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Some H-Godunova–Levin Function Inequalities Using Center Radius (Cr) Order Relation

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Abstract: Interval analysis distinguishes between different types of order relations. As a result of these order relations, convexity and nonconvexity contribute to different kinds of inequalities. Despite this, convex theory is commonly known to rely on Godunova–Levin functions because their properties make it more efficient for determining inequality terms than convex ones. The purpose of this study is to introduce the notion of cr-h-Godunova–Levin functions by using total order relation between two intervals. Considering their properties and widespread use, center-radius order relation appears to be ideally suited for the study of inequalities. In this paper, various types of inequalities are introduced using center-radius order (cr) relation. The cr-order relation enables us firstly to derive some Hermite–Hadamard ($\mathcal{H}.\mathcal{H}$) inequalities, and then to present Jensen-type inequality for h-Godunova–Levin interval-valued functions (GL-$\mathcal{IVFS}$) using a Riemann integral operator. This kind of convexity unifies several new and well-known convex functions. Additionally, the study includes useful examples to support its findings. These results confirm that this new concept is useful for addressing a wide range of inequalities. We hope that our results will encourage future research into fractional versions of these inequalities and optimization problems associated with them.

Keywords: Jensen inequality; Hermite–Hadamard inequality; Godunova–Levin function; cr-order relation; interval-valued function

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1. Introduction

The results of uncertainty problems in real life may be invalid if a specific number is used for describing the results. This is why avoiding such errors and getting effective results is so important. Interval analysis was first applied to automatic error analysis by Moore [1] in 1969 for resulting in an improvement of calculation results and attracting the attention of many researchers. In interval analysis, interval numbers are used as variables, and interval operations are used instead of numbers. The interval is widely used in uncertain problems because it can be expressed as an uncertain variable, such as in computer graphics [2], decision-making analysis [3], multi-objective optimization [4], and error analysis [5]. Thus, interval analysis research has produced numerous excellent results, and interested readers can consult Refs. [6–8].

Mathematicians and other scientists have long acknowledged the significance of convexity in fields like economics, probability theory, optimal control theory. In addition, several inequalities have been documented in the literature as well, see Refs. [9–12]. Convexity and inequality have played a crucial role in many disciplines and applications in recent decades, which has led to extensive research and application in the field of generalized convexity of interval-valued functions, see Refs. [13–18]. Here are a few recent applications related to these inequalities, see Refs. [19–22]. The continuity of interval-valued functions is described by Breckner [23], who defined (A, s)-convex and (A, s)-concave mapping. Moreover, some other inequalities concerning $\mathcal{IVFS}$ have been established in the last decade. In the context of $\mathcal{IVFS}$, Chalco-Cano et al. [24], derived some Ostrowski inequalities using the generalized Hukuhara derivative. Inequalities of Opial type were established for generalized Hukuhara differentiable $\mathcal{IVFS}$ by Costa et al. [25]. Matkowski et al. [26], established the interval version of Jensen inequality. It was Zhao [27] and his co-authors who first established the $\mathcal{H}.\mathcal{H}$ and Jensen inequality by using h-convexity for $\mathcal{IVFS}$. Generally, a classical Hermite–Hadamard inequality is defined as the following:

$$\frac{\eta(t)+\eta(u)}{2} \geq \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \geq \eta\left(\frac{t+u}{2}\right). \tag{1}$$

where $\eta : K \subseteq R \to R$ is a convex function on K with $v < w$ such that $v, w \in K$. Due to the fact that the Hermite–Hadamard inequality is the first geometrical interpretation of convex mappings in elementary mathematics, this inequality has attracted a lot of attention. The following are some generalizations and extensions of this inequality, see Refs. [28–31]. First, h-convex was developed by Varoşanec [32], in 2007. Hermite–Hadamard-based inequalities have been developed by several authors using h-convex functions, see Refs. [33–36]. Currently, these results are based on inclusion relations and interval LU-order relations, which are flawed. The inequalities obtained by using these old partial order relations are less precise than those obtained by using center-radius order relation, which can be verified by comparing the examples defined in this literature. Furthermore, we observe that, in examples, the interval difference between endpoints is much closer than for other convexity classes. In light of this, it is of great importance to be able to use a total order relation to study the convexity and inequalities of $\mathcal{IVFS}$. As a result, we will focus the whole paper on Bhunia et al. [37], total interval order relation that is, cr-order. In 2020, Rahman [38], defined cr-convex functions and used cr-order to study nonlinear constrained optimization problems. Using the notions of cr-convexity and cr-order relation, Wei Liu and his co-authors developed a refined version of $\mathcal{H}.\mathcal{H}$ and Jensen-type inequalities in 2022, see Refs. [39,40].

Theorem 1 (See [39])**.** *Let $\eta : [t,u] \to R_I{}^+$. Consider $h : (0,1) \to R^+$ and $h(\frac{1}{2}) \neq 0$. If $\eta \in SX(cr\text{-}h, [t,u], R_I{}^+)$ and $\eta \in IR_{[t,u]}$, then*

$$\frac{1}{2h(\frac{1}{2})}\eta\left(\frac{t+u}{2}\right) \preceq_{cr} \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \preceq_{cr} [\eta(t)+\eta(u)]\int_0^1 h(x)dx. \tag{2}$$

By using cr-convexity, a Jensen-type inequality is also proven.

Theorem 2 (See [39])**.** *Let $d_i \in R^+$, $z_i \in [t,u]$ with $k \geq 2$. If h is super multiplicative non-negative function and $\eta \in SX(cr\text{-}h, [t,u], R_I{}^+)$, then the inequality become as :*

$$\eta\left(\frac{1}{D_k}\sum_{i=1}^k d_i z_i\right) \preceq_{cr} \sum_{i=1}^k h\left(\frac{d_i}{D_k}\right)\eta(z_i), \tag{3}$$

Based on the h-GL function, Ohud Almutairi and Adem Kiliman proved the following result in 2019, see Ref. [35].

