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Exact Solutions for the KMM System in (2+1)-Dimensions and Its Fractional Form with Beta-Derivative

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Abstract: Fractional calculus is useful in studying physical phenomena with memory effects. In this paper, the fractional KMM (FKMM) system with beta-derivative in (2+1)-dimensions was studied for the first time. It can model short-wave propagation in saturated ferromagnetic materials, which has many applications in the high-tech world, especially in microwave devices. Using the properties of beta-derivatives and a proper transformation, the FKMM system was initially changed into the KMM system, which is a (2+1)-dimensional generalization of the sine-Gordon equation. Lie symmetry analysis and the optimal system for the KMM system were investigated. Using the optimal system, we obtained eight (1+1)-dimensional reduction equations. Based on the reduction equations, new soliton solutions, oblique analytical solutions, rational function solutions and power series solutions for the KMM system and FKMM system were derived. Using the properties of beta-derivatives and another transformation, the FKMM system was changed into a system of ordinary differential equations. Based on the obtained system of ordinary differential equations, Jacobi elliptic function solutions and solitary wave solutions for the FKMM system were derived. For the KMM system, the results about Lie symmetries, optimal system, reduction equations, and oblique traveling wave solutions are new, since Lie symmetry analysis method has not been applied to such a system before. For the FKMM system, all of the exact solutions are new. The main novelty of the paper lies in the fact that beta-derivatives have been used to change fractional differential equations into classical differential equations. The technique can also be extended to other fractional differential equations.

Keywords: KMM system; Lie symmetries; optimal system; exact solutions; conservation laws; power series solutions; conservation laws; fractional KMM system; beta-derivative

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1. Introduction

During the past three decades, fractional calculus achieved significant popularity and importance as a result of its applications in numerous fields of science and engineering [1–7]. For example, it has been successfully applied to problems in physics [8], hydrology [9,10], and chaos theory [11,12]. Fractional differential equations (FDEs), which are generalizations of classical differential equations of integer order, can describe physical phenomena that depend on both the time instant and time history. Although these fractional differential equations are often difficult to solve analytically, many methods have been proposed and proven efficient, such as homotopy analysis technique [13,14], variational iteration method [15], Sumudu decomposition method [16], Lie symmetry analysis method [17,18], and so on [19]. At the same time, many definitions of fractional derivative have been proposed and studied, such as the Caputo fractional derivative, Riemann–Liouville derivative, Grünwald–Letnikov derivative, Hadamard derivative, and the conformable derivative [20]. Among these, beta-derivatives [21–24] can be used to change fractional differential equations into partial differential equations (PDEs) or ordinary differential equations (ODEs). Therefore, approaches for finding exact solutions of nonlinear PDEs such as the Bäcklund transformation method [25], Hirota's bilinear method [26], **Painlevé** expansion method [27], and so on [28–30], can be employed to find exact solutions to fractional differential equations.

Ferrite materials, which have many special magnetic and electrical properties such as high resistivity, high magnetic permeability, moderate saturation magnetization, and excellent thermal stability, have been widely used in many high-tech fields for over half a century. Recently, certain nanofabrication techniques have made it possible to manufacture ferromagnetic particles to lengths of 20–30 nm. Since the original work by Kraenkel, Manna, and Merle in 2000 [31], researchers have constructed a series of Kraenkel–Manna–Merle (KMM) systems in order to study microwave propagation behavior in ferrite media, which are of importance in explaining and predicting nonlinear phenomena that occur in ferrite materials. Up to now, there have been (1+1)-dimensional KMM systems [32–41], their complex forms [42–44], (2+1)-dimensional KMM systems [45,46], and various generalizations [47–53], when considering Gilbert damping or inhomogeneous exchange. For the (1+1)-dimensional KMM systems, loop-like solutions [32,33], rogue wave solutions [36,39], and interactional behaviors such as twining behaviors between solitons have been studied [34,38,40,41]. Particularly, the fractional KMM system in (1+1)-dimensions with beta-derivative has been proposed and studied [24]. It has been shown that the wave profile changes for different values of the fractional parameter. Inspired by this, we wanted to investigate the fractional KMM (FKMM) system with beta-derivative in (2+1)-dimensions. From the point of view of mathematical physics, solitons or stable solitary waves that are localized in more than (1+1)-dimensions are of great interest.

To the best of our knowledge, the fractional forms of (2+1)-dimensional KMM systems have not been reported in the existing literature. In this paper, we investigated the following (2+1)-dimensional fractional KMM (FKMM) system:

$$\begin{aligned} \left(u_T^{\beta}\right)_x &= -vv_x + v_y + u_{yy}, \\ \left(v_T^{\beta}\right)_x &= vu_x + v_{yy} - u_y, \end{aligned} \tag{1}$$

where the physical observables $u = u(x, y, T)$ and $v = v(x, y, T)$ describe the external magnetization and the magnetic field, respectively. $D_T^{\beta}(.)$ is the beta-derivative [21–24], and is defined as follows:

$$D_T^{\beta}(f(T)) = \frac{d^{\beta} f(T)}{dT^{\beta}} = \lim_{\varepsilon \to 0} \frac{f(T + \varepsilon(T + \frac{1}{\Gamma(\beta)})^{1-\beta}) - f(T)}{\varepsilon}, 0 < \beta \leq 1.$$

The beta-derivative has the following properties [21]:

$$\begin{aligned} D_T^{\beta}(f(T)) &= f'(T)(T + \tfrac{1}{\Gamma(\beta)})^{1-\beta}, \\ D_T^{\beta}(f \circ g(T)) &= f'(g(T))g'(T)(T + \tfrac{1}{\Gamma(\beta)})^{1-\beta}. \end{aligned}$$

In fact, the beta-derivative can build the relationship between fractional differential equations and classical differential equations. If we take the following transformation:

$$u = u(x, y, t), v = v(x, y, t),\ t = \frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta}, \tag{2}$$

then FKMM system (1) can be changed into the following (2+1)-dimensional KMM system [45,46]:

$$\begin{aligned} u_{xt} &= -vv_x + v_y + u_{yy}, \\ v_{xt} &= vu_x + v_{yy} - u_y. \end{aligned} \tag{3}$$

In [45], based on Maxwell's equations and the Landau–Lifshitz equation, system (3) was proposed to describe the electromagnetic wave propagation in a saturated, nonconducting ferromagnetic medium. KMM system (3) is a (2+1)-dimensional generalization of the well-known, completely integrable (1+1)-dimensional sine-Gordon model [45]. In [46], the transverse stability of short line-solitons for system (3) has been researched, and it has been found that the unstable line solitons of system (3) could decay into stable two-dimensional solitary waves. To the best of our knowledge, there is no further research that studied KMM system (3).

The main purpose of this paper is two-fold: on the one hand, through performing Lie symmetry analysis, we obtained group-invariant solutions to KMM system (3). In the process of applying Lie symmetries to achieve group-invariant solutions, each symmetry sub-algebra corresponds to a group-invariant solution. In order to classify all of the group-invariant solutions, the optimal system of the symmetry algebra will be considered by a direct algorithm. Then, exact solutions to system (3) can be constructed by the optimal system. On the other hand, we derived exact solutions to the FKMM system (1) by means of transform (2) and the solutions of system (3).

The framework of the remainder of this paper is organized as follows. In Section 2, we initially perform Lie symmetry analysis on KMM system (3), then find the optimal system of the symmetries. In Section 3, using the obtained optimal system, all of the reduction equations and many new exact solutions for the KMM system and FKMM system are obtained. In Section 4, beginning from a reduction equation in the previous section, power series solutions for the KMM system and FKMM system are obtained. In Section 5, exact solutions of the FKMM system are further studied by means of a transformation. Section 6 is devoted to analysis and discussion of the methods and results in this paper. In Section 7, some closing words and future directions of the research are presented.

