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Hermite–Hadamard, Fejér and Pachpatte-Type Integral Inequalities for Center-Radius Order Interval-Valued Preinvex Functions

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Abstract: The objective of this manuscript is to establish a link between the concept of inequalities and Center-Radius order functions, which are intriguing due to their properties and widespread use. We introduce the notion of the $\mathcal{CR}$ (Center-Radius)-order interval-valued preinvex function with the help of a total order relation between two intervals. Furthermore, we discuss some properties of this new class of preinvexity and show that the new concept unifies several known concepts in the literature and also gives rise to some new definitions. By applying these new definitions, we have amassed many classical and novel special cases that serve as applications of the key findings of the manuscript. The computations of $\mathtt{cr}$-order intervals depend upon the following concept $\mathfrak{B} = \left\langle \mathfrak{B}_c, \mathfrak{B}_r \right\rangle = \left\langle \frac{\overline{\mathfrak{B}}+\underline{\mathfrak{B}}}{2}, \frac{\overline{\mathfrak{B}}-\underline{\mathfrak{B}}}{2} \right\rangle$. Then, for the first time, inequalities such as Hermite–Hadamard, Pachpatte, and Fejér type are established for $\mathcal{CR}$-order in association with the concept of interval-valued preinvexity. Some numerical examples are given to validate the main results. The results confirm that this new concept is very useful in connection with various inequalities. A fractional version of the Hermite–Hadamard inequality is also established to show how the presented results can be connected to fractional calculus in future developments. Our presented results will motivate further research on inequalities for fractional interval-valued functions, fuzzy interval-valued functions, and their associated optimization problems.

Keywords: total-order relation; $\mathcal{CR}$-preinvexity; center-radius ($\mathcal{CR}$)-order; interval-valued functions; Hermite–Hadamard inequality; Fejér inequality

MSC: 26A51; 26A33; 26D10

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1. Introduction

Interval analysis is a subset of set-valued analysis, which is the study of sets in the context of mathematics and general topology. A historical example of interval enclosure is Archimede's method, which included calculating the circumference of a circle. The interval uncertainty that is present in many mathematical and computational models of deterministic real-world processes was addressed by this idea. This is a method that studies interval variables instead of point variables and expresses computation results as intervals, thereby eliminating errors that lead to erroneous conclusions. One of the initial objectives of the interval-valued analysis was to consider the error estimations of finite state machines' numerical solutions. Moore [1] is credited with being the first user of intervals in computer mathematics, having published the first book on interval analysis in 1966. Following the publication of this book, several scientists began to research the theory and applications of interval arithmetic.

There is no doubt that interval analysis is extremely important in both pure and applied sciences. However, the interval analysis method, which has been used in mathematical models in engineering for over fifty years as one of the approaches to solve interval uncertain structural systems, is a critical cornerstone. It is worth noting that applications in robotics [2], scientific computations [3], signal processing [4], optimization [5], automatic error analysis [6], computer graphics [7], and neural network [8] have all been considered. We refer interested readers to [9–16] and the bibliographies cited in them for recent developments in the field of interval-valued mappings.

A real valued function $\mathfrak{D} : \mathrm{K} \subseteq \mathbb{R} \to \mathbb{R}$ (set of real numbers) is said to be convex if the following inequality satisfies

$$\mathfrak{D}(\mathfrak{t}\mathfrak{p} + (1-\mathfrak{t})\mathfrak{r}) \leq \mathfrak{t}\mathfrak{D}(\mathfrak{p}) + (1-\mathfrak{t})\mathfrak{D}(\mathfrak{r}), \tag{1}$$

for all $\mathfrak{p}, \mathfrak{r} \in \mathrm{K}, \mathfrak{t} \in [0,1]$.

The generalized convexity of mappings on the other hand is a powerful tool for dealing with a wide range of challenges in nonlinear analysis and applied analysis, including numerous problems in mathematical physics. Several generalizations of convex functions have recently been studied rigorously. The concept of integral inequalities is an interesting topic of mathematical analytic research. The concept of convexity is well-known in the theory of inequality. Inequalities and various types of extended convex mappings have also been considered important in the study of differential and integral equations. Their significant influence is visible in electrical networks, symmetry analysis, probability theory, operations research, finance, decision making, numerical analysis, and equilibrium. The use of several fundamental integral inequalities as a method to encourage the subjective features of convexity is studied.

Let $\mathfrak{D} : \mathrm{K} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function with $\mathfrak{p} < \mathfrak{r}$ and $\mathfrak{p}, \mathfrak{r} \in \mathrm{K}$. Then, the Hermite–Hadamard inequality is expressed as follows: (see [17]):

$$\mathfrak{D}\left(\frac{\mathfrak{p}+\mathfrak{r}}{2}\right) \leq \frac{1}{\mathfrak{r}-\mathfrak{p}} \int_{\mathfrak{p}}^{\mathfrak{r}} \mathfrak{D}(x)dx \leq \frac{\mathfrak{D}(\mathfrak{p}) + \mathfrak{D}(\mathfrak{r})}{2}. \tag{2}$$

Numerous mathematicians have recently generalized and extended the standard Hermite–Hadamard inequality under the premise of some interesting new definitions as a generalization of a convex function. On the other hand, many scholars have contributed to the development of inequalities and properties associated with generalized convexity in a variety of directions, as evidenced by the published publications [18–20] and the references cited therein. As a very practical technique, fractional calculus is an essential cornerstone in both applied sciences and mathematics. Academics have recommended that many scholars take into account the applications of fractional calculus to solve many real-life problems. Several researchers investigated the Hermite–Hadamard-type integral inequalities [21], Hermite–Hadamard–Mercer inequalities [22], Simpson-type inequality [23], and Ostrowski inequality [24] using the Riemann–Liouville fractional integral operators. In [25], the Simpson–Mercer integral inequality was studied via the Atangana–Baleanu fractional operator, in [26], the Hermite–Hadamard inequality, and the Fejér-type integral inequalities were presented via Katugampola-type fractional integral operators. Additionally, the Hermite–Hadamard inequality and its Mercer counterpart were explored via Caputo–

Fabrizio fractional integrals [27,28]. The research mentioned above demonstrates the close connection that exists between integral inequalities and fractional integral operators. In the past few years, several scientists have connected integral inequalities with interval-valued functions (IVFs), yielding numerous important results. Costa [29] proposed the Opial-type inequalities, Chalco-Cano [30] used the generalized Hukuhara derivative to investigate Ostrowski-type inequalities, and Roman-Flores [31] determined the Minkowski type inequalities and Beckenbach's type inequalities. In the year 2018, Zhao et al. [32] defined the interval-valued h-convex function and developed the generalization of the $\mathrm{H}-\mathrm{H}$ inequality in the context of interval-valued analysis. An et al. [33] introduced the interval (h_1, h_2)-convex function. Recently, Zhao et al. [34] enhanced this notion by introducing interval-valued coordinated convex functions and establishing corresponding $\mathrm{H}-\mathrm{H}$ type inequalities. It was further used to reinforce the $\mathrm{H}-\mathrm{H}$ and Fejer-type inequalities for n-polynomial convex interval-valued function [35] and preinvex functions [36,37]. The concept of interval-valued preinvex functions was recently expanded to interval-valued coordinated preinvex functions by Lai et al. [38].

However, these results are predicated on the inclusion and interval Lower-Upper (LU) or Left-Right (LR) order relations, which are partial orders. As a result, determining how to employ a total order relation to investigate the convexity and inequality of interval-valued functions is a critical issue. Similarly, we will start to deal with the total interval order relation i.e., $\mathcal{CR}$-order as proposed by Bhunia et al. [39] throughout this study. The primary goal of this research is to investigate the $\mathcal{CR}$-h-preinvexity of interval-valued functions in terms of $\mathcal{CR}$-order. As a result, we begin by introducing the new notion of interval-valued $\mathcal{CR}$-h-preinvex functions and studying their fundamental features. The definition and basic features of $\mathcal{CR}$-h-preinvex functions are used in the following chapters to construct several inequalities for interval-valued functions.

The advantage of the present study is that we introduce a new notion of interval-valued preinvexity concerning a total order relation, i.e., Center-Radius order, which is very new in the literature. This article opens a new direction in the field of inequalities as to how we can incorporate the concepts of CR-interval valued function with integral inequalities such as Hermite–Hadamard, Pachpatte, and Fejér type. It is here to note that the concept of CR-order interval-valued analysis is different from the classical interval-valued analysis. Here, we calculate the intervals using the concept of Center and Radius given as $\mathfrak{B}_c = \frac{\overline{\mathfrak{B}}+\underline{\mathfrak{B}}}{2}$ and $\mathfrak{B}_r = \frac{\overline{\mathfrak{B}}-\underline{\mathfrak{B}}}{2}$, respectively, where $\overline{\mathfrak{B}}$ and $\underline{\mathfrak{B}}$ are endpoints of an interval $\mathfrak{B}$.

Motivated by the concepts of interval valued analysis as presented in Section 2, the above literature about integral inequalities and the concepts of $\mathcal{CR}$-order and $\mathcal{CR}$-convex function presented by Bhunia and Samanta [39], Rahman et al. [40], Shi et al. [41] and Liu et al. [42], we define a new notion of $\mathcal{CR}-\mathtt{h}$-preinvex function and present related inequalities using this.

The manuscript is structured as follows: The prerequisite and relevant facts regarding the related inequalities and the interval-valued analysis are discussed in Section 2. Next, we introduce a new notion of preinvex function involving Center-Radius order, i.e., $\mathcal{CR}-\mathtt{h}$-preinvex function and discuss its basic properties in Section 3. In Section 4, we derive some new versions of the Hermite–Hadamard and Pachpatte and Fejér-type inequalities for $\mathcal{CR}-\mathtt{h}$-preinvex function and for the product of two $\mathcal{CR}-\mathtt{h}$-preinvex functions. Our plans and recommendations for researchers are presented in Section 5. A brief conclusion and potential scopes for further research, which are linked to the results presented in this paper, are explored in Section 6.

2. Preliminaries

The Center and Radius ($\mathcal{CR}$) preinvex interval-valued functions, the theory of preinvexity, and interval-valued integration, all of which are used extensively throughout the work, are presented in this section.

Hanson defined the notion of invex functions concerning the bifunction $\xi(\cdot,\cdot)$ in mathematical programming in 1981. Soon after Hanson's work [43] was published, Ben-Israel and Mond [44] looked into invex sets and preinvex functions. Preinvexity encompasses a broader term than convexity. Weir and Mond [45] used the concept of invex sets to investigate the theory of preinvexity in 1988.

