



Article

# Well-Posedness and Global Attractors for Viscous Fractional Cahn–Hilliard Equations with Memory

Eylem Öztürk<sup>1</sup> and Joseph L. Shomberg<sup>2,\*</sup> <sup>1</sup> Department of Mathematics, Hacettepe University, Beytepe, Ankara 06800, Turkey<sup>2</sup> Department of Mathematics and Computer Science, Providence College, Providence, RI 02918, USA

\* Correspondence: jshomber@providence.edu

**Abstract:** We examine a viscous Cahn–Hilliard phase-separation model with memory and where the chemical potential possesses a nonlocal fractional Laplacian operator. The existence of global weak solutions is proven using a Galerkin approximation scheme. A continuous dependence estimate provides uniqueness of the weak solutions and also serves to define a precompact pseudometric. This, in addition to the existence of a bounded absorbing set, shows that the associated semigroup of solution operators admits a compact connected global attractor in the weak energy phase space. The minimal assumptions on the nonlinear potential allow for arbitrary polynomial growth.

**Keywords:** Cahn–Hilliard equation; fractional Laplacian; memory

## 1. Introduction

Let  $\Omega$  be a smooth (at least Lipschitz) bounded domain in  $\mathbb{R}^N$ ,  $N = 3, 2, 1$ , with boundary  $\partial\Omega$  and let  $T > 0$ . We consider the following viscous fractional Cahn–Hilliard equation in the unknown (order parameter)  $u$  satisfying

$$\partial_t u(t, x) = \int_0^\infty k(s) \Delta^\beta u(t-s, x) ds \quad \text{in } \Omega \times (0, T), \quad (1)$$

$k$  is a so-called relaxation kernel, with a chemical potential  $\mu$  given by

$$\mu(t, x) = \alpha \partial_t u(t, x) + (-\Delta)^\beta u(t, x) + F'(u(t, x)) \quad \text{in } \Omega \times \mathbb{R}, \quad (2)$$

$\alpha > 0$ ,  $\beta \in (0, 1)$ , and typically,  $F$  is a double-well potential (the precise assumptions on  $F$  are stated in (N1)–(N3) below), subject to the boundary conditions

$$u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega \times (0, T) \quad \text{and} \quad \partial_n \mu = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (3)$$

with the given initial and past conditions

$$u(0) = u_0(0) \quad \text{in } \Omega \quad \text{and} \quad u(-t) = u_0(-t) \quad \text{in } \Omega \times [0, T), \quad (4)$$

for

$$u_0 : \Omega \times (-\infty, 0) \rightarrow \mathbb{R}.$$

Here, we define  $(-\Delta)^\beta$  with  $0 < \beta < 1$  as the (nonlocal) fractional Laplace operator. In other words, let  $\Omega \subset \mathbb{R}^N$  be an arbitrary open set and fix

$$\mathcal{L}^1(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable, } \int_\Omega \frac{|u(x)|}{(1+|x|)^{N+2\beta}} dx < \infty \right\}.$$

For  $u \in \mathcal{L}^1(\mathbb{R}^N)$ ,  $x \in \mathbb{R}^N$ , and  $\varepsilon > 0$ , we write

$$(-\Delta)_\varepsilon^\beta u(x) = C_{N,\beta} \int_{\{y \in \mathbb{R}^N, |y-x| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2\beta}} dy$$



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with the normalized constant  $C_{N,\beta}$  given by

$$C_{N,\beta} = \frac{\beta 2^{2\beta} \Gamma\left(\frac{N+2\beta}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-\beta)}, \quad (5)$$

where  $\Gamma$  denotes the usual gamma function. The (restricted) fractional Laplacian  $(-\Delta)^\beta u$  of the function  $u$  is defined by the formula

$$(-\Delta)^\beta u(x) = C_{N,\beta} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\beta}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^\beta u(x), \quad x \in \mathbb{R}^N, \quad (6)$$

provided that the limit exists. We call  $A^\beta$  the self-adjoint realization of the fractional Laplacian  $(-\Delta)^\beta$  with Dirichlet boundary condition (3)<sub>1</sub>, see, e.g., [1] (Section 2.2) (see also [2]).

Some remarks: First, observe the chemical potential (2) involves the Neumann (no-flux) condition described by (3). Hence, when the memory function  $k$  is close to the Dirac delta function, we recover the usual parabolic equation associated with the Cahn–Hilliard equation with the flux-free chemical potential.

Naturally, we are also interested in the closely related problem to (1)–(4) whereby the fractional Laplace operator  $(-\Delta)^\beta$  is replaced with the regional fractional Laplacian,  $A_{\Omega,\varepsilon}^\beta$ , defined by first setting

$$A_{\Omega,\varepsilon}^\beta u(x) = C_{N,\beta} \int_{\{y \in \Omega, |y-x| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2\beta}} dy,$$

where  $C_{N,\beta}$  is given by (5), then

$$A_\Omega^\beta u(x) = C_{N,\beta} \text{P.V.} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2\beta}} dy = \lim_{\varepsilon \downarrow 0} A_{\Omega,\varepsilon}^\beta u(x), \quad x \in \Omega, \quad (7)$$

provided that the limit exists. Assuming  $u \in \mathcal{D}(\Omega)$  (see [1] (page 1280)) then the two fractional Laplacian operators are related by

$$(-\Delta)^\beta u(x) = A_\Omega^\beta u(x) + V_\Omega(x)u(x), \quad \forall u \in \mathcal{D}(\Omega) \quad (8)$$

with the following potential

$$V_\Omega(x) := C_{N,\beta} \int_{\mathbb{R}^N \setminus \Omega} \frac{dy}{|x-y|^{N+2\beta}}, \quad x \in \Omega. \quad (9)$$

The comparable Cahn–Hilliard problem with the regional fractional Laplacian is then (1) with the chemical potential

$$\mu = \alpha \partial_t u + A_\Omega^\beta u + F'(u) \quad \text{in } \Omega \times (0, T), \quad (10)$$

now subject to the boundary conditions

$$u = 0 \quad \text{on } \partial\Omega \times (0, T) \quad \text{and} \quad \partial_n \mu = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (11)$$

with the above initial and past conditions in (4). Our focus here is on obtaining results for the restricted fractional Laplacian, of which the regional counterpart can be viewed as a perturbation thanks to (8). The restricted fractional Laplacian appears in the context of nonlocal phase transitions with Dirichlet boundary conditions in [3,4]. On the other hand, the regional fractional Laplacian is generally better suited to treat problems with nonhomogeneous boundary data and even dynamic boundary conditions (see [1,5] and the references therein).

It should also be noted that we only consider the viscous case, where  $\alpha > 0$ , since the nonviscous counterpart  $\alpha = 0$  inherits no added regularity for  $\alpha \partial_t \phi$ .

Inside a bounded container  $\Omega \subset \mathbb{R}^3$ , the Cahn–Hilliard equation (see [6]) is a phase separation model for a binary solution (e.g., a cooling alloy, glass, or polymer),

$$\partial_t u = \nabla \cdot [\kappa(u) \nabla \mu],$$

where  $u$  is the *order-parameter* (the relative difference of the two phases),  $\kappa$  is the *mobility function* (we set  $\kappa \equiv 1$  throughout this article), and  $\mu$  is the *chemical potential* (the first variation of the free-energy  $E$  with respect to  $u$ ). In the classical model,

$$\mu = -\Delta u + F'(u) \quad \text{and} \quad E(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) dx,$$

where  $F$  describes the density of potential energy in  $\Omega$  (e.g., the double-well potential  $F(s) = (1 - s^2)^2$ ).

Recently the nonlocal free-energy functional has appeared in the literature [7],

$$E(\phi) = \int_{\Omega} \int_{\Omega} \frac{1}{4} J(x - y) (\phi(x) - \phi(y))^2 dx dy + \int_{\Omega} F(\phi) dx,$$

hence, the *chemical potential* is,

$$\mu = a\phi - J * \phi + F'(\phi), \tag{12}$$

where

$$a(x) = \int_{\Omega} J(x - y) dy \quad \text{and} \quad (J * \phi)(x) = \int_{\Omega} J(x - y) \phi(y) dy. \tag{13}$$

In view of [8,9], the nonlocality expressed in (12)–(13) (see also [10–19]) is termed *weak* while the type under consideration here in (2) and (6) is called *strong*. Under certain conditions the strong type reduces to the weak (see [8], and also see [7]). Recently there has been much interest in the nonlocal Cahn–Hilliard equation with strong interactions of the restricted fractional Laplacian type (6) and the regional fractional Laplacian type (7) (see [3,5,8,9,20]). The results in these references concern global well-posedness, and when available, the existence of finite dimensional global attractors and regularity.

Additionally, there has been exceptional growth concerning dissipative infinite-dimensional systems with *memory* including models arising in the theory of heat conduction in special materials (see, e.g., [21–25]) and the theory of phase-transitions (see, e.g., [26–34]). One feature of equations that undergo “memory relaxation” is the admissibility of a so-called *inertia* term. For example, (see, e.g., [35]) the first-order equation with memory

$$u_t(t) + \int_0^{\infty} k_{\varepsilon}(s) f(u(t - s)) ds = 0$$

for

$$k_{\varepsilon}(s) = \frac{1}{\varepsilon} e^{-s/\varepsilon}$$

leads us (formally) to the “hyperbolic relaxation” equation

$$\varepsilon u_{tt}(t) + u_t(t) + f(u(t)) = 0.$$

In this way, our model also includes the viscous Cahn–Hilliard equation with inertial term (see [36]). Hence, the novelty in the present work is a relaxation of a phase-field model with a strongly interacting nonlocal diffusion mechanism.

In this article, our aims were:

- To provide a framework to establish the global (in time) well-posedness of the model problems (1)–(4) and (1), (4), (10), and (11).
- To prove the semigroup of solution operators admits a compact global attractor.

In order to reach these aims, we require sufficient growth conditions on  $F$  (given below) in order to employ a Galerkin scheme with suitable a priori estimates. With a finite energy phase space identified, a one-parameter family of solution operators is defined, hence generating a semidynamical system. This semigroup is dissipative on the energy phase space and also defines an  $\alpha$ -contraction on the phase space. The existence of a compact global attractor follows.

## 2. Past History Formulation and Functional Setup

We now introduce the well-established past history approach from [37] (see also [27,29]) by defining the past history variable, for all  $s > 0$  and  $t > 0$ ,

$$\eta^t(x, s) = \int_0^s -\Delta\mu(x, t - \sigma)d\sigma. \quad (14)$$

Observe that  $\eta$  satisfies the boundary condition

$$\eta^t(x, 0) = 0 \quad \text{on} \quad \Omega \times (0, \infty). \quad (15)$$

When  $k$  is sufficiently smooth and vanishes at  $+\infty$  (these assumptions will be made more precise below), then integration by parts yields

$$\int_0^\infty k(s)\Delta\mu(x, t - s)ds = - \int_0^\infty v(s)\eta^t(x, s)ds$$

where  $v(s) = -k'(s)$ .

We may now formulate the model problem (1)–(4) as:

**Problem P.** Find  $(u, \eta) = (u(x, t), \eta^t(x, s))$  on  $(0, \infty)$  such that

$$\partial_t u(x, t) + \int_0^\infty v(s)\eta^t(x, s)ds = 0 \quad \text{in} \quad \Omega \times (0, \infty) \quad (16)$$

$$\mu(x, t) = \alpha\partial_t u(x, t) + (-\Delta)^\beta u(x, t) + F'(u(x, t)) \quad \text{in} \quad \Omega \times (0, \infty) \quad (17)$$

$$\partial_t \eta^t(x, s) + \partial_s \eta^t(x, s) = -\Delta\mu(x, t) \quad \text{in} \quad \Omega \times (0, \infty) \times (0, \infty) \quad (18)$$

held subject to (3) and (15), and satisfying the initial conditions (4)<sub>1</sub> and

$$\eta^0(x, s) = \eta_0(x, s) \quad \text{in} \quad \Omega \times (0, \infty), \quad (19)$$

whereby with (14),

$$\eta_0(x, s) = \int_0^s -\Delta\mu_0(x, -y)dy \quad \text{in} \quad \Omega \times (0, \infty), \quad (20)$$

where in light of (4)<sub>2</sub>,

$$\mu_0(x, t) = \alpha\partial_t u_0(x, t) + (-\Delta)^\beta u_0(x, t) + F'(u_0(x, t)) \quad \text{for} \quad t \leq 0. \quad (21)$$

Additionally, we are also interested in treating the related problem where the above fractional Laplace operator  $(-\Delta)^\beta$  is replaced with the regional counterpart  $A_\Omega^\beta$ . Hence, the formulation of the related regional Problem P is based on (1), (4), (10), and (11).

Here, we introduce some notation. From now on, we denote by  $\|\cdot\|_X$  the norm in the specified (real) Banach space  $X$ , and  $(\cdot, \cdot)_Y$  denotes the product on the specified (real) Hilbert space  $Y$ . The dual pairing between  $Y$  and the dual  $Y^*$  is denoted by  $\langle u, v \rangle_{Y^* \times Y}$ . The set  $\Omega$  is omitted from the space when we indicate the norm. We denote the measure of the domain  $\Omega$  by  $|\Omega|$ . In many calculations, functional notation indicating dependence on

the variable  $t$  is dropped; for example, we write  $u$  in place of  $u(t)$  or  $\eta^t$  in place of  $\eta^t(s)$ . Throughout the paper,  $C$  denotes a *generic* positive constant, while  $Q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  denotes a *generic* increasing function. Such generic terms may or may not indicate dependencies on the (physical) parameters of the model problem, and may even change from line to line.