2. Lie Symmetry Analysis and Optimal System of KMM System (3)

Generally speaking, Lie symmetry denotes a transformation that leaves the solution manifold of a system invariant, i.e., it maps any solution of the system into a solution of the same system, hence it is also called geometric symmetry. In this section, we will perform Lie symmetry analysis on KMM system (3). Suppose that Lie symmetry of system (3) is expressed as follows:

$$V = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + W\frac{\partial}{\partial v}, \tag{4}$$

where ξ, η, τ, U and W are functions of x, y, t, u and v, respectively. According to [29], the Lie symmetry (4) can be determined by the following invariant condition equations:

$$\begin{aligned} &U^{xt} + vW^x + v_x W - W^y - U^{yy} = 0, \\ &W^{xt} - vU^x - u_x W - W^{yy} + U^y = 0, \end{aligned} \tag{5}$$

where

$$\begin{aligned} &U^x = D_x(U - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xx} + \eta u_{xy} + \tau u_{xt}, \\ &U^y = D_y(U - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xy} + \eta u_{yy} + \tau u_{ty}, \\ &U^{xt} = D_{xt}(U - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxt} + \eta u_{xyt} + \tau u_{xtt}, \\ &U^{yy} = D_{yy}(U - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xyy} + \eta u_{yyy} + \tau u_{yyt}, \\ &W^x = D_x(W - \xi v_x - \eta v_y - \tau v_t) + \xi v_{xx} + \eta v_{xy} + \tau v_{xt}, \\ &W^y = D_y(W - \xi v_x - \eta v_y - \tau v_t) + \xi v_{xy} + \eta v_{yy} + \tau v_{ty}, \\ &W^{xt} = D_{xt}(W - \xi v_x - \eta v_y - \tau v_t) + \xi v_{xxt} + \eta v_{xyt} + \tau v_{xtt}, \\ &W^{yy} = D_{yy}(W - \xi v_x - \eta v_y - \tau v_t) + \xi v_{xyy} + \eta v_{yyy} + \tau v_{yyt}. \end{aligned} \tag{6}$$

Substituting (6) into (5), with u, v being solutions of (3), we collect the coefficients of the derivatives of u and v, and set them to zero to obtain the following:

$$\begin{aligned}&\xi = -C_1 x + C_3 y + C_4,\\ &\eta = 2C_3 t + C_5,\\ &\tau = C_1 t + C_2,\\ &U = -C_1 u + f(t),\\ &W = -C_1 v - C_3,\end{aligned}$$

where C_1, C_2, C_3, C_4, and C_5 are arbitrary constants, and $f(t)$ is an arbitrary function of t. Thus, the Lie algebra of (3) is spanned by the following:

$$V_f = f(t)\frac{\partial}{\partial u},$$

and

$$V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial y}, V_3 = \frac{\partial}{\partial t}, V_4 = x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, V_5 = y\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y} - \frac{\partial}{\partial v}. \quad (7)$$

For (7), the commutation relations are obtained by the following Lie bracket:

$$[V_i, V_j] = V_iV_j - V_jV_i, (i, j = 1, 2, \ldots, 5) \quad (8)$$

and they are listed in the following Table 1.

Table 1. Commutator table of (7).

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	V_1	0
V_2	0	0	0	0	V_1
V_3	0	0	0		$2V_2$
V_4	$-V_1$	0	V_3	0	$-V_5$
V_5	0	$-V_1$	$-2V_2$	V_5	0

From Table 1, we know that the symmetries $V_i\,(i = 1, 2, \ldots, 5)$ form a closed five-dimensional Lie algebra. The five-dimensional Lie algebra has many sub-algebras; theoretically, one sub-algebra can derive a group-invariant solution. In order to classify all of the group-invariant solutions, we find the optimal system of (7) by the method proposed in [54] and used in [55].

In order to construct the optimal system, invariants will initially be derived. Taking $V = \sum_{i=1}^{5} a_iV_i, W = \sum_{j=1}^{5} b_jV_j$, where a_i and $b_j(i, j = 1, 2, \ldots, 5)$ are constants, we have the following:

$$\begin{aligned}&Ad_{\exp(\varepsilon W)}(V)\\ &= V - \varepsilon[W, V] + o(\varepsilon^2)\\ &= V - \varepsilon[b_1V_1 + b_2V_2 + b_3V_3 + b_4V_4 + b_5V_5, a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5] + o(\varepsilon^2)\\ &= V - \varepsilon\sum_{j=1}^{5}\sum_{i=1}^{5} b_ja_i[V_j, V_i] + o(\varepsilon^2)\\ &= V - \varepsilon(\theta_1V_1 + \theta_2V_2 + \theta_3V_3 + \theta_4V_4 + \theta_5V_5) + o(\varepsilon^2),\end{aligned} \quad (9)$$

with

$$\theta_1 = b_1a_4 + b_2a_5 - b_4a_1 - b_5a_2,\ \theta_2 = 2b_3a_5 - 2b_5a_3,$$
$$\theta_3 = -b_3a_4 + b_4a_3,\ \ \theta_4 = 0,\ \ \theta_5 = -b_4a_5 + b_5a_4.$$

For any $b_j(j = 1, 2, 3, 4, 5)$, the function ϕ with regard to a_1, a_2, a_3, a_4 and a_5 needs to satisfy the following condition:

$$\theta_1 \frac{\partial\phi}{\partial a_1} + \theta_2 \frac{\partial\phi}{\partial a_2} + \theta_3 \frac{\partial\phi}{\partial a_3} + \theta_4 \frac{\partial\phi}{\partial a_4} + \theta_5 \frac{\partial\phi}{\partial a_5} = 0. \tag{10}$$

Collecting the coefficients of all b_i in (10), we will obtain five differential equations as follows:

$$\begin{aligned} &a_4 \tfrac{\partial\phi}{\partial a_1} = 0,\ a_5 \tfrac{\partial\phi}{\partial a_1} = 0,\ 2a_5 \tfrac{\partial\phi}{\partial a_2} - a_4 \tfrac{\partial\phi}{\partial a_3} = 0, \\ &-a_1 \tfrac{\partial\phi}{\partial a_1} + a_3 \tfrac{\partial\phi}{\partial a_3} - a_5 \tfrac{\partial\phi}{\partial a_5} = 0, \\ &-a_2 \tfrac{\partial\phi}{\partial a_1} - 2a_3 \tfrac{\partial\phi}{\partial a_2} + a_4 \tfrac{\partial\phi}{\partial a_5} = 0. \end{aligned} \tag{11}$$

By searching for the solutions to (11), we can obtain $\phi(a_1, a_2, a_3, a_4, a_5) = F(a_4, \frac{a_4a_2+2a_3a_5}{2})$, with F being an arbitrary function of a_4 and $\frac{a_4a_2+2a_3a_5}{2}$. Hence, KMM system (3) has the following two basic invariants: $\Delta_1 = a_4$ and $\Delta_2 = \frac{a_4a_2+2a_3a_5}{2}$.

According to the theory in [29,54], the adjoint representation can be obtained by the following:

$$Ad_{\exp(\varepsilon V_i)}(V_j) = V_j - \varepsilon[V_i, V_j] + o(\varepsilon^2)$$

and the results are shown in the following Table 2.

Table 2. Adjoint representation table of (7).

Ad	V_1	V_2	V_3	V_4	V_5
V_1	V_1	V_2	V_3	$V_4 - \varepsilon V_1$	V_5
V_2	V_1	V_2	V_3	V_4	$V_5 - \varepsilon V_1$
V_3	V_1	V_2	V_3	$V_4 + \varepsilon V_3$	$V_5 - 2\varepsilon V_2$
V_4	$e^{\varepsilon} V_1$	V_2	$e^{-\varepsilon} V_3$	V_4	$e^{\varepsilon} V_5$
V_5	V_1	$V_2 + \varepsilon V_1$	$V_3 + 2\varepsilon V_2$	$V_4 - \varepsilon V_5$	V_5

Applying the adjoint action of V_1 on $V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5$, we have the following:

$$Ad_{\exp(\varepsilon V_1)}(V) = V - \varepsilon[V_1, V] + o(\varepsilon^2) = (a_1, a_2, a_3, a_4, a_5) \cdot A_1 \cdot (a_1, a_2, a_3, a_4, a_5)^T,$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\varepsilon_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly, one can obtain A_2, A_3, A_4 and A_5 as follows:

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\varepsilon_2 & 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & 1 & 0 \\ 0 & -2\varepsilon_3 & 0 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} e^{\varepsilon_4} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-\varepsilon_4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\varepsilon_4} \end{bmatrix}, A_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \varepsilon_5 & 1 & 0 & 0 & 0 \\ 0 & 2\varepsilon_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\varepsilon_5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, the adjoint transformation equation for KMM system (3) is the following:

$$(\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3, \widetilde{a}_4, \widetilde{a}_5) = (a_1, a_2, a_3, a_4, a_5) \cdot A, \tag{12}$$

where $A = A_1 A_2 A_3 A_4 A_5$ and

$$A = \begin{bmatrix} e^{\varepsilon_4} & 0 & 0 & 0 & 0 \\ \varepsilon_5 & 1 & 0 & 0 & 0 \\ 0 & 2e^{-\varepsilon_4}\varepsilon_5 & e^{-\varepsilon_4} & 0 & 0 \\ -\varepsilon_1 e^{\varepsilon_4} & 2\varepsilon_3 e^{-\varepsilon_4}\varepsilon_5 & \varepsilon_3 e^{-\varepsilon_4} & 1 & -\varepsilon_5 \\ -\varepsilon_2 e^{\varepsilon_4} - 2\varepsilon_3\varepsilon_5 & -2\varepsilon_3 & 0 & 0 & e^{\varepsilon_4} \end{bmatrix}.$$

Now, we can construct the optimal system of (7). From (12), we can obtain the following cases:

Case 1. $\Delta_1 = a_4 = 1, \Delta_2 = \frac{a_2 + 2a_3 a_5}{2} = C$, *where C is an arbitrary constant.*

Taking $\varepsilon_1 = 2a_3 a_5^2 + a_2 a_5 - a_5\varepsilon_2 + a_1, \varepsilon_3 = -a_3, \varepsilon_4 = \ln\left(\frac{\varepsilon_5}{a_5}\right)$, *we find that* $a_1 V_1 + a_2 V_2 + a_3 V_3 + V_4 + a_5 V_5$ *is equivalent to* $V_4 + \gamma V_2$, *with* $\gamma = 2C$.