Definition 1. *Let* $\mathfrak{p} \in \mathrm{K} \subset \mathbb{R}^n$*; then,* K *is said to be invex at* $\mathfrak{p}$ *with respect to* $\xi : \mathrm{K} \times \mathrm{K} \to \mathbb{R}^n$*, if for each* $\mathfrak{r} \in \mathrm{K}$*,*

$$\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}) \in \mathrm{K}, \qquad \mathfrak{t} \in [0,1].$$

Definition 2 (see [45])**.** *Let* $\mathrm{K} \neq \emptyset \in \mathbb{R}$ *be an invex set with respect to* $\xi : \mathrm{K} \times \mathrm{K} \neq \emptyset \to \mathbb{R}$*. Then, the function* $\mathfrak{D} : \mathrm{K} \to \mathbb{R}$ *is said to be preinvex with respect to* ξ *if*

$$\mathfrak{D}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \leqq \mathfrak{t}\mathfrak{D}(\mathfrak{r}) + (1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{p}), \qquad (\forall \mathfrak{p},\mathfrak{r} \in \mathrm{K};\ \mathfrak{t} \in [0,1]).$$

Definition 3 (see [46])**.** *Let* $\mathrm{K} \neq \emptyset \in \mathbb{R}$ *be an invex set with respect to* $\xi : \mathrm{K} \times \mathrm{K} \neq \emptyset \to \mathbb{R}$ *and* $\mathtt{h} \neq 0$*. Then, the function* $\mathfrak{D} : \mathrm{K} \to \mathbb{R}$ *is said to be* $\mathtt{h}$*-preinvex with respect to* ξ *if*

$$\mathfrak{D}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \leqq \mathtt{h}(\mathfrak{t})\mathfrak{D}(\mathfrak{r}) + \mathtt{h}(1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{p}), \qquad (\forall \mathfrak{p},\mathfrak{r} \in \mathrm{K};\ \mathfrak{t} \in [0,1]).$$

Condition C. (see [47]). Let $\mathrm{K} \subset \mathbb{R}^n$ be an open invex subset with respect to $\xi : \mathrm{K} \times \mathrm{K} \to \mathbb{R}$. For any $\mathfrak{p},\mathfrak{r} \in \mathrm{K}$ and $\mathfrak{t} \in [0,1]$,

$$\xi(\mathfrak{r},\mathfrak{r} + \mathfrak{t}\,\xi(\mathfrak{p},\mathfrak{r})) = -\mathfrak{t}\,\xi(\mathfrak{p},\mathfrak{r}) \tag{3}$$

and

$$\xi(\mathfrak{p},\mathfrak{r} + \mathfrak{t}\,\xi(\mathfrak{p},\mathfrak{r})) = (1-\mathfrak{t})\,\xi(\mathfrak{p},\mathfrak{r}). \tag{4}$$

In fact, for any $\mathfrak{p},\mathfrak{r} \in \mathrm{K}$ and $\mathfrak{t}_1,\mathfrak{t}_2 \in [0,1]$, we find from Condition C that

$$\xi(\mathfrak{r} + \mathfrak{t}_2\,\xi(\mathfrak{p},\mathfrak{r}),\mathfrak{r} + \mathfrak{t}_1\,\xi(\mathfrak{p},\mathfrak{r})) = (\mathfrak{t}_2 - \mathfrak{t}_1)\xi(\mathfrak{p},\mathfrak{r}).$$

Basic Properties of Interval-Valued Functions

Here, in this subsection, we present some basic arithmetic about interval analysis, which will be very helpful throughout the paper.

$$[\mathfrak{B}] = [\underline{\mathfrak{B}},\overline{\mathfrak{B}}] \qquad (x \in \mathbb{R},\ \underline{\mathfrak{B}} \leqq x \leqq \overline{\mathfrak{B}};\ \underline{\mathfrak{B}},\overline{\mathfrak{B}} \in \mathbb{R})$$

$$[\mathfrak{C}] = [\underline{\mathfrak{C}},\overline{\mathfrak{C}}] \qquad (x \in \mathbb{R},\ \underline{\mathfrak{C}} \leqq x \leqq \overline{\mathfrak{C}};\ \underline{\mathfrak{C}},\overline{\mathfrak{C}} \in \mathbb{R})$$

$$[\mathfrak{B}] + [\mathfrak{C}] = [\underline{\mathfrak{B}},\overline{\mathfrak{B}}] + [\underline{\mathfrak{C}},\overline{\mathfrak{C}}] = [\underline{\mathfrak{B}} + \underline{\mathfrak{C}},\overline{\mathfrak{B}} + \overline{\mathfrak{C}}]$$

$$\gamma\mathfrak{B} = \gamma[\underline{\mathfrak{B}},\overline{\mathfrak{B}}] = \begin{cases} [\gamma\underline{\mathfrak{B}},\gamma\overline{\mathfrak{B}}] & (\gamma > 0) \\ \{0\} & (\gamma = 0) \\ [\gamma\overline{\mathfrak{B}},\gamma\underline{\mathfrak{B}}] & (\gamma < 0), \end{cases}$$

where $\gamma \in \mathbb{R}$.

Let $\mathbb{R}_I, \mathbb{R}_I^+$ and $\mathbb{R}_I^-$ be the set of all closed intervals of $\mathbb{R}$, the set of all positive closed intervals of $\mathbb{R}$ and the set of all negative closed intervals of $\mathbb{R}$, respectively. We now discuss some algebraic properties of interval arithmetic.

Let $\mathfrak{B} = [\underline{\mathfrak{B}}, \bar{\mathfrak{B}}] \in \mathbb{R}_I$, then $\mathfrak{B}_c = \frac{\overline{\mathfrak{B}}+\underline{\mathfrak{B}}}{2}$ and $\mathfrak{B}_r = \frac{\overline{\mathfrak{B}}-\underline{\mathfrak{B}}}{2}$ are the center and radius of interval $\mathfrak{B}$, respectively. The center-radius form of interval $\mathfrak{B}$ can be represented as:

$$\mathfrak{B} = \left\langle \mathfrak{B}_c, \mathfrak{B}_r \right\rangle = \left\langle \frac{\overline{\mathfrak{B}}+\underline{\mathfrak{B}}}{2}, \frac{\overline{\mathfrak{B}}-\underline{\mathfrak{B}}}{2} \right\rangle.$$

Now, we present the order relation of center and radius of the interval as follows:

Definition 4. *The center–radius order relation for* $\mathfrak{B} = [\underline{\mathfrak{B}}, \overline{\mathfrak{B}}] = \langle \mathfrak{B}_c, \mathfrak{B}_r \rangle$, $\mathfrak{C} = [\underline{\mathfrak{C}}, \overline{\mathfrak{C}}] = \langle \mathfrak{C}_c, \mathfrak{C}_r \rangle \in \mathbb{R}_I$ *defined as:*

$$\mathfrak{B} \preceq_{cr} \mathfrak{C} \Longleftrightarrow \begin{cases} \mathfrak{B}_c < \mathfrak{C}_c, & if \quad \mathfrak{B}_c \neq \mathfrak{C}_c \\ \mathfrak{B}_r \leq \mathfrak{C}_r, & if \quad \mathfrak{B}_c = \mathfrak{C}_c \end{cases}$$

NOTE: For any two intervals $\mathfrak{B}, \mathfrak{C} \in \mathbb{R}_I$, we have either $\mathfrak{B} \preceq_{cr} \mathfrak{C}$ or $\mathfrak{C} \preceq_{cr} \mathfrak{B}$.

Here, in this subsection, we present the Riemann integral operator for interval-valued functions.

Definition 5 (see [48])**.** *Let* $\mathfrak{D} : [\mathfrak{p}, \mathfrak{r}]$ *be an interval-valued function such that* $\mathfrak{D} = [\underline{\mathfrak{D}}, \overline{\mathfrak{D}}]$. *Then,* $\mathfrak{D}$ *is Riemann integrable on* $[\mathfrak{p}, \mathfrak{r}]$ *if and only if* $\underline{\mathfrak{D}}$ *and* $\overline{\mathfrak{D}}$ *are Riemann integrable on* $[\mathfrak{p}, \mathfrak{r}]$, *that is,*

$$(IR) \int_{\mathfrak{p}}^{\mathfrak{r}} \mathfrak{D}(z)dz = \left[(R) \int_{\mathfrak{p}}^{\mathfrak{r}} \underline{\mathfrak{D}}(z)dz, (R) \int_{\mathfrak{p}}^{\mathfrak{r}} \overline{\mathfrak{D}}(z)dz\right].$$

The set of all Riemann integrable interval-valued functions on $[\mathfrak{p}, \mathfrak{r}]$ *is represented by* $\mathcal{IR}_{([\mathfrak{p},\mathfrak{r}])}$.

Shi et al. [41] proved that the integral is order preserving with respect to the $\mathcal{CR}$-order relations.

Theorem 1. *Let* $\mathfrak{D}, \mathsf{G} : [\mathfrak{p}, \mathfrak{r}]$ *be interval-valued functions given by* $\mathfrak{D} = [\underline{\mathfrak{D}}, \overline{\mathfrak{D}}]$ *and* $\mathsf{G} = [\underline{\mathsf{G}}, \overline{\mathsf{G}}]$. *If* $\mathfrak{D}(z) \preceq_{\mathcal{CR}} \mathsf{G}(z)$ *for all* $u \in [\mathfrak{p}, \mathfrak{r}]$, *then*

$$\int_{\mathfrak{p}}^{\mathfrak{r}} \mathfrak{D}(z)dz \preceq_{\mathcal{CR}} \int_{\mathfrak{p}}^{\mathfrak{r}} \mathsf{G}(z)dz.$$

Now, we will give an example to validate the above theorem. Figures 1–3 show the graphical representations of theorem 1.

Example 1. *Let* $\mathfrak{D} = [z, 2z]$ *and* $\mathsf{G} = [z^2, z^2 + 2]$. *Then, for* $z \in [0, 1]$

$$\mathfrak{D}_{\mathcal{C}} = \frac{3z}{2}, \mathfrak{D}_{\mathcal{R}} = \frac{z}{2}, \mathsf{G}_{\mathcal{C}} = z^2 + 1 \text{ and } \mathsf{G}_{\mathcal{R}} = 1.$$

So, by using the Definition 4, we have $\mathfrak{D}(z) \preceq_{\mathcal{CR}} \mathsf{G}(z)$, $z \in [0, 1]$.
Since,

$$\int_0^1 [z, 2z]dz = \left[\frac{1}{2}, 1\right]$$

and

$$\int_0^1 [z^2, z^2 + 2]dz = \left[\frac{1}{3}, \frac{7}{3}\right].$$

Now, again using the Definition 4, we have

$$\int_0^1 \mathfrak{D}(z)dz \preceq_{\mathcal{CR}} \int_0^1 \mathsf{G}(z)dz.$$

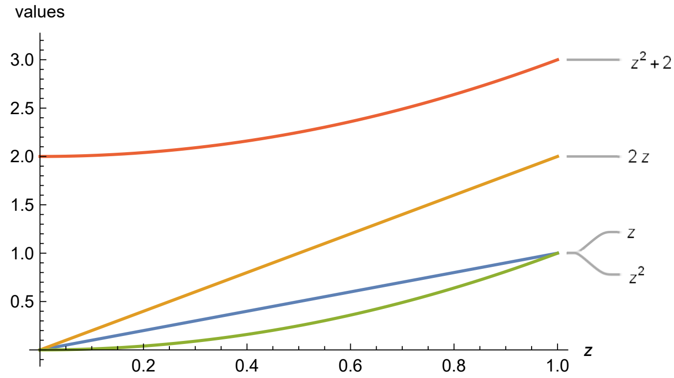

Figure 1. The graph shows the validity of $\mathcal{CR}$-order relations.

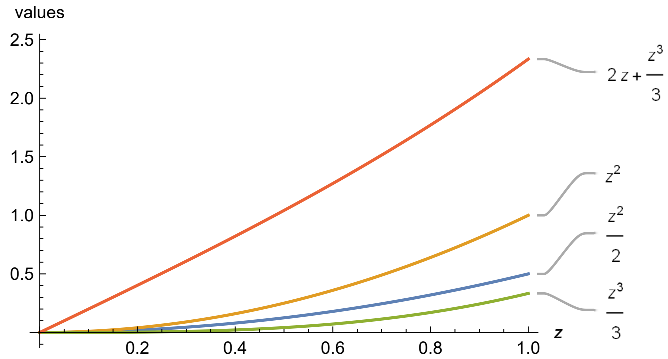

Figure 2. The graph shows the validity of Theorem 1.