Let us define the linear operator  $A_N := -\Delta$  on  $D(A_N) = \{\psi \in H^2(\Omega) : \partial_n \psi = 0 \text{ on } \partial\Omega\}$ , as the realization in  $L^2(\Omega)$  of the Laplace operator endowed with Neumann boundary conditions. Here,  $-\Delta$  denotes the usual (local) Laplace operator. It is well-known that  $A_N$  is the generator of a bounded analytic semigroup  $e^{-A_N t}$  on  $L^2(\Omega)$ . Additionally,  $A_N$  is nonnegative and self-adjoint on  $L^2(\Omega)$ . With  $H^{-r}(\Omega) := (H^r(\Omega))^*$ ,  $r \in \mathbb{N}_+$ , denote by  $\langle \cdot \rangle$  the spatial average over  $\Omega$ , i.e.,

$$\langle \psi \rangle := \frac{1}{|\Omega|} \langle \psi, 1 \rangle_{H^{-r} \times H^r}.$$

We set  $H^r_{(0)}(\Omega) = \{\psi \in H^r(\Omega) : \langle \psi \rangle = 0\}$ ,  $H^0(\Omega) = L^2(\Omega)$ , and we know that  $A_N^{-1} : H^0_{(0)}(\Omega) \rightarrow H^0_{(0)}(\Omega)$  is a well-defined mapping. We refer to the following norms in  $H^{-r}(\Omega)$  (which are equivalent to the usual norms)

$$\|\psi\|_{H^{-r}}^2 = \|A_N^{-r/2}(\psi - \langle \psi \rangle)\|^2 + |\langle \psi \rangle|^2. \tag{22}$$

The Sobolev space  $H^1(\Omega)$  is endowed with the norm,

$$\|\psi\|_{H^1}^2 := \|\nabla \psi\|^2 + \langle \psi \rangle^2. \tag{23}$$

Denote by  $\lambda_\Omega > 0$  the constant in the Poincaré–Wirtinger inequality,

$$\|\psi - \langle \psi \rangle\| \leq \sqrt{\lambda_\Omega} \|\nabla \psi\|. \tag{24}$$

Whence, for  $\lambda_\Omega^* := \max\{\lambda_\Omega, 1\}$ , there holds, for all  $\psi \in H^1(\Omega)$ ,

$$\begin{aligned} \|\psi\|^2 &\leq \lambda_\Omega \|\nabla \psi\|^2 + \langle \psi \rangle^2 \\ &\leq \lambda_\Omega^* \|\psi\|_{H^1}^2. \end{aligned} \tag{25}$$

We now more rigorously describe the fractional Laplacian with Dirichlet boundary conditions. For an arbitrary bounded domain  $\Omega \subset \mathbb{R}^N$  and for  $\beta \in (0, 1)$ , denote the fractional-order Sobolev space by,

$$W^{\beta,2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\beta}} dx dy < \infty \right\},$$

to be equipped with the norm

$$\|u\|_{W^{\beta,2}} := \left( \int_\Omega |u(x)|^2 dx + \frac{C_{N,\beta}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\beta}} dx dy \right)^{1/2},$$

where  $C_{N,\beta}$  is given by (5). Let

$$W_0^{\beta,2}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{\beta,2}(\Omega)}.$$

Hence,  $W_0^{\beta,2}(\Omega)$  is a closed subspace of  $W^{\beta,2}(\Omega)$  containing  $\mathcal{D}(\Omega)$ . Moreover, thanks to [38] (Theorem 10.1.1),

$$W_0^{\beta,2}(\Omega) = \{u \in W^{\beta,2}(\mathbb{R}^N) : \tilde{u} = 0 \text{ on } \mathbb{R}^N \setminus \Omega\},$$

where  $\tilde{u}$  is the quasi-continuous version (with respect to the capacity defined with the space  $W^{\beta,2}(\Omega)$ ) of  $u$ . One may easily show that the following defines an equivalent norm on the space  $W_0^{\beta,2}(\Omega)$ ,

$$\begin{aligned} \|u\|_{W_0^{\beta,2}}^2 &= \frac{C_{N,\beta}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\beta}} dx dy + \int_{\Omega} V_{\Omega}(x) |u(x)|^2 dx \\ &= \frac{C_{N,\beta}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\beta}} dx dy. \end{aligned} \tag{26}$$

Here,  $V_{\Omega}$  is the potential (9).

**Remark 1.** Either definition of the space  $W_0^{\beta,2}(\Omega)$  makes sense for any arbitrary open set  $\Omega \subset \mathbb{R}^3$  (not necessarily bounded). Furthermore, if  $\Omega$  has a Lipschitz boundary, then by [39],  $W_0^{\beta,2}(\Omega) = W^{\beta,2}(\Omega)$  for every  $0 < \beta \leq \frac{1}{2}$ .

From now on, we write  $u \in W_0^{\beta,2}(\Omega)$  to mean  $u \in W^{\beta,2}(\mathbb{R}^N)$  and  $u = 0$  on  $\mathbb{R}^N \setminus \Omega$ . Let  $a_{E,\beta}$  be the bilinear symmetric closed form with domain  $D(a_{E,\beta}) = W_0^{\beta,2}(\Omega)$  and defined for  $u, v \in W_0^{\beta,2}(\Omega)$  by

$$\begin{aligned} a_{E,\beta}(u, v) &= \frac{C_{N,\beta}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2\beta}} dx dy + \int_{\Omega} V_{\Omega}(x) u(x) v(x) dx \\ &= \frac{C_{N,\beta}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2\beta}} dx dy. \end{aligned} \tag{27}$$

Let  $A_{E,\beta}$  be the closed linear self-adjoint operator on  $L^2(\Omega)$  associated with  $a_{E,\beta}$  by

$$\begin{cases} D(A_{E,\beta}) := \{u \in W_0^{\beta,2}(\Omega) : \exists v \in L^2(\Omega), a_{E,\beta}(u, \varphi) = (v, \varphi) \forall \varphi \in W_0^{\beta,2}(\Omega)\} \\ A_{E,\beta}u = v. \end{cases} \tag{28}$$

According to [1] (Proposition 2.2), the operator  $A_{E,\beta}$  on  $L^2(\Omega)$  associated with the bilinear form  $a_{E,\beta}$  is given by

$$D(A_{E,\beta}) := \{u \in W_0^{\beta,2}(\Omega) : (-\Delta)_E^{\beta} u \in L^2(\Omega)\} \quad \text{and} \quad \forall u \in D(A_{E,\beta}), \quad A_{E,\beta}u := (-\Delta)_E^{\beta} u. \tag{29}$$

Observe that comparing (6) and (26)–(29) shows, for all  $u \in D(A_{E,\beta})$ ,

$$((-\Delta)_E^{\beta} u, u) = a_{E,\beta}(u, u) = \|u\|_{W_0^{\beta,2}}^2. \tag{30}$$

Concerning the related regional problem discussed above, we let  $a_{D,\beta}$  be the bilinear symmetric closed form with domain  $D(a_{D,\beta}) = W_0^{\beta,2}(\Omega)$  and defined for  $u, v \in W_0^{\beta,2}(\Omega)$  by

$$a_{D,\beta}(u, v) = \frac{C_{N,\beta}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2\beta}} dx dy. \tag{31}$$

Let  $A_{D,\beta}$  be the closed linear self-adjoint operator on  $L^2(\Omega)$  associated with  $a_{D,\beta}$  by

$$\begin{cases} D(A_{D,\beta}) := \{u \in W_0^{\beta,2}(\Omega) : \exists v \in L^2(\Omega), a_{D,\beta}(u, \varphi) = (v, \varphi) \forall \varphi \in W_0^{\beta,2}(\Omega)\} \\ A_{D,\beta}u = v. \end{cases} \tag{32}$$

Then, by [1] (Proposition 2.3), the operator  $A_{D,\beta}$  on  $L^2(\Omega)$  associated with the bilinear form  $a_{D,\beta}$  is given by

$$D(A_{D,\beta}) := \{u \in W_0^{\beta,2}(\Omega) : A_\Omega^\beta u \in L^2(\Omega)\} \quad \text{and} \quad \forall u \in D(A_{D,\beta}), \quad A_{D,\beta}u := A_\Omega^\beta u. \tag{33}$$

We introduce the spaces for the memory variable  $\eta$ . First, the product in  $H^\sigma(\Omega)$  for  $\sigma \in \mathbb{R}$  and  $u_1, u_2 \in H^\sigma(\Omega)$  is defined by

$$(u_1, u_2)_{H^\sigma} = (A_N^{\sigma/2}u_1, A_N^{\sigma/2}u_2). \tag{34}$$

For a nonnegative measurable function  $\theta$  defined on  $\mathbb{R}_+$  and for a Hilbert space  $W$  (with inner-product  $(\cdot, \cdot)_W$ ), let  $L_\theta^2(\mathbb{R}_+; W)$  be the Hilbert space of  $W$ -valued functions on  $\mathbb{R}_+$  equipped with the following product,

$$(\phi_1, \phi_2)_{L_\theta^2(\mathbb{R}_+; W)} = \int_0^\infty \theta(s)(\phi_1(s), \phi_2(s))_W ds.$$

Thus, we set

$$\mathcal{M}_\sigma = L_\nu^2(\mathbb{R}_+; H^\sigma(\Omega)) \quad \text{and} \quad \mathcal{M}_\sigma^{(0)} = L_\nu^2(\mathbb{R}_+; H_\Omega^\sigma(\Omega)) \quad \text{for } \sigma \in \mathbb{R},$$

where  $\nu = \nu(s)$  is the kernel from (16). Hence, for  $\sigma \in \mathbb{R}$  and  $\phi_1, \phi_2 \in \mathcal{M}_\sigma$ , using (34) the product in  $\mathcal{M}_\sigma$  (and  $\mathcal{M}_\sigma^{(0)}$ ) can be expressed as

$$(\phi_1, \phi_2)_{\mathcal{M}_\sigma} = \int_0^\infty \nu(s)(A_N^{\sigma/2}\phi_1(s), A_N^{\sigma/2}\phi_2(s)) ds.$$

Naturally, we may also consider spaces of the form  $H_\nu^k(\mathbb{R}_+; H^\sigma(\Omega))$  for  $k \in \mathbb{N}$ .

We mention that solutions of Problem **P** must also satisfy the mass conservation constraints,

$$\langle u(t) \rangle = \langle u_0(0) \rangle \quad \text{and} \quad \langle \eta^t(s) \rangle = 0 \quad \forall t > 0, \forall s > 0. \tag{35}$$

With this, it is important to realize that the norm of  $\eta^t$  in the space  $\mathcal{M}_{-1}^{(0)}$  may be expressed *without* writing the average value of  $\eta_0$  in (22) by virtue of the second constraint of (35). Indeed, for  $\eta^t \in \mathcal{M}_{-1}^{(0)}$ ,

$$\begin{aligned} \|\eta^t\|_{\mathcal{M}_{-1}} &= \left( \int_0^\infty \nu(s) \|\eta^t(s)\|_{H^{-1}}^2 ds \right)^{1/2} \\ &= \left( \int_0^\infty \nu(s) \|A_N^{-1/2}\eta^t(s)\|^2 ds \right)^{1/2}. \end{aligned}$$

We now state the basic function spaces we intend to study Problem **P** in. For each  $\beta \in (0, 1)$  and  $\sigma \in \mathbb{R}$ , define the following (weak) energy Hilbertian phase-space  $\mathcal{H}_{\beta,\sigma} := W_0^{\beta,2}(\Omega) \times \mathcal{M}_{\sigma-1}^{(0)}$ , equipped with the norm on  $W_0^{\beta,2}(\Omega) \times \mathcal{M}_{\sigma-1}^{(0)}$  whose square is given by, for all  $\phi = (u, \eta)^{tr} \in \mathcal{H}_{\beta,\sigma}$ ,

$$\|\phi\|_{\mathcal{H}_{\beta,\sigma}}^2 := \|u\|_{W_0^{\beta,2}}^2 + \|\eta^t\|_{\mathcal{M}_{\sigma-1}}^2.$$

Then, for each  $M \geq 0$ , define the closed subset

$$\mathcal{H}_{\beta,\sigma}^M = \{\phi = (u, \eta)^{tr} \in \mathcal{H}_{\beta,\sigma} : |\langle u \rangle| \leq M\}. \tag{36}$$

When we are concerned with the dynamical system associated with the model Problem **P**, we utilize the following metric space,

$$\mathcal{X}_{\beta,\sigma}^M := \left\{ \phi = (u, \eta)^{tr} \in \mathcal{H}_{\beta,\sigma}^M : F(u) \in L^1(\Omega) \right\},$$

endowed with the metric

$$d_{\mathcal{X}_{\beta,\sigma}^M}(\phi_1, \phi_2) := \|\phi_1 - \phi_2\|_{\mathcal{H}_{\beta,\sigma}^M} + \left| \int_{\Omega} F(u_1)dx - \int_{\Omega} F(u_2)dx \right|^{1/2}.$$

**Remark 2.** The embedding  $\mathcal{H}_{\beta,1}^M \hookrightarrow \mathcal{H}_{\beta,0}^M$  is continuous but not compact, due to the presence of the second component  $\mathcal{M}_{\sigma-1}^{(0)}$ . Indeed, see [40] for a counterexample.

It is appropriate for us to state the various assumptions that may be used on the kernel  $\nu$ .

**(K1)**  $\nu \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$  and  $\nu(s) \geq 0$  for all  $s \in \mathbb{R}_+$ .

**(K2)**  $\nu'(s) \leq 0$  for all  $s \in \mathbb{R}_+$ .

**(K3)**  $k_0 = \int_0^\infty \nu(s)ds > 0$ . (For the sake of simplicity, we now assume  $k_0 = 1$  throughout the rest of the paper.)

**(K4)**  $\nu_0 = \lim_{s \rightarrow 0^+} \nu(s) < \infty$ .

**(K5)**  $\nu'(s) + \lambda\nu(s) \leq 0$  for a.e.  $s \in \mathbb{R}_+$ , for some  $\lambda > 0$ .

Some remarks for these assumptions: By assumption (K2), the inequality holds for all  $\eta^t \in D(T_r)$

$$(T_r \eta^t, \eta^t)_{\mathcal{M}_{-1}} \leq 0. \tag{37}$$

We remind the reader that the assumption (K5) is only required when we examine the asymptotic behavior of the solutions (and in that case, (K2) is redundant).