Case 2. $\Delta_1 = a_4 = 0, \Delta_2 = a_3 a_5 = 1$.

Let $a_5 = 1$, *then* $a_3 = 1$. *Taking*$\varepsilon_2 = -\frac{1}{2}a_2^2 + 2a_2\varepsilon_3 - 2\varepsilon_3^2 + a_1, \varepsilon_4 = 0, \varepsilon_5 = -\frac{1}{2}a_2 + \varepsilon_3$, *we find that* $a_1 V_1 + a_2 V_2 + a_3 V_3 + a_5 V_5$ *is equivalent to*$V_3 + V_5$.

Case 3. $\Delta_1 = a_4 = 0, \Delta_2 = a_3 a_5 = -1$.

Let $a_3 = 1$, *then* $a_5 = -1$. *Taking* $\varepsilon_4 = 0, \varepsilon_5 = -\frac{a_2}{2} - \varepsilon_3, \varepsilon_2 = \frac{a_2^2}{2} + 2a_2\varepsilon_3 + 2\varepsilon_3^2 - a_1$, *we find that* $a_1 V_1 + a_2 V_2 + a_3 V_3 + a_5 V_5$ *is equivalent to* $V_3 - V_5$.

In the following, we discuss the case when $\Delta_1 = a_4 = 0, \Delta_2 = a_3 a_5 = 0$.

Case 4. $a_5 = 1, a_3 = 0$. *Taking* $\varepsilon_2 = a_1, \varepsilon_3 = \frac{1}{2}a_2, \varepsilon_4 = 0$, *we find that* $a_1 V_1 + a_2 V_2 + V_5$ *is equivalent to* V_5.

Case 5. $a_5 = 0, a_3 = 1$. *Taking* $\varepsilon_4 = 0, \varepsilon_5 = -\frac{1}{2}a_2$, *we find that* $a_1 V_1 + a_2 V_2 + V_3$ *is equivalent to* $V_3 + \alpha V_1$, *where* $\alpha = a_1 - \frac{a_2^2}{2}$.

Case 6. $a_5 = 0, a_3 = 0, a_2 = 1$. *Taking*$\varepsilon_5 = -a_1 e^{\varepsilon_4}$, *we find that* $a_1 V_1 + V_2$ *is equivalent to* V_2.

Case 7. $a_5 = 0, a_3 = 0, a_2 = 0, a_1 = 1$. *Thus,* V_1 *is equivalent to* V_1.

In summary, an optimal system of (7) is

$$V_1, V_2, V_3 + \alpha V_1, V_4 + \gamma V_2, V_3 + V_5, V_3 - V_5, V_5, \tag{13}$$

where α and γ are arbitrary constants.

3. Reduction Equations and Group-Invariant Solutions to (3) and (1)

Based on the optimal system (13), we can reduce KMM system (3) to eight PDEs in (1+1)-dimensions. For some reduction systems in (1+1)-dimensions, it is still difficult to find their exact solutions. Hence, we perform Lie symmetry analysis on them for a second time, and reduce them to ODEs.

Case 8. V_1

For the symmetry $V_1 = \frac{\partial}{\partial x}$*, we can obtain the following group-invariant solution to system (3):*

$$\begin{cases} u = F(y,t), \\ v = G(y,t), \end{cases}$$

where F and G satisfy the following reduction equations:

$$\begin{cases} G_y + F_{yy} = 0, \\ F_y - G_{yy} = 0. \end{cases}$$

The above linear differential equations can be solved, and then an analytical solution to KMM system (3) can be obtained as follows:

$$\begin{cases} lu = f_1(t) + f_2(t)\sin y + f_3(t)\cos y, \\ v = -f_2(t)\cos y + f_3(t)\sin y + f_4(t), \end{cases} \tag{14}$$

where $f_1(t), f_2(t), f_3(t)$ *and* $f_4(t)$ *are arbitrary functions of* t*.*

Case 9. V_2

For the symmetry$V_2 = \frac{\partial}{\partial y}$*,we can obtain a group-invariant solution as follows:*

$$\begin{cases} u = F(x,t), \\ v = G(x,t), \end{cases} \tag{15}$$

where F and G satisfy the following reduction equations:

$$\begin{cases} F_{xt} + GG_x = 0, \\ G_{xt} - GF_x = 0. \end{cases} \tag{16}$$

The equations in (16) constitute the (1+1)-dimensional KMM system [36].

Case 10. $V_3 + \alpha V_1$ *with* $\alpha = 0$

For the symmetry $V_3 + \alpha V_1$ *with* $\alpha = 0$*, we can obtain a group-invariant solution as follows:*

$$\begin{cases} u = F(x,y), \\ v = G(x,y), \end{cases}$$

where F and G satisfy the following reduction equations:

$$\begin{cases} GG_x - G_y - F_{yy} = 0, \\ -GF_x - G_{yy} + F_y = 0. \end{cases}$$

Applying the variable separating method [56] *to the above equations, we can find two exact solutions to system (3) as shown below:*

$$\begin{cases} u = -x - \frac{1}{6}C_1y^3 + \frac{1}{2}C_2y^2 + C_3y + C_4, \\ v = \frac{1}{2}C_1y^2 - C_2y - C_3 + C_1, \end{cases} \tag{17}$$

and

$$\begin{cases} u = C_2 x - \frac{C_5 \cos(\sqrt{C_2+1}y)}{\sqrt{C_2+1}} + \frac{C_4 \sin(\sqrt{C_2+1}y)}{\sqrt{C_2+1}} + \frac{C_2 C_1 y}{C_2+1} + C_3, \\ v = -\sin(\sqrt{C_2+1}y) C_5 - \cos(\sqrt{C_2+1}y) C_4 + \frac{C_1}{C_2+1}, \end{cases} \quad (C_2 > -1) \tag{18}$$

where C_1, C_2, C_3, C_4 *and* C_5*are constants.*

Case 11. $V_3 + \alpha V_1$ *with* $\alpha \neq 0$

For the symmetry $V_3 + \alpha V_1$ *with* $\alpha \neq 0$*, we can obtain a group-invariant solution as follows:*

$$\begin{cases} u = F(y, \theta), \\ v = G(y, \theta), \end{cases}$$

where $\theta = t - \frac{x}{\alpha}$*, F and G satisfy the following reduction equations:*

$$\begin{cases} GG_\theta + \alpha G_y + \alpha F_{yy} + F_{\theta\theta} = 0, \\ GF_\theta - \alpha G_{yy} + \alpha F_y - G_{\theta\theta} = 0. \end{cases} \tag{19}$$

From (19), we obtain a dark soliton solution of (3) via *the consistent Riccati expansion (CRE) method* [57,58]*, as follows:*

$$\begin{cases} u = u_0 - \frac{2(A_1^2\alpha + A_2^2)}{A_2} \tanh(A_1 y + A_2(t - \frac{x}{\alpha})), \\ v = \frac{-A_1\alpha}{A_2} - \frac{2I(A_1^2\alpha + A_2^2)}{A_2} \tanh(A_1 y + A_2(t - \frac{x}{\alpha})), \end{cases} \tag{20}$$

where u_0*,* A_1 *and* A_2 *are constants,* $A_2 \neq 0$*. This is a new soliton solution to KMM system (3), and it is different from the soliton solution in* [45,46]*. Usually, the CRE method can be used to find various interaction solutions between different types of excitations. The CRE method has been successfully applied to (1+1)-dimensional KMM systems and exact solutions, including breather soliton, periodic oscillation soliton as well as multipole instanton* [36]*. However, for the (2+1)-dimensional KMM system (3), (20) is the only result we can achieve.*