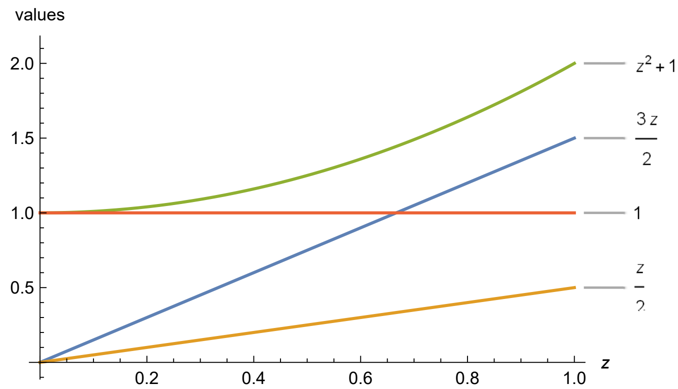

Figure 3. The graph of $\mathfrak{D}_{\mathcal{C}} = \frac{3z}{2}, \mathfrak{D}_{\mathcal{R}} = \frac{z}{2}, \mathsf{G}_{\mathcal{C}} = z^2 + 1$ and $\mathsf{G}_{\mathcal{R}} = 1$.

3. The Concept of Interval Valued $\mathcal{CR}$-h-Preinvex function

Definition 6. *Let $\mathfrak{D} : [\mathfrak{p}, \mathfrak{r}]$ be non-negative interval-valued functions given by $\mathfrak{D} = [\underline{\mathfrak{D}}, \overline{\mathfrak{D}}]$ and $\mathsf{h} : [0,1] \to \mathbb{R}^+$ be a non-negative function. Then, the function $\mathfrak{D}$ on the invex set $\mathbb{K}$ is said to be interval valued $\mathcal{CR} - \mathsf{h}$-preinvex with respect to ξ if*

$$\mathfrak{D}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p})\big) \preceq_{\mathcal{CR}} \mathsf{h}(\mathfrak{t})\mathfrak{D}(\mathfrak{r}) + \mathsf{h}(1-\mathfrak{t})\, \mathfrak{D}(\mathfrak{p}) \qquad (\forall \mathfrak{p}, \mathfrak{r} \in \mathrm{K};\ \mathfrak{t} \in [0,1]).$$

Remark 1. *If we put $h(\mathfrak{t}) = \mathfrak{t}$, then the Definition 6 reduces to interval valued $\mathcal{CR}$-preinvex functions.*

$$\mathfrak{D}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p})\big) \preceq_{\mathcal{CR}} \mathfrak{t}\mathfrak{D}(\mathfrak{r}) + (1-\mathfrak{t})\, \mathfrak{D}(\mathfrak{p}).$$

Remark 2. *If we put $h(\mathfrak{t}) = \mathfrak{t}^s$, then the Definition 6 reduces to interval valued $\mathcal{CR}$-s-preinvex functions.*

$$\mathfrak{D}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p})\big) \preceq_{\mathcal{CR}} \mathfrak{t}^s\mathfrak{D}(\mathfrak{r}) + (1-\mathfrak{t})^s\, \mathfrak{D}(\mathfrak{p}).$$

Remark 3. *If we put $h(\mathfrak{t}) = \mathfrak{t}(1-\mathfrak{t})$, then the Definition 6 reduces to interval valued $\mathcal{CR}$-tgs-preinvex functions.*

$$\mathfrak{D}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \preceq_{\mathcal{CR}} \mathfrak{t}(1-\mathfrak{t})[\mathfrak{D}(\mathfrak{r}) + \mathfrak{D}(\mathfrak{p})].$$

Remark 4. *If we put $h(\mathfrak{t}) = \frac{1}{\mathfrak{t}}$, then the Definition 6 reduces to interval valued $\mathcal{CR}$-Goudunova–Levin-preinvex functions.*

$$\mathfrak{D}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \preceq_{\mathcal{CR}} \frac{\mathfrak{D}(\mathfrak{r})}{\mathfrak{t}} + \frac{\mathfrak{D}(\mathfrak{p})}{1-\mathfrak{t}}.$$

Remark 5. *If we choose $\xi(\mathfrak{r},\mathfrak{p}) = \mathfrak{r} - \mathfrak{p}$, then the Definition 6 reduces to interval valued $\mathcal{CR}$-h-convex functions.*

$$\mathfrak{D}\big(\mathfrak{t}\mathfrak{r} + (1-\mathfrak{t})\mathfrak{p}\big) \preceq_{\mathcal{CR}} h(\mathfrak{t})\mathfrak{D}(\mathfrak{r}) + h(1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{p}).$$

Remark 6. *If we choose $h(\mathfrak{t}) = \mathfrak{t}$, then the Remark 5 reduces to interval valued $\mathcal{CR}$-convex functions.*

$$\mathfrak{D}\big(\mathfrak{t}\mathfrak{r} + (1-\mathfrak{t})\mathfrak{p}\big) \preceq_{\mathcal{CR}} \mathfrak{t}\mathfrak{D}(\mathfrak{r}) + (1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{p}).$$

Remark 7. *If we choose $\underline{\mathfrak{D}} = \overline{\mathfrak{D}}$, then Definition 6 reduces to h-preinvex functions.*

$$\mathfrak{D}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \leq \mathsf{h}(\mathfrak{t})\mathfrak{D}(\mathfrak{r}) + \mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{p}).$$

If we choose $\underline{\mathfrak{D}} = \overline{\mathfrak{D}}$, and $h(\mathfrak{t}) = \mathfrak{t}$, then the Definition 6 reduces to preinvex functions.

$$\mathfrak{D}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \leq \mathfrak{t}\mathfrak{D}(\mathfrak{r}) + (1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{p}).$$

If we choose $\underline{\mathfrak{D}} = \overline{\mathfrak{D}}$, then Remarks 2–4 reduces to s-preinvex, tgs-preinvex, and Goudunova–Levin-preinvex functions, respectively. In addition, if we choose $\xi(\mathfrak{r},\mathfrak{p}) = (\mathfrak{r}-\mathfrak{p})$, then we have many convexities, i.e., h-convex function, convex function, s-convex function, tgs-convex function, Goudunova–Levin convex functions, etc.

The definitions of our manuscript can be extended to fractal sets, and hence, these results can also be established for functions defined on fractal sets. From the above definition and remarks, we can conclude that the presented new notion of $\mathcal{CR} - \mathsf{h}$-preinvex functions gives rise to some new notions of $\mathcal{CR}$-preinvexities such as $\mathcal{CR}$-h-preinvex function, $\mathcal{CR}$-s-preinvex function, $\mathcal{CR}$-preinvex functions and $\mathcal{CR}$-convexities such as $\mathcal{CR}$-h-convex function, $\mathcal{CR}$-s-convex function, and $\mathcal{CR}$-convex function. It also recovers many existing concepts such as preinvex function, h-preinvex function, s-preinvex function, tgs-preinvex function, Goudunova–Levin-preinvex functions, h-convex function, convex function, s-convex function, tgs-convex function, Goudunova–Levin convex functions, and many more in the literature. This proves the novelty and importance of our results. We believe these new concepts and results of this paper will guide some innovative research ideas for scholars/researchers working in the field of inequalities.

Proposition 1. *Let $\mathfrak{D} : [\mathfrak{p},\mathfrak{r}] \to \mathbb{R}_I$ be interval-valued functions given by $\mathfrak{D} = [\underline{\mathfrak{D}},\overline{\mathfrak{D}}] = \langle \mathfrak{D}_{\mathcal{C}}, \mathfrak{D}_{\mathcal{R}} \rangle$. If $\mathfrak{D}_{\mathcal{C}}$ and $\mathfrak{D}_{\mathcal{R}}$ are h-preinvex functions, then $\mathfrak{D}$ is an interval valued $\mathcal{CR} - \mathsf{h}$-preinvex function.*

Proof. Since $\mathfrak{D}_{\mathcal{C}}$ and $\mathfrak{D}_{\mathcal{R}}$ are h-preinvex function, then for each $\mathfrak{t} \in [0,1]$, we have

$$\mathfrak{D}_{\mathcal{C}}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \leq \mathsf{h}(\mathfrak{t})\mathfrak{D}_{\mathcal{C}}(\mathfrak{r}) + \mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}_{\mathcal{C}}(\mathfrak{p})$$

and

$$\mathfrak{D}_{\mathcal{R}}\big(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \leq \mathsf{h}(\mathfrak{t})\mathfrak{D}_{\mathcal{R}}(\mathfrak{r}) + \mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}_{\mathcal{R}}(\mathfrak{p}).$$

If $\mathfrak{D}_{\mathcal{C}}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \neq \mathsf{h}(\mathfrak{t})\mathfrak{D}_{\mathcal{C}}(\mathfrak{r})+\mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}_{\mathcal{C}}(\mathfrak{p})$, then

$$\mathfrak{D}_{\mathcal{C}}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) < \mathsf{h}(\mathfrak{t})\mathfrak{D}_{\mathcal{C}}(\mathfrak{r})+\mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}_{\mathcal{C}}(\mathfrak{p}).$$

This implies

$$\mathfrak{D}_{\mathcal{C}}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \preceq_{\mathcal{CR}} \mathsf{h}(\mathfrak{t})\mathfrak{D}_{\mathcal{C}}(\mathfrak{r})+\mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}_{\mathcal{C}}(\mathfrak{p}).$$

Otherwise, $\mathfrak{D}_{\mathcal{R}}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \leq \mathsf{h}(\mathfrak{t})\mathfrak{D}_{\mathcal{R}}(\mathfrak{r})+\mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}_{\mathcal{R}}(\mathfrak{p})$ implies

$$\mathfrak{D}_{\mathcal{R}}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \preceq_{\mathcal{CR}} \mathsf{h}(\mathfrak{t})\mathfrak{D}_{\mathcal{R}}(\mathfrak{r})+\mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}_{\mathcal{R}}(\mathfrak{p}).$$

Now, from Definition 4, we can clearly see that

$$\mathfrak{D}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big) \preceq_{\mathcal{CR}} \mathsf{h}(\mathfrak{t})\mathfrak{D}(\mathfrak{r})+\mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{p}).$$

This proves that if $\mathfrak{D}_{\mathcal{C}}$ and $\mathfrak{D}_{\mathcal{R}}$ are h-preinvex functions, then $\mathfrak{D}$ is interval valued $\mathcal{CR}-\mathsf{h}$-preinvex functions. □

4. Application of $\mathcal{CR}$-Preinvexity to Inequalities

In this section, we will discuss the application of the new concept i.e., $\mathcal{CR}-\mathsf{h}$-preinvex functions to present some Hermite–Hadamard, Pachppate and Fejér-type inequalities.