In order to formulate a suitable (abstract) evolution equation for  $\eta^t$ , we define the linear operator  $T_r = -\partial_s$  with the domain

$$D(T_r) = \{\eta^t \in \mathcal{M}_{-1}^{(0)} : \partial_s \eta^t \in \mathcal{M}_{-1}^{(0)}, \eta^t(0) = 0\}.$$

It is well-known that  $T_r$  is the infinitesimal generator of the right-translation semigroup on  $\mathcal{M}_{-1}$ ; indeed, the following result comes from [37] (Theorem 3.1).

**Proposition 1.** The operator  $T_r$  with domain  $D(T_r)$  is an infinitesimal generator of a strongly continuous semigroup of contractions on  $\mathcal{M}_{-1}$ , denoted  $e^{T_r t}$ .

As a consequence, we also have (see, e.g., [41] (Corollary IV.2.2)).

**Corollary 1.** Let  $T > 0$  and assume  $g \in L^1(0, T; H^{-1}(\Omega))$ . Then, for every  $\eta_0 \in \mathcal{M}_{-1}$ , the Cauchy problem for  $\eta^t$ ,

$$\begin{cases} \partial_t \eta^t = T_r \eta^t + g(t), & \text{for } t > 0, \\ \eta^0 = \eta_0, \end{cases} \tag{38}$$

has a unique (mild) solution  $\eta \in C([0, T]; \mathcal{M}_{-1})$  which can be explicitly given as

$$\eta^t(s) = \begin{cases} \int_0^s g(t-y)dy, & \text{for } 0 < s \leq t, \\ \eta_0(s-t) + \int_0^t g(t-y)dy, & \text{for } s > t, \end{cases} \tag{39}$$

see also [21] (Section 3.2) and [37] (Section 3).

### 3. Variational Formulation and Well-Posedness

To begin this section, we state the assumptions on the nonlinear term  $F$  and report some important consequences of these assumptions. These assumptions on  $F$  are based on [13,15] and can be found in [5] (Section 3).



(N1)  $F \in C^2_{loc}(\mathbb{R})$  and there exists  $c_F > 0$  such that, for all  $r \in \mathbb{R}$ ,

$$F''(r) \geq -c_F.$$

(N2) There exist  $c_F > 0$  and  $p \in (1, 2]$  such that, for all  $r \in \mathbb{R}$ ,

$$|F'(r)|^p \leq c_F(|F(r)| + 1).$$

(N3) There exist  $C_1, C_2 > 0$  such that, for all  $r \in \mathbb{R}$ ,

$$F(r) \geq C_1|r|^{p/(p-1)} - C_2.$$

The last assumption is not needed to obtain the existence of weak solutions, but it is relied upon later when we seek the existence of strong/regular solutions and uniqueness of these solutions.

(N4) There exist  $\rho \geq 2$  and  $C_3 > 0$  such that, for all  $r \in \mathbb{R}$ ,

$$|F''(r)| \leq C_3(1 + |r|^{\rho-2}). \tag{40}$$

The following remarks are from [5]. Assumption (N1) implies that the potential  $F$  is a quadratic perturbation of some strictly convex function; i.e., there holds,

$$F(r) = G(r) - \frac{c_F}{2}r^2, \tag{41}$$

with  $G \in C^2(\mathbb{R})$  strictly convex as  $G'' \geq 0$  in  $\Omega$ . Furthermore, with (N1), for each  $M \geq 0$  there are constants  $C_i > 0, i = 3, \dots, 6$ , (with  $C_4$  and  $C_5$  depending on  $M$  and  $F$ ) such that, for all  $r \in \mathbb{R}$ ,

$$F(r) - C_3 \leq C_4(r - M)^2 + F'(r)(r - M), \tag{42}$$

$$\frac{1}{2}|F'(s)|(1 + |r|) \leq F'(r)(r - M) + C_5, \tag{43}$$

(see [26] (Equations (4.7) and (4.8))) and

$$|F(r)| - C_6 \leq |F'(r)|(1 + |r|). \tag{44}$$

The last inequality appears in [42] (page 8). With the positivity condition (N3), it follows that, for all  $r \in \mathbb{R}$ ,

$$|F'(r)| \leq c_F(|F(r)| + 1). \tag{45}$$

Assumption (N2) allows for arbitrary polynomial growth  $\bar{p} = p/(p - 1)$  in the potential  $F$ . Significantly, the double-well potential  $F(r) = (r^2 - 1)^2$  satisfies (N2) with  $p = 4/3$  and (N4) with  $p = 2$ .

We are now ready to introduce the variational/weak formulation of Problem P.

**Definition 1.** Let  $T > 0$  and  $\phi_0 = (u_0, \eta_0)^{tr} \in \mathcal{H}_{\beta,0}^M = W_0^{\beta,2}(\Omega) \times \mathcal{M}_{-1}^{(0)}$  be such that  $F(u_0) \in L^1(\Omega)$ . A pair  $\phi = (u, \eta)$  satisfying

$$\phi = (u, \eta) \in L^\infty(0, T; \mathcal{H}_{\beta,0}^M), \tag{46}$$

$$\partial_t u \in L^2(0, T; H^{-1}(\Omega)), \tag{47}$$

$$\partial_t \eta \in L^2(0, T; H_v^{-1}(\mathbb{R}_+; H_{(0)}^{-1}(\Omega))), \tag{48}$$

$$\mu \in L^2(0, T; W^{-\beta,2}(\Omega)), \tag{49}$$

$$F'(u) \in L^\infty(0, T; L^p(\Omega)) \tag{50}$$

is called a WEAK SOLUTION to Problem **P** on  $[0, T]$  with initial data  $\phi_0 = (u_0, \eta_0) \in \mathcal{H}_{\beta,0}^M$  if the following identities hold almost everywhere in  $(0, T)$ , and for all  $v \in H^1(\Omega)$ ,  $\xi \in W_0^{\beta,2}(\Omega) \cap L^p(\Omega)$ , and  $\zeta \in \mathcal{M}_1$ :

$$\langle \partial_t u, v \rangle_{H^{-1} \times H^1} + \int_0^\infty v(s) \langle \eta^t(s), v \rangle_{H^{-1} \times H^1} ds = 0, \tag{51}$$

$$a_{E,\beta}(u, \xi) + \langle F'(u), \xi \rangle_{W^{-\beta,2} \times W_0^{\beta,2}} + \alpha \langle \partial_t u, \xi \rangle_{W^{-\beta,2} \times W_0^{\beta,2}} = \langle \mu, \xi \rangle_{W^{-\beta,2} \times W_0^{\beta,2}}, \tag{52}$$

$$(\partial_t \eta^t, \zeta)_{\mathcal{M}_{-1}} - (T_r \eta^t, \zeta)_{\mathcal{M}_{-1}} = (\mu, \zeta)_{\mathcal{M}_0}. \tag{53}$$

Furthermore, the initial conditions hold in the  $L^2$ -sense

$$u(0) = u_0 \quad \text{and} \quad \eta^0 = \eta_0. \tag{54}$$

Finally, we say that  $\phi = (u, \eta)^{tr}$  is a GLOBAL WEAK SOLUTION of Problem **P** if it is a weak solution on  $[0, T]$ , for any  $T > 0$ .

**Remark 3.** It is important to note that although  $\eta_0$  is defined by (14) and (21),  $\eta_0$  may be taken to be initial data independent of  $u$ . Henceforth we consider a more general problem with respect to the original one.

**Remark 4.** Concerning Equation (53) and the representation Formula (39), we have

$$T_r \eta^t(s) = -\partial_s \eta^t(s) = \begin{cases} \Delta \mu(t-s) & \text{for } 0 < s \leq t, \\ -\partial_s \eta_0(s-t) & \text{for } s > t. \end{cases}$$

Thus, when given  $\eta_0 \in \mathcal{M}_{-1}^{(0)}$ , then  $T_r \eta^t \in H_v^{-1}(\mathbb{R}_+; H^{-1}(\Omega))$ , for each  $t \in (0, T)$ , by virtue of (49). Moreover, taking  $\zeta = 1$  in the variational equation

$$(\partial_t \eta^t, \zeta)_{\mathcal{M}_{-1}} - (T_r \eta^t, \zeta)_{\mathcal{M}_{-1}} = - \int_0^\infty v(s) (-\Delta \mu, \zeta)_{H^{-1} \times H^1} ds,$$

we find, for all  $s > t$ ,

$$\frac{\partial}{\partial t} \langle \eta^t(s) \rangle + \frac{\partial}{\partial s} \langle \eta_0(s-t) \rangle = k_0 \langle \Delta \mu(t-s) \rangle.$$

We know that  $\eta_0 \in \mathcal{M}_{-1}^{(0)}$  and  $k_0 = 1$ , hence

$$\frac{\partial}{\partial t} \langle \eta^t(s) \rangle = 0,$$

and it follows that

$$\langle \eta^t(s) \rangle = 0 \quad \forall t \geq 0.$$

**Remark 5.** In the Cahn–Hilliard model, it is well-known that the average value of  $u$  is conserved (see, e.g., [43] (Section III.4.2)). A similar property holds here for our problem. Indeed, we may choose the test function  $v = 1$  in (51) which yields

$$\frac{\partial}{\partial t} \langle u(t) \rangle + \int_0^\infty v(s) \langle \eta^t(s) \rangle ds = 0.$$

By (4), there holds  $\langle \eta^t(s) \rangle = 0$  for all  $t > 0$  and for all  $s > 0$ . Hence, we recover the conservation of mass

$$\langle u(t) \rangle = \langle u_0 \rangle \quad \text{and} \quad \langle \partial_t u(t) \rangle = 0, \quad \forall t \geq 0. \tag{55}$$

**Remark 6.** Before we continue to the existence statement, it is worthwhile to recall Theorem A1 (d) in Appendix A for which the following embedding holds

$$D(A_{E,\beta}) \hookrightarrow L^\infty(\Omega), \quad \forall \beta \in \left(\frac{N}{4}, 1\right), \quad \text{for } N = 1, 2, 3. \tag{56}$$

**Theorem 1.** Let  $T > 0$  and  $\phi_0 = (u_0, \eta_0)^{tr} \in \mathcal{H}_{\beta,0}^M = W_0^{\beta,2}(\Omega) \times \mathcal{M}_{-1}^{(0)}$  for  $\beta \in (\frac{N}{4}, 1)$ ,  $N = 1, 2, 3$ , be such that  $F(u_0) \in L^1(\Omega)$ . Assume  $\alpha > 0$  and that (K1)–(K4) and (N1)–(N3) hold. Problem P admits at least one weak solution  $\phi = (u, \eta)$  on  $(0, T)$  according to Definition 1 with the additional regularity

$$u \in L^\infty(0, T; L^{p/(p-1)}(\Omega)), \tag{57}$$

$$\sqrt{\alpha} \partial_t u \in L^2(\Omega \times (0, T)), \tag{58}$$

$$\eta \in L^2(0, T; L^2_{-v'}(\mathbb{R}_+; H_0^{-1}(\Omega))), \tag{59}$$

$$F(u) \in L^\infty(0, T; L^1(\Omega)), \quad F'(u) \in L^\infty(0, T; L^1(\Omega)). \tag{60}$$

for any  $T > 0$ . Furthermore, setting

$$\mathcal{E}(t) := \| \|u(t)\| \|_{W_0^{\beta,2}}^2 + 2(F(u(t)), 1) + \|\eta^t\|_{\mathcal{M}_{-1}}^2 + C \tag{61}$$

for some  $C > 0$  sufficiently large, the following energy equality holds for every such weak solution,

$$\mathcal{E}(t) + 2 \int_0^t \left( \alpha \|\partial_t u(\tau)\|^2 d\tau - \int_0^\infty v'(s) \|\eta^\tau(s)\|_{H^{-1}}^2 ds \right) d\tau = \mathcal{E}(0). \tag{62}$$

**Proof.** The proof proceeds in several steps. The existence proof begins with a Faedo–Galerkin approximation procedure in which we later pass to the limit. We first assume that  $u_0 \in D(A_{E,\beta})$ . (This assumption will be used to show that there is a sequence  $\{u_{0n}\}_{n=1}^\infty$  such that  $u_{0n} \rightarrow u_0$  in  $D(A_{E,\beta})$  as well as  $L^\infty(\Omega)$  per (56), which will be important in light of the fact that  $F(u_{0n})$  is of arbitrary polynomial growth per assumptions (N1)–(N3).) The existence of a weak solution for  $u_0 \in W_0^{\beta,2}(\Omega)$  with  $F(u_0) \in L^1(\Omega)$  follows from a density argument. To establish the equality in the energy identity, we exploit the fact that the potential  $F$  is a quadratic perturbation of some strictly convex function.