From (20) and (2), a dark soliton solution to FKMM system (1) is as follows:

$$\begin{cases} u = u_0 - \frac{2(A_1^2\alpha + A_2^2)}{A_2} \tanh\Big(A_1 y + A_2\Big(\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^\beta - \frac{x}{\alpha}\Big)\Big), \\ v = \frac{-A_1\alpha}{A_2} - \frac{2I(A_1^2\alpha + A_2^2)}{A_2} \tanh\Big(A_1 y + A_2\Big(\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^\beta - \frac{x}{\alpha}\Big)\Big). \end{cases} \tag{21}$$

Case 12. $V_4 + \gamma V_2$

For the symmetry$V_4 + \gamma V_2 = x\frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$*, we can obtain a group-invariant solution as follows:*

$$\begin{cases} u = xF(\xi, \theta), \\ v = xG(\xi, \theta), \end{cases} \tag{22}$$

where $\xi = -\gamma \ln x + y, \theta = tx$*, F and G satisfy the following reduction equations:*

$$\begin{cases} \theta GG_\theta + \theta F_{\theta\theta} + G^2 - F_{\xi\xi} - G_\xi + 2F_\theta - \gamma GG_\xi - \gamma F_{\xi\theta} = 0, \\ -\theta GF_\theta + \theta G_{\theta\theta} - GF - G_{\xi\xi} + F_\xi + 2G_\theta + \gamma GF_\xi - \gamma G_{\xi\theta} = 0. \end{cases} \tag{23}$$

The equations in (23) are variable-coefficient (1+1)-dimensional PDEs, and it is very difficult to solve them. We perform Lie symmetry analysis on (23). After calculations, Lie symmetry of (23) is as shown below:

$$V = C_{11}\frac{\partial}{\partial \xi} + \frac{C_{12}}{\theta}\frac{\partial}{\partial F}, \tag{24}$$

where C_{11} *and* C_{12} *are arbitrary constants. From (24), we obtain a group-invariant solution of* (23)

$$\begin{cases} F = \frac{C_{12}\xi}{C_{11}\theta} + f(\theta), \\ G = g(\theta), \end{cases} \tag{25}$$

where $f(\theta)$ and $g(\theta)$ are solutions of the following reduction equations:

$$\begin{cases} \theta g(\theta)g'(\theta) + \theta f''(\theta) + g^2(\theta) + 2f'(\theta) + \frac{C_{12}\gamma}{C_{11}\theta^2} = 0, \\ -\theta g(\theta)f'(\theta) + \theta g''(\theta) - g(\theta)f(\theta) + 2g'(\theta) + \frac{C_{12}(\gamma g(\theta)+1)}{C_{11}\theta} = 0. \end{cases} \tag{26}$$

From (22) *and* (25), *an exact solution for* (3) *is as shown below:*

$$\begin{cases} u = x\left(\frac{C_{12}\xi}{C_{11}\theta} + f(\theta)\right), \\ v = xg(\theta), \end{cases} \tag{27}$$

where $f(\theta)$ and $g(\theta)$ are determined by (26).

Case 13. $V_3 + V_5$

For the symmetry $V_3 + V_5 = y\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y} + \frac{\partial}{\partial t} - \frac{\partial}{\partial v}$, we can obtain a group-invariant solution to (3) *as follows*:

$$\begin{cases} u = F(\xi, \theta), \\ v = -t + G(\xi, \theta), \end{cases} \tag{28}$$

where $\xi = t^2 - y, \theta = \frac{2}{3}t^3 - ty + x$, F and G satisfy the following reduction equations:

$$\begin{cases} GG_\theta + \xi F_{\theta\theta} + G_\xi - F_{\xi\xi} = 0, \\ -GF_\theta + \xi G_{\theta\theta} - G_{\xi\xi} - F_\xi = 0. \end{cases} \tag{29}$$

After performing Lie symmetry analysis on (29), we find that the Lie symmetry of (29) is the following:

$$V = C_{21}\frac{\partial}{\partial\theta} + C_{22}\frac{\partial}{\partial F}, \tag{30}$$

where C_{21} and C_{22} are arbitrary constants. When $C_{21} = 1$, we can obtain the following group-invariant solution to (29):

$$\begin{cases} G = P(\xi), \\ F = C_{22}\theta + Q(\xi), \end{cases} \tag{31}$$

where P and Q satisfy the following reduction equations:

$$\begin{cases} P'(\xi) - Q''(\xi) = 0, \\ C_{22}P(\xi) + P''(\xi) + Q'(\xi) = 0. \end{cases} \tag{32}$$

The equations in (32) are solvable. When $C_{22} > -1$, an oblique analytical solution of (3) can be obtained by (28) and (31) as follows:

$$\begin{cases} u = C_{22}\left(\frac{2}{3}t^3 - ty + x\right) - \frac{N_2\cos\left(\sqrt{C_{22}+1}\left(t^2-y\right)\right)}{\sqrt{C_{22}+1}} + \frac{N_1\sin\left(\sqrt{C_{22}+1}\left(t^2-y\right)\right)}{\sqrt{C_{22}+1}} - \frac{N_0C_{22}\left(t^2-y\right)}{C_{22}+1} + N_3, \\ v = -t + N_2\sin\left(\sqrt{C_{22}+1}\left(t^2-y\right)\right) + N_1\cos\left(\sqrt{C_{22}+1}\left(t^2-y\right)\right) - \frac{N_0C_{22}}{C_{22}+1}. \end{cases} \tag{33}$$

When $C_{22} = -1$, a rational function solution to (3) can also be obtained by (28) and (31) as shown below:

$$\begin{cases} u = -\frac{2}{3}t^3 + ty - x + \frac{1}{6}N_0t^6 - \frac{1}{2}N_0t^4y + \frac{1}{2}N_0t^2y^2 - \frac{1}{6}N_0y^3 + \frac{1}{2}N_1t^4 - N_1t^2y + \frac{1}{2}N_1y^2 + N_2t^2 - N_2y + N_3, \\ v = -t + \frac{1}{2}N_0t^4 - N_0yt^2 + \frac{1}{2}N_0y^2 + N_1t^2 - N_1y + N_2 + N_0. \end{cases} \tag{34}$$

Here N_0, N_1, N_2 and N_3 are arbitrary constants. From (33), we can obtain an oblique traveling wave solution for FKMM system (1) by replacing t with $\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^\beta$. Then, from (34), a rational function solution for (1) can be derived by replacing t with $\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^\beta$.

Case 14. $V_3 - V_5$

For the symmetry $V_3 - V_5 = -y\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + \frac{\partial}{\partial v}$*, we can obtain a group-invariant solution to (3) as follows:*

$$\begin{cases} u = F(\xi, \theta), \\ v = t + G(\xi, \theta), \end{cases} \tag{35}$$

where $\xi = t^2 + y, \theta = \frac{2}{3}t^3 + ty + x$*, F and G satisfy the following reduction equations:*

$$\begin{cases} GG_\theta + \xi F_{\theta\theta} - G_\xi - F_{\xi\xi} = 0, \\ -GF_\theta + \xi G_{\theta\theta} - G_{\xi\xi} + F_\xi = 0. \end{cases} \tag{36}$$

After performing Lie symmetry analysis on (36), we find that the Lie symmetry of (36) is the same as for (30). From (30), we can obtain a group-invariant solution to (36) when $C_{21} = 1$*:*

$$\begin{cases} G = P(\xi), \\ F = C_{22}\theta + Q(\xi), \end{cases} \tag{37}$$

where P and Q satisfy the following reduction equations:

$$\begin{cases} P'(\xi) + Q''(\xi) = 0, \\ C_{22}P(\xi) + P''(\xi) - Q'(\xi) = 0. \end{cases} \tag{38}$$

The equations in (38) are also solvable. When $C_{21} > -1$ *and* $C_{22} \neq 0$*, an oblique traveling wave solution of (3) can be obtained by (35) and (37):*

$$\begin{cases} u = C_{22}(\frac{2}{3}t^3 + ty + x) + N_1 + N_2(t^2 + y) + N_3\sin(\sqrt{C_{22}+1}(t^2 + y)) + N_4\cos(\sqrt{C_{22}+1}(t^2 + y)), \\ v = t - N_3\cos(\sqrt{C_{22}+1}(t^2 + y))\sqrt{C_{22}+1} + N_4\sin(\sqrt{C_{22}+1}(t^2 + y))\sqrt{C_{22}+1} + \frac{N_2}{C_{22}}. \end{cases} \tag{39}$$

When $C_{22} = -1$*, another rational function solution to (3) can also be obtained by (35) and (37)*

$$\begin{cases} u = -\frac{2}{3}t^3 - ty - x + \frac{1}{6}N_1(t^2 + y)^3 + \frac{1}{2}N_2(t^2 + y)^2 + N_3(t^2 + y) + N_4, \\ v = t - N_1 - \frac{1}{2}N_1(t^2 + y)^2 - N_2(t^2 + y) - N_3. \end{cases} \tag{40}$$

Here N_1, N_2, N_3 *and* N_4 *are arbitrary constants. From (39) and (40), two analytical solutions to FKMM system (1) can be obtained by replacing t with* $\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^\beta$*. The characteristics of these solutions will be further discussed in* Section 6.