Theorem 2. *Suppose that $\mathfrak{D} : [\mathfrak{p},\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})] \longrightarrow \mathbb{R}$ is an interval-valued function which is given by*

$$\mathfrak{D}(\mathrm{z}) = \big[\underline{\mathfrak{D}}(\mathrm{z}),\ \overline{\mathfrak{D}}(\mathrm{z})\big],$$

for all $\mathrm{z}\in[\mathfrak{p},\mathfrak{r}]$. If $\mathfrak{D} : [\mathfrak{p},\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})] \longrightarrow \mathbb{R}$ is $\mathcal{CR}-\mathsf{h}$-preinvex function and satisfies Condition C. Then, for $\mathsf{h}(\frac{1}{2})>0$, the following inequalities hold true:

$$\frac{1}{2\mathsf{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \preceq_{\mathcal{CR}} \frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathrm{z})d\mathrm{z} \preceq_{\mathcal{CR}} [\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 \mathsf{h}(\mathfrak{t})d\mathfrak{t}.$$

Proof. From the definition of $\mathcal{CR}-\mathsf{h}$-preinvex functions and employing the Condition C, we have

$$\mathfrak{D}\left(x+\frac{1}{2}\xi(y,x)\right) \preceq_{\mathcal{CR}} \mathsf{h}\left(\frac{1}{2}\right)[\mathfrak{D}(x)+\mathfrak{D}(y)].$$

Choosing $x=\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})$ and $y=\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p})$. It is seen that

$$\begin{aligned}\mathfrak{D}&\left(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})+\frac{1}{2}\xi(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}),\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\right)\\ &\preceq_{\mathcal{CR}} \mathsf{h}\left(\frac{1}{2}\right)[\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))].\end{aligned}$$

This implies

$$\frac{1}{\mathsf{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \preceq_{\mathcal{CR}} [\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))]. \tag{5}$$

Integrating both sides of the above inequality (17) over the closed inteval [0, 1], we obtain

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\\
&\preceq_{\mathcal{CR}}\left[\int_0^1 \mathfrak{D}(\mathfrak{p}+t\xi(\mathfrak{r},\mathfrak{p}))dt+\int_0^1 \mathfrak{D}(\mathfrak{p}+(1-t)\xi(\mathfrak{r},\mathfrak{p}))dt\right]\\
&=\Bigg[\int_0^1 (\underline{\mathfrak{D}}(\mathfrak{p}+t\xi(\mathfrak{r},\mathfrak{p}))+\underline{\mathfrak{D}}(\mathfrak{p}+(1-t)\xi(\mathfrak{r},\mathfrak{p})))dt,\\
&\qquad \int_0^1 (\overline{\mathfrak{D}}(\mathfrak{p}+t\xi(\mathfrak{r},\mathfrak{p}))+\overline{\mathfrak{D}}(\mathfrak{p}+(1-t)\xi(\mathfrak{r},\mathfrak{p})))dt\Bigg]\\
&=\left[\frac{2}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\underline{\mathfrak{D}}(z)dz,\frac{2}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\overline{\mathfrak{D}}(z)dz\right]\\
&=\frac{2}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(z)dz.
\end{aligned}$$

From the above developments, we can conclude that

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\preceq_{\mathcal{CR}}\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(z)dz. \tag{6}$$

This completes the proof of the first inequality. Next, to prove the second inequality, from the definition of $\mathcal{CR}-\mathsf{h}$-preinvex functions, we have

$$\mathfrak{D}\big(\mathfrak{p}+t\xi(\mathfrak{r},\mathfrak{p})\big)\preceq_{\mathcal{CR}} h(t)\mathfrak{D}(\mathfrak{r})+h(1-t)\,\mathfrak{D}(\mathfrak{p}).$$

Integrating the above inequality over [0, 1], we have

$$\int_0^1 \mathfrak{D}\big(\mathfrak{p}+t\xi(\mathfrak{r},\mathfrak{p})\big)dt\preceq_{\mathcal{CR}}\mathfrak{D}(\mathfrak{r})\int_0^1 h(t)dt+\mathfrak{D}(\mathfrak{p})\int_0^1 h(1-t)dt.$$

This implies,

$$\begin{aligned}
\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(z)dz&\preceq_{\mathcal{CR}}[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 [h(t)+h(1-t)]dt\\
&=[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 h(t)dt.
\end{aligned} \tag{7}$$

Consequently, from Equations (18) and (19), we conclude the desired result, i.e.,

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\preceq_{\mathcal{CR}}\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(z)dz\preceq_{\mathcal{CR}}[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 h(t)dt.$$

This completes the proof. □

Remark 8. *If $\underline{\mathfrak{D}}=\overline{\mathfrak{D}}$, then it is clearly see that Theorem 2 yields the following result for the h-preinvex function.*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\leq\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(z)dz\leq[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 h(t)dt.$$

Remark 9. *When we choose* $\xi(\mathfrak{r},\mathfrak{p}) = \mathfrak{r}-\mathfrak{p}$*, Theorem 2 yields results for* $\mathcal{CR}-\mathsf{h}$*-convex functions, i.e.,*

$$\frac{1}{2\mathsf{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\frac{\mathfrak{p}+\mathfrak{r}}{2}\right) \preceq_{\mathcal{CR}} \frac{1}{\mathfrak{r}-\mathfrak{p}}\int_{\mathfrak{p}}^{\mathfrak{r}}\mathfrak{D}(\mathsf{z})d\mathsf{z} \preceq_{\mathcal{CR}} [\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 \mathsf{h}(\mathfrak{t})d\mathfrak{t}.$$

Remark 10. *If we choose* $\mathsf{h}(\mathfrak{t}) = \mathfrak{t}$ *in Theorem 2, we have new results for* $\mathcal{CR}$*-preinvex functions*

$$\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \preceq_{\mathcal{CR}} \frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathsf{z})d\mathsf{z} \preceq_{\mathcal{CR}} \frac{\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})}{2}.$$

Remark 11. *If we choose* $\mathsf{h}(\mathfrak{t}) = \mathfrak{t}$ *and* $\xi(\mathfrak{r},\mathfrak{p}) = \mathfrak{r}-\mathfrak{p}$ *in Theorem 2, we have new results for* $\mathcal{CR}$*-convex functions*

$$\mathfrak{D}\left(\frac{\mathfrak{p}+\mathfrak{r}}{2}\right) \preceq_{\mathcal{CR}} \frac{1}{\mathfrak{r}-\mathfrak{p}}\int_{\mathfrak{p}}^{\mathfrak{r}}\mathfrak{D}(\mathsf{z})d\mathsf{z} \preceq_{\mathcal{CR}} \frac{\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})}{2}.$$

Example 2. *Let* $\mathfrak{D}(\mathsf{z}) = [2-\mathsf{z}^{\frac{1}{2}}, 3(2-\mathsf{z}^{\frac{1}{2}})]$*,* $\xi(\mathfrak{r},\mathfrak{p}) = \mathfrak{r}-\mathfrak{p}$*,* $\mathfrak{p} = 0$ *and* $\mathfrak{r} = 2$*. Then, for* $\mathsf{h}(\mathfrak{t}) = \mathfrak{t}$*, we have*

$$\frac{1}{2\mathsf{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \approx [1,3].$$
$$\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathsf{z})d\mathsf{z} \approx [1.06, 3.17].$$
$$[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 \mathsf{h}(\mathfrak{t})d\mathfrak{t} \approx [1.29, 3.88].$$

Thus,

$$[1,3] \preceq_{\mathcal{CR}} [1.06, 3.17] \preceq_{\mathcal{CR}} [1.29, 3.88].$$

This eventually validates the accuracy of Theorem 2.

Theorem 3. *Suppose that* $\mathfrak{D}, \mathsf{G} : [\mathfrak{p}, \mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})] \longrightarrow \mathbb{R}$ *is an interval-valued function given by*

$$\mathfrak{D}(\mathsf{z}) = \left[\underline{\mathfrak{D}}(\mathsf{z}), \overline{\mathfrak{D}}(\mathsf{z})\right] \text{ and } \mathsf{G}(\mathsf{z}) = \left[\underline{\mathsf{G}}(\mathsf{z}), \overline{\mathsf{G}}(\mathsf{z})\right]$$

for all $\mathsf{z} \in [\mathfrak{p},\mathfrak{r}]$ *and* $\mathfrak{D}, \mathsf{G} \in \mathcal{IR}_{([\mathfrak{p},\mathfrak{r}])}$*. If* $\mathfrak{D} : [\mathfrak{p}, \mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})] \longrightarrow \mathbb{R}$ *is a* $\mathcal{CR}-\mathsf{h}_1$*-preinvex function and* $\mathsf{G} : [\mathfrak{p}, \mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})] \longrightarrow \mathbb{R}$ *is a* $\mathcal{CR}-\mathsf{h}_2$*-preinvex function. Then, the following inequalities hold true:*

$$\begin{aligned}&\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathsf{z})\mathsf{G}(\mathsf{z})d\mathsf{z}\\ &\preceq_{\mathcal{CR}} \mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})d\mathfrak{t} + \mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})d\mathfrak{t},\end{aligned} \tag{8}$$

where

$$\mathcal{M}(\mathfrak{p},\mathfrak{r}) = \mathfrak{D}(\mathfrak{p})\mathsf{G}(\mathfrak{p}) + \mathfrak{D}(\mathfrak{r})\mathsf{G}(\mathfrak{r})$$

and

$$\mathcal{N}(\mathfrak{p},\mathfrak{r}) = \mathfrak{D}(\mathfrak{p})\mathsf{G}(\mathfrak{r}) + \mathfrak{D}(\mathfrak{r})\mathsf{G}(\mathfrak{p}).$$

Proof. Since $\mathfrak{D}$ is interval-valued $\mathcal{CR}-\mathsf{h}_1$-preinvex functions and G is interval-valued $\mathcal{CR}-\mathsf{h}_2$-preinvex functions, we have

$$\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})) \preceq_{\mathcal{CR}} \mathsf{h}_1(\mathfrak{t})\mathfrak{D}(\mathfrak{r}) + \mathsf{h}_1(1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{p})$$

and

$$\mathsf{G}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})) \preceq_{\mathcal{CR}} \mathsf{h}_2(\mathfrak{t})\mathsf{G}(\mathfrak{r})+\mathsf{h}_2(1-\mathfrak{t})\ \mathsf{G}(\mathfrak{p}).$$

Multiplying both the above inequalities, it is readily seen that

$$\begin{aligned}
&\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\\
&\preceq_{\mathcal{CR}} [\mathsf{h}_1(\mathfrak{t})\mathfrak{D}(\mathfrak{r})+\mathsf{h}_1(1-\mathfrak{t})\ \mathfrak{D}(\mathfrak{p})]\cdot[\mathsf{h}_2(\mathfrak{t})\mathsf{G}(\mathfrak{r})+\mathsf{h}_2(1-\mathfrak{t})\ \mathsf{G}(\mathfrak{p})]\\
&= \mathsf{h}_1(\mathfrak{t})\mathsf{h}_2(\mathfrak{t})[\mathfrak{D}(\mathfrak{r})\mathsf{G}(\mathfrak{r})]+\mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})[\mathfrak{D}(\mathfrak{p})\mathsf{G}(\mathfrak{p})]\\
&+\mathsf{h}_1(\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})[\mathfrak{D}(\mathfrak{r})\mathsf{G}(\mathfrak{p})]+\mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})[\mathfrak{D}(\mathfrak{p})\mathsf{G}(\mathfrak{r})].
\end{aligned} \tag{9}$$

Integrating both sides of the above inequality (9) over the closed interval $[0,1]$, we find that

$$\begin{aligned}
&\int_0^1 \mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}\\
&\preceq_{\mathcal{CR}} [\mathfrak{D}(\mathfrak{r})\mathsf{G}(\mathfrak{r})]\int_0^1 \mathsf{h}_1(\mathfrak{t})\mathsf{h}_2(\mathfrak{t})d\mathfrak{t}+[\mathfrak{D}(\mathfrak{p})\mathsf{G}(\mathfrak{p})]\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})d\mathfrak{t}\\
&+[\mathfrak{D}(\mathfrak{r})\mathsf{G}(\mathfrak{p})]\int_0^1 \mathsf{h}_1(\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})d\mathfrak{t}+[\mathfrak{D}(\mathfrak{p})\mathsf{G}(\mathfrak{r})]\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})d\mathfrak{t}.
\end{aligned}$$

By using Definition 5, we thus obtain

$$\begin{aligned}
&\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})} \mathfrak{D}(z)\mathsf{G}(z)dz\\
&\preceq_{\mathcal{CR}} [\mathfrak{D}(\mathfrak{p})\mathsf{G}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})\mathsf{G}(\mathfrak{r})]\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})d\mathfrak{t}\\
&\quad+[\mathfrak{D}(\mathfrak{p})\mathsf{G}(\mathfrak{r})+\mathfrak{D}(\mathfrak{r})\mathsf{G}(\mathfrak{p})]\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})d\mathfrak{t}\\
&= \mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})d\mathfrak{t}+\mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})d\mathfrak{t}.
\end{aligned}$$

Finally, we are led to the desired result as asserted by Theorem 3. □

Remark 12. *If $\underline{\mathfrak{D}}=\overline{\mathfrak{D}}$, then it is clearly seen that Theorem 3 yields the result for the* h*-preinvex function.*

$$\begin{aligned}
&\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})} \mathfrak{D}(z)\mathsf{G}(z)dz\\
&\le \mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})d\mathfrak{t}+\mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})d\mathfrak{t}.
\end{aligned}$$