*Step 1: The Galerkin approximation.* To begin, we introduce the family  $\{v_j\}_{j \geq 1}$  of eigenvectors of the fractional Laplacian  $A_{E,\beta}$ , which exist thanks to Theorem A1 in Appendix A. Moreover, there is a family  $\{w_j\}_{j \geq 1}$  consisting of the eigenvectors of the Neumann–Laplacian  $A_N$ , and with this, we define the smooth sequence of  $\{z_j\}_{j \geq 1} \subset D(T_r) \cap W_v^{1,2}(\mathbb{R}_+; H_0^1(\Omega))$  by  $z_j = b_j w_j$  such that  $\{b_j\}_{j \geq 1} \subset C_c^\infty(\mathbb{R}_+)$  is an orthonormal basis for  $L_v^2(\mathbb{R}_+)$ . Using these, we define the following finite-dimensional spaces:

$$V^n = \text{span}\{v_1, v_2, \dots, v_n\}, \quad W^n = \text{span}\{w_1, w_2, \dots, w_n\}, \quad \mathcal{M}^n = \text{span}\{z_1, z_2, \dots, z_n\}, \tag{63}$$

and set

$$V^\infty = \bigcup_{n=1}^\infty V^n, \quad W^\infty = \bigcup_{n=1}^\infty W^n, \quad \mathcal{M}^\infty = \bigcup_{n=1}^\infty \mathcal{M}^n.$$

Clearly,  $V^\infty$  is a dense subspace of  $W_0^{\beta,2}(\Omega)$  and  $W^\infty$  is a dense subspace of  $H^1(\Omega)$ . In addition,  $\mathcal{M}^\infty$  is a dense subspace of  $\mathcal{M}_{-1}^{(0)}$ . For  $T > 0$  fixed, we look for two functions of the form on  $(0, T)$ ,

$$u_n(t) = \sum_{k=1}^n a_k^{(n)}(t)v_k \quad \text{and} \quad \eta_n^t(s) = \sum_{k=1}^n c_k^{(n)}(t)z_k, \tag{64}$$

where  $a_j^{(n)}$  and  $c_j^{(n)}$  are assumed to be (at least)  $C^2([0, T])$  for each  $j = 1, 2, \dots$  and for each  $n = 1, 2, \dots$ , which solve the following approximating Problem  $\mathbf{P}_n$ :

$$(\partial_t u_n, v) + \int_0^\infty v(s)(\eta_n^t(s), v)ds = 0 \tag{65}$$

$$a_{E,\beta}(u_n, \xi) + (F'(u_n), \xi) + \alpha(\partial_t u_n, \xi) = (\mu_n, \xi) \tag{66}$$

$$(\partial_t \eta_n^t, \zeta)_{\mathcal{M}_{-1}} - (Tr \eta_n^t, \zeta)_{\mathcal{M}_{-1}} = (\mu_n, \zeta)_{\mathcal{M}_0} \tag{67}$$

$$u_n(0) = u_{0n}, \quad \eta_n^0 = \eta_{0n} \tag{68}$$

for every  $v \in V^n$ ,  $\xi \in W^n$  and  $\zeta \in \mathcal{M}^n$ , and where  $u_{0n}$ , and  $\eta_{0n}$  denote the finite-dimensional projections of  $u_0$  and  $\eta_0$  onto  $V^n$  and  $\mathcal{M}^n$ , respectively. This approximating problem is equivalent to solving a Cauchy problem for a system of ordinary differential equations (indeed, see, e.g., [26] (page 131)). Hence, the Cauchy–Lipschitz theorem ensures that there exists a  $T_n \in (0, \infty]$  such that this approximating system has a unique maximal solution.

*Step 2: A priori estimates.* We now derive some a priori estimates in order to show that  $T_n = \infty$  for every  $n \geq 1$  and that the sequences of  $u_n, \eta_n^t, \mu_n$  are bounded in suitable functional spaces. By using  $v = \mu_n$  as a test function in (65) and  $\xi = \partial_t u_n$  as a test function in (66) we obtain

$$(\partial_t u_n, \mu_n) + \int_0^\infty v(s)(\eta_n^t(s), \mu_n)ds = 0 \tag{69}$$

$$(\mu_n, \partial_t u_n) = ((-\Delta)_E^\beta u_n, \partial_t u_n) + (F'(u_n), \partial_t u_n) + \alpha \|\partial_t u_n\|^2, \tag{70}$$

and taking  $\zeta = \eta_n^t$  as a test function in (67) yields (for the products in  $\mathcal{M}_{-1}$ , this is a multiplication by  $(-\Delta)^{-1}\eta_n^t$  in  $\mathcal{M}_0$ )

$$\begin{aligned} \int_0^\infty v(s) \left( \int_\Omega \partial_t \eta_n^t(x, s) (-\Delta)^{-1} \eta_n^t(x, s) dx \right) ds + \int_0^\infty v(s) \left( \int_\Omega \partial_s \eta_n^t(x, s) (-\Delta)^{-1} \eta_n^t(x, s) dx \right) ds \\ = \int_0^\infty v(s) \left( \int_\Omega (-\Delta) \mu_n(x, t) (-\Delta)^{-1} \eta_n^t(x, s) dx \right) ds, \end{aligned}$$

which is, after an integration by parts,

$$(\partial_t \eta_n^t, \eta_n^t)_{\mathcal{M}_{-1}} + (\partial_s \eta_n^t, \eta_n^t)_{\mathcal{M}_{-1}} = (\mu_n, \eta_n^t)_{\mathcal{M}_0}. \tag{71}$$

Then, combining the results produces the differential identity, which holds for almost all  $t \in (0, T)$ ,

$$\frac{1}{2} \frac{d}{dt} \left\{ \|u_n\|_{W_0^{\beta,2}}^2 + 2(F(u_n), 1) + \|\eta^t\|_{\mathcal{M}_{-1}}^2 \right\} + \alpha \|\partial_t u_n\|^2 - (Tr \eta_n^t, \eta_n^t)_{\mathcal{M}_{-1}} = 0. \tag{72}$$

For all  $t \in (0, T_n)$ , set

$$\mathcal{E}_n(t) := \|u_n(t)\|_{W_0^{\beta,2}}^2 + 2(F(u_n(t)), 1) + \|\eta_n^t\|_{\mathcal{M}_{-1}}^2 + C \tag{73}$$

where in light of (N3), the functional  $\mathcal{E}_n(t)$  is nonnegative for all  $t \in (0, T_n)$ . We have

$$\frac{d}{dt}\mathcal{E}_n + 2\alpha\|\partial_t u_n\|^2 - 2\int_0^\infty v'(s)\|\eta_n^t(s)\|_{H^{-1}}^2 ds = 0 \tag{74}$$

for almost all  $t \in (0, T_n)$ . Hence, integrating the equation above with respect to time in  $(0, t)$ , we are led to the following integral equality (which does hold for the approximate solutions)

$$\mathcal{E}_n(t) + 2\int_0^t \left( \alpha\|\partial_t u_n(\tau)\|^2 - \int_0^\infty v'(s)\|\eta_n^\tau(s)\|_{H^{-1}}^2 ds \right) d\tau = \mathcal{E}_n(0). \tag{75}$$

Furthermore, from (73) and assumption (N3), we find the lower bound

$$\|u_n(t)\|_{W_0^{\beta,2}}^2 + 2C_1\|u_n(t)\|_{L^{p/(p-1)}}^{p/(p-1)} + \|\eta_n^t\|_{\mathcal{M}_{-1}}^2 \leq \mathcal{E}_n(t). \tag{76}$$

Using the fact that  $F(u_0) \in L^1(\Omega)$ , we also obtain the upper bound

$$\begin{aligned} \mathcal{E}_n(t) &\leq \mathcal{E}_n(0) \leq \|u_n(0)\|_{W_0^{\beta,2}}^2 + (F(u_n(0)), 1) + \|\eta_n^0\|_{\mathcal{M}_{-1}}^2 \\ &\leq Q(\|\phi_n(0)\|_{\mathcal{H}_{\beta,0}^M}) + C. \end{aligned} \tag{77}$$

In particular, the uniform bound derived from (75)–(77) implies that the local solution to Problem  $\mathbf{P}_n$  can be extended up to time  $T$ , that is  $T_n = T$ , for every  $n$ . Moreover, from (75) and (76) we deduce the following bounds for the approximate solution

$$\|u_n\|_{L^\infty(0,T;W_0^{\beta,2})} \leq C \tag{78}$$

$$\|\eta_n\|_{L^\infty(0,T;\mathcal{M}_{-1})} \leq C \tag{79}$$

$$\|F(u_n)\|_{L^\infty(0,T;L^1)} \leq C \tag{80}$$

$$\sqrt{\alpha}\|\partial_t u_n\|_{L^2(\Omega \times (0,T))} \leq C \tag{81}$$

$$\|\eta_n\|_{L^2(0,T;L^2_{-v'}(\mathbb{R}_+;H^{-1}))} \leq C \tag{82}$$

$$\|u_n\|_{L^\infty(0,T;L^{p/(p-1)})} \leq C. \tag{83}$$

Obviously, (45) and (80) immediately show us

$$\|F'(u_n)\|_{L^\infty(0,T;L^1)} \leq C. \tag{84}$$

Next, since  $\langle A_N^{-1}\partial_t u_n \rangle = 0$  (recall (55)), we may (and do) take  $v = A_N^{-1}\partial_t u_n$  in (65) which leads us to the estimate,

$$\|A_N^{-\frac{1}{2}}\partial_t u_n\|^2 \leq \int_0^\infty v(s)\|A_N^{-\frac{1}{2}}\eta_n^t(s)\|\|A_N^{-\frac{1}{2}}\partial_t u_n(t)\| ds, \tag{85}$$

that is,

$$\|\partial_t u_n\|_{H^{-1}}^2 \leq \int_0^\infty v(s)\|\eta_n^t(s)\|_{H^{-1}}\|\partial_t u_n\|_{H^{-1}} ds. \tag{86}$$

Using the Cauchy–Schwartz inequality and assumption (K3), we can write

$$\|\partial_t u_n\|_{H^{-1}} \leq \|\eta_n^t\|_{\mathcal{M}_{-1}}. \tag{87}$$

Thus, (79) and (87) yield

$$\|\partial_t u_n\|_{L^\infty(0,T;H^{-1})} \leq C. \tag{88}$$

We need to bound  $F'(u_n)$ , then  $\mu_n$ . In light of (66), we apply (84), (88), and the fact that operator  $A_{E,\beta}$  is bounded from  $W_0^{\beta,2}(\Omega)$  into  $W^{-\beta,2}(\Omega)$  (in particular,  $\|A_{E,\beta}u_n\|_{L^2(0,T;W^{-\beta,2}(\Omega))} \leq C$ ), to obtain the following uniform bounds for  $\mu_n$

$$|\langle \mu_n \rangle| \leq C, \tag{89}$$

and

$$\|\mu_n\|_{L^2(0,T;W^{-\beta,2}(\Omega))} \leq C. \tag{90}$$

This completes Step 2.

*Step 3: Passage to the limit.* On account of the above uniform inequalities, we can argue that there are functions  $u, \eta, \mu$ , such that, up to subsequences,

$$u_n \rightharpoonup u \quad \text{weakly-* in } L^\infty(0, T; W_0^{\beta,2}(\Omega)), \tag{91}$$

$$u_n \rightharpoonup u \quad \text{weakly-* in } L^\infty(0, T; L^{p/(p-1)}(\Omega)), \tag{92}$$

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{weakly-* in } L^\infty(0, T; H^{-1}(\Omega)), \tag{93}$$

$$\sqrt{\alpha} \partial_t u_n \rightharpoonup \sqrt{\alpha} \partial_t u \quad \text{weakly in } L^2(\Omega \times (0, T)), \tag{94}$$

$$\eta_n \rightharpoonup \eta \quad \text{weakly-* in } L^\infty(0, T; \mathcal{M}_{-1}), \tag{95}$$

$$\eta_n \rightharpoonup \eta \quad \text{weakly in } L^2(0, T; L^2_{-v'}(\mathbb{R}_+; H^{-1}(\Omega))), \tag{96}$$

$$\partial_t \eta_n \rightharpoonup \partial_t \eta \quad \text{weakly in } L^2(0, T; H_v^{-1}(\mathbb{R}_+; H^{-1}(\Omega))), \tag{97}$$

$$\mu_n \rightharpoonup \mu \quad \text{weakly in } L^2(0, T; W^{-\beta,2}(\Omega)). \tag{98}$$

(Note that (97) is due to (67) and the definition of the operator  $T_r$ .) Using the above convergences (91) and (93), as well as the fact that the injection  $W_0^{\beta,2}(\Omega) \hookrightarrow L^2(\Omega)$  is compact for any  $\beta \in (0, 1)$ , we draw upon the conclusion of the Aubin–Lions Lemma (see Lemma A1 in Appendix A) to deduce the following embedding is compact

$$W := \{\chi \in L^2(0, T; W_0^{\beta,2}(\Omega)) : \partial_t \chi \in L^2(0, T; H^{-1}(\Omega))\} \hookrightarrow L^2(\Omega \times (0, T)). \tag{99}$$

Hence,

$$u_n \rightarrow u \quad \text{strongly in } L^2(\Omega \times (0, T)), \tag{100}$$

and we deduce that  $u_n$  converges to  $u$ , almost everywhere in  $\Omega \times (0, T)$ . Using assumption (N1) with (100), we deduce

$$F'(u_n) \rightarrow F'(u) \quad \text{strongly in } L^2(0, T; L^1(\Omega)). \tag{101}$$

Thus, we now have all the sufficient convergence results to pass to the limit in Equations (65) and (66) in order to recover (16) and (17), respectively. It remains to recover Equation (67) after we pass to the limit. An integration by parts on the first term in (67) and then an application of (95) yields, for any  $\zeta \in C_0^\infty((0, T); C_0^\infty((0, T); H^1(\Omega)))$

$$\int_0^T (\partial_t \eta_n^\tau, \zeta)_{\mathcal{M}_{-1}} d\tau = - \int_0^T (\eta_n^\tau, \partial_t \zeta)_{\mathcal{M}_{-1}} d\tau \rightarrow - \int_0^T (\eta^\tau, \partial_t \zeta)_{\mathcal{M}_{-1}} d\tau. \tag{102}$$

With this, we have

$$\partial_t \eta_n^t \rightharpoonup \partial_t \eta^t \quad \text{weakly in } L^2(0, T; H_v^{-1}(\mathbb{R}_+; H^{-1}(\Omega))) \tag{103}$$

and that  $\eta^t \in L^\infty(0, T; H_v^{-1}(\mathbb{R}_+; H^{-1}(\Omega)))$ . Furthermore, with the help of (96), we have

$$-\int_0^T (T_r \eta_n^\tau, \zeta)_{\mathcal{M}_{-1}} d\tau = -\int_0^T v'(s)(\eta_n^\tau, \zeta)_{H^{-1}} d\tau \rightarrow -\int_0^T v'(s)(\eta^\tau, \zeta)_{H^{-1}} d\tau. \tag{104}$$

By using a density argument (see [37]) and the following distributional equality

$$-\int_0^T (\eta_n^\tau, \partial_t \zeta)_{\mathcal{M}_{-1}} d\tau - \int_0^T v'(s)(\eta^\tau, \zeta)_{H^{-1}(\Omega)} d\tau = \int_0^T (\partial_t \eta^\tau - T_r \eta^\tau, \zeta)_{\mathcal{M}_{-1}} d\tau, \tag{105}$$

we also get (67) on account of (95) and (98). This completes Step 3 of the proof.