Theorem 3. *Let $\eta : [t,u] \to R$. If η is* h-*Godunova–Levin function and $h(\frac{1}{2}) \neq 0$, then*

$$\frac{h(\frac{1}{2})}{2}\eta\left(\frac{t+u}{2}\right) \leq \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \leq [\eta(t)+\eta(u)]\int_0^1 \frac{dx}{h(x)}. \tag{4}$$

It has the advantage of introducing a new concept of interval-valued Godunova–Levin functions pertaining to a total order relation, namely, a center-radius order, which is very novel in the literature. The discussion in this article opens up a new avenue in the field of inequalities, showing how cr-interval-valued functions can be incorporated with various integral inequalities. It is important to note that cr-order interval-valued analysis differs from classical interval-valued analysis. To calculate intervals, we use the centre and radius concept given as $t_c = \frac{\underline{t}+\overline{t}}{2}$ and $t_r = \frac{\overline{t}-\underline{t}}{2}$, respectively, where $\overline{t}$ and $\underline{t}$ are endpoints of interval t.

Inspired by Refs. [35,37,39,40], we introduce a novel class of convexity by using centre-radius order relation for $\mathcal{IVFS}$ which are known as cr-h-GL functions. First, we derived new variants of $\mathcal{H}.\mathcal{H}$ inequality, then we represented the Jensen inequality by using this new class. Additionally, the study includes useful examples to support its findings.

Lastly, the structure of the paper is as follows. In Section 2, preliminary information is provided. The key problems are described in Section 3. There is a conclusion at the end of Section 5.

2. Preliminaries

For the notions which are used in this paper and are not defined here, see Refs. [39,40]. The space of intervals is denoted by the following R_I of R. In addition, the collection of all positive intervals can be represented by R_I^+. For $\nu \in R$, the Minkowski addition and scalar multiplication are defined by

$$t + u = [\underline{t}, \overline{t}] + [\underline{u}, \overline{u}] = [\underline{t} + \underline{u}, \overline{t} + \overline{u}];$$

$$\nu t = \nu.[\underline{t}, \overline{t}] = \begin{cases} [\nu\underline{t}, \mu\overline{t}], \, if \; \nu > 0, \\ \{0\}, \, if \; \nu = 0, \\ [\nu\overline{t}, \nu\underline{t}], \, if \; \nu < 0, \end{cases}$$

respectively.

Let t= $[\underline{t}, \overline{t}] \in R_I$, $t_c = \frac{\underline{t}+\overline{t}}{2}$ is center while $t_r = \frac{\overline{t}-\underline{t}}{2}$ is known as radius of t. Center-radius form of interval t is represented by

$$t = \left(\frac{\underline{t} + \overline{t}}{2}, \frac{\overline{t} - \underline{t}}{2} \right) = (t_c, t_r).$$

Here is the definition of order relation for radii and centers.

Definition 1 (See [37]). *Consider t= $[\underline{t}, \overline{t}] = (t_c, t_r)$, $u = [\underline{u}, \overline{u}] = (u_c, u_r) \in R_I$, then centre-radius order (In short cr-order) relation is defined as*

$$t \preceq_{cr} u \Leftrightarrow \begin{cases} t_c < u_c, t_c \neq u_c \\ \\ t_c \leq u_c, t_c = u_c \end{cases}$$

In addition, we describe the concept of Riemann integrable (In short IR) as it pertains to interval-valued functions in [41].

Theorem 4 (See [41]). *Let $\eta : [t, u] \to R_I$ be $\mathcal{IVF}$ given by $\eta(\nu) = [\underline{\eta}(\nu), \overline{\eta}(\nu)]$ for each $\nu \in [t, u]$ and $\underline{\eta}, \overline{\eta}$ are IR over interval $[t, u]$. In that case, we would say that our function η is IR over interval $[t, u]$, and*

$$\int_t^u \eta(\nu) d\nu = \left[\int_t^u \underline{\eta}(\nu) d\nu, \int_t^u \overline{\eta}(\nu) d\nu \right].$$

Assign $IR_{[t,u]}$, to all Riemann integrables (IR) $\mathcal{IVFS}$ over the interval.

Theorem 5 (See [39]). *Let $\eta, \zeta : [t,u] \to R_I^+$ given by $\eta = [\underline{\eta}, \overline{\eta}]$, and $\zeta = [\underline{\zeta}, \overline{\zeta}]$. If $\eta, \zeta \in IR_{[t,u]}$, and $\eta(\nu) \preceq_{cr} \zeta(\nu), \forall \nu \in [t,u]$, then*

$$\int_t^u \eta(\nu) d\nu \preceq_{cr} \int_t^u \zeta(\nu) d\nu.$$

For a more detailed explanation of interval analysis notations, see Refs. [41].

Definition 2 (See [39]). *Consider $h : [0,1] \to R^+$. We say that $\eta : [t,u] \to R^+$ is known h-convex function, or that $\eta \in SX(h, [t,u], R^+)$, if $\forall\, t_1, u_1 \in [t,u]$ and $\nu \in [0,1]$, we have*

$$\eta(\nu t_1 + (1-\nu)u_1) \leq h(\nu)\eta(t_1) + h(1-\nu)\eta(u_1). \tag{5}$$

If in (5) $\leq$ replaced with $\geq$ it is referred to as h-concave function or $\eta \in SV(h, [t,u], R^+)$.