Remark 13. *When we choose $\xi(\mathfrak{r},\mathfrak{p})=\mathfrak{r}-\mathfrak{p}$, Theorem 3 yields a result for the $\mathcal{CR}-$* h*-convex function:*

$$\begin{aligned}
&\frac{1}{\mathfrak{r}-\mathfrak{p}}\int_{\mathfrak{p}}^{\mathfrak{r}} \mathfrak{D}(z)\mathsf{G}(z)dz\\
&\preceq_{\mathcal{CR}} \mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})d\mathfrak{t}+\mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})d\mathfrak{t}.
\end{aligned}$$

Remark 14. *When we choose $\mathsf{h}_1(\mathfrak{t})=\mathsf{h}_2(\mathfrak{t})=\mathfrak{t}$, Theorem 3 yields a result for the $\mathcal{CR}$-preinvex function:*

$$\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})} \mathfrak{D}(z)\mathsf{G}(z)dz \preceq_{\mathcal{CR}} \frac{\mathcal{M}(\mathfrak{p},\mathfrak{r})}{3}+\frac{\mathcal{N}(\mathfrak{p},\mathfrak{r})}{6}.$$

Remark 15. *When we choose* $\mathsf{h}_1(\mathfrak{t}) = \mathsf{h}_2(\mathfrak{t}) = \mathfrak{t}$ *and* $\xi(\mathfrak{r}, \mathfrak{p}) = \mathfrak{r} - \mathfrak{p}$, *then Theorem 3 yields the following result for the* $\mathcal{CR}$*-convex function:*

$$\frac{1}{\mathfrak{r}-\mathfrak{p}}\int_{\mathfrak{p}}^{\mathfrak{r}} \mathfrak{D}(z)\mathsf{G}(z)dz \preceq_{\mathcal{CR}} \frac{\mathcal{M}(\mathfrak{p},\mathfrak{r})}{3} + \frac{\mathcal{N}(\mathfrak{p},\mathfrak{r})}{6}.$$

Example 3. *Let* $\mathfrak{D}(z) = [2 - z^{\frac{1}{2}}, 3(2 - z^{\frac{1}{2}})]$, $\mathsf{G}(z) = [e^z - z, e^z + z]$, $\xi(\mathfrak{r}, \mathfrak{p}) = \mathfrak{r} - \mathfrak{p}$, $\mathfrak{p} = 0$ *and* $\mathfrak{r} = 2$. *Then, for* $\mathsf{h}_1(\mathfrak{t}) = \mathsf{h}_2(\mathfrak{t}) = \mathfrak{t}$, *we have*

$$\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})} \mathfrak{D}(z)\mathsf{G}(z)dz \approx \left[\frac{3.935}{2}, \frac{22.228}{2}\right].$$
$$\mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})d\mathfrak{t} + \mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})d\mathfrak{t} \approx \left[\frac{27.364}{6}, \frac{97.404}{6}\right].$$

Thus, it can be easily seen that

$$\left[\frac{3.935}{2}, \frac{22.228}{2}\right] \preceq_{\mathcal{CR}} \left[\frac{27.364}{6}, \frac{97.404}{6}\right].$$

This eventually validates the accuracy of Theorem 3.

Theorem 4. *Suppose that* $\mathfrak{D}, \mathsf{G} : [\mathfrak{p}, \mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p})] \longrightarrow \mathbb{R}$ *is an interval-valued function given by*

$$\mathfrak{D}(z) = \left[\underline{\mathfrak{D}}(z), \overline{\mathfrak{D}}(z)\right] \text{ and } \mathsf{G}(z) = \left[\underline{\mathsf{G}}(z), \overline{\mathsf{G}}(z)\right]$$

for all $z \in [\mathfrak{p}, \mathfrak{r}]$ *and* $\mathfrak{D}, \mathsf{G} \in \mathcal{IR}_{([\mathfrak{p},\mathfrak{r}])}$. *If* $\mathfrak{D} : [\mathfrak{p}, \mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p})] \longrightarrow \mathbb{R}$ *is a* $\mathcal{CR} - \mathsf{h}_1$*-preinvex function and* $\mathsf{G} : [\mathfrak{p}, \mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p})] \longrightarrow \mathbb{R}$ *is* $\mathcal{CR} - \mathsf{h}_2$*-is a preinvex function. Then, the following inequalities hold true:*

$$\begin{aligned} &\frac{1}{2\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \cdot \mathsf{G}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \\ &\preceq_{\mathcal{CR}} \frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})} \mathfrak{D}(z)\mathsf{G}(z)dz \\ &\quad + \mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})d\mathfrak{t} + \mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})d\mathfrak{t}, \end{aligned}$$

where $\mathcal{M}(\mathfrak{p}, \mathfrak{r})$ *and* $\mathcal{N}(\mathfrak{p}, \mathfrak{r})$ *are the same as defined in earlier theorems.*

Proof. From the definition of $\mathcal{CR} - \mathsf{h}$-preinvex functions and employing the condition C, we have

$$\frac{1}{\mathsf{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \preceq_{\mathcal{CR}} [\mathfrak{D}(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})) + \mathfrak{D}(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))].$$

Since $\mathfrak{D}$ and G are interval-valued $\mathcal{CR} - \mathsf{h}_1$-preinvex function and $\mathcal{CR} - \mathsf{h}_2$-preinvex functions, respectively, then using Condition C, we have

$$\begin{aligned} \mathfrak{D}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) &= \mathfrak{D}\left(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}) + \frac{1}{2}\xi(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}), \mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\right) \\ &\preceq_{\mathcal{CR}} \mathsf{h}_1\left(\frac{1}{2}\right)[\mathfrak{D}(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})) + \mathfrak{D}(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))]. \end{aligned} \quad (10)$$

Similarly,

$$\mathsf{G}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)=\mathsf{G}\left(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})+\frac{1}{2}\xi(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}),\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\right)$$
$$\preceq_{\mathcal{CR}}\mathsf{h}_2\left(\frac{1}{2}\right)[\mathsf{G}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\mathsf{G}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))]. \quad (11)$$

Now, multiplying the Equations (10) and (11) side by side, we obtain

$$\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\cdot\mathsf{G}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)$$
$$\preceq_{\mathcal{CR}}\mathsf{h}_1\left(\frac{1}{2}\right)[\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))]\cdot\mathsf{h}_2\left(\frac{1}{2}\right)[\mathsf{G}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\mathsf{G}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))]$$
$$=\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\underline{\mathfrak{D}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\underline{\mathsf{G}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\underline{\mathfrak{D}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\underline{\mathsf{G}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))$$
$$+\underline{\mathfrak{D}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\underline{\mathsf{G}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))+\underline{\mathfrak{D}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\underline{\mathsf{G}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})),$$
$$\overline{\mathfrak{D}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\overline{\mathsf{G}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\overline{\mathfrak{D}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\overline{\mathsf{G}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))$$
$$+\overline{\mathfrak{D}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\overline{\mathsf{G}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))+\overline{\mathfrak{D}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\overline{\mathsf{G}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\Big]$$
$$=\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\underline{\mathfrak{D}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\underline{\mathsf{G}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})),\overline{\mathfrak{D}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\overline{\mathsf{G}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\Big]$$
$$+\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\underline{\mathfrak{D}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\underline{\mathsf{G}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p})),\overline{\mathfrak{D}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\overline{\mathsf{G}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\Big]$$
$$+\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\underline{\mathfrak{D}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\underline{\mathsf{G}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p})),\overline{\mathfrak{D}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\overline{\mathsf{G}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\Big]$$
$$+\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\underline{\mathfrak{D}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\underline{\mathsf{G}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})),\overline{\mathfrak{D}}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\overline{\mathsf{G}}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\Big]$$
$$=\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\Big]$$
$$+\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))+\mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\Big]$$
$$\preceq_{\mathcal{CR}}\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\Big]$$
$$+\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[(\mathsf{h}_1(\mathfrak{t})\mathfrak{D}(\mathfrak{r})+\mathsf{h}_1(1-\mathfrak{t})\ \mathfrak{D}(\mathfrak{p}))\cdot(\mathsf{h}_2(\mathfrak{t})\mathsf{G}(\mathfrak{p})+\mathsf{h}_2(1-\mathfrak{t})\ \mathsf{G}(\mathfrak{r}))$$
$$+(\mathsf{h}_1(\mathfrak{t})\mathfrak{D}(\mathfrak{p})+\mathsf{h}_1(1-\mathfrak{t})\ \mathfrak{D}(\mathfrak{r}))\cdot\mathsf{h}_2(\mathfrak{t})\mathsf{G}(\mathfrak{r})+\mathsf{h}_2(1-\mathfrak{t})\ \mathsf{G}(\mathfrak{p})\Big]$$
$$=\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))+\mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathsf{G}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\Big]$$
$$+\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\Big[\mathcal{M}(\mathfrak{p},\mathfrak{r})[\mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(\mathfrak{t})+\mathsf{h}_1(\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})]+\mathcal{N}(\mathfrak{p},\mathfrak{r})[\mathsf{h}_1(\mathfrak{t})\mathsf{h}_2(\mathfrak{t})+\mathsf{h}_1(1-\mathfrak{t})\mathsf{h}_2(1-\mathfrak{t})]\Big].$$

Upon integration over [0, 1], we have

$$\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\cdot\mathsf{G}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)$$
$$\preceq_{\mathcal{CR}} 2\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)\left\{\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathsf{z})\mathsf{G}(\mathsf{z})d\mathsf{z}\right.$$
$$\left.+\mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1\mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(\mathsf{t})d\mathsf{t}+\mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1\mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(1-\mathsf{t})d\mathsf{t}\right\}.$$

This readily gives,

$$\frac{1}{2\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\cdot\mathsf{G}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\preceq_{\mathcal{CR}}\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathsf{z})\mathsf{G}(\mathsf{z})d\mathsf{z}$$
$$+\mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1\mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(\mathsf{t})d\mathsf{t}+\mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1\mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(1-\mathsf{t})d\mathsf{t}.$$

This completes the proof of the desired result. □

Remark 16. *If $\underline{\mathfrak{D}}=\overline{\mathfrak{D}}$, then it is clearly seen that Theorem 4 yields a result for the* h*-preinvex function.*

$$\frac{1}{2\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\cdot\mathsf{G}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)$$
$$\leq\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathsf{z})\mathsf{G}(\mathsf{z})d\mathsf{z}+\mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1\mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(\mathsf{t})d\mathsf{t}+\mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1\mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(1-\mathsf{t})d\mathsf{t}.$$

Remark 17. *When we choose $\xi(\mathfrak{r},\mathfrak{p})=\mathfrak{r}-\mathfrak{p}$, Theorem 4 yields a result for the $\mathcal{CR}-$*h*-convex function:*

$$\frac{1}{2\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)}\mathfrak{D}\left(\frac{\mathfrak{p}+\mathfrak{r}}{2}\right)\cdot\mathsf{G}\left(\frac{\mathfrak{p}+\mathfrak{r}}{2}\right)$$
$$\preceq_{\mathcal{CR}}\frac{1}{\mathfrak{r}-\mathfrak{p}}\int_{\mathfrak{p}}^{\mathfrak{r}}\mathfrak{D}(\mathsf{z})\mathsf{G}(\mathsf{z})d\mathsf{z}+\mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1\mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(\mathsf{t})d\mathsf{t}+\mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1\mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(1-\mathsf{t})d\mathsf{t}.$$

Remark 18. *If $\mathsf{h}_1(\mathsf{t})=\mathsf{h}_2(\mathsf{t})=\mathsf{t}$, then it is clearly seen that Theorem 4 yields a result for the $\preceq_{\mathcal{CR}}$-preinvex function.*

$$2\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\cdot\mathsf{G}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)$$
$$\preceq_{\mathcal{CR}}\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathsf{z})\mathsf{G}(\mathsf{z})d\mathsf{z}+\frac{\mathcal{M}(\mathfrak{p},\mathfrak{r})}{6}+\frac{\mathcal{N}(\mathfrak{p},\mathfrak{r})}{3}.$$