Case 15. V_5

For the symmetry $V_5 = y\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y} - \frac{\partial}{\partial v}$*, we can obtain a group-invariant solution as shown below:*

$$\begin{cases} u = F(\theta, t), \\ v = G(\theta, t) - \frac{y}{2t}, \end{cases} \tag{41}$$

where $\theta = -4xt + y^2$*, F and G satisfy the following reduction equations:*

$$\begin{cases} -4tGG_\theta - 4\theta F_{\theta\theta} - 4tF_{\theta t} - 6F_\theta + \frac{1}{2t} = 0, \\ 4tGF_\theta - 4\theta G_{\theta\theta} - 4tG_{\theta t} - 6G_\theta = 0. \end{cases} \tag{42}$$

The equations in (42) are variable-coefficient PDEs, and their solutions are difficult to construct. Therefore, we seek Lie symmetry for (42), which are shown as follows:

$$\begin{aligned} &W_1 = \frac{\theta}{\sqrt{t}}\frac{\partial}{\partial\theta} + \sqrt{t}\frac{\partial}{\partial t} - \frac{F}{2\sqrt{t}}\frac{\partial}{\partial F} - \frac{G}{2\sqrt{t}}\frac{\partial}{\partial G}, W_2 = t\frac{\partial}{\partial t} - F\frac{\partial}{\partial F} - G\frac{\partial}{\partial G}, \\ &W_3 = t\frac{\partial}{\partial\theta}, W_{h(t)} = h(t)\frac{\partial}{\partial F}. \end{aligned} \tag{43}$$

By means of W_1, we can obtain the following group-invariant solution to (42):

$$\begin{cases} F = \frac{Q(\Omega)}{\sqrt{\theta}}, \\ G = \frac{P(\Omega)}{\sqrt{\theta}}, \end{cases} \tag{44}$$

where $\Omega = \frac{t}{\theta}$, P and Q satisfy the following reduction equations:

$$\begin{cases} 4\Omega^3 P(\Omega)P'(\Omega) + 2\Omega^2 P^2(\Omega) + \frac{1}{2} = 0, \\ 4\Omega^3 P(\Omega)Q'(\Omega) + 2\Omega^2 P(\Omega)Q(\Omega) = 0. \end{cases} \tag{45}$$

Exact solutions to (45) can be found, and they are as follows:

$$\begin{cases} P(\Omega) = \pm\frac{1}{2\Omega}\sqrt{1+4K_2\Omega}, \\ Q(\Omega) = \frac{K_1}{\sqrt{\Omega}}. \end{cases} \tag{46}$$

Based on (41) and (44), the following rational function solution to (3) can be obtained:

$$\begin{cases} u = \frac{K_1}{\sqrt{t}}, \\ v = \pm\frac{1}{2t}\sqrt{-4xt+y^2+4K_2t} - \frac{y}{2t}, \end{cases} \tag{47}$$

where K_1 and K_2 are constants. From (47), a rational function solution to (1) can be obtained by replacing t with $\frac{1}{\beta}(T+\frac{1}{\Gamma(\beta)})^{\beta}$. By means of W_2, we can find a group-invariant solution to (42) as shown below:

$$\begin{cases} F = \frac{Q(z)}{t}, \\ G = \frac{P(z)}{t}, \end{cases} \tag{48}$$

where $z = -4xt + y^2$, P and Q satisfy the following reduction equations:

$$\begin{cases} 8PP' + 8zQ'' + 4Q' - 1 = 0, \\ -2PQ' + 2zP'' + P' = 0. \end{cases} \tag{49}$$

This is a system of ODEs, and we will study its power series solution in the next section.

Remark 1. *Using optimal system (13), we reduce KMM system (3) to eight reduction equations in (1+1)-dimensions. These equations represent the complete classification of all (1+1)-dimensional reduction equations. For the reduction equations (23), (29), (36) and (42), we perform Lie symmetry analysis on them for a second time, and reduce them to ordinary differential equations (26), (32), (38), (45) and (49). The solutions to (32), (38) and (45) have been found; solutions to (49) will be studied in Section 4. However, the solutions of (26) have not been found.*

Remark 2. *We should mention that all of the solutions for KMM system (3) in this section have not been reported in the existing literature. The oblique analytical solutions expressed by (33) and (39) are difficult to achieve usingother methods, for example, the CRE method in [57] and the MAE method in [24].*

4. Power Series Solutions to Systems (3) and (1)

The power series method is very useful for solving ODEs [59]. We will seek a solution for system (49) in the following form:

$$P(z) = \sum_{n=0}^{\infty} p_n z^n, Q(z) = \sum_{n=0}^{\infty} q_n z^n, \tag{50}$$

where p_n and $q_n (n = 0, 1, 2, \ldots)$ are all undetermined constants. Taking (50) into (49), we obtain the following:

$$\begin{aligned}
&(8p_0p_1+4q_1-1)+8(2p_0p_2+p_1^2+3q_2)z+\sum_{n=2}^{\infty}\sum_{j=0}^{n}8(n+1-j)p_jp_{n+1-j}z^n+\sum_{n=2}^{\infty}4(n+1)(2n+1)q_{n+1}z^n=0,\\
&(-2p_0q_1+p_1)+(6p_2-2p_1q_1-4p_0q_2)z+\sum_{n=2}^{\infty}\sum_{j=0}^{n}\left(-2(n+1-j)p_jq_{n+1-j}\right)z^n+\sum_{n=2}^{\infty}(2n+1)(n+1)p_{n+1}z^n=0.
\end{aligned} \tag{51}$$

For arbitrary constants p_0 and q_0, we can obtain the following coefficients from (51):

$$\begin{cases} p_1=\frac{p_0}{2\left(4p_0^2+1\right)},\\ q_1=\frac{1}{4\left(4p_0^2+1\right)}, \end{cases} \tag{52}$$

$$\begin{cases} p_2=-\frac{p_1}{4p_0^2+9}(2p_0p_1-3q_1),\\ q_2=-\frac{p_1}{4p_0^2+9}(2p_0q_1+3p_1), \end{cases} \tag{53}$$

$$\begin{cases} p_{n+1}=\frac{2}{(n+1)(4n^2+4n+4p_0^2+1)}\left((2n+1)\sum_{j=1}^{n}(n+1-j)p_jq_{n+1-j}-2p_0\sum_{j=1}^{n}(n+1-j)p_jp_{n+1-j}\right),\\ q_{n+1}=\frac{-2}{(n+1)(4n^2+4n+4p_0^2+1)}\left(2p_0\sum_{j=1}^{n}(n+1-j)p_jq_{n+1-j}+(2n+1)\sum_{j=1}^{n}(n+1-j)p_jp_{n+1-j}\right), \end{cases} \tag{54}$$

for all $n=2,3,\ldots$.

Thus, from (54) we can obtain the following:

$$\begin{cases} p_3=\frac{-2}{3(4p_0^2+25)}(-10p_1q_2-5p_2q_1+6p_0p_1p_2),\\ q_3=\frac{-2}{3(4p_0^2+25)}(2p_0p_2q_1+4p_0p_1q_2+15p_1p_2), \end{cases} \tag{55}$$

$$\begin{cases} p_4=\frac{-1}{2(4p_0^2+49)}\left(-21p_1q_3-14p_2q_2-7p_3q_1+8p_1p_3p_0+4p_0p_2^2\right),\\ q_4=\frac{-1}{4p_0^2+49}\left(3p_0p_1q_3+2p_0p_2q_2+p_0p_3q_1+14p_1p_3+7p_2^2\right), \end{cases} \tag{56}$$

$$\begin{cases} p_5=\frac{-2}{5(4p_0^2+81)}(10p_0p_1p_4+10p_0p_2p_3-36p_1q_4-27p_2q_3-18p_3q_2-9p_4q_1),\\ q_5=\frac{-2}{5(4p_0^2+81)}(8p_0p_1q_4+6p_0p_2q_3+4p_0p_3q_2+2p_0p_4q_1+45p_1p_4+45p_2p_3), \end{cases} \tag{57}$$

and so on.