Definition 3 (See [39]). *Consider $h : (0,1) \to R^+$. We say that $\eta : [t,u] \to R^+$ is known as h-GL function, or that $\eta \in SGX(h, [t,u], R^+)$, if $\forall\, t_1, u_1 \in [t,u]$ and $\nu \in (0,1)$, we have*

$$\eta(\nu t_1 + (1-\nu)u_1) \leq \frac{\eta(t_1)}{h(\nu)} + \frac{\eta(u_1)}{h(1-\nu)}. \tag{6}$$

If in (6) $\leq$ replaced with $\geq$ it is referred to as h-*Godunova–Levin concave function or $\eta \in SGV(h, [t,u], R^+)$.*

Now let's introduce the interval-valued function concept for cr-convexity.

Definition 4 (See [39]). *Consider $h : [0,1] \to R^+$. We say that $\eta = [\underline{\eta}, \overline{\eta}] : [t,u] \to R_I^+$ is called cr-h-convex function, or that $\eta \in SX(cr\text{-}h, [t,u], R_I^+)$, if $\forall\, t_1, u_1 \in [t,u]$ and $\nu \in [0,1]$, we have*

$$\eta(\nu t_1 + (1-\nu)u_1) \preceq_{cr} h(\nu)\eta(t_1) + h(1-\nu)\eta(u_1). \tag{7}$$

If in (7) $\leq$ replaced with $\geq$ it is referred to as cr-h-concave function or $\eta \in SV(cr\text{-}h, [t,u], R_I^+)$.

Definition 5 (See [39]). *Consider $h : (0,1) \to R^+$. We say that $\eta = [\underline{\eta}, \overline{\eta}] : [t,u] \to R_I^+$ is called cr*-h-*Godunova–Levin convex function, or that $\eta \in SGX(cr\text{-}h, [t,u], R_I^+)$, if $\forall\, t_1, u_1 \in [t,u]$ and $\nu \in (0,1)$, we have*

$$\eta(\nu t_1 + (1-\nu)u_1) \preceq_{cr} \frac{\eta(t_1)}{h(\nu)} + \frac{\eta(u_1)}{h(1-\nu)}. \tag{8}$$

If in (2) $\leq$ replaced with $\geq$ it is referred to as cr-h-*Godunova–Levin concave function or $\eta \in SGV(cr\text{-}h, [t,u], R_I^+)$.*

Remark 1.
- *If $h(\nu) = 1$, Definition 5 becomes a cr-P-function [39].*
- *If $h(\nu) = \frac{1}{h(\nu)}$, Definition 5 becomes a cr h-convex function [39].*
- *If $h(\nu) = \nu$, Definition 5 becomes a cr-Godunova–Levin function [39].*
- *If $h(\nu) = \frac{1}{\nu^s}$, Definition 5 becomes a cr-s-convex function [39].*
- *If $h(\nu) = \nu^s$, Definition 5 becomes a cr-s-GL function [39].*

3. Main Results

Proposition 1. *Consider $h_1, h_2 : (0,1) \to R^+$ be non-negative functions and*

$$\frac{1}{h_2} \leq \frac{1}{h_1}, \nu \in (0,1).$$

If $\eta \in SGX(cr\text{-}h_2, [t,u], {R_I}^+)$, then $\eta \in SGX(cr\text{-}h_1, [t,u], {R_I}^+)$.

Proof. Since $\eta \in SGX(cr\text{-}h_2, [t,u], R_I{}^+)$, then for all $t_1, u_1 \in [t,u], \nu \in (0,1)$, we have

$$\eta(\nu t_1 + (1-\nu)u_1) \preceq_{cr} \frac{\eta(t_1)}{h_2(\nu)} + \frac{\eta(u_1)}{h_2(1-\nu)}$$

$$\preceq_{cr} \frac{\eta(t_1)}{h_1(\nu)} + \frac{\eta(u_1)}{h_1(1-\nu)}.$$

Hence, $\eta \in SGX(cr\text{-}h_1, [t,u], R_I{}^+)$. □

Proposition 2. *Let $\eta : [t,u] \to R_I$ given by $[\underline{\eta}, \overline{\eta}] = (\eta_c, \eta_r)$. If η_c and η_r are h-GL over $[t,u]$, then η is known as cr-h-GL function over $[t,u]$.*

Proof. Since η_c and η_r are h-GL over $[t,u]$, then for each $\nu \in (0,1)$ and for all $t_1, u_1 \in [t,u]$, we have

$$\eta_c(\nu t_1 + (1-\nu)u_1) \preceq_{cr} \frac{\eta_c(t_1)}{h(\nu)} + \frac{\eta_c(u_1)}{h(1-\nu)},$$

and

$$\eta_r(\nu t_1 + (1-\nu)u_1) \preceq_{cr} \frac{\eta_r(t_1)}{h(\nu)} + \frac{\eta_r(u_1)}{h(1-\nu)},$$

Now, if

$$\eta_c(\nu t_1 + (1-\nu)u_1) \neq \frac{\eta_c(t_1)}{h(\nu)} + \frac{\eta_c(u_1)}{h(1-\nu)},$$

then for each $\nu \in (0,1)$ and for all $t_1, u_1 \in [t,u]$,

$$\eta_c(\nu t_1 + (1-\nu)u_1) < \frac{\eta_c(t_1)}{h(\nu)} + \frac{\eta_c(u_1)}{h(1-\nu)},$$

Accordingly,

$$\eta_c(\nu t_1 + (1-\nu)u_1) \preceq_{cr} \frac{\eta_c(t_1)}{h(\nu)} + \frac{\eta_c(u_1)}{h(1-\nu)}.$$

Otherwise, for each $\nu \in (0,1)$ and for all $t_1, u_1 \in [t,u]$,

$$\eta_r(\nu t_1 + (1-\nu)u_1) \leq \frac{\eta_r(t_1)}{h(\nu)} + \frac{\eta_r(u_1)}{h(1-\nu)} \Rightarrow \eta(\nu t_1 + (1-\nu)u_1) \preceq_{cr} \frac{\eta(t_1)}{h(\nu)} + \frac{\eta(u_1)}{h(1-\nu)}.$$

Taking all of the above into account, and Definition 1 this can be written as

$$\eta(\nu t_1 + (1-\nu)u_1) \preceq_{cr} \frac{\eta(t_1)}{h(\nu)} + \frac{\eta(u_1)}{h(1-\nu)}$$

for each $\nu \in (0,1)$ and for all $t_1, u_1 \in [t,u]$.
This completes the proof. □

This section establishes $\mathcal{H}.\mathcal{H}$ inequalities for cr-h-GL functions.