Remark 19. *If $\mathsf{h}_1(\mathsf{t})=\mathsf{h}_2(\mathsf{t})=\mathsf{t}$ and $\xi(\mathfrak{r},\mathfrak{p})=\mathfrak{r}-\mathfrak{p}$, then it is clearly seen that, Theorem 4 yields result for $\preceq_{\mathcal{CR}}$-convex function.*

$$2\mathfrak{D}\left(\frac{\mathfrak{p}+\mathfrak{r}}{2}\right)\cdot\mathsf{G}\left(\frac{\mathfrak{p}+\mathfrak{r}}{2}\right)\preceq_{\mathcal{CR}}\frac{1}{\mathfrak{r}-\mathfrak{p}}\int_{\mathfrak{p}}^{\mathfrak{r}}\mathfrak{D}(\mathsf{z})\mathsf{G}(\mathsf{z})d\mathsf{z}+\frac{\mathcal{M}(\mathfrak{p},\mathfrak{r})}{6}+\frac{\mathcal{N}(\mathfrak{p},\mathfrak{r})}{3}.$$

Example 4. *Let* $\mathfrak{D}(z) = [2 - z^{\frac{1}{2}}, 3(2 - z^{\frac{1}{2}})]$, $\mathsf{G}(z) = [e^z - z, e^z + z]$, $\xi(\mathfrak{r}, \mathfrak{p}) = \mathfrak{r} - \mathfrak{p}$, $\mathfrak{p} = 0$ *and* $\mathfrak{r} = 2$. *Then, for* $\mathsf{h}_1(\mathsf{t}) = \mathsf{h}_2(\mathsf{t}) = \mathsf{t}$, *we have*

$$\frac{1}{2\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \cdot \mathsf{G}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \approx [3.44, 22.31].$$

$$\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})} \mathfrak{D}(z)\mathsf{G}(z)dz$$
$$+ \mathcal{M}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(1-\mathsf{t})d\mathsf{t} + \mathcal{N}(\mathfrak{p},\mathfrak{r})\int_0^1 \mathsf{h}_1(1-\mathsf{t})\mathsf{h}_2(\mathsf{t})d\mathsf{t} \approx [7.0855, 33.7531].$$

Thus, it can be easily seen that

$$[3.44, 22.31] \preceq_{\mathcal{CR}} [7.0855, 33.7531].$$

This eventually validates the accuracy of Theorem 4.

Theorem 5. (Second Hermite–Hadamard-Fejér inequality for $\mathcal{CR} - \mathsf{h}$-preinvex function). *Suppose that* $\mathfrak{D} : [\mathfrak{p}, \mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p})] \longrightarrow \mathbb{R}$ *is an interval-valued function which is given by*

$$\mathfrak{D}(z) = \left[\underline{\mathfrak{D}}(z), \overline{\mathfrak{D}}(z)\right],$$

for all $z \in [\mathfrak{p}, \mathfrak{r}]$. *If* $\mathfrak{D} : [\mathfrak{p}, \mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p})] \longrightarrow \mathbb{R}$ *is a* $\mathcal{CR} - \mathsf{h}$*-preinvex function and* $\mathcal{P} : [\mathfrak{p}, \mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p})] \longrightarrow \mathbb{R}$, $\mathcal{P} > 0$ *is symmetric with respect to* $\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r}, \mathfrak{p})$. *Then, the following inequalities hold true:*

$$\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})} \mathfrak{D}(z)\mathcal{P}(z)dz \preceq_{\mathcal{CR}} [\mathfrak{D}(\mathfrak{p}) + \mathfrak{D}(\mathfrak{r})]\int_0^1 \mathsf{h}(\mathsf{t})\mathcal{P}(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathsf{t}.$$

Proof. Since $\mathfrak{D}$ is interval-valued $\mathcal{CR} - \mathsf{h}$-preinvex functions and $\mathcal{P}$ is symmetric with respect to $\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r}, \mathfrak{p})$, we have

$$\mathfrak{D}\big(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p})\big) \cdot \mathcal{P}\big(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p})\big) \preceq_{\mathcal{CR}} [\mathsf{h}(\mathsf{t})\mathfrak{D}(\mathfrak{r}) + \mathsf{h}(1-\mathsf{t})\,\mathfrak{D}(\mathfrak{p})] \cdot \mathcal{P}\big(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p})\big)$$

and

$$\mathfrak{D}\big(\mathfrak{p} + (1-\mathsf{t})\xi(\mathfrak{r},\mathfrak{p})\big) \cdot \mathcal{P}\big(\mathfrak{p} + (1-\mathsf{t})\xi(\mathfrak{r},\mathfrak{p})\big)$$
$$\preceq_{\mathcal{CR}} [\mathsf{h}(\mathsf{t})\mathfrak{D}(\mathfrak{p}) + \mathsf{h}(1-\mathsf{t})\,\mathfrak{D}(\mathfrak{r})] \cdot \mathcal{P}\big(\mathfrak{p} + (1-\mathsf{t})\xi(\mathfrak{r},\mathfrak{p})\big).$$

Adding both the above inequalities and then integrating over [0, 1], it is readily seen that

$$\int_0^1 \mathfrak{D}\big(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p})\big) \cdot \mathcal{P}\big(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p})\big)d\mathsf{t} + \int_0^1 \mathfrak{D}\big(\mathfrak{p} + (1-\mathsf{t})\xi(\mathfrak{r},\mathfrak{p})\big) \cdot \mathcal{P}\big(\mathfrak{p} + (1-\mathsf{t})\xi(\mathfrak{r},\mathfrak{p})\big)d\mathsf{t}$$
$$\preceq_{\mathcal{CR}} \int_0^1 \Big[\mathfrak{D}(\mathfrak{p})\big(\mathsf{h}(1-\mathsf{t})\mathcal{P}(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p})) + \mathsf{h}(\mathsf{t})\mathcal{P}(\mathfrak{p} + (1-\mathsf{t})\xi(\mathfrak{r},\mathfrak{p}))\big)$$
$$+ \mathfrak{D}(\mathfrak{r})\big(\mathsf{h}(\mathsf{t})\mathcal{P}(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p})) + \mathsf{h}(1-\mathsf{t})\mathcal{P}(\mathfrak{p} + (1-\mathsf{t})\xi(\mathfrak{r},\mathfrak{p}))\big)\Big]d\mathsf{t}$$
$$= \quad 2\mathfrak{D}(\mathfrak{p})\int_0^1 \mathsf{h}(\mathsf{t})\mathcal{P}(\mathfrak{p} + (1-\mathsf{t})\xi(\mathfrak{r},\mathfrak{p}))d\mathsf{t} + 2\mathfrak{D}(\mathfrak{r})\int_0^1 \mathsf{h}(\mathsf{t})\mathcal{P}(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathsf{t}.$$

Now, using the symmetry property of $\mathcal{P}$, we have

$$= \quad 2[\mathfrak{D}(\mathfrak{p}) + \mathfrak{D}(\mathfrak{r})]\int_0^1 \mathsf{h}(\mathsf{t})\mathcal{P}(\mathfrak{p} + \mathsf{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathsf{t}. \tag{12}$$

Since,

$$\int_0^1 \mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathcal{P}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}+\int_0^1 \mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))\cdot\mathcal{P}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}$$
$$=\frac{2}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathrm{z})\mathcal{P}(\mathrm{z})d\mathrm{z}. \tag{13}$$

The proof of Theorem 5 readily follows from the above two developments (12) and (13). □

Remark 20. *If* $\underline{\mathfrak{D}}=\overline{\mathfrak{D}}$, *then it is clearly seen that Theorem 5 yields a result for the* h*-preinvex function.*

$$\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathrm{z})\mathcal{P}(\mathrm{z})d\mathrm{z}\leq[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 \mathsf{h}(\mathfrak{t})\mathcal{P}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}.$$

Remark 21. *If* $\mathsf{h}(\mathfrak{t})=\mathfrak{t}$, *then it is clearly seen that Theorem 5 yields a result for the* $\preceq_{\mathcal{CR}}$*-preinvex function.*

$$\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathrm{z})\mathcal{P}(\mathrm{z})d\mathrm{z}\preceq_{\mathcal{CR}}[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 \mathfrak{t}\mathcal{P}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}.$$

Remark 22. *If* $\xi(\mathfrak{r},\mathfrak{p})=\mathfrak{r}-\mathfrak{p}$, *then it is clearly seen that Theorem 5 yields a result for the* $\preceq_{\mathcal{CR}}-\mathfrak{h}$*-convex function.*

$$\frac{1}{\mathfrak{r}-\mathfrak{p}}\int_{\mathfrak{p}}^{\mathfrak{r}}\mathfrak{D}(\mathrm{z})\mathcal{P}(\mathrm{z})d\mathrm{z}\preceq_{\mathcal{CR}}[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 \mathfrak{h}(\mathfrak{t})\mathcal{P}((1-\mathfrak{t})\mathfrak{p}+\mathfrak{t}\mathfrak{r})d\mathfrak{t}.$$

Remark 23. *If* $\mathsf{h}(\mathfrak{t})=\mathfrak{t}$ *and* $\xi(\mathfrak{r},\mathfrak{p})=\mathfrak{r}-\mathfrak{p}$, *then it is clearly seen that Theorem 5 yields a result for the* $\preceq_{\mathcal{CR}}$*-convex function.*

$$\frac{1}{\mathfrak{r}-\mathfrak{p}}\int_{\mathfrak{p}}^{\mathfrak{r}}\mathfrak{D}(\mathrm{z})\mathcal{P}(\mathrm{z})d\mathrm{z}\preceq_{\mathcal{CR}}[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 \mathfrak{t}\mathcal{P}((1-\mathfrak{t})\mathfrak{p}+\mathfrak{t}\mathfrak{r})d\mathfrak{t}.$$

Example 5. *Let* $\mathfrak{D}(\mathrm{z})=[2-\mathrm{z}^{\frac{1}{2}},3(2-\mathrm{z}^{\frac{1}{2}})]$, $\xi(\mathfrak{r},\mathfrak{p})=\mathfrak{r}-\mathfrak{p}$, $\mathfrak{p}=0$ *and* $\mathfrak{r}=2$. *Then, for* $\mathsf{h}(\mathfrak{t})=\mathfrak{t}$ *and symmetric function* $\mathcal{P}(\mathrm{z})=\mathrm{z}$ *for* $\mathrm{z}\in[0,1]$ *and* $\mathcal{P}(\mathrm{z})=-\mathrm{z}+2$ *for* $\mathrm{z}\in[1,2]$, *we have*

$$\begin{aligned}&\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathrm{z})\mathcal{P}(\mathrm{z})d\mathrm{z}\\&=\frac{1}{2}\int_0^2\mathfrak{D}(\mathrm{z})\mathcal{P}(\mathrm{z})d\mathrm{z}\\&=\frac{1}{2}\int_0^1\left[\left(2-\mathrm{z}^{\frac{1}{2}}\right)\mathrm{z},3\mathrm{z}\left(2-\mathrm{z}^{\frac{1}{2}}\right)\right]d\mathrm{z}\\&+\frac{1}{2}\int_1^2\left[\left(2-\mathrm{z}^{\frac{1}{2}}\right)(-\mathrm{z}+2),3(-\mathrm{z}+2)\left(2-\mathrm{z}^{\frac{1}{2}}\right)\right]d\mathrm{z}\\&\approx[0.512419,1.53726].\end{aligned}$$

$$
\begin{aligned}
&[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{r})]\int_0^1 \mathsf{h}(\mathfrak{t})\mathcal{P}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}\\
&=\left([2,6]+\left[2-2^{\frac{1}{2}},3\left(2-2^{\frac{1}{2}}\right)\right]\right)\int_0^1 \mathfrak{t}\mathcal{P}(2\mathfrak{t})d\mathfrak{t}\\
&=\left[4-2^{\frac{1}{2}},6+3\left(2-2^{\frac{1}{2}}\right)\right]\left(\int_0^{\frac{1}{2}}2\mathfrak{t}^2d\mathfrak{t}+\int_{\frac{1}{2}}^1 \mathfrak{t}(-2\mathfrak{t}+2)d\mathfrak{t}\right)\\
&\approx[0.646447,1.93934].
\end{aligned}
$$

Thus, it can be easily seen that

$$[0.512419,1.53726]\preceq_{\mathcal{CR}}[0.646447,1.93934].$$

This eventually validates the accuracy of Theorem 5.