*Step 4: Energy equality.* To begin, let  $u_0 \in D(A_{E,\beta})$ ,  $\eta_0 \in \mathcal{M}_{-1}^{(0)}$  and let  $\phi = (u, \eta)^{tr}$  be the corresponding weak solution. Recall from (100), we have, for almost all  $t \in (0, T)$ ,

$$u_n(t) \rightarrow u(t) \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{106}$$

Since  $F$  is measurable and positive (see (N1) and (N3), respectively), Fatou’s lemma implies

$$\int_\Omega F(u(t)) dx \leq \liminf_{n \rightarrow +\infty} \int_\Omega F(u_n(t)) dx. \tag{107}$$

Passing to the limit in (75), and while keeping in mind (91), (94), (95), (97), (98), and (101), as well as the weak lower-semicontinuity of the norm, we arrive at the integral inequality which holds for any weak solution

$$\mathcal{E}(t) + 2 \int_0^t \left( \alpha \|\partial_t u(\tau)\|^2 d\tau - \int_0^\infty v'(s) \|\eta^\tau(s)\|_{H^{-1}}^2 ds \right) d\tau \leq \mathcal{E}(0).$$

We argue as in the proof of [12] (Corollary 2) to establish the energy equality. Indeed, take  $\xi = \mu$  in (51). By (17), we need to treat the dual pairing  $\langle F'(u), \partial_t u \rangle_{W^{-\beta,2} \times W_0^{\beta,2}}$ . It is here where we employ (41), where  $F'(u) = G'(u) - c_F u$  and  $G' \in C^1(\mathbb{R})$  is monotone increasing. Define the functional  $\mathcal{G} : L^2(\Omega) \rightarrow \mathbb{R}$  by

$$\mathcal{G}(\phi) := \begin{cases} \int_\Omega G(u) dx & \text{if } G(u) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Now, by [44] (Proposition 2.8, Chapter II), it follows that  $\mathcal{G}$  is convex, lower-semicontinuous on  $L^2(\Omega)$ , and  $\chi \in \partial \mathcal{G}(u)$  if and only if  $\chi = G'(u)$  almost everywhere in  $\Omega$ . Since we have (47), we apply [45] (Proposition 4.2) to find that there holds, for almost all  $t \in (0, T)$ ,

$$\begin{aligned} \langle \partial_t u, F'(u) \rangle_{W^{-\beta,2} \times W_0^{\beta,2}} &= \langle \partial_t u, G'(u) \rangle_{W^{-\beta,2} \times W_0^{\beta,2}} - c_F \langle \partial_t u, u \rangle_{W^{-\beta,2} \times W_0^{\beta,2}} \\ &= \frac{d}{dt} \left\{ \mathcal{G}(u) - \frac{c_F}{2} \|u\|^2 \right\} \\ &= \frac{d}{dt} \int_\Omega F(u) dx. \end{aligned}$$

Similar to Step 2 above, take  $v = \mu$ ,  $\xi = \partial_t u$ , and  $\zeta = \eta^t$  (now without the index  $n$ ) in (51)–(53), respectively. Using the above result on the dual product with  $F'(u)$  and (47), we are led to the differential identity (74) with  $E$ ,  $u$ , and  $\eta$  in place of  $\mathcal{E}_n$ ,  $u_n$ , and  $\eta_n$ , respectively. Integrating the resulting differential identity on  $(0, t)$  produces (62) as claimed. This completes Step 4.

*Step 5:  $(u, \eta)$  weak solution to Problem P.* Now let us take  $\phi_0 = (u_0, \eta_0)^{tr} \in \mathcal{H}_{\beta,0}^M$  where  $F(u_0) \in L^1(\Omega)$ . Proceeding exactly as in [12] (page 440) the bounds (78)–(84) and (88)–(90) hold. Moreover, with the aid of the Aubin–Lions compact embedding (again see Lemma A1 in Appendix A below) we deduce the existence of functions  $u$ ,  $\eta$ , and  $\mu$  that satisfy (46), (49),

(57), and (59). Thus, passing to the limit in the variational formulation for  $\phi_k = (u_k, \eta_k)^{tr}$ , we find  $\phi = (u, \eta)^{tr}$  is a solution corresponding to the initial data  $\phi_0 = (u_0, \eta_0)^{tr} \in \mathcal{H}_{\beta,0}^M$  for which  $F(u_0) \in L^1(\Omega)$ . This finishes the proof of the theorem.  $\square$

Before we continue, we make some important remarks.

**Remark 7.** *The continuity property*

$$u \in C([0, T]; W_0^{\beta-\iota, 2}(\Omega)),$$

for any  $\iota > 0$  sufficiently small follows from the conditions in Definition 1 after an application of the Aubin–Lions Lemma (see Lemma A1 in Appendix A). In addition, the property

$$\eta \in C([0, T]; \mathcal{M}_{-1}^{(0)})$$

follows from the density argument in [37]. Thus, we deduce the continuity properties

$$\phi = (u, \eta) \in C([0, T]; \mathcal{H}_{\beta,0}^M).$$

**Remark 8.** *From (62), we see that if there is a  $t^* > 0$  in which*

$$\mathcal{E}(t^*) = \mathcal{E}(0),$$

then, for all  $t \in (0, t^*)$ ,

$$\int_0^t \left( \alpha \|\partial_t u(\tau)\|^2 + \|\eta^\tau\|_{L_{-\nu}^2(\mathbb{R}_+; H^{-1})}^2 \right) d\tau = 0. \tag{108}$$

We deduce  $\partial_t u(t) = 0$  for all  $t \in (0, t^*)$ . Additionally, since  $u(t) = u_0$  for all  $t \in (0, t^*)$ , Equation (17) shows

$$\mu(t) = A_{E,\beta} u_0 + F'(u_0) \quad \forall t \in (0, t^*),$$

i.e.,  $\mu(t) = \mu^*$  is also stationary. Thus, by the definition of  $\eta^t$  given in (14), we find here that, for each  $t \in (0, t^*)$

$$\eta^t(s) = s A_N \mu^* \quad \forall s \geq 0.$$

Therefore,  $\phi = (u, \eta)^{tr}$  is a fixed point of the trajectory  $\phi(t) = \mathcal{S}(t)\phi_0$ , where  $\mathcal{S}$  is the solution operator defined below in Corollary 2.

The following result (see [26] (Theorem 3.4)) concerns the existence of strong/regular solutions which is utilized in the proof of the continuous dependence estimate. Note that we now employ the added assumption on the nonlinear term.

**Theorem 2.** *Let  $T > 0$ ,  $\beta \in (0, 1)$ , and  $\phi_0 = (u_0, \eta_0)^{tr} \in \mathcal{H}_{\beta+1, \beta+1}^M := W_0^{\beta+1, 2}(\Omega) \times L_v^2(\mathbb{R}_+; W_0^{\beta, 2}(\Omega))$  be such that  $F(u_0) \in L^1(\Omega)$  and  $\eta_0 \in D(T_r)$ . Assume  $\alpha > 0$  and that (K1)–(K4) and (N1)–(N3) hold. Additionally, assume that (N4) holds. Problem **P** admits at least one weak solution  $\phi = (u, \eta)$  on  $(0, T)$  according to Definition (1) with the additional regularity, for any  $T > 0$ ,*

$$\phi = (u, \eta) \in L^\infty(0, T; \mathcal{H}_{\beta+1, \beta+1}^M) \cap W^{1, \infty}(0, T; \mathcal{H}_{\beta, 0}^M), \tag{109}$$

$$\sqrt{\alpha} \partial_t u \in L^2(0, T; H^1(\Omega)) \tag{110}$$

$$\partial_{tt} u \in L^\infty(0, T; H^{-1}(\Omega)), \tag{111}$$

$$\sqrt{\alpha} \partial_{tt} u \in L^2(\Omega \times (0, T)), \tag{112}$$

$$\mu \in L^\infty(0, T; H^1(\Omega)), \tag{113}$$

$$\eta \in L^\infty(0, T; D(T_r)). \tag{114}$$



**Proof.** The proof relies on the Galerkin approximation scheme developed in the proof of Theorem 1. We seek  $\phi_n = (u_n, \eta_n)$  of the form (64) satisfying Problem  $\mathbf{P}_n$ :

$$(\partial_{tt}u_n, v) + \int_0^\infty v(s)(\partial_t \eta_n^t(s), v) ds = 0 \tag{115}$$

$$a_{E,\beta}(\partial_t u_n, \xi) + (F''(u_n)\partial_t u, \xi) + \alpha(\partial_{tt}u_n, \xi) = (\partial_t \mu_n, \xi) \tag{116}$$

$$(\partial_{tt}\eta_n^t, \zeta)_{\mathcal{M}_{-1}} - (T_r \partial_t \eta_n^t, \zeta)_{\mathcal{M}_{-1}} = (\partial_t \mu_n, \zeta)_{\mathcal{M}_0} \tag{117}$$

for every  $t \in (0, T)$ ,  $v \in V^n$ ,  $\xi \in W^n$ , and  $\zeta \in \mathcal{M}^n$ , and which satisfy the initial conditions

$$u_n(0) = \tilde{u}_{0n} \quad \text{and} \quad \eta_n^0 = \tilde{\eta}_{0n}, \tag{118}$$

where we set

$$\tilde{u}_{0n} := - \int_0^\infty v(s)\eta_{0n}(s)ds, \tag{119}$$

and

$$\tilde{\eta}_{0n} := T_r \eta_{0n} + A_N \mu_{0n}, \tag{120}$$

and also

$$\mu_{0n} = -\alpha \int_0^\infty v(s)\eta_{0n}(s)ds + A_{E,\beta}u_{0n} + F'(u_{0n}). \tag{121}$$

It is important to note that when  $\phi_0 = (u_0, \eta_0)$  satisfies the assumptions of Theorem 2, then it is guaranteed that  $(\tilde{u}_0, \tilde{\eta}_0) \in \mathcal{H}_{1,0}^M$ . Indeed, relying on the fact that  $\|(u_{0n}, \eta_{0n})\|_{\mathcal{H}_{\beta,0}^M} \leq \|(u_0, \eta_0)\|_{\mathcal{H}_{\beta,0}^M}$ , we easily obtain the estimate  $\|(\partial_t u_n(0), \partial_t \eta_n^0)\|_{\mathcal{H}_{\beta,0}^M} \leq Q(\|(u_0, \eta_0)\|_{\mathcal{H}_{\beta+1, \beta+1}^M})$ . Now, for any fixed  $n \in \mathbb{N}$ , we find a unique local maximal solution  $\phi_n = (u_n, \eta_n) \in C^2([0, T_n]; \mathcal{H}_{\beta+1, 2}^M)$ . Next, we integrate (115) and (116) with respect to time on  $(0, t)$  and argue as in the proof of Theorem 1 to find the uniform bounds (78)–(84), (88), and (90). In order to obtain the required higher-order estimates, let us begin by labeling

$$\tilde{u}(t) = \partial_t u(t), \quad \tilde{\eta}^t = \partial_t \eta^t, \quad \tilde{\mu}(t) = \partial_t \mu(t),$$

where we are also dropping the index  $n$  for the sake of simplicity. Then,  $(\tilde{u}, \tilde{\eta})$  solves the system

$$\langle \partial_t \tilde{u}, v \rangle_{H^{-1} \times H^1} + \int_0^\infty v(s) \langle \tilde{\eta}^t(s), v \rangle_{H^{-1} \times H^1} ds = 0, \tag{122}$$

$$a_{E,\beta}(\tilde{u}, \xi) + (F''(u)\tilde{u}, \xi) + \alpha(\partial_t \tilde{u}, \xi) = \langle \tilde{\mu}, \xi \rangle_{W^{-\beta, 2} \times W_0^{\beta, 2}}, \tag{123}$$

$$(\partial_t \tilde{\eta}^t, \zeta)_{\mathcal{M}_{-1}} - (T_r \tilde{\eta}^t, \zeta)_{\mathcal{M}_{-1}} = (\tilde{\mu}, \zeta)_{\mathcal{M}_0}, \tag{124}$$

for all  $v \in H^1(\Omega)$ ,  $\xi \in W_0^{\beta, 2}(\Omega)$ , and  $\zeta \in \mathcal{M}_1$ , with the initial conditions

$$\tilde{u}(0) = \tilde{u}_0 \quad \text{and} \quad \tilde{\eta}^0 = \tilde{\eta}_0.$$

Let us now take  $v = \tilde{\mu}$ ,  $\xi = \partial_t \tilde{u}$ , and  $\zeta = \tilde{\eta}^t$  in (122)–(124), respectively. Summing the resulting identities together, we obtain, for all  $t \in (0, T)$ ,

$$\frac{1}{2} \frac{d}{dt} \left\{ \|\tilde{u}\|_{W_0^{\beta, 2}}^2 + \|\tilde{\eta}^t\|_{\mathcal{M}_{-1}}^2 \right\} - \int_0^\infty v'(s) \|\tilde{\eta}^t(s)\|_{H^{-1}}^2 ds + \alpha \|\partial_t \tilde{u}\|^2 = -(F''(u)\tilde{u}, \partial_t \tilde{u}).$$