Theorem 6. *Consider $h : (0,1) \to R^+$ and $h(\frac{1}{2}) \neq 0$. Let $\eta : [t,u] \to R_I{}^+$, if $\eta \in SGX(cr\text{-}h, [t,u], R_I{}^+)$ and $\eta \in IR_{[t,u]}$, we have*

$$\frac{h(\frac{1}{2})}{2}\eta\Big(\frac{t+u}{2}\Big) \preceq_{cr} \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \preceq_{cr} [\eta(t)+\eta(u)]\int_0^1 \frac{dx}{h(x)}.$$

Proof. Since $\eta \in SGX(\text{cr-h}, [t,u], {R_I}^+)$, we have

$$h\left(\frac{1}{2}\right)\eta\left(\frac{t+u}{2}\right) \preceq_{cr} \eta(xt+(1-x)u)+\eta((1-x)t+xu)$$

Integration over (0,1), we have

$$\begin{aligned} h\left(\frac{1}{2}\right)\eta\left(\frac{t+u}{2}\right) &\preceq_{cr} \left[\int_0^1 \eta(xt+(1-x)u)dx+\int_0^1 \eta((1-x)t+xu)dx\right] \\ &= \Bigg[\int_0^1 \underline{\eta}(xt+(1-x)u)dx+\int_0^1 \underline{\eta}((1-x)t+xu)dx, \\ &\quad \int_0^1 \overline{\eta}(xt+(1-x)u)dx+\int_0^1 \overline{\eta}((1-x)t+xu)dx\Bigg] \\ &= \left[\frac{2}{u-t}\int_t^u \underline{\eta}(\nu)d\nu, \frac{2}{u-t}\int_t^u \overline{\eta}(\nu)d\nu\right] \\ &= \frac{2}{u-t}\int_t^u \eta(\nu)d\nu. \end{aligned} \tag{9}$$

By Definition 5, we have

$$\eta(xt+(1-x)u) \preceq_{cr} \frac{\eta(t)}{h(x)}+\frac{\eta(u)}{h(1-x)}$$

With integration over (0,1), we have

$$\int_0^1 \eta(xt+(1-x)u)dx \preceq_{cr} \eta(t)\int_0^1 \frac{dx}{h(x)}+\eta(u)\int_0^1 \frac{dx}{h(1-x)}$$

Accordingly,

$$\frac{1}{u-t}\int_t^u \eta(\nu)d\nu \preceq_{cr} [\eta(t)+\eta(u)]\int_0^1 \frac{dx}{h(x)} \tag{10}$$

Now, combining (9) and (10), we get required result

$$\frac{h(\frac{1}{2})}{2}\eta\left(\frac{t+u}{2}\right) \preceq_{cr} \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \preceq_{cr} [\eta(t)+\eta(u)]\int_0^1 \frac{dx}{h(x)}.$$

□

Remark 2.
- *If $h(x)=1$, Theorem 6 becomes result for cr- P-functions:*

$$\frac{1}{2}\eta\left(\frac{t+u}{2}\right) \preceq_{cr} \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \preceq_{cr} [\eta(t)+\eta(u)].$$

- *If $h(x)=\frac{1}{x}$, Theorem 6 becomes result for cr-convex functions:*

$$\eta\left(\frac{t+u}{2}\right) \preceq_{cr} \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \preceq_{cr} \frac{[\eta(t)+\eta(u)]}{2}.$$

- *If $h(x)=\frac{1}{(x)^s}$, Theorem 6 becomes result for cr-s-convex function:*

$$2^{s-1}\eta\left(\frac{t+u}{2}\right) \preceq_{cr} \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \preceq_{cr} \frac{[\eta(t)+\eta(u)]}{s+1}.$$

Example 1. *Let* $[t,u] = [0,1]$, $h(x) = \frac{1}{x}$, $\forall\, x \in\ (0,1)$. $\eta : [t,u] \to {R_I}^+$ *is defined as*

$$\eta(\nu) = [-\nu^2 + 1, 2\nu^2 + 2].$$

where

$$\frac{h(\frac{1}{2})}{2}\eta\Big(\frac{t+u}{2}\Big) = \eta\Big(\frac{1}{2}\Big) = \Big[\frac{3}{4}, \frac{5}{2}\Big],$$

$$\frac{1}{u-t}\int_t^u \eta(\nu)d\nu = \Big[\int_0^1 (-\nu^2+1)d\nu, \int_0^1 (2\nu^2+2)d\nu\Big] = \Big[\frac{2}{3}, \frac{8}{3}\Big],$$

$$[\eta(t)+\eta(u)]\int_0^1 \frac{dx}{h(x)} = \Big[\frac{1}{2}, 3\Big].$$

As a result,

$$\Big[\frac{3}{4}, \frac{5}{2}\Big] \preceq_{cr} \Big[\frac{2}{3}, \frac{8}{3}\Big] \preceq_{cr} \Big[\frac{1}{2}, 3\Big].$$

This proves the above theorem.