Theorem 6. (First Hermite–Hadamard-Fejér inequality for $\mathcal{CR}-\mathbf{h}$-preinvex function). *Suppose that* $\mathfrak{D}:[\mathfrak{p},\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})]\longrightarrow\mathbb{R}$ *is an interval-valued function which is given by*

$$\mathfrak{D}(\mathrm{z})=\left[\underline{\mathfrak{D}}(\mathrm{z}),\ \overline{\mathfrak{D}}(\mathrm{z})\right]$$

for all $\mathrm{z}\in[\mathfrak{p},\mathfrak{r}]$. *If* $\mathfrak{D}:[\mathfrak{p},\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})]\longrightarrow\mathbb{R}$ *is* $\mathcal{CR}-\mathsf{h}$*-preinvex function and* $\mathcal{P}:[\mathfrak{p},\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})]\longrightarrow\mathbb{R}$, $\mathcal{P}>0$ *is symmetric with respect to* $\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})$. *Then, assuming* $\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathcal{P}(\mathrm{z})d\mathrm{z}>0$, *the following inequalities hold true:*

$$\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\preceq_{\mathcal{CR}}\frac{2\mathsf{h}\left(\frac{1}{2}\right)}{\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathcal{P}(\mathrm{z})d\mathrm{z}}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathrm{z})\mathcal{P}(\mathrm{z})d\mathrm{z}.$$

Proof. Since $\mathfrak{D}$ is an interval-valued $\mathcal{CR}-\mathsf{h}$-preinvex function and with the help of Condition C, we have

$$\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\preceq_{\mathcal{CR}}\mathsf{h}\left(\frac{1}{2}\right)[\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}+\mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}].$$

Multiplying the above inequality by $\mathcal{P}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big)=\mathcal{P}\big(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p})\big)$ and then integrating over [0, 1], it is readily seen that

$$
\begin{aligned}
&\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)\int_0^1\mathcal{P}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big)d\mathfrak{t}\\
&\qquad\preceq_{\mathcal{CR}}\mathsf{h}\left(\frac{1}{2}\right)\bigg[\int_0^1\mathfrak{D}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big)\cdot\mathcal{P}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big)d\mathfrak{t}\\
&\qquad\qquad+\int_0^1\mathfrak{D}\big(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p})\big)\cdot\mathcal{P}\big(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p})\big)d\mathfrak{t}\bigg].
\end{aligned}\tag{14}
$$

Since,

$$
\begin{aligned}
\int_0^1\mathfrak{D}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big)\cdot\mathcal{P}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big)d\mathfrak{t}&=\int_0^1\mathfrak{D}\big(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p})\big)\cdot\mathcal{P}\big(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p})\big)d\mathfrak{t}\\
&=\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(\mathrm{z})\mathcal{P}(\mathrm{z})d\mathrm{z}
\end{aligned}\tag{15}
$$

and

$$\int_0^1\mathcal{P}\big(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p})\big)d\mathfrak{t}=\frac{1}{\xi(\mathfrak{r},\mathfrak{p})}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathcal{P}(\mathrm{z})d\mathrm{z}.\tag{16}$$

Using (15) and (16) in (14), we have

$$\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \preceq_{\mathcal{CR}} \frac{2\mathsf{h}\left(\frac{1}{2}\right)}{\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathcal{P}(z)dz}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(z)\mathcal{P}(z)dz.$$

This concludes the proof of the desired Theorem 6. □

Remark 24. *If $\underline{\mathfrak{D}} = \overline{\mathfrak{D}}$, then it is clearly seen that Theorem 6 yields a result for the* h*-preinvex function.*

$$\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \leq \frac{2\mathsf{h}\left(\frac{1}{2}\right)}{\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathcal{P}(z)dz}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(z)\mathcal{P}(z)dz.$$

Remark 25. *If $\mathsf{h}(\mathfrak{t}) = \mathfrak{t}$, then it is clearly seen that Theorem 6 yields a result for the $\preceq_{\mathcal{CR}}$-preinvex function.*

$$\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) \preceq_{\mathcal{CR}} \frac{1}{\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathcal{P}(z)dz}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(z)\mathcal{P}(z)dz.$$

Remark 26. *If $\xi(\mathfrak{r},\mathfrak{p}) = \mathfrak{r}-\mathfrak{p}$, then it is clearly seen that Theorem 6 yields a result for the $\preceq_{\mathcal{CR}}$ $-\mathsf{h}$-convex function.*

$$\mathfrak{D}\left(\frac{\mathfrak{p}+\mathfrak{r}}{2}\right) \preceq_{\mathcal{CR}} \frac{2\mathsf{h}\left(\frac{1}{2}\right)}{\int_{\mathfrak{p}}^{\mathfrak{r}}\mathcal{P}(z)dz}\int_{\mathfrak{p}}^{\mathfrak{r}}\mathfrak{D}(z)\mathcal{P}(z)dz.$$

Remark 27. *If $\mathsf{h}(\mathfrak{t}) = \mathfrak{t}$ and $\xi(\mathfrak{r},\mathfrak{p}) = \mathfrak{r}-\mathfrak{p}$, then it is clearly seen that Theorem 6 yields a result for the $\preceq_{\mathcal{CR}}$-convex function.*

$$\mathfrak{D}\left(\frac{\mathfrak{p}+\mathfrak{r}}{2}\right) \preceq_{\mathcal{CR}} \frac{1}{\int_{\mathfrak{p}}^{\mathfrak{r}}\mathcal{P}(z)dz}\int_{\mathfrak{p}}^{\mathfrak{r}}\mathfrak{D}(z)\mathcal{P}(z)dz.$$

Remark 28. *Combining Theorems 5 and 6 for $\mathcal{P}(z) = 1$, we have Theorem 2.*

Example 6. *Let $\mathfrak{D}(z) = [2-z^{\frac{1}{2}}, 3(2-z^{\frac{1}{2}})]$, $\xi(\mathfrak{r},\mathfrak{p}) = \mathfrak{r}-\mathfrak{p}$, $\mathfrak{p} = 0$ and $\mathfrak{r} = 2$. Then, for $\mathsf{h}(\mathfrak{t}) = \mathfrak{t}$ and symmetric function $\mathcal{P}(z) = z$ for $z \in [0,1]$ and $\mathcal{P}(z) = -z+2$ for $z \in [1,2]$, we have*

$$\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right) = \mathfrak{D}(1) = [2-1, 3(2-1)] = [1,3].$$

$$\begin{aligned}
&\frac{2\mathsf{h}\left(\frac{1}{2}\right)}{\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathcal{P}(z)dz}\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\mathfrak{D}(z)\mathcal{P}(z)dz \\
&= \frac{1}{\int_0^2\mathcal{P}(z)dz}\int_0^2\left[2-z^{\frac{1}{2}}, 3(2-z^{\frac{1}{2}})\right]\mathcal{P}(z)dz \\
&= \int_0^1\left[z\left(2-z^{\frac{1}{2}}\right), 3z(2-z^{\frac{1}{2}})\right]dz + \int_1^2\left[(-z+2)\left(2-z^{\frac{1}{2}}\right), 3(-z+2)(2-z^{\frac{1}{2}})\right]\mathcal{P}(z)dz \\
&\approx [1.02484, 3.07452].
\end{aligned}$$

Thus, it can be easily seen that

$$[1,3] \preceq_{\mathcal{CR}} [1.02484, 3.07452].$$

This eventually validates the accuracy of Theorem 6.

5. Future Recommendations Associated with Fractional Integral Operators

In this section, we will provide the connection between our results and fractional inetgral inequalities, which will bring researchers working in fractional calculus into play. Before discussing the key conclusions, we will go over the definitions of the k-Riemann–Liouville fractional operators given as follows:

Definition 7 (see [49] for details)**.** *Let $k > 0$ and $\mathfrak{D} \in \mathcal{L}[\mathfrak{p}, \mathfrak{r}]$ be the set of all Lebesgue measurable functions on $[\mathfrak{p}, \mathfrak{r}]$. Then, for the order $\alpha > 0$, the left and right k-Riemann–Liouville (R-L) fractional integrals are defined as follows:*

$$\mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\mathfrak{D}(x) := \frac{1}{k\Gamma_k(\alpha)}\int_{\mathfrak{p}}^{x}(x-\mathrm{z})^{\frac{\alpha}{k}-1}\,\mathfrak{D}(\mathrm{z})d\mathrm{z} \qquad (0 \leqq \mathfrak{p} < x < \mathfrak{r})$$

and

$$\mathrm{I}^{\alpha,k}_{\mathfrak{r}^-}\mathfrak{D}(x) := \frac{1}{k\Gamma(\alpha)}\int_{x}^{\mathfrak{r}}(\mathrm{z}-x)^{\frac{\alpha}{k}-1}\,\mathfrak{D}(\mathrm{z})d\mathrm{z} \qquad (0 \leqq \mathfrak{r} < x < \mathfrak{r}),$$

respectively, where $\Gamma_k(\alpha) = \int_0^\infty \mu^{\alpha-1}exp\left(-\frac{\mu^k}{k}\right)d\mu$ is the k-gamma function given in [50].

Corollary 1. *Let $\mathfrak{D} : [\mathfrak{p}, \mathfrak{r}]$ be an interval-valued function such that $\mathfrak{D} = [\underline{\mathfrak{D}}, \overline{\mathfrak{D}}]$ with $\underline{\mathfrak{D}}, \overline{\mathfrak{D}} \in \mathfrak{R}_{[\mathfrak{p},\mathfrak{r}]}$. Then*

$$\mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\mathfrak{D}(x) = \left[\mathrm{I}^{\alpha}_{\mathfrak{p}^+}\underline{\mathfrak{D}}(x), \mathrm{I}^{\alpha}_{\mathfrak{p}^+}\overline{\mathfrak{D}}(x)\right]$$

and

$$\mathrm{I}^{\alpha,k}_{\mathfrak{r}^-}\mathfrak{D}(x) = \left[\mathrm{I}^{\alpha}_{\mathfrak{r}^-}\underline{\mathfrak{D}}(x), \mathrm{I}^{\alpha}_{\mathfrak{r}^-}\overline{\mathfrak{D}}(x)\right].$$

