-4}
(0.9, 0.9)	6.3208×10^{-1}	4.0652×10^{-2}	3.7896×10^{-3}	1.1447×10^{-2}	3.4362×10^{-3}	7.5174×10^{-4}
(1.0, 1.0)	4.0413×10^{-1}	4.7204×10^{-3}	4.7920×10^{-5}	2.8127×10^{-3}	6.8379×10^{-3}	7.3464×10^{-4}

6. Conclusions and Remarks

In this paper, we provided some new fractional reproducing kernel spaces. The analytical solution of the two-sided space-fractional super-diffusive model was represented in the form of series based on these new reproducing kernel spaces. The N-term numerical solution of the two-sided space-fractional super-diffusive model was obtained. For more accuracy, we used the piecewise reproducing kernel method.

In Figure 1, the exact solution and N-term numerical solution solution are compared. In Figure 2, comparisons are made between the traditional reproducing kernel method and the piecewise reproducing kernel method. Figure 3 and Table 2 show the advantages of the piecewise reproducing kernel method. We found that these new fractional reproducing kernel spaces were efficient for the two-sided space-fractional super-diffusive model.

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