Theorem 7. *Consider* $h : (0,1) \to R^+$ *and* $h(\frac{1}{2}) \neq 0$. *Let* $\eta : [t,u] \to {R_I}^+$, *if* $\eta \in SGX(cr\text{-}h, [t,u], {R_I}^+)$ *and* $\eta \in\ IR_{[t,u]}$, *we have*

$$\frac{\big[h(\frac{1}{2})\big]^2}{4}\eta\Big(\frac{t+u}{2}\Big) \preceq_{cr} \triangle_1 \preceq_{cr} \frac{1}{u-t}\int_t^u \eta(\nu)d\nu \preceq_{cr} \triangle_2$$

$$\preceq_{cr} \Bigg\{[\eta(t)+\eta(u)]\Big[\frac{1}{2} + \frac{1}{h(\frac{1}{2})}\Big]\Bigg\}\int_0^1 \frac{dx}{h(x)},$$

where

$$\triangle_1 = \frac{[h(\frac{1}{2})]}{4}\Big[\eta\Big(\frac{3t+u}{4}\Big) + \eta\Big(\frac{3u+t}{4}\Big)\Big],$$

$$\triangle_2 = \Big[\eta\Big(\frac{t+u}{2}\Big) + \frac{\eta(t)+\eta(u)}{2})\Big]\int_0^1 \frac{dx}{h(x)}.$$

Proof. Consider $[t, \frac{t+u}{2}]$, we have

$$\eta\Big(\frac{t+\frac{t+u}{2}}{2}\Big) = \eta\Big(\frac{3t+u}{2}\Big) \preceq_{cr} \frac{\eta\big(xt+(1-x)\frac{t+u}{2}\big)}{h(\frac{1}{2})} + \frac{\eta\big((1-x)t+x\frac{t+u}{2}\big)}{h(\frac{1}{2})}$$

Integration over (0,1), we have

$$\eta\Big(\frac{3t+u}{2}\Big) \preceq_{cr} \frac{1}{h(\frac{1}{2})}\Bigg[\int_0^1 \eta(xt+(1-x)\frac{t+u}{2})dx + \int_0^1 \eta(x\frac{t+u}{2}+(1-x)u)dx\Bigg]$$

$$= \frac{1}{h(\frac{1}{2})}\Big[\frac{2}{u-t}\int_u^{\frac{t+u}{2}} \eta(\nu)d\nu + \frac{2}{u-t}\int_u^{\frac{t+u}{2}} \eta(\nu)d\nu\Big]$$

$$= \frac{4}{h(\frac{1}{2})}\Big[\frac{1}{w-v}\int_v^{\frac{v+w}{2}} \varphi(\mu)d\mu\Big].$$

Accordingly,

$$\frac{[h(\frac{1}{2})]}{4}\eta\Big(\frac{3t+u}{2}\Big) \preceq_{cr} \frac{1}{u-t}\int_t^{\frac{t+u}{2}} \eta(\nu)d\nu. \tag{11}$$

Similarly for interval $[\frac{t+u}{2}, u]$, we have

$$\frac{[h(\frac{1}{2})]}{4}\eta\Big(\frac{3u+t}{2}\Big) \preceq_{cr} \frac{1}{u-t}\int_t^{\frac{t+u}{2}} \eta(\nu)d\nu. \tag{12}$$

Adding inequalities (11) and (12), we get

$$\triangle_1 = \frac{[h(\frac{1}{2})]}{4}\bigg[\eta\Big(\frac{3t+u}{4}\Big) + \eta\Big(\frac{3u+t}{4}\Big)\bigg] \preceq_{cr} \Big[\frac{1}{u-t}\int_t^u \eta(\nu)d\nu\Big].$$

Now

$$\begin{aligned}
&\frac{\big[h(\frac{1}{2})\big]^2}{4}\eta\Big(\frac{t+u}{2}\Big) \\
&= \frac{\big[h(\frac{1}{2})\big]^2}{4}\eta\bigg(\frac{1}{2}\Big(\frac{3t+u}{4}\Big) + \frac{1}{2}\Big(\frac{3u+t}{4}\Big)\bigg) \\
&\preceq_{cr} \frac{\big[h(\frac{1}{2})\big]^2}{4}\left[\frac{\eta\big(\frac{3t+u}{4}\big)}{h(\frac{1}{2})} + \frac{\eta\big(\frac{3u+t}{4}\big)}{h(\frac{1}{2})}\right] \\
&= \frac{[h(\frac{1}{2})]}{4}\bigg[\eta\Big(\frac{3t+u}{4}\Big) + \eta\Big(\frac{3u+t}{4}\Big)\bigg] \\
&= \triangle_1 \\
&\preceq_{cr} \frac{[h(\frac{1}{2})]}{4}\bigg\{\frac{1}{h(\frac{1}{2})}\Big[\eta(t) + \eta\Big(\frac{t+u}{2}\Big)\Big] + \frac{1}{h(\frac{1}{2})}\Big[\eta(u) + \eta\Big(\frac{t+u}{2}\Big)\Big]\bigg\} \\
&= \frac{1}{2}\Big[\frac{\eta(t)+\eta(u)}{2} + \eta\Big(\frac{t+u}{2}\Big)\Big] \\
&\preceq_{cr} \Big[\frac{\eta(t)+\eta(u)}{2} + \eta\Big(\frac{t+u}{2}\Big)\Big]\int_0^1 \frac{dx}{h(x)} \\
&= \triangle_2 \\
&\preceq_{cr} \Big[\frac{\eta(t)+\eta(u)}{2} + \frac{\eta(t)}{h(\frac{1}{2})} + \frac{\eta(u)}{h(\frac{1}{2})}\Big]\int_0^1 \frac{dx}{h(x)} \\
&\preceq_{cr} \bigg[\frac{\eta(t)+\eta(u)}{2} + \frac{1}{h(\frac{1}{2})}\big[\eta(t)+\eta(u)\big]\bigg]\int_0^1 \frac{dx}{h(x)} \\
&\preceq_{cr} \bigg\{\big[\eta(t)+\eta(u)\big]\Big[\frac{1}{2} + \frac{1}{h(\frac{1}{2})}\Big]\bigg\}\int_0^1 \frac{dx}{h(x)}.
\end{aligned}$$