Here, we apply (K5) as well as (N4) with (83) and the embedding  $W_0^{\beta,2}(\Omega) \hookrightarrow L^2(\Omega)$  to find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \|\tilde{u}\|_{W_0^{\beta,2}}^2 + \|\tilde{\eta}^t\|_{\mathcal{M}_{-1}}^2 \right\} + \lambda \|\tilde{\eta}^t\|_{\mathcal{M}_{-1}}^2 + \alpha \|\partial_t \tilde{u}\|^2 &\leq C_\alpha \|\tilde{u}\|^2 + \frac{\alpha}{2} \|\partial_t \tilde{u}\| \\ &\leq C_\alpha \|\tilde{u}\|_{W_0^{\beta,2}}^2 + \frac{\alpha}{2} \|\partial_t \tilde{u}\|, \end{aligned} \tag{125}$$

where  $C_\alpha \sim \alpha^{-1}$  is a positive constant. Integrating (125) over  $(0, t)$  produces

$$\begin{aligned} \|\tilde{u}(t)\|_{W_0^{\beta,2}}^2 + \|\tilde{\eta}^t\|_{\mathcal{M}_{-1}}^2 + \int_0^t \left( 2\lambda \|\tilde{\eta}^\tau\|_{\mathcal{M}_{-1}}^2 + \alpha \|\partial_t \tilde{u}(\tau)\|^2 \right) d\tau \\ \leq \|\tilde{u}(0)\|_{W_0^{\beta,2}}^2 + \|\tilde{\eta}^0\|_{\mathcal{M}_{-1}}^2 + C_\alpha \int_0^t \|\tilde{u}(\tau)\|_{W_0^{\beta,2}}^2 d\tau, \end{aligned} \tag{126}$$

and an application of Grönwall’s (integral) inequality shows, for all  $t \geq 0$ ,

$$\|(\tilde{u}(t), \tilde{\eta}^t)\|_{\mathcal{H}_{\beta,0}^M} \leq Q(\|(\tilde{u}_0, \tilde{\eta}_0)\|_{\mathcal{H}_{\beta,0}^M}) \tag{127}$$

and

$$\sqrt{\alpha} \|\partial_t \tilde{u}(t)\|_{L^2(\Omega \times (0,T))} \leq Q(\|(\tilde{u}_0, \tilde{\eta}_0)\|_{\mathcal{H}_{\beta,0}^M}). \tag{128}$$

Through (119)–(121), we find  $\|(\tilde{u}_0, \tilde{\eta}_0)\|_{\mathcal{H}_{\beta,0}^M}$  depends on

$$\int_0^\infty v(s) \|\eta_0(s)\|_{W_0^{\beta,2}}^2 ds, \quad \|A_N \mu_0\|_{\mathcal{M}_{-1}} \quad \text{and} \quad \|T_r \eta_0\|_{\mathcal{M}_{-1}},$$

hence the assumption on the initial data is justified.

Furthermore, we now consider (67) and take  $\zeta = A_N \bar{\mu}(t)$  where  $\bar{\mu} = \mu - \langle \mu \rangle$ , so that, with (79), (82), and (127), we obtain, for all  $t \geq 0$  and for every  $\varepsilon > 0$ ,

$$\|\nabla \mu\|^2 = (\partial_t \eta^t, \mu)_{\mathcal{M}_0} - (T_r \eta^t, \mu)_{\mathcal{M}_0} \tag{129}$$

$$= \int_0^\infty v(s) (\partial_t \eta^t(s), \mu(t)) ds - \int_0^\infty v'(s) (\eta^t(s), \mu(t)) ds \tag{130}$$

$$\leq C_\varepsilon \left( \|\partial_t \eta^t\|_{\mathcal{M}_{-1}}^2 - \int_0^\infty v'(s) \|\eta^t(s)\|_{H^{-1}}^2 ds \right) + \varepsilon \|\nabla \mu\|^2 \tag{131}$$

$$\leq C_\varepsilon \left( 1 - \int_0^\infty v'(s) \|\eta^t(s)\|_{H^{-1}}^2 ds \right) + \varepsilon \|\nabla \mu\|^2 \tag{132}$$

$$\leq C_\varepsilon + \varepsilon \|\nabla \mu\|^2 \tag{133}$$

where  $C_\varepsilon \sim \varepsilon^{-1}$ . Together (89) and (133) show us, for all  $t \geq 0$ ,

$$\|\mu(t)\|_{H^1} \leq C. \tag{134}$$

At this point we can reason as is in the proof of Theorem 1 to find that there is a solution  $\phi = (u, \eta) \in W^{1,\infty}(0, T; \mathcal{H}_{\beta,0}^M)$  to Problem P satisfying (111) and (112). Additionally, thanks to (134), the condition (113) holds. It remains to show that

$$\phi = (u, \eta) \in L^\infty \left( 0, T; \left[ W_0^{\beta+1,2}(\Omega) \times L_v^2(\mathbb{R}_+; W_0^{\beta,2}(\Omega)) \right] \right).$$

First, in light of (127), we multiply (16) by  $A_{E,\beta} \eta^t$  in  $L^2(\Omega)$  which yields

$$\|\eta^t\|_{L_v^2(\mathbb{R}_+; W_0^{\beta,2}(\Omega))}^2 = - \int_0^\infty v(s) (A_{E,\beta}^{\frac{1}{2}} \partial_t u(t), A_{E,\beta}^{\frac{1}{2}} \eta^t(s)) ds.$$

Hence,  $\eta \in L^\infty(0, T; L^2_v(\mathbb{R}_+; W_0^{\beta,2}(\Omega)))$ . Next, we consider the identity (52) whereby we may now rely on the regularity properties of  $\partial_t u$  and  $\mu$ . We take  $\zeta = A_N \partial_t u$  to produce

$$\frac{1}{2} \frac{d}{dt} \|u\|_{W_0^{\beta+1,2}}^2 + \langle F''(u) \nabla u, \nabla u \rangle + \alpha \|\partial_t u\|_{H^1}^2 = \langle \nabla \mu, \nabla u \rangle.$$

After applying (N1) and integrating the resulting differential inequality with respect to  $t$  over  $(0, t)$ , we obtain for all  $t \geq 0$ ,

$$\|u(t)\|_{W_0^{\beta+1,2}}^2 + 2 \int_0^t \alpha \|\partial_t u(\tau)\|_{H^1}^2 d\tau \leq \|u(0)\|_{W_0^{\beta+1,2}}^2 + Q(\|(u_0, \eta_0)\|_{\mathcal{H}_{\beta,0}^M}).$$

We now deduce

$$u \in L^\infty(0, T; W_0^{\beta+1,2}(\Omega)) \quad \text{and} \quad \sqrt{\alpha} \partial_t u \in L^2(0, T; H^1(\Omega)).$$

This completes the proof.  $\square$

The following proposition provides continuous dependence and uniqueness for the solutions constructed above.

**Proposition 2.** *Let the assumptions of Theorem 1 hold. Additionally, assume (N4) holds. Let  $T > 0$  and let  $\phi_i = (u_i, \eta_i)^{tr}$ ,  $i = 1, 2$ , be two solutions to Problem P on  $(0, T)$  corresponding to the initial data  $\phi_{0i} = (u_{0i}, \eta_{0i})^{tr} \in \mathcal{H}_{\beta,0}^M = W_0^{\beta,2}(\Omega) \times \mathcal{M}_{-1}^{(0)}$ , such that  $F(u_{0i}) \in L^1(\Omega)$ ,  $i = 1, 2$ . Then, for each  $\alpha > 0$ , there is a positive constant  $C_\alpha \sim \alpha^{-1}$  such that the following estimate holds, for any  $t \in (0, T)$ ,*

$$\begin{aligned} \|\phi_1(t) - \phi_2(t)\|_{\mathcal{H}_{\beta,0}^M}^2 + \int_0^t \left( \alpha \|\partial_t u_1(\tau) - \partial_t u_2(\tau)\|^2 + \|\eta_1^\tau - \eta_2^\tau\|_{L^2_{-v'}(\mathbb{R}_+; H^{-1})}^2 \right) d\tau \\ \leq e^{C_\alpha t} \|\phi_{01} - \phi_{02}\|_{\mathcal{H}_{\beta,0}^M}^2. \end{aligned} \tag{135}$$

**Proof.** To begin, we assume  $(u_{0i}, \eta_{0i})$ ,  $i = 1, 2$ , satisfy the assumptions of Theorem 2 (recall, above we are assuming (N4) holds), and we work with the more regular solutions to obtain (135). For all  $t \in [0, T]$ , we then set

$$\phi(t) := \phi_1(t) - \phi_2(t), \quad u(t) := u_1(t) - u_2(t), \quad \eta^t := \eta_1^t - \eta_2^t \quad \text{and} \quad \mu := \mu_1 - \mu_2$$

where  $\phi_i(t) = (u_i(t), \eta_i^t)$  is a solution corresponding to  $(u_{0i}, \eta_{0i})$ ,  $i = 1, 2$ . Then, formally,  $\phi = (u, \eta)$  solves the equations for all  $v \in H^1(\Omega)$ ,  $\xi \in W_0^{\beta,2}(\Omega) \cap L^p(\Omega)$ , and  $\zeta \in \mathcal{M}_1$ :

$$\langle \partial_t u, v \rangle_{H^{-1} \times H^1} + \int_0^\infty v(s) \langle \eta^t(s), v \rangle_{H^{-1} \times H^1} ds = 0, \tag{136}$$

$$a_{E,\beta}(u, \xi) + \langle F'(u_1) - F'(u_2), \xi \rangle_{W^{-\beta,2} \times W_0^{\beta,2}} + \alpha \langle \partial_t u, \xi \rangle_{W^{-\beta,2} \times W_0^{\beta,2}} = \langle \mu, \xi \rangle_{W^{-\beta,2} \times W_0^{\beta,2}}, \tag{137}$$

$$(\partial_t \eta^t, \zeta)_{\mathcal{M}_{-1}} - (T_r \eta^t, \zeta)_{\mathcal{M}_{-1}} = (\mu, \zeta)_{\mathcal{M}_0} \tag{138}$$

with the initial data

$$u(0) = u_{01} - u_{02}, \quad \eta^0 = \eta_{01} - \eta_{02}.$$

In (136), we choose  $v = \mu$  and in (137), we choose  $\xi = \partial_t u$ . Owing to Theorem 2, for each  $t \in [0, T]$ , these elements are in  $H^1(\Omega)$  and  $W_0^{\beta,2}(\Omega)$ , respectively, then we sum the results to obtain

$$(A_{E,\beta} u, \partial_t u) + (F'(u_1) - F'(u_2), \partial_t u) + \alpha \|\partial_t u\|^2 + \int_0^\infty v(s) (\mu, \eta^t(s)) ds = 0. \tag{139}$$

Further, multiplying (138) by  $A_N^{-1}\eta^t$  in  $\mathcal{M}_0$ , then adding the obtained relation to (139), we have

$$\frac{1}{2} \frac{d}{dt} \{ \|u\|_{W_0^{\beta,2}}^2 + \|\eta^t\|_{\mathcal{M}_{-1}}^2 \} + \alpha \|\partial_t u\|^2 - \int_0^\infty v'(s) \|\eta^t(s)\|_{H^{-1}}^2 ds + (F'(u_1) - F'(u_2), \partial_t u) = 0. \tag{140}$$

Using Hölder’s inequality, (N4), Young’s inequality, and the embedding  $L^\infty(\Omega) \hookrightarrow W_0^{\beta,2}(\Omega)$ , we estimate the remaining product as

$$\begin{aligned} |(F'(u_1) - F'(u_2), \partial_t u)| &\leq \|F'(u_1) - F'(u_2)\| \|\partial_t u\| \\ &\leq C \|(1 + |u_1|^{\rho-2} + |u_2|^{\rho-2})u\| \|\partial_t u\| \\ &\leq C(1 + \|u_1\|_{L^{2(\rho-2)}}^{\rho-2} + \|u_2\|_{L^{2(\rho-2)}}^{\rho-2}) \|u\|_{L^\infty} \|\partial_t u\| \\ &\leq Q_\alpha(\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta,0}^M}) \|u\|_{W_0^{\beta,2}}^2 + \frac{\alpha}{2} \|\partial_t u\|^2, \end{aligned} \tag{141}$$

where the positive monotone increasing function  $Q_\alpha(\cdot) \sim \alpha^{-1}$  (we remind the reader  $\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta+1,\beta+1}^M} \leq Q\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta,0}^M}$ , for  $i = 1, 2$  and the bounds on  $u_1$  and  $u_2$  follow from (61) and (62)). With (140) and (141), we obtain the following differential inequality which holds for almost all  $t \in [0, T]$

$$\begin{aligned} \frac{d}{dt} \{ \|u\|_{W_0^{\beta,2}}^2 + \|\eta^t\|_{\mathcal{M}_{-1}}^2 \} + \alpha \|\partial_t u\|^2 + \|\eta^t\|_{L_{-\nu'}^2(\mathbb{R}_+; H^{-1})}^2 &\leq Q_\alpha(\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta,0}^M}) \|u\|_{W_0^{\beta,2}}^2 \\ &\leq Q_\alpha(\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta,0}^M}) \left( \|u\|_{W_0^{\beta,2}}^2 + \|\eta^t\|_{\mathcal{M}_{-1}}^2 \right). \end{aligned} \tag{142}$$

Applying a Grönwall inequality to (142), we obtain, for all  $t \in [0, T]$ ,

$$\begin{aligned} \|u(t)\|_{W_0^{\beta,2}}^2 + \|\eta^t\|_{\mathcal{M}_{-1}}^2 + \int_0^t \left( \alpha \|\partial_t u(\tau)\|^2 + \|\eta^\tau\|_{L_{-\nu'}^2(\mathbb{R}_+; H^{-1})}^2 \right) d\tau \\ \leq e^{C_\alpha} \left( \|u(0)\|_{W_0^{\beta,2}}^2 + \|\eta^0\|_{\mathcal{M}_{-1}}^2 \right). \end{aligned} \tag{143}$$

This shows the claim (135) holds for the regular solutions. Since none of the above constants due to the above estimate actually depend on the assumptions of Theorem 2, then standard approximation arguments can be employed to obtain (135) for the weak solutions as well.  $\square$

**Remark 9.** It is quite important to remark that when  $N = 3$ , the uniqueness for the nonviscous problem (where  $\alpha = 0$ ) remains an open problem (indeed, see [36,46,47]).

We now formalize the semidynamical system generated by Problem P.