Theorem 7. *Suppose that $\mathfrak{D} : [\mathfrak{p}, \mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p})] \longrightarrow \mathbb{R}$ is an interval-valued function which is given by*

$$\mathfrak{D}(\mathrm{z}) = \left[\underline{\mathfrak{D}}(\mathrm{z}),\ \overline{\mathfrak{D}}(\mathrm{z})\right],$$

for all $\mathrm{z} \in [\mathfrak{p}, \mathfrak{r}]$. If $\mathfrak{D} : [\mathfrak{r} + \xi(\mathfrak{p}, \mathfrak{r}), \mathfrak{r}] \longrightarrow \mathbb{R}$ is $\mathtt{cr} - \mathsf{h}$-preinvex function and satisfies Condition C. Then, for $\mathsf{h}(\frac{1}{2}) > 0$, the following fractional inequalities hold true for $\alpha > 0$:

$$\begin{aligned}
&\frac{1}{\alpha \mathsf{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r}, \mathfrak{p})\right) \\
&\quad \preceq_{\mathtt{cr}} \frac{\Gamma_k(\alpha)}{(\xi(\mathfrak{r}, \mathfrak{p}))^{\frac{\alpha}{k}}}\left[\mathrm{I}^{\alpha,k}_{(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))^-}\mathfrak{D}(\mathfrak{p}) + \mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\mathfrak{D}(\mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p}))\right] \\
&\quad\quad \preceq_{\mathtt{cr}} \frac{[\mathfrak{D}(\mathfrak{p}) + \mathfrak{D}(\mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p}))]}{k}\int_0^1 \mathrm{t}^{\frac{\alpha}{k}-1}[\mathsf{h}(\mathrm{t}) + \mathsf{h}(1-\mathrm{t})]d\mathrm{t}.
\end{aligned}$$

Proof. From the definition of $\mathtt{cr} - \mathsf{h}$-preinvex functions and employing the Condition C, we have

$$\mathfrak{D}\left(x + \frac{1}{2}\xi(y, x)\right) \preceq_{\mathtt{cr}} \mathsf{h}\left(\frac{1}{2}\right)[\mathfrak{D}(x) + \mathfrak{D}(y)].$$

Choosing $x = \mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p})$ and $y = \mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r}, \mathfrak{p})$. It is seen that

$$\mathfrak{D}\left(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p}) + \frac{1}{2}\xi(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r}, \mathfrak{p}), \mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p}))\right)$$
$$\preceq_{cr} \mathsf{h}\left(\frac{1}{2}\right)[\mathfrak{D}(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p})) + \mathfrak{D}(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r}, \mathfrak{p}))].$$

which implies,

$$\frac{1}{\mathsf{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r}, \mathfrak{p})\right) \preceq_{cr} [\mathfrak{D}(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p})) + \mathfrak{D}(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r}, \mathfrak{p}))]. \tag{17}$$

Multiplying both sides of the above inequality (17) by $\mathfrak{t}^{\frac{\alpha}{k}-1}$ and then integrating over the closed inteval [0, 1], we obtain

$$\begin{aligned}
&\frac{1}{\mathsf{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r}, \mathfrak{p})\right)\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}d\mathfrak{t} \\
&\preceq_{cr} \left[\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}\mathfrak{D}(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p}))d\mathfrak{t} + \int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}\mathfrak{D}(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r}, \mathfrak{p}))d\mathfrak{t}\right] \\
&= \left[\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}(\underline{\mathfrak{D}}(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p})) + \underline{\mathfrak{D}}(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r}, \mathfrak{p})))d\mathfrak{t},\right. \\
&\quad \left.\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}(\overline{\mathfrak{D}}(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p})) + \overline{\mathfrak{D}}(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r}, \mathfrak{p})))d\mathfrak{t}\right] \\
&= \left[\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\left(\frac{\mathsf{z}-\mathfrak{p}}{\xi(\mathfrak{r}, \mathfrak{p})}\right)^{\frac{\alpha}{k}-1}\underline{\mathfrak{D}}(\mathsf{z})\frac{d\mathsf{z}}{\xi(\mathfrak{r}, \mathfrak{p})} + \int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\left(\frac{\mathfrak{p}+\xi(\mathfrak{r}, \mathfrak{p})-\mathsf{z}}{\xi(\mathfrak{r}, \mathfrak{p})}\right)^{\frac{\alpha}{k}-1}\underline{\mathfrak{D}}(\mathsf{z})\frac{d\mathsf{z}}{\xi(\mathfrak{r}, \mathfrak{p})},\right. \\
&\quad \left.\int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\left(\frac{\mathsf{z}-\mathfrak{p}}{\xi(\mathfrak{r}, \mathfrak{p})}\right)^{\frac{\alpha}{k}-1}\overline{\mathfrak{D}}(\mathsf{z})\frac{d\mathsf{z}}{\xi(\mathfrak{r}, \mathfrak{p})} + \int_{\mathfrak{p}}^{\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p})}\left(\frac{\mathfrak{p}+\xi(\mathfrak{r}, \mathfrak{p})-\mathsf{z}}{\xi(\mathfrak{r}, \mathfrak{p})}\right)^{\frac{\alpha}{k}-1}\overline{\mathfrak{D}}(\mathsf{z})\frac{d\mathsf{z}}{\xi(\mathfrak{r}, \mathfrak{p})}\right] \\
&= \left[\frac{k\Gamma_k(\alpha)}{(\xi(\mathfrak{r}, \mathfrak{p}))^{\frac{\alpha}{k}}}\left[\mathrm{I}^{\alpha,k}_{(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))^-}\underline{\mathfrak{D}}(\mathfrak{p}) + \mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\underline{\mathfrak{D}}(\mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p}))\right],\right. \\
&\quad \left.\frac{k\Gamma_k(\alpha)}{(\xi(\mathfrak{r}, \mathfrak{p}))^{\frac{\alpha}{k}}}\left[\mathrm{I}^{\alpha,k}_{(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))^-}\overline{\mathfrak{D}}(\mathfrak{p}) + \mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\overline{\mathfrak{D}}(\mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p}))\right]\right] \\
&= \frac{k\Gamma_k(\alpha)}{(\xi(\mathfrak{r}, \mathfrak{p}))^{\frac{\alpha}{k}}}\left[\mathrm{I}^{\alpha,k}_{(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))^-}\mathfrak{D}(\mathfrak{p}) + \mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\mathfrak{D}(\mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p}))\right]
\end{aligned}$$

From the above developments, we can conclude that

$$\frac{1}{\alpha\mathsf{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p} + \frac{1}{2}\xi(\mathfrak{r}, \mathfrak{p})\right) \preceq_{cr} \frac{\Gamma_k(\alpha)}{(\xi(\mathfrak{r}, \mathfrak{p}))^{\frac{\alpha}{k}}}\left[\mathrm{I}^{\alpha,k}_{(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))^-}\mathfrak{D}(\mathfrak{p}) + \mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\mathfrak{D}(\mathfrak{p} + \xi(\mathfrak{r}, \mathfrak{p}))\right]. \tag{18}$$

This completes the proof of the first inequality. Next, to prove the second inequality, from the definition of $cr - \mathsf{h}$-preinvex functions, we have

$$\mathfrak{D}(\mathfrak{p} + \mathfrak{t}\xi(\mathfrak{r}, \mathfrak{p})) \preceq_{cr} \mathsf{h}(\mathfrak{t})\mathfrak{D}(\mathfrak{r}) + \mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{p}).$$

and

$$\mathfrak{D}(\mathfrak{p} + (1-\mathfrak{t})\xi(\mathfrak{r}, \mathfrak{p})) \preceq_{cr} \mathsf{h}(\mathfrak{t})\mathfrak{D}(\mathfrak{p}) + \mathsf{h}(1-\mathfrak{t})\,\mathfrak{D}(\mathfrak{r}).$$

Adding the above two inequalities and multiplying by $\mathfrak{t}^{\frac{\alpha}{k}-1}$, then integrating over the closed inteval [0, 1], we have

$$\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}\mathfrak{D}(\mathfrak{p}+\mathfrak{t}\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}+\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}\mathfrak{D}(\mathfrak{p}+(1-\mathfrak{t})\xi(\mathfrak{r},\mathfrak{p}))d\mathfrak{t}$$
$$\preceq_{cr} [\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))]\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}[\mathrm{h}(\mathfrak{t})+\mathrm{h}(1-\mathfrak{t})]d\mathfrak{t}$$

This implies,

$$\frac{\Gamma_k(\alpha)}{(\xi(\mathfrak{r},\mathfrak{p}))^{\frac{\alpha}{k}}}\left[\mathrm{I}^{\alpha,k}_{(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))^-}\mathfrak{D}(\mathfrak{p})+\mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\mathfrak{D}(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))\right]$$
$$\preceq_{cr} \frac{[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))]}{k}\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}[\mathrm{h}(\mathfrak{t})+\mathrm{h}(1-\mathfrak{t})]d\mathfrak{t} \quad (19)$$

Consequently, from Equations (18) and (19), we conclude the desired result, i.e.,

$$\frac{1}{\alpha \mathrm{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)$$
$$\preceq_{cr} \frac{\Gamma_k(\alpha)}{(\xi(\mathfrak{r},\mathfrak{p}))^{\frac{\alpha}{k}}}\left[\mathrm{I}^{\alpha,k}_{(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))^-}\mathfrak{D}(\mathfrak{p})+\mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\mathfrak{D}(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))\right]$$
$$\preceq_{cr} \frac{[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))]}{k}\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}[\mathrm{h}(\mathfrak{t})+\mathrm{h}(1-\mathfrak{t})]d\mathfrak{t}.$$

This completes the proof. □

Remark 29. *If we choose $\alpha = k = 1$ in the above Theorem 7, then we recover the classical inequality for preinvex function as given in Theorem 2.*

Note: This shows the connection of our results with fractional integral inequalities. In the future, researchers are encouraged to establish such results for different types of fractional operators.

Remark 30. *If $\underline{\mathfrak{D}} = \overline{\mathfrak{D}}$, then it is clearly seen that Theorem 2 yields the following result for the h-preinvex function.*

$$\frac{1}{\alpha \mathrm{h}\left(\frac{1}{2}\right)}\mathfrak{D}\left(\mathfrak{p}+\frac{1}{2}\xi(\mathfrak{r},\mathfrak{p})\right)$$
$$\leq \frac{\Gamma_k(\alpha)}{(\xi(\mathfrak{r},\mathfrak{p}))^{\frac{\alpha}{k}}}\left[\mathrm{I}^{\alpha,k}_{(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))^-}\mathfrak{D}(\mathfrak{p})+\mathrm{I}^{\alpha,k}_{\mathfrak{p}^+}\mathfrak{D}(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))\right]$$
$$\leq \frac{[\mathfrak{D}(\mathfrak{p})+\mathfrak{D}(\mathfrak{p}+\xi(\mathfrak{r},\mathfrak{p}))]}{k}\int_0^1 \mathfrak{t}^{\frac{\alpha}{k}-1}[\mathrm{h}(\mathfrak{t})+\mathrm{h}(1-\mathfrak{t})]d\mathfrak{t}.$$

In the future, this new concept can be incorporated to present different inequalities such as Hermite–Hadamard, Ostrowski, Hadamard–Mercer, Simpson, Fejér, and Bullen type. The above-mentioned inequalities can be proved for various interval-valued CR convexities such as:

- CR-h convex function.
- CR-(h_1, h_2) convex function.
- CR Godunova–Levin Functions.
- CR-Harmonic convex function.
- CR-Harmonially-h convex function.
- CR-Harmonically-(h_1, h_2) convex function.

- CR-Harmonically Godunova–Levin Functions.
- CR-Harmonically (h_1, h_2)-Godunova–Levin Functions.

We intend to also generalize these results in connection with fractional calculus, quantum calculus, coordinated interval-valued functions, etc. As these are the hot topics of research in the field of integral inequalities, they will attract many mathematicians to explore the incorporation of CR-order interval-valued analysis with the above-mentioned concepts.

6. Conclusions

Any integral inequality can be a very effective tool for applications. In particular, using integral operators as a predictive tool, inequalities can be crucial for measuring, computing errors, and defining such processes. Incorporating the uncertainty into the prediction procedures, the C-R (Center-Radius) interval-valued functions might be a useful substitute. Using the integral operator and the center-radius order on the space of the real and compact intervals as a foundation, we demonstrate the Hermite–Hadamard, the Pachpatte, and the Hadamard-Fejér types of inequalities. By doing this, we contribute to the set-valued context's generalization of numerous classical integral inequalities. Additionally, examples and graphical presentations are provided to clarify the outcomes attained.

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