□

Example 2. *Thanks to Example 1, we have*

$$\frac{\big[h(\frac{1}{2})\big]^2}{4}\eta\Big(\frac{t+u}{2}\Big) = \eta\Big(\frac{1}{2}\Big) = \Big[\frac{3}{4}, \frac{5}{2}\Big],$$

$$\triangle_1 = \frac{1}{2}\bigg[\eta\Big(\frac{1}{4}\Big) + \eta\Big(\frac{3}{4}\Big)\bigg] = \Big[\frac{11}{16}, \frac{21}{8}\Big],$$

$$\triangle_2 = \bigg[\frac{\eta(0)+\eta(1)}{2} + \eta\Big(\frac{1}{2}\Big)\bigg]\int_0^1 \frac{dx}{h(x)},$$

$$\triangle_2 = \frac{1}{2}\left(\left[\frac{1}{2}, 3\right] + \left[\frac{3}{4}, \frac{5}{2}\right]\right),$$

$$\triangle_2 = \left[\frac{5}{8}, \frac{11}{4}\right],$$

$$\left\{[\eta(t) + \eta(u)]\left[\frac{1}{2} + \frac{1}{h(\frac{1}{2})}\right]\right\}\int_0^1 \frac{dx}{h(x)} = \left[\frac{1}{2}, 3\right].$$

Thus, we obtain

$$\left[\frac{3}{4}, \frac{5}{2}\right] \preceq_{cr} \left[\frac{11}{16}, \frac{21}{8}\right] \preceq_{cr} \left[\frac{2}{3}, \frac{8}{3}\right] \preceq_{cr} \left[\frac{5}{8}, \frac{11}{4}\right] \preceq_{cr} \left[\frac{1}{2}, 3\right].$$

This proves the above theorem.

Theorem 8. *Let* $\eta, \zeta : [t, u] \to {R_I}^+, h_1, h_2 : (0, 1) \to R^+$ *such that* $h_1, h_2 \neq 0$. *If* $\eta \in SGX$ $(cr\text{-}h_1, [t, u], {R_I}^+)$, $\zeta \in SGX(cr\text{-}h_2, [t, u], {R_I}^+)$ *and* $\eta, \zeta \in IR_{[v,w]}$ *then, we have*

$$\frac{1}{u-t}\int_t^u \eta(\nu)\zeta(\nu)d\nu \preceq_{cr} M(t, u)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx + N(t, u)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx$$

where

$$M(v, w) = \eta(t)\zeta(t) + \eta(u)\zeta(u), N(t, u) = \eta(t)\zeta(u) + \eta(u)\zeta(t).$$

Proof. Conider $\eta \in SGX(cr\text{-}h_1, [t, u], {R_I}^+)$, $\zeta \in SGX(cr\text{-}h_2, [t, u], {R_I}^+)$ then, we have

$$\eta\Big(tx + (1-x)u\Big) \preceq_{cr} \frac{\eta(t)}{h_1(x)} + \frac{\eta(u)}{h_1(1-x)},$$

$$\zeta\Big(tx + (1-x)u\Big) \preceq_{cr} \frac{\zeta(t)}{h_2(x)} + \frac{\zeta(u)}{h_2(1-x)}.$$

Then,

$$\eta\Big(tx + (1-x)u\Big)\zeta\Big(tx + (1-x)u\Big)$$

$$\preceq_{cr} \frac{\eta(t)\zeta(t)}{h_1(x)h_2(x)} + \frac{\eta(t)\zeta(u)}{h_1(x)h_2(1-x)} + \frac{\eta(u)\zeta(t)}{h_1(1-x)h_2(x)} + \frac{\eta(u)\zeta(u)}{h_1(1-x)h_2(1-x)}.$$

Integration over (0,1), we have

$$\int_0^1 \eta\Big(tx + (1-x)u\Big)\zeta\Big(tx + (1-x)u\Big)dx$$

$$= \left[\int_0^1 \underline{\eta}\Big(tx + (1-x)u\Big)\underline{\zeta}\Big(tx + (1-x)u\Big)dx, \int_0^1 \overline{\eta}\Big(tx + (1-x)u\Big)\overline{\zeta}\Big(tx + (1-x)u\Big)dx\right]$$

$$= \left[\frac{1}{u-t}\int_t^u \underline{\eta}(\nu)\underline{\zeta}(\nu)d\nu, \frac{1}{u-t}\int_t^u \overline{\eta}(\nu)\overline{\zeta}(\nu)d\nu\right] = \frac{1}{u-t}\int_t^u \eta(\nu)\zeta(\nu)d\nu$$

$$\preceq_{cr} \int_0^1 \frac{[\eta(t)\zeta(t) + \eta(u)\zeta(u)]}{h_1(x)h_2(x)}dx + \int_0^1 \frac{[\eta(t)\zeta(u) + \eta(u)\zeta(t)]}{h_1(x)h_2(1-x)}dx$$

It follows that

$$\frac{1}{u-t}\int_t^u \eta(\nu)\zeta(\nu)d\nu \preceq_{cr} M(t, u)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx + N(t, u)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx.$$

Theorem is proved. □

Example 3. *Let* $[t,u]=[1,2]$, $h_1(x)=h_2(x)=\frac{1}{x}$, $\forall\, x\in\ (0,1)$. $\eta,\zeta:[t,u]\to {R_I}^+$ *be defined as*

$$\eta(\nu)=[-\nu^2+1,2\nu^2+2],\zeta(\nu)=[-\nu,\nu+1].$$

Then,

$$\frac{1}{u-t}\int_t^u \eta(\nu)\zeta(\nu)d\nu=\left[\frac{9}{4},\frac{103}{6}\right],$$

$$M(t,u)\int_0^1\frac{1}{h_1(x)h_2(x)}dx=M(1,2)\int_0^1 x^2dx=\left[\frac{-140}{3},70\right],$$

$$N(t,u)\int_0^1\frac{1}{h_1(x)h_2(1-x)}dx=N(1,2)\int_0^1 x^2dx=\left[-42,\frac{224}{3}\right].$$

It follows that

$$\left[\frac{9}{4},\frac{103}{6}\right]\preceq_{cr}\left[\frac{-140}{3},70\right]+\left[-42,\frac{224}{3}\right]=\left[\frac{-266}{3},\frac{434}{3}\right].$$

It follows that the theorem above is true.