**Corollary 2.** Let the assumptions of Theorem 1 be satisfied. Additionally, assume (N4) holds. We can define a strongly continuous semigroup of solution operators  $\mathcal{S} = (\mathcal{S}(t))_{t \geq 0}$ , for each  $\alpha > 0$  and  $\beta \in (0, 1)$ ,

$$\mathcal{S}(t) : \mathcal{X}_{\beta,0}^M \rightarrow \mathcal{X}_{\beta,0}^M$$

by setting, for all  $t \geq 0$ ,

$$\mathcal{S}(t)\phi_0 := \phi(t)$$

where  $\phi(t) = (u(t), \eta^t)$  is the unique global weak solution to Problem P. Furthermore, as a consequence of (135), the semigroup  $\mathcal{S}(t) : \mathcal{X}_{\beta,0}^M \rightarrow \mathcal{X}_{\beta,0}^M$  is Lipschitz continuous on  $\mathcal{X}_{\beta,0}^M$  uniformly in  $t$  on compact intervals.

#### 4. Absorbing Sets and Global Attractors

We now give a dissipation estimate for Problem P from which we deduce the existence of a bounded absorbing set and an important uniform bound on the solutions of Problem P. The existence of an absorbing set is also used later to show that the semigroup of solution operators  $\mathcal{S}$  admits a compact global attractor in the metric space  $\mathcal{X}_{\beta,0}^M$ .

**Lemma 1.** Let  $\phi_0 = (u_0, \eta_0)^{tr} \in \mathcal{H}_{\beta,0}^M = W_0^{\beta,2}(\Omega) \times \mathcal{M}_{-1}^{(0)}$  for  $\beta \in (\frac{N}{4}, 1)$ ,  $N = 1, 2, 3$ , be such that  $F(u_0) \in L^1(\Omega)$ . Assume (K1), (K3)–(K5), and (N1)–(N3) hold. Assume  $\phi = (u, \eta)^{tr}$  is a weak solution to Problem P. There are positive constants  $\kappa_1$  and  $C$ , each depending on  $\Omega$  but independent of  $t, \alpha$ , and  $\phi_0$ , such that, for all  $t \geq 0$ , the following holds

$$\|\phi(t)\|_{\mathcal{H}_{\beta,0}^M}^2 + \int_t^{t+1} \alpha \|\partial_t u(\tau)\|^2 d\tau \leq Q(\|\phi_0\|_{\mathcal{H}_{\beta,0}^M}) e^{-\kappa_1 t} + C, \tag{144}$$

for some monotonically increasing function  $Q$  independent of  $t$  and  $\alpha$ .

**Proof.** The idea of the proof is from [26]. We give a formal calculation that can be justified by a suitable Faedo–Galerkin approximation based on the proof of Theorem 1 above. To begin, define the functional, for all  $t \geq 0$ ,

$$\mathcal{Y}(t) := \mathcal{E}(t) + \varepsilon \alpha \|u(t)\|^2 - 2\varepsilon \int_0^\infty v(s) \left( u(t), A_N^{-1} \eta^t(s) \right) ds, \tag{145}$$

where  $\varepsilon \in (0, \lambda)$  will be chosen sufficiently small later. From (16)–(18), we find

$$\begin{aligned} & -\frac{d}{dt} \int_0^\infty v(s) \left( u, A_N^{-1} \eta^t(s) \right) ds \\ &= \|\partial_t u\|_{H^{-1}}^2 - \int_0^\infty v(s) \left( u, A_N^{-1} \partial_t \eta^t(s) \right) ds \\ &= \|\partial_t u\|_{H^{-1}}^2 - \int_0^\infty v'(s) \left( u, A_N^{-1} \eta^t(s) \right) ds - \int_0^\infty v(s) \left( u, \mu \right) ds \\ &= \|\partial_t u\|_{H^{-1}}^2 - \int_0^\infty v'(s) \left( u, A_N^{-1} \eta^t(s) \right) ds - \frac{\alpha}{2} \frac{d}{dt} \|u\|^2 - \|u\|_{W_0^{\beta,2}}^2 - (F'(u), u). \end{aligned} \tag{146}$$

Differentiating  $\mathcal{Y}$  with respect to  $t$  while keeping in mind (73), (74) (without the index  $n$ ), and (146), we find

$$\frac{d}{dt} \mathcal{Y} + \varepsilon_0 \mathcal{Y} - 2 \int_0^\infty v'(s) \|\eta^t(s)\|_{H^{-1}}^2 ds = h(t), \tag{147}$$

for  $\varepsilon_0 \in (0, \varepsilon)$  where

$$\begin{aligned} h(t) = & -2\alpha \|\partial_t u(t)\|^2 + 2\varepsilon \|\partial_t u(t)\|_{H^{-1}}^2 - 2\varepsilon \int_0^\infty v'(s) \left( u(t), A_N^{-1} \eta^t(s) \right) ds \\ & - 2\varepsilon_0 (F'(u(t))u(t) - F(u(t)), 1) - 2(\varepsilon - \varepsilon_0) (F'(u(t)), u(t)) + \varepsilon_0 \|\eta^t\|_{\mathcal{M}_{-1}}^2 \\ & - (2\varepsilon - \varepsilon_0) \|u(t)\|_{W_0^{\beta,2}}^2 + \varepsilon_0 \varepsilon \alpha \|u(t)\|^2 - 2\varepsilon_0 \varepsilon \int_0^\infty v(s) \left( u(t), A_N^{-1} \eta^t(s) \right) ds + \varepsilon_0 C. \end{aligned} \tag{148}$$

From (42) and (43) (with  $M = 0$ ), it follows that

$$\begin{aligned} & -2\varepsilon_0 (F'(u(t))u(t) - F(u(t)), 1) - 2(\varepsilon - \varepsilon_0) (F'(u(t)), u(t)) \\ & \leq -(\varepsilon - \varepsilon_0) (|F(u)|, 1) + \varepsilon_0 C \|u\|_{W_0^{\beta,2}}^2. \end{aligned} \tag{149}$$

Next, using assumption (K4) and the embeddings  $H^{-1}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W_0^{\beta,2}(\Omega)$ , we find

$$\begin{aligned}
 -2\varepsilon \int_0^\infty v'(s)(u, A_N^{-1}\eta^t(s))ds &= -2\varepsilon \int_0^\infty v'(s)(A_N^{-1/2}u, A_N^{-1/2}\eta^t(s))ds \\
 &\leq -\varepsilon \int_0^\infty v'(s) \left( \frac{1}{\nu_0} \|u\|_{W_0^{\beta,2}}^2 + C\nu_0 \|\eta^t(s)\|_{H^{-1}}^2 \right) ds \\
 &\leq \varepsilon \|u\|_{W_0^{\beta,2}}^2 - \varepsilon C \int_0^\infty v'(s) \|\eta^t(s)\|_{H^{-1}}^2 ds,
 \end{aligned}
 \tag{150}$$

and, with (K3) and (87) (without the index  $n$ ),

$$-2\varepsilon_0\varepsilon \int_0^\infty v(s)(u, A_N^{-1}\eta^t(s))ds \leq \varepsilon_0\varepsilon C \|u\|_{W_0^{\beta,2}}^2 + \varepsilon_0\varepsilon \|\eta^t\|_{\mathcal{M}_{-1}}^2.
 \tag{151}$$

Together, (148)–(151) make the following estimate

$$\begin{aligned}
 h &\leq -2\alpha \|\partial_t u\|^2 + 2\varepsilon \|\partial_t u\|_{H^{-1}}^2 - (\varepsilon - \varepsilon_0(1 + C + \varepsilon\alpha C)) \|u\|_{W_0^{\beta,2}}^2 + 2\varepsilon_0 \|\eta^t\|_{\mathcal{M}_{-1}}^2 \\
 &\quad - \varepsilon C \int_0^\infty v'(s) \|\eta^t(s)\|_{H^{-1}}^2 ds + C.
 \end{aligned}
 \tag{152}$$

Here, we employ assumption (K5) so that from (147) and (152), we are able to fix  $\varepsilon \in (0, \lambda)$  and  $\varepsilon_0 \in (0, \varepsilon)$  sufficiently small to, in turn, find positive constants  $\varepsilon_1, \varepsilon_2, \varepsilon_3$  so that there holds

$$\frac{d}{dt} \mathcal{Y} + \varepsilon_1 \mathcal{Y} + 2\|\eta^t\|_{\mathcal{M}_{-1}}^2 + \varepsilon_2 \alpha \|\partial_t u\|^2 + \varepsilon_3 \|u\|_{W_0^{\beta,2}}^2 \leq C.
 \tag{153}$$

It is important to note that  $C$  on the right-hand side of (153) is independent of  $t$  and  $\phi_0$ . One can readily show (see (73), (76)–(77)) that there holds, for all  $t \geq 0$ ,

$$C_1 \|\phi(t)\|_{\mathcal{H}_{\beta,0}^M}^2 - C_2 \leq \mathcal{Y}(t) \leq Q(\|\phi_0\|_{\mathcal{H}_{\beta,0}^M}),
 \tag{154}$$

for some positive constants  $C_1, C_2$ , and for some monotone nondecreasing function  $Q$  independent of  $t$ . Finally, by applying a Grönwall type inequality to (153) (see, e.g., [34] (Lemma 2.5)), then integrating the result and applying (154) yield the claim (144). This finishes the proof.  $\square$

We immediately deduce the existence of a bounded absorbing set from Lemma 1.

**Proposition 3.** *Let the assumptions of Lemma 1 hold. Additionally, assume (N4) holds. Then, there exists  $R_0 > 0$ , independent of  $t$  and  $\phi_0$ , such that  $\mathcal{S}(t)$  possesses an absorbing ball  $\mathcal{B}_{\beta,0}^M(R_0) \subset \mathcal{H}_{\beta,0}^M$ , bounded in  $\mathcal{H}_{\beta,0}^M$ . Precisely, for any bounded subset  $B \subset \mathcal{H}_{\beta,0}^M$ , there exists  $t_0 = t_0(B) > 0$  such that  $\mathcal{S}(t)B \subset \mathcal{B}_{\beta,0}^M(R_0)$ , for all  $t \geq t_0$ . Moreover, for every  $R > 0$ , there exists  $C_* = C_*(R) \geq 0$ , such that, for any  $\phi_0 \in \mathcal{B}_{\beta,0}^M(R)$ ,*

$$\sup_{t \geq 0} \|\mathcal{S}(t)\phi_0\|_{\mathcal{H}_{\beta,0}^M} + \int_0^\infty \|\partial_t u(\tau)\|^2 d\tau \leq C_*,
 \tag{155}$$

where  $\mathcal{B}_{\beta,0}^M(R)$  denotes the ball in  $\mathcal{H}_{\beta,0}^M$  of radius  $R$ , centered at  $\mathbf{0}$ .

Throughout the remainder of the article, we simply write  $\mathcal{B}_{\beta,0}^M$  in place of  $\mathcal{B}_{\beta,0}^M(R_0)$  to denote the bounded absorbing set admitted by the semigroup of solution operators  $\mathcal{S}(t)$ .

For the rest of this section, our aim is to prove the following.

**Theorem 3.** *Let the assumptions of Lemma 1 hold. Additionally, assume (N4) holds. The dynamical system  $(\mathcal{X}_{\beta,0}^M, \mathcal{S}(t))$  (see Corollary 2) possesses a connected global attractor  $\mathcal{A}_{\beta,0}^M$  in  $\mathcal{H}_{\beta,0}^M$ . Precisely:*

- 1 For each  $t \geq 0$ ,  $\mathcal{S}(t)\mathcal{A}_{\beta,0}^M = \mathcal{A}_{\beta,0}^M$ ;
- 2 For every nonempty bounded subset  $B$  of  $\mathcal{H}_{\beta,0}^M$ ,

$$\lim_{t \rightarrow \infty} \text{dist}_{\mathcal{H}_{\beta,0}^M}(\mathcal{S}(t)B, \mathcal{A}_{\beta,0}^M) := \lim_{t \rightarrow \infty} \sup_{\zeta \in B} \inf_{\tilde{\zeta} \in \mathcal{A}_{\beta,0}^M} \|\mathcal{S}(t)\zeta - \tilde{\zeta}\|_{\mathcal{H}_{\beta,0}^M} = 0.$$

Additionally:

- 3 The global attractor is the unique maximal compact invariant subset in  $\mathcal{H}_{\beta,0}^M$  given by

$$\mathcal{A}_{\beta,0}^M := \omega(\mathcal{B}_{\beta,0}^M) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \mathcal{S}(t)\mathcal{B}_{\beta,0}^M}^{\mathcal{H}_{\beta,0}^M}.$$

Furthermore:

- 4 The global attractor  $\mathcal{A}_{\beta,0}^M$  is connected and given by the union of the unstable manifolds connecting the equilibria of  $\mathcal{S}(t)$ .
- 5 For each  $\zeta_0 = (\phi_0, \theta_0)^{tr} \in \mathcal{H}_{\beta,0}^M$ , the set  $\omega(\zeta_0)$  is a connected compact invariant set, consisting of the fixed points of  $\mathcal{S}(t)$ .

With the existence of a bounded absorbing set  $\mathcal{B}_{\beta,0}^M$  (in Lemma 1), the existence of a global attractor now depends on the precompactness of the semigroup of solution operators  $\mathcal{S}$ . To this end we show there is a  $t_* > 0$  such that the map  $\mathcal{S}(t_*)$  is a so-called  $\alpha$ -contraction on  $\mathcal{B}_{\beta,0}^M$ ; that is, there exist a time  $t_* > 0$ , a constant  $0 < \kappa < 1$ , and a precompact pseudometric  $M_*$  on  $\mathcal{B}_{\beta,0}^M$  such that, for all  $\phi_{01}, \phi_{02} \in \mathcal{B}_{\beta,0}^M$ ,

$$\|\mathcal{S}(t_*)\phi_{01} - \mathcal{S}(t_*)\phi_{02}\|_{\mathcal{H}_{\beta,0}^M} \leq \kappa \|\phi_{01} - \phi_{02}\|_{\mathcal{H}_{\beta,0}^M} + M_*(\phi_{01}, \phi_{02}). \tag{156}$$

Such a contraction is commonly used in connection with phase-field-type equations as an alternative to establish the precompactness of a semigroup; for some particular recent results see [16,48,49].