Therefore, we can obtain a power series solution (50) with the coefficients given by (52)–(54). Moreover, we can show the convergence of (50).

From (54), one can obtain the following:

$$\begin{cases} |p_{n+1}|\le\sum_{j=1}^{n}\left|p_jq_{n+1-j}\right|+\sum_{j=1}^{n}\left|p_jp_{n+1-j}\right|,\\ |q_{n+1}|\le\sum_{j=1}^{n}\left|p_jq_{n+1-j}\right|+\sum_{j=1}^{n}\left|p_jp_{n+1-j}\right|. \end{cases}$$

Suppose that

$$r_i=|p_i|,s_i=|q_i|,i=0,1,2,$$

and

$$\begin{cases} r_{n+1}=\sum_{j=1}^{n}\left|r_js_{n+1-j}\right|+\sum_{j=1}^{n}\left|r_jr_{n+1-j}\right|,\\ s_{n+1}=\sum_{j=1}^{n}\left|r_js_{n+1-j}\right|+\sum_{j=1}^{n}\left|r_jr_{n+1-j}\right|, \end{cases}$$

where $n = 2, 3, \ldots$. Then, it is easily seen that $|p_n| \le r_n, |q_n| \le s_n, n = 0, 1, 2, 3, \ldots$. In other words, the two series $R = R(z) = \sum_{n=0}^{\infty} r_n z^n$ and $S = S(z) = \sum_{n=0}^{\infty} s_n z^n$ are majorant series of $P(z) = \sum_{n=0}^{\infty} p_n z^n$ and $Q(z) = \sum_{n=0}^{\infty} q_n z^n$ in (50), respectively.

Next, we prove that $R = R(z)$ and $S = S(z)$ have a positive radius of convergence. After calculation, we have the following:

$$\begin{aligned} R &= r_0 + r_1 z + r_2 z^2 + \sum_{n=2}^{\infty}\sum_{j=1}^{n} \left|r_j s_{n+1-j}\right| z^{n+1} + \sum_{n=2}^{\infty}\sum_{j=1}^{n} \left|r_j r_{n+1-j}\right| z^{n+1} \\ &= r_0 + r_1 z + r_2 z^2 + RS + r_0 s_0 - r_0 S - s_0 R - r_1 s_1 z + R^2 + {r_0}^2 - {r_1}^2 z^2 - 2r_0 R, \end{aligned}$$

$$\begin{aligned} S &= s_0 + s_1 z + s_2 z^2 + \sum_{n=2}^{\infty}\sum_{j=1}^{n} \left|r_j s_{n+1-j}\right| z^{n+1} + \sum_{n=2}^{\infty}\sum_{j=1}^{n} \left|r_j r_{n+1-j}\right| z^{n+1} \\ &= s_0 + s_1 z + s_2 z^2 + RS + r_0 s_0 - r_0 S - s_0 R - r_1 s_1 z + R^2 + {r_0}^2 - {r_1}^2 z^2 - 2r_0 R. \end{aligned}$$

Consider the following implicit functional system of z:

$$\begin{aligned} H_1(z, R, S) &= r_0 + r_1 z + r_2 z^2 + RS + r_0 s_0 - r_0 S - s_0 R - r_1 s_1 z + R^2 + {r_0}^2 - {r_1}^2 z^2 - 2r_0 R - R = 0, \\ H_2(z, R, S) &= s_0 + s_1 z + s_2 z^2 + RS + r_0 s_0 - r_0 S - s_0 R - r_1 s_1 z + R^2 + {r_0}^2 - {r_1}^2 z^2 - 2r_0 R - S = 0. \end{aligned}$$

Since H_1 and H_2 are analytic in the neighborhood of $(0, r_0, s_0)$, and $H_1(0, r_0, s_0) = 0$ and $H_2(0, r_0, s_0) = 0$, the following Jacobian determinant:

$$J = \left.\frac{\partial(H_1, H_2)}{\partial(R, S)}\right|_{(0, r_0, s_0)} = 1 \ne 0,$$

thus $R = R(z)$ and $S = S(z)$ are analytic in the neighborhood of $(0, r_0, s_0)$, and with a positive convergence radius by the implicit function theorem [60]. This means that the two power series in (50) converge in the neighborhood of $(0, r_0, s_0)$. Thus, the power series solution (50) is an analytical solution to (49). Therefore, the analytical solution to (3) is the following:

$$\begin{cases} u = \frac{1}{t}\left(q_0 + q_1(-4xt + y^2) + q_2(-4xt + y^2)^2 + \sum_{n=2}^{\infty} q_{n+1}(-4xt + y^2)^{n+1}\right), \\ v = \frac{-y}{2t} + \frac{1}{t}\left(p_0 + p_1(-4xt + y^2) + p_2(-4xt + y^2)^2 + \sum_{n=2}^{\infty} p_{n+1}(-4xt + y^2)^{n+1}\right), \end{cases} \tag{58}$$

where the coefficients p_i and $q_i (i = 1, 2, \ldots)$ are determined by (52), (53) and (54). From (58), a power series solution to (1) can be obtained by replacing t with $\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta}$.

5. Traveling Wave Solutions of (1)

For the fractional KMM system (1), we can change it into ODEs by the following transformation:

$$u = u(\varepsilon), v = v(\varepsilon), \varepsilon = k_1 x + k_2 y + \frac{k_3}{\beta}\left(T + \frac{1}{\Gamma(\beta)}\right)^{\beta}, \tag{59}$$

where k_1, k_2 and k_3 are constants to be determined, $0 < \beta \le 1$. Substituting (59) into (1) and applying the properties of the beta-derivative [21–24], we obtain the following:

$$\begin{aligned} u_T^{\beta} &= \left(T + \tfrac{1}{\Gamma(\beta)}\right)^{1-\beta} u' \varepsilon_T = \left(T + \tfrac{1}{\Gamma(\beta)}\right)^{1-\beta} u' \tfrac{k_3}{\beta} \beta \left(T + \tfrac{1}{\Gamma(\beta)}\right)^{\beta-1} = k_3 u', \\ v_T^{\beta} &= \left(T + \tfrac{1}{\Gamma(\beta)}\right)^{1-\beta} v' \varepsilon_T = \left(T + \tfrac{1}{\Gamma(\beta)}\right)^{1-\beta} v' \tfrac{k_3}{\beta} \beta \left(T + \tfrac{1}{\Gamma(\beta)}\right)^{\beta-1} = k_3 v', \end{aligned}$$

thus $\left(u_T^\beta\right)_x = k_3 k_1 u''$ and $\left(v_T^\beta\right)_x = k_3 k_1 v''$. Then, the fractional KMM system (1) reduces to the following system of ODEs after simplification:

$$u' = \frac{k_2 v - \frac{k_1}{2} v^2}{k_1 k_3 - k_2^2} + k_0, \tag{60}$$

$$v'^2 = C_4 v^4 + C_3 v^3 + C_2 v^2 + C_1 v + C_0, \tag{61}$$

where the coefficients C_0, C_1, C_2, C_3 and C_4 are the following:

$$C_0 = -\frac{k_4}{(k_1 k_3 - k_2^2)^2}, C_1 = -\frac{2k_0 k_1 k_2 k_3 - 2k_0 k_2^3}{(k_1 k_3 - k_2^2)^2},$$

$$C_2 = \frac{-k_0 k_1 k_2^2 - k_2^2 + k_0 k_1^2 k_3}{(k_1 k_3 - k_2^2)^2}, C_3 = \frac{k_1 k_2}{(k_1 k_3 - k_2^2)^2}, C_4 = -\frac{k_1^2}{4(k_1 k_3 - k_2^2)^2}.$$

Here, k_0 and k_4 are integral constants.

Equation (61) is called a Jacobi elliptic equation, and its solutions have been studied in [28]. Substituting the expression of v into (60) and integrating with respect to ε, we can obtain the explicit expression of u. Next, we choose suitable values of k_0, k_1, k_2, k_3, and k_4 in order to derive exact traveling wave solutions for (1).