Theorem 9. *Let* $\eta,\zeta:[t,u]\to {R_I}^+,h_1,h_2:(0,1)\to R^+$ *such that* $h_1,h_2\neq 0$. *If* $\eta\in SGX(cr\text{-}h_1,[t,u],{R_I}^+)$, $\zeta\in SGX(cr\text{-}h_2,[t,u],{R_I}^+)$ *and* $\eta,\zeta\in\ IR_{[v,w]}$ *then, we have*

$$\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{2}\eta\Big(\frac{t+u}{2}\Big)\zeta\Big(\frac{t+u}{2}\Big)\preceq_{cr}\frac{1}{u-t}\int_t^u\eta(\nu)\zeta(\nu)d\mu$$

$$+M(t,u)\int_0^1\frac{1}{h_1(x)h_2(x)}dx+N(t,u)\int_0^1\frac{1}{h_1(x)h_2(1-x)}dx.$$

Proof. Since $\eta\in SGX(cr\text{-}h_1,[t,u],{R_I}^+)$, $\zeta\in SGX(cr\text{-}h_2,[t,u],{R_I}^+)$, we have

$$\eta\left(\frac{t+u}{2}\right)\preceq_{cr}\frac{\eta\Big(tx+(1-x)u\Big)}{h_1(\frac{1}{2})}+\frac{\eta\Big(t(1-x)+xu\Big)}{h_1(\frac{1}{2})},$$

$$\zeta\left(\frac{t+u}{2}\right)\preceq_{cr}\frac{\zeta\Big(tx+(1-x)u\Big)}{h_2(\frac{1}{2})}+\frac{\zeta\Big(t(1-x)+xu\Big)}{h_2(\frac{1}{2})}.$$

Then,

$$\eta\left(\frac{t+u}{2}\right)\zeta\left(\frac{t+u}{2}\right)$$

$$\preceq_{cr}\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[\eta\Big(tx+(1-x)u\Big)\zeta\Big(tx+(1-x)u\Big)+\eta\Big(t(1-x)+xu\Big)\zeta\Big(t(1-x)+xu\Big)\right]$$

$$+\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[\eta\Big(tx+(1-x)u\Big)\zeta\Big(t(1-x)+xu\Big)+\eta\Big(t(1-x)+xu\Big)\zeta\Big(tx+(1-x)u\Big)\right]$$

$$\preceq_{cr}\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[\eta\Big(tx+(1-x)u\Big)\zeta\Big(tx+(1-x)u\Big)+\eta\Big(t(1-x)+(xu\Big)\zeta\Big(t(1-x)+xu\Big)\right]$$

$$+\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[\left(\frac{\eta(t)}{h_1(x)}+\frac{\eta(u)}{h_1(1-x)}\right)\left(\frac{\zeta(u)}{h_2(1-x)}+\frac{\zeta(u)}{h_2(x)}\right)\right.$$

$$+\left(\frac{\eta(t)}{h_1(1-x)}+\frac{\eta(u)}{h_1(x)}\right)\left(\frac{\zeta(t)}{h_2(x)}+\frac{\zeta(u)}{h_2(1-x)}\right)\Bigg]$$

$$\preceq_{cr} \frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[\eta\Big(tx+(1-x)u\Big)\zeta\Big(tx+(1-x)u\Big)+\eta\Big(t(1-x)+ux\Big)\zeta\Big(t(1-x)+ux\Big)\right]$$

$$+\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[\left(\frac{1}{h_1(x)h_2(1-x)}+\frac{1}{h_1(1-x)h_2(x)}\right)M(t,u)\right.$$

$$\left.+\left(\frac{1}{h_1(x)h_2(x)}+\frac{1}{h_1(1-x)h_2(1-x)}\right)N(t,u)\right].$$

Integration over $(0,1)$, we have

$$\int_0^1 \eta\Big(\frac{t+u}{2}\Big)\zeta\Big(\frac{t+u}{2}\Big)dx=\left[\int_0^1 \underline{\eta}\Big(\frac{t+u}{2}\Big)\underline{\zeta}\Big(\frac{t+u}{2}\Big)dx,\int_0^1 \overline{\eta}\Big(\frac{t+u}{2}\Big)\overline{\zeta}\Big(\frac{t+u}{2}\Big)dx\right]$$

$$=\eta\Big(\frac{t+u}{2}\Big)\zeta\Big(\frac{t+u}{2}\Big)dx\preceq_{cr}\frac{2}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[\frac{1}{u-t}\int_t^u \eta(\nu)\zeta(\nu)d\nu\right]$$

$$+\frac{2}{h(\frac{1}{2})h(\frac{1}{2})}\left[M(t,u)\int_0^1\frac{1}{h_1(x)h_2(1-x)}dx+N(t,u)\int_0^1\frac{1}{h_1(x)h_2(x)}dx\right]$$

Multiply both sides by $\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{2}$ above equation, we get required result

$$\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{2}\eta\Big(\frac{t+u}{2}\Big)\zeta\Big(\frac{t+u}{2}\Big)\preceq_{cr}\frac{1}{u-t}\int_t^u \eta(\nu)\zeta(\nu)d\nu$$

$$+M(t,u)\int_0^1\frac{1}{h_1(x)h_2(1-x)}dx+N(t,u)\int_0^1\frac{1}{h_1(x)h_2(x)}dx.$$