**Lemma 2.** Under the assumptions of Proposition 2 where  $\phi_{01}, \phi_{02} \in \mathcal{B}_{\beta,0}^M$ , there are positive constants  $\kappa_2, C_1$ , and  $C_{2\alpha} \sim \alpha^{-1}$ , each depending on  $\Omega$  but independent of  $t$  and  $\phi_{01}, \phi_{02}$ , such that, for all  $t \geq 0$ ,

$$\begin{aligned} \|\phi_1(t) - \phi_2(t)\|_{\mathcal{H}_{\beta,0}^M}^2 &\leq C_1 e^{-\kappa_2 t} \|\phi_1(0) - \phi_2(0)\|_{\mathcal{H}_{\beta,0}^M}^2 \\ &\quad + C_{2\alpha} \int_0^t \left( \|\nabla \mu_1(\tau) - \nabla \mu_2(\tau)\|^2 + \|u_1(\tau) - u_2(\tau)\|^2 \right) d\tau. \end{aligned} \tag{157}$$

**Proof.** The proof is based on the proof of Proposition 2. We begin by recovering (140) by multiplying (136) and (137) by  $\mu$  and  $\partial_t u$ , respectively, in  $L^2(\Omega)$ , and multiplying (138) by  $A_N^{-1} \eta^t$  in  $\mathcal{M}_0$ , then adding the obtained relations together to find

$$\frac{1}{2} \frac{d}{dt} \{ \|u\|_{W_0^{\beta,2}}^2 + \|\eta^t\|_{\mathcal{M}_{-1}}^2 \} + \alpha \|\partial_t u\|^2 - \int_0^\infty v'(s) \|\eta^t(s)\|_{H^{-1}}^2 ds + (F'(u_1) - F'(u_2), \partial_t u) = 0. \tag{158}$$

Recall  $\phi_1 = (u_1, \eta_1)$ ,  $\phi_2 = (u_2, \eta_2)$  are the unique weak solutions corresponding to the initial data  $\phi_{01}$  and  $\phi_{02}$ , respectively; also,  $u = u_1 - u_2$  and  $\eta^t = \eta_1^t - \eta_2^t$  formally satisfy (136) and (137). Applying Assumption (K5) and the estimate based on (N4),

$$\begin{aligned} |(F'(u_1) - F'(u_2), \partial_t u)| &\leq \|F'(u_1) - F'(u_2)\| \|\partial_t u\| \\ &\leq C \|(1 + |u_1|^{\rho-2} + |u_2|^{\rho-2})u\| \|\partial_t u\| \\ &\leq C(1 + \|u_1\|_{L^{2(\rho-2)}}^{\rho-2} + \|u_2\|_{L^{2(\rho-2)}}^{\rho-2}) \|u\|_{L^\infty} \|\partial_t u\| \\ &\leq Q_\alpha(\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta,0}^M}) \|u\|_{W_0^{\beta,2}}^2 + \frac{\alpha}{2} \|\partial_t u\|^2 \end{aligned} \tag{159}$$

$$\leq Q_\alpha(\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta,0}^M}) + \frac{\alpha}{2} \|\partial_t u\|^2, \tag{160}$$

where the positive monotone increasing function  $Q_\alpha(\cdot) \sim \alpha^{-1}$ , and we find the differential inequality

$$\frac{1}{2} \frac{d}{dt} \{ \|u\|_{W_0^{\beta,2}}^2 + \|\eta^t\|_{\mathcal{M}_{-1}}^2 \} + \frac{\alpha}{2} \|\partial_t u\|^2 + \lambda \|\eta^t\|_{\mathcal{M}_{-1}}^2 \leq Q_\alpha(\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta,0}^M}). \tag{161}$$

In addition, we now multiply (137) by  $u$  in  $L^2(\Omega)$  to obtain

$$\|u\|_{W_0^{\beta,2}}^2 + (F'(u_1) - F'(u_2), u) + \frac{\alpha}{2} \frac{d}{dt} \|u\|^2 = (\mu, u). \tag{162}$$

Estimating the first product above using (N1) yields

$$(F'(u_1) - F'(u_2), u) \geq -c_F \|u\|^2. \tag{163}$$

We also estimate with Young’s inequality

$$(\mu, u) \leq \frac{1}{2} \|\mu\|^2 + \frac{1}{2} \|u\|^2. \tag{164}$$

Combining (161)–(164) yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \|u\|_{W_0^{\beta,2}}^2 + \|\eta^t\|_{\mathcal{M}_{-1}}^2 + \frac{\alpha}{2} \|u\|^2 \right\} + \frac{\alpha}{2} \|\partial_t u\|^2 + \|u\|_{W_0^{\beta,2}}^2 + \lambda \|\eta^t\|_{\mathcal{M}_{-1}}^2 \\ \leq \frac{1}{2} \|\mu\|^2 + Q_\alpha(\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta,0}^M}) \|u\|_{W_0^{\beta,2}}^2. \end{aligned} \tag{165}$$

Then, adding  $\frac{\alpha}{2} \|u\|^2$  to each side of (165), we find

$$\frac{d}{dt} \mathcal{N} + c\mathcal{N} + \alpha \|\partial_t u\|^2 \leq \|\mu\|^2 + Q_\alpha(\|(u_{0i}, \eta_{0i})\|_{\mathcal{H}_{\beta,0}^M}), \tag{166}$$

where  $c = \min\{2, 2\lambda, \alpha\}$  and

$$\mathcal{N}(t) := \|u(t)\|_{W_0^{\beta,2}}^2 + \|\eta^t\|_{\mathcal{M}_{-1}}^2 + \frac{\alpha}{2} \|u(t)\|^2. \tag{167}$$

Applying Grönwall’s inequality to (166) after omitting the term  $\alpha \|\partial_t u\|^2$ , we obtain the claim (157). □

Consequently, we deduce the following precompactness result for the semigroup  $\mathcal{S}$ .



**Proposition 4.** Let the assumptions of Lemma 2 hold. There is  $t_* > 0$  such that the operator  $\mathcal{S}(t_*)$  is a strict contraction up to the precompact pseudometric on  $\mathcal{B}_{\beta,0}^M$ , in the sense of (156), where

$$M_*(\phi_{01}, \phi_{02}) := C_{2\alpha} \left( \int_0^{t_*} \left( \|\nabla \mu_1(\tau) - \nabla \mu_2(\tau)\|^2 + \|u_1(\tau) - u_2(\tau)\|^2 \right) d\tau \right)^{1/2}, \quad (168)$$

with  $C_\alpha \sim \alpha^{-1}$ . Furthermore,  $\mathcal{S}$  is precompact on  $\mathcal{B}_{\beta,0}^M$ .

**Proof.** Naturally, we follow from the conclusion of Lemma 2. Clearly, there is a  $t_* > 0$  so that  $C_1 e^{-\kappa_2 t_* / 2} < 1$ . Thus, the operator  $\mathcal{S}(t_*)$  is a strict contraction up to the pseudometric  $M_*$  defined by (168). The pseudometric  $M_*$  is precompact thanks to the Aubin–Lions compact embedding (99). This completes the proof.  $\square$

**Proof of Theorem 3.** The precompactness of the solution operators  $\mathcal{S}$  follows via the method of precompact pseudometrics (see Proposition 4). With the existence of a bounded absorbing set  $\mathcal{B}_{\beta,0}^M$  in  $\mathcal{H}_{\beta,0}^M$  (Lemma 1), the existence of a global attractor in  $\mathcal{H}_{\beta,0}^M$  is well-known and can be found in [50,51] for example. Additional characteristics of the attractor follow thanks to the gradient structure of Problem P (Remark 8). In particular, the first three claims in the statement of Theorem 3 are a direct result of the existence of an absorbing set, a Lyapunov functional  $\mathcal{E}$ , and the fact that the system  $(\mathcal{X}_{\beta,0}^M, \mathcal{S}(t), \mathcal{E})$  is a gradient. The fourth property is a direct result of [51] (Theorem VII.4.1), and the fifth follows from [52] (Theorem 6.3.2). This concludes the proof.  $\square$

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## Appendix A

The following is reported from [1] (Theorem 2.5).

**Theorem A1.** Let  $0 < \beta < 1$ . For  $K \in \{E, D\}$ , the following assertions hold:

- The operator  $-A_{K,\beta}$  generates a submarkovian semigroup  $(e^{-A_{K,\beta}t})_{t \geq 0}$  on  $L^2(\Omega)$  and hence can be extended to a strongly continuous contraction semigroup on  $L^p(\Omega)$  for every  $p \in [1, \infty)$ , and to a contraction semigroup on  $L^\infty(\Omega)$ .
- The operator  $A_{K,\beta}$  has a compact resolvent, and hence has a discrete spectrum. The spectrum of  $A_{K,\beta}$  may be ordered as an increasing sequence of real numbers  $0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots$  that diverges to  $+\infty$ . Moreover, 0 is not an eigenvalue for  $A_{K,\beta}$ , and if  $\phi_k$  is an eigenfunction associated with the eigenvalue  $\lambda_k$ , then  $\phi_k \in D(A_{K,\beta}) \cap L^\infty(\Omega)$ .
- Denoting the generator of the semigroup on  $L^p(\Omega)$  by  $A_{K,p}$  so that  $A_K = A_{K,2}$ , then the spectrum of  $A_{K,p}$  is independent of  $p$  for every  $p \in [1, \infty]$ .
- There holds  $D(A_{K,\beta}) \subset L^\infty(\Omega)$  provided that  $N < 4\beta$ . Let  $p \in (2, \infty)$  and assume that  $N < 4\beta p / (p - 2)$ . Then,  $D(A_{K,\beta}) \subset L^p(\Omega)$ .

**Remark A1.** From [1] (page 1284, after Equation (2.3)), we know the following embedding is compact

$$W_0^{\beta,2}(\Omega) \hookrightarrow L^p(\Omega) \quad \text{when } 1 \leq p < \star \quad \text{for } \star = \begin{cases} \frac{2N}{N-2\beta} & \text{if } N > 2\beta \\ +\infty & \text{if } N = 2\beta. \end{cases} \quad (\text{A1})$$

Furthermore,

$$W_0^{\beta,2}(\Omega) \hookrightarrow C^{0,h}(\overline{\Omega}) \quad \text{with } h := \beta - \frac{N}{2} \quad \text{if } N < 2\beta \quad \text{and } 2 < p < \infty.$$

The following result is the classical Aubin–Lions Lemma, reported here for the reader’s convenience (see [53], and, e.g., [54] (Lemma 5.51) or [52] (Theorem 3.1.1)).

**Lemma A1.** Let  $X, Y, Z$  be Banach spaces where  $Z \hookrightarrow Y \hookrightarrow X$  with continuous injections, the second being compact. Then, the following embeddings are compact:

$$W := \{\chi \in L^2(0, T; X), \partial_t \chi \in L^2(0, T; Z)\} \hookrightarrow L^2(0, T; Y),$$

and

$$W' := \{\chi \in L^\infty(0, T; X), \partial_t \chi \in L^2(0, T; Z)\} \hookrightarrow C([0, T]; Y).$$

Here, we recall the notion of  $\alpha$ -contraction and provide the main propositions which guarantee the existence of a global attractor for the semigroup of solution operators  $S(t)$ .

**Definition A1.** Let  $X$  be a Banach space and  $\alpha$  be a measure of compactness in  $X$  (see, e.g., [49] (Definition A.1)). Let  $B \subset X$ . A continuous map  $T : B \rightarrow B$  is an  $\alpha$ -contraction on  $B$ , if there exists a number  $q \in (0, 1)$  such that for every subset  $A \subset B$ ,  $\alpha(T(A)) \leq q\alpha(A)$ .

**Proposition A1.** Assume that  $B \subset X$  is closed and bounded, and that  $T : B \rightarrow B$  is an  $\alpha$ -contraction on  $B$ . Define the semigroup generated by the iterations of  $T$ , i.e.,  $S := (T^n)_{n \in \mathbb{N}}$ . Then, the set

$$\omega(B) := \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} T^m(B)}^X$$

is compact, invariant, and attracts  $B$ .

**Proposition A2.** Assume that  $S$  is a continuous semigroup of operators on  $X$  admitting a bounded, positively invariant absorbing set  $B$ , and that there exists  $t_* > 0$  such that the operator  $S_* := S(t_*)$  is an  $\alpha$ -contraction on  $B$ . Let

$$A_* := \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} S_*^m(B)}^X = \omega_*(B)$$

be the  $\omega$ -limit set of  $B$  under the map  $S_*$ , and set

$$A := \bigcup_{0 \leq t \leq t_*} S(t)A_*$$

Assume further that for all  $t \in [0, t_*]$ , the map  $x \rightarrow S(t)x$  is Lipschitz continuous from  $B$  to  $B$ , with Lipschitz constant  $L(t)$ ,  $L : [0, t_*] \rightarrow (0, +\infty)$  being a bounded function. Then,  $A = \omega(B)$ , and this set is the global attractor of  $S$  in  $B$ .

Theorems 3.1 and 3.2 are motivated by [55] (Sections II.2 and III.2), but appear in the above form in [49] (Appendix A) and [56] (Sections II.7). We also rely on the following.

**Definition A2.** A pseudometric  $d$  in  $X$  is precompact in  $X$  if every bounded sequence has a subsequence which is a Cauchy sequence relative to  $d$ .

**Proposition A3.** Let  $B \subset X$  be bounded, let  $d$  be a precompact pseudometric in  $X$ , and let  $T : B \rightarrow B$  be a continuous map. Suppose  $T$  satisfies the estimate

$$\|Tx - Ty\|_X \leq q\|x - y\|_X + d(x, y)$$

for all  $x, y \in B$  and some  $q \in (0, 1)$  independent of  $x$  and  $y$ . Then,  $T$  is an  $\alpha$ -contraction.

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