Case 16. *When* $k_0 = \frac{2m}{m-1}, k_1 = 2k_2, k_3 = \frac{k_2 m - k_2 + 1}{2m-2}, k_4 = -\frac{k_2^2 m}{(m-1)^2}$, *there are two Jacobi elliptic function solutions to (1) as shown below:*

$$\begin{cases} u_1 = \frac{1}{1-m}(\text{EllipticE}(\text{sn}(\varepsilon), m) - (m+1)\varepsilon) + \frac{\sqrt{m}\text{cn}(\varepsilon)\text{dn}(\varepsilon)}{(1-m)\left(\sqrt{m}\text{sn}(\varepsilon)+1\right)}, \\ v_1 = \frac{\sqrt{m}\text{sn}(\varepsilon)+1}{2\sqrt{m}\text{sn}(\varepsilon)+m\text{sn}^2(\varepsilon)+1}, \end{cases} \tag{62}$$

and

$$\begin{cases} u_2 = \frac{1}{1-m}(\text{EllipticE}(\text{sn}(\varepsilon), m) - (m+1)\varepsilon) + \frac{\sqrt{m}\text{cn}(\varepsilon)\text{dn}(\varepsilon)}{(1-m)\left(\sqrt{m}\text{sn}(\varepsilon)+1\right)}, \\ v_2 = \frac{m^{3/2}\text{sn}(\varepsilon)}{m^{3/2}\text{sn}(\varepsilon)-\text{dn}^2(\varepsilon)+1}, \end{cases} \tag{63}$$

where $m(0 < m < 1)$ *denotes the modulus of the Jacobi elliptic function, EllipticE is the incomplete elliptic integral, and* $\text{Elliptic}(z, m) = \int_0^z \sqrt{1-m^2t^2}/\sqrt{1-t^2}dt$.

Case 17. *When* $k_0 = \frac{m^2}{2(k_2-2k_3)(m^2-1)}, k_1 = 2k_2, k_4 = \frac{k_2^2 m^2}{4(m^2-1)}$, *there are three solutions to (1) as follows:*

$$\begin{cases} u_3 = \frac{-1}{2(k_2-2k_3)(m^2-1)(-1+m\text{sn}(a\varepsilon))}\Big(m\sqrt{1-m^2}(k_2-2k_3)\text{cn}(a\varepsilon)\text{dn}(a\varepsilon) \\ \quad -\varepsilon m\text{sn}(a\varepsilon) + \varepsilon + \sqrt{1-m^2}(k_2-2k_3)\text{EllipticE}(\text{sn}(a\varepsilon), m)(m\text{sn}(a\varepsilon)-1)\Big), \\ v_3 = \frac{m\text{sn}(a\varepsilon)}{m\text{sn}(a\varepsilon)+\text{dn}(a\varepsilon)-1}, \end{cases} \tag{64}$$

$$\begin{cases} u_4 = \frac{-1}{2(k_2-2k_3)(m^2-1)}\left(\sqrt{1-m^2}(k_2-2k_3)(\text{EllipticE}(\text{sn}(a\varepsilon), m) - m\text{sn}(a\varepsilon)) - \varepsilon\right), \\ v_4 = \frac{m\text{cn}(a\varepsilon)}{m\text{cn}(a\varepsilon)+\text{dn}(a\varepsilon)+\sqrt{1-m^2}}, \end{cases} \tag{65}$$

$$\begin{cases} u_5 = \frac{-1}{2(k_2-2k_3)(m^2-1)}\left(\sqrt{1-m^2}(k_2-2k_3)(\text{EllipticE}(\text{sn}(a\varepsilon), m) - m\text{sn}(a\varepsilon)) - \varepsilon\right), \\ v_5 = \frac{\text{dn}(a\varepsilon)+\sqrt{1-m^2}}{\text{dn}(a\varepsilon)+m\text{cn}(a\varepsilon)+\sqrt{1-m^2}}, \end{cases} \tag{66}$$

where $a = \frac{1}{\sqrt{1-m^2}(k_2-2k_3)}$, $m\,(0 < m < 1)$ *denotes the modulus of the Jacobi elliptic function,* $\text{Elliptic}(z, m) = \int_0^z \frac{\sqrt{1-m^2t^2}}{\sqrt{1-t^2}}dt$.

Case 18. *When$k_1 = 1$, there is one soliton solution to (1), shown below:*

$$\begin{aligned} u_6 &= 2(k_2^2 - k_3)\mathrm{sech}(\varepsilon), \\ v_6 &= (k_2^2 - k_3)(2\tanh(\varepsilon) - \varepsilon). \end{aligned} \tag{67}$$

When $\varepsilon = k_1 x + k_2 y + k_3 t$, solution (67) is the same as the plane solitary wave solution in [45]*, with $k_3 = -w + p^2$ and $k_2 = p$.*

6. Results and Discussion

In this paper, FKMM system (1) with beta-derivative in (2+1)-dimensions were studied from the point of analytical solutions. The beta-derivative, which is a new proposed definition of the fractional derivative, has been used to change fractional differential equations into PDEs for the first time. The beta-derivative may not be seen as a fractional derivative but can be considered to be a natural extension of the classical derivative. Using one of the advantages of the beta-derivative in changing fractional differential equations into classical differential equations, we changed the (2+1)-dimensional FKMM system (1) into the (2+1)-dimensional KMM system (3) and a system of ODEs named (60) and (61). Taking a suitable transformation, FKMM system (1) can be changed into (1+1)-dimensional PDEs. For example, if we set the following:

$$u = u(x+y, t), v = v(x+y, t),\ t = \frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta}, \tag{68}$$

then FKMM system (1) can be changed into a system of (1+1)-dimensional PDEs.

Through the optimal system (13), we reduced the (2+1)-dimensional KMM system (3) to (1+1)-dimensional PDEs. For some of the derived (1+1)-dimensional PDEs, we performed Lie symmetry analysis for a second time and reduced them to ODEs. These ODEs are different from those in (60) and (61), and they possess novel solutions. For example, from (39), a solution to FKMM system (1) is expressed as follows:

$$\begin{cases} u = C_{22}\left(\frac{2}{3}(\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta})^3 + (\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta})y + x\right) + N_1 + N_2\left((\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta})^2 + y\right) \\ \quad + N_3 \sin\left(\sqrt{C_{22}+1}\left((\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta})^2 + y\right)\right) + N_4 \cos\left(\sqrt{C_{22}+1}\left((\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta})^2 + y\right)\right), \\ v = \frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta} - N_3 \cos\left(\sqrt{C_{22}+1}\left((\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta})^2 + y\right)\right)\sqrt{C_{22}+1} \\ \quad + N_4 \sin\left(\sqrt{C_{22}+1}\left((\frac{1}{\beta}(T + \frac{1}{\Gamma(\beta)})^{\beta})^2 + y\right)\right)\sqrt{C_{22}+1} + \frac{N_2}{C_{22}}. \end{cases} \tag{69}$$

This is a triangular periodic solution expressed by sine and cosine functions, and it is different from the triangular periodic solutions in (1+1)-dimensional FKMM, which are expressed by tangent or cotangent functions [24]. In [61], the authors also studied these kinds of oblique traveling wave solutions for the Heisenberg models of ferromagnetic spin chains with beta-derivative by implementing the generalized exponential expansion method. However, the structures of oblique traveling wave solutions in [61] differ from our results.

In addition to the oblique traveling wave solutions, we also obtained two soliton solutions for FKMM system (1). Solution (21) is a dark soliton solution, while solution (67) is a combo soliton solution. In (67), u_6 is a bright soliton, and v_6 is a dark soliton. Soliton solutions have very important uses in describing wave propagation in various media. We take as an example to illustrate the behaviors of the electromagnetic wave in a saturated ferromagnetic medium. Figures 1–3 depict the solution u_6 for different β at $T = 1$ when taking $k_2 = 1, k_3 = 0.25$. Figure 4 shows the different locations of the solution with different β at $T = 1, y = 0$. As can be seen from the figures, the wave profiles and locations of the solutions change as the fractional order parameter β changes. The larger the value of β, the more forward the solution is located; conversely, the smaller the value of β, the more backward the solution is located.

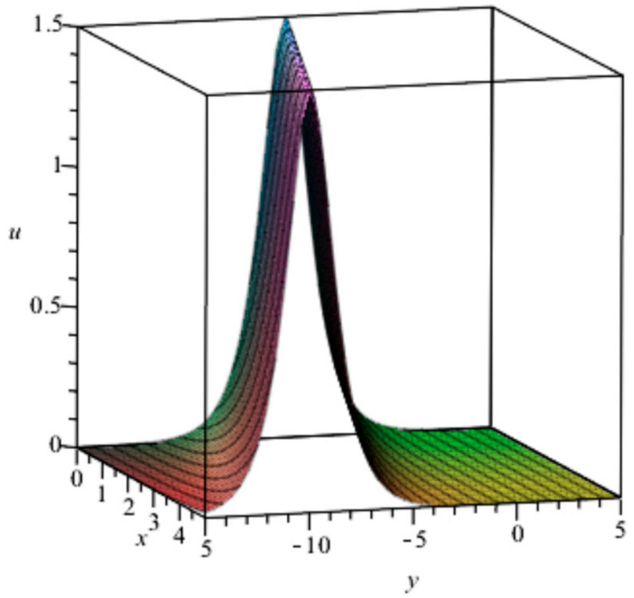

Figure 1. Solution locations for β = 0.05.