As a result, the proof is complete. □

Example 4. *Let* $[t,u]=[1,2]$, $h_1(x)=h_2(x)=\frac{1}{x}$, $\forall\ x\in(0,1)$. $\eta,\zeta:[t,u]\to {R_I}^+$ *be defined as*

$$\eta(\nu)=[-\nu^2+1,2\nu^2+2],\zeta(\nu)=[-\nu,\nu+1].$$

Then,

$$\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{2}\eta\left(\frac{t+u}{2}\right)\zeta\left(\frac{t+u}{2}\right)=2\eta\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)=\left[\frac{-78}{4},130\right],$$

$$\frac{1}{u-t}\int_t^u \eta(\nu)\zeta(\nu)d\nu=\left[\frac{9}{4},\frac{103}{6}\right],$$

$$M(t,u)\int_0^1\frac{1}{h_1(x)h_2(1-x)}dx=M(1,2)\int_0^1 x^2dx=\left[\frac{-140}{3},70\right],$$

$$N(t,u)\int_0^1\frac{1}{h_1(x)h_2(x)}dx=N(1,2)\int_0^1 x^2dx=\left[-42,\frac{224}{3}\right].$$

It follows that

$$\left[\frac{-78}{4},130\right]\preceq_{cr}\left[\frac{9}{4},\frac{103}{6}\right]+\left[\frac{-140}{3},70\right]+\left[-42,\frac{224}{3}\right]=\left[\frac{-1037}{12},\frac{971}{6}\right].$$

This proves the above theorem.

4. Jensen-Type Inequality

This section establishes cr-h-GL version of Jensen-type inequality.

Theorem 10. *Let $d_i \in R^+$, $z_i \in [t, u]$ with $k \geq 2$. If h is non-negative and super multiplicative function or$\eta \in SGX(cr\text{-}h, [t, u], R_I{}^+)$. Then the inequality become as :*

$$\eta\left(\frac{1}{D_k}\sum_{i=1}^{k} d_i z_i\right) \preceq_{cr} \sum_{i=1}^{k}\left[\frac{\eta(z_i)}{h\left(\frac{d_i}{D_k}\right)}\right], \tag{13}$$

where $D_k = \sum_{i=1}^{k} d_i$.

Proof. When $k = 2$, inequality (13) holds. Assume that inequality (13) is also true for $k-1$, then

$$\begin{aligned}
\eta\left(\frac{1}{D_k}\sum_{i=1}^{k} d_i z_i\right) &= \eta\left(\frac{d_k}{D_k} z_k + \sum_{i=1}^{k-1}\frac{d_i}{D_k} z_i\right) \\
&= \eta\left(\frac{d_k}{D_k} z_k + \frac{D_{k-1}}{D_k}\sum_{i=1}^{k-1}\frac{d_i}{D_{k-1}} z_i\right) \\
&\preceq_{cr} \frac{\eta(z_k)}{h(\frac{d_k}{D_k})} + \frac{\eta\left(\sum_{i=1}^{k-1}\frac{d_i}{D_{k-1}} z_i\right)}{h(\frac{D_{k-1}}{D_k})} \\
&\preceq_{cr} \frac{\eta(z_k)}{h(\frac{d_k}{D_k})} + \sum_{i=1}^{k-1}\left[\frac{\eta(z_i)}{h\left(\frac{d_i}{D_{k-1}}\right)}\right]\frac{1}{h(\frac{D_{k-1}}{D_k})} \\
&\preceq_{cr} \frac{\eta(z_k)}{h(\frac{d_k}{D_k})} + \sum_{i=1}^{k-1}\left[\frac{\eta(z_i)}{h(\frac{d_i}{D_k})}\right] \\
&\preceq_{cr} \sum_{i=1}^{k}\left[\frac{\eta(z_i)}{h\left(\frac{d_i}{D_k}\right)}\right].
\end{aligned}$$

Therefore, the result can be proved by mathematical induction. □

Remark 3.
- *If $h(x) = 1$, Theorem 10 becomes result for cr-P-functions:*

$$\eta\left(\frac{1}{D_k}\sum_{i=1}^{k} d_i z_i\right) \preceq_{cr} \sum_{i=1}^{k}\eta(z_i).$$

- *If $h(x) = \frac{1}{x}$, Theorem 10 becomes result for cr-convex functions:*

$$\eta\left(\frac{1}{D_k}\sum_{i=1}^{k} d_i z_i\right) \preceq_{cr} \sum_{i=1}^{k}\frac{d_i}{D_k}\eta(z_i).$$

- *If $h(x) = \frac{1}{(x)^s}$, Theorem 10 becomes result for cr-s-convex function:*

$$\eta\left(\frac{1}{D_k}\sum_{i=1}^{k} d_i z_i\right) \preceq_{cr} \sum_{i=1}^{k}\left(\frac{d_i}{D_k}\right)^s\eta(z_i).$$

5. Conclusions

In this study, we introduce the h-GL concept for $\mathcal{IVFS}$ using cr-order. The purpose of this concept was to study Jensen and $\mathcal{H}.\mathcal{H}$ inequalities for $\mathcal{IVFS}$. Recent results developed by Wei Liu [39,40] and Adem Kiliman [35] are generalized in this study. As a further support for our main findings, we provide a few relevant examples. We can explore this topic in the future by determining equivalent inequalities for different types of convexity. A new direction begins to emerge in convex optimization theory under the influence of this concept. As part of our future research, we will be interested in the study of differential equations with intervals and application of cr-h-GL functions to optimize problems using cr-order. It is hoped that other scientists in various scientific disciplines will benefit from this concept.

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