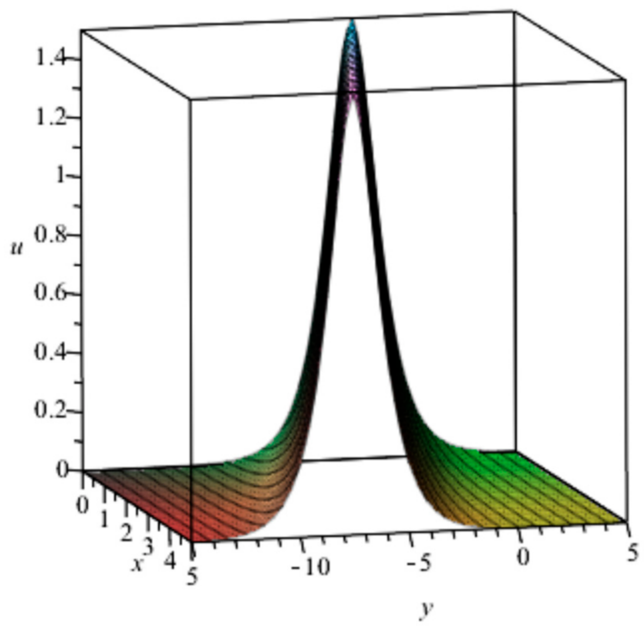

Figure 2. Solution locations for β = 0.3.

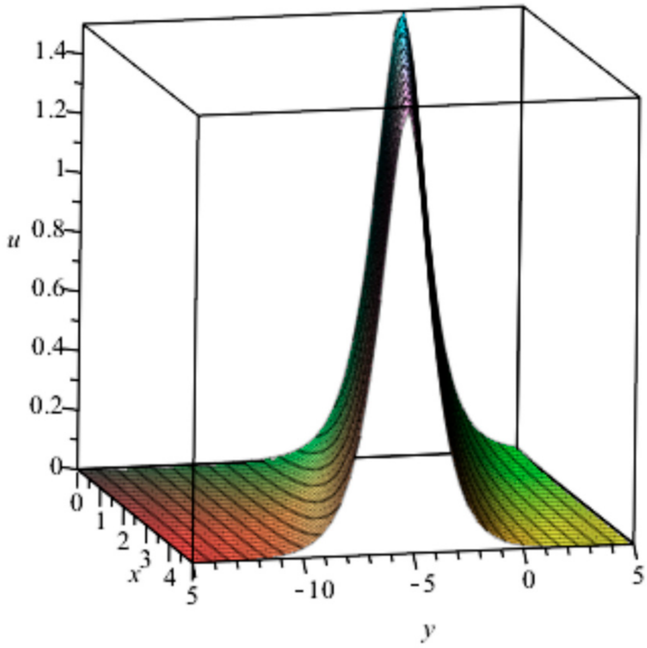

Figure 3. Solution locations for $\beta = 0.9$.

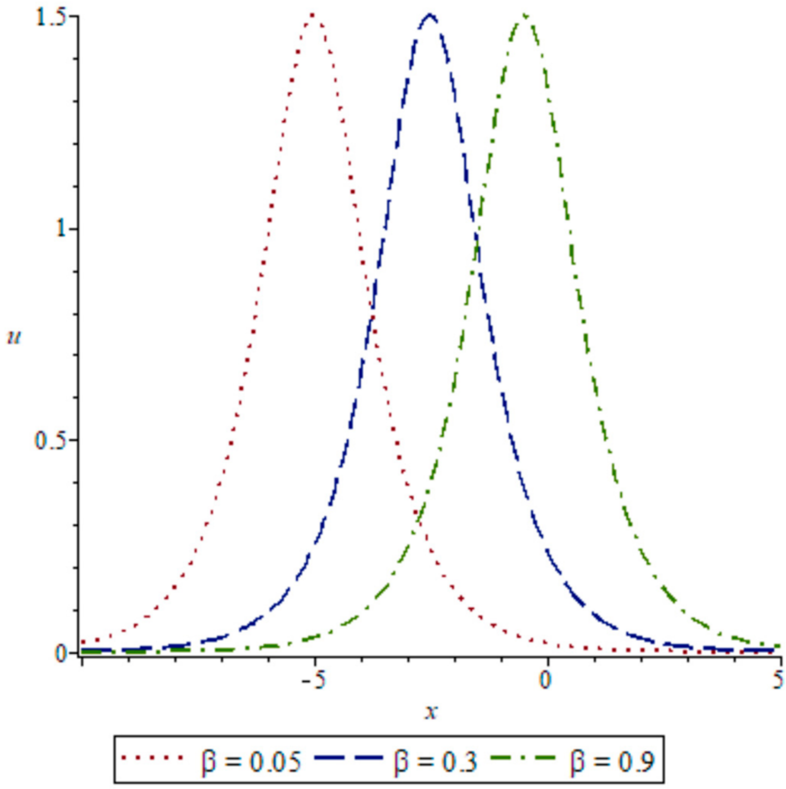

Figure 4. Locations of the solutions with different β.

7. Conclusions

Recently, beta-derivative, which is a newly introduced fractional derivative, was applied to a (2+1)-dimensional KMM system. The new model can describe electromagnetic wave propagation in a saturated nonconducting ferromagnetic medium when considering the effects of memory. Applying the properties of beta-derivatives, the KMM system with beta-derivative (FKMM system (1)) can be changed into KMM system (3). To the best of our knowledge, this is the first time that beta-derivative has been used to change fractional differential equations into PDEs. Through constructing exact solutions for KMM system (3), exact solutions for FKMM system (1) can be derived through a transformation.

In this paper, we first investigated new exact solutions to KMM system (3) by the Lie symmetry analysis method. Lie symmetries and their optimal system were derived. By the optimal system, all of the (1+1)-dimensional reduction equations of KMM system (3) were obtained. Through performing Lie symmetry analysis on some reduction equations, we reduced them to ODEs. Based on the reduced (1+1)-dimensional PDEs and ODEs, many new analytical solutions for KMM system (3), including soliton solutions, oblique analytical solutions, rational function solutions, and power series solutions, were obtained. The solitary wave solution of (3) that was obtained in [45] was derived as well. In addition, we obtained a novel dark soliton solution (20) for (3). In particular, we obtained novel oblique analytical solutions (33) and (39), which are triangular periodic solutions of (3) and are expressed by sine and cosine functions. Those oblique solutions are difficult to obtain by other algebraic methods, for example, the CRE method in [57] and the MAE method [24]. For FKMM system (1), dark soliton solutions and oblique analytical solutions were constructed at the same time. Furthermore, one can still investigate the solutions to the KMM system and FKMM system via the reduced (1+1)-dimensional PDEs and ODEs. The explicit solutions for (26) remain an open problem.

In addition, the FKMM system is also changed into a system of ODEs by means of a transformation and the properties of beta-derivatives. Making use of the known solutions to a Jacobi elliptic equation, Jacobi elliptic function solutions and soliton solutions for the FKMM system were constructed. To the best of our knowledge, a (2+1)-dimensional FKMM system with beta-derivative has been proposed and studied for the first time, and all the solutions for the FKMM system are new. Soliton solutions (21) as well as (67), and oblique analytical solution (69) have important physical applications. The properties of (67) have been illustrated by Figures 1–4.

For the (1+1)-dimensional KMM system, it has rich soliton structures [36,41]. However, some similar results have not been derived for the (2+1)-dimensional KMM system. In the future, rogue wave solutions for the (2+1)-dimensional KMM system and FKMM system will be studied using the truncated Painlevé analysis method or Hirota's bilinear method. Moreover, new propagation structures such as the breather soliton and periodic oscillation soliton may be studied.

There have been many definitions of fractional derivative, such as the Riemann–Liouville derivative, the Atangana–Baleanu derivative, the conformable derivative, and so on. In this paper, we only studied the fractional KMM system with beta-derivative, which is a weakness of our research. In the future, we will study the KMM system with other derivatives, and compare the results between different fractional derivatives.

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