fractal and fractional

Article

Solutions of a Nonlinear Diffusion Equation with a Regularized Hyper-Bessel Operator

Nguyen Hoang Luc [1], Donal O'Regan [2] and Anh Tuan Nguyen [1,*]

[1] Division of Applied Mathematics, Thu Dau Mot University, Thu Dau Mot City 75000, Vietnam
[2] School of Mathematical and Statistical Sciences, National University of Ireland, H91 TK33 Galway, Ireland
* Correspondence: nguyenanhtuan@tdmu.edu.vn

Abstract: We investigate the Cauchy problem for a nonlinear fractional diffusion equation, which is modified using the time-fractional hyper-Bessel derivative. The source function is a gradient source of Hamilton–Jacobi type. The main objective of our current work is to show the existence and uniqueness of mild solutions. Our desired goal is achieved using the Picard iteration method, and our analysis is based on properties of Mittag–Leffler functions and embeddings between Hilbert scales spaces and Lebesgue spaces.

Keywords: gradient nonlinearity; fractional diffusion equation; hyper-Bessel; fractional partial differential equations

MSC: 35K20; 35K58

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1. Introduction

Fractional partial differential equations (FPDEs) arise naturally in modeling since fractional derivatives help to describe phenomena efficiently [1], and FPDEs arise in many fields of applied science [2–8]; see also [9–29].

In this study, we consider a Cauchy problem for a time-space fractional hyper-Bessel differential equation as follows:

$$\begin{cases} {}_{C}\mathbb{D}_t^{\alpha,\beta}\varphi(t,x) + (-\Delta)^{\sigma}\varphi(t,x) = \left|\nabla\varphi(t,x)\right|^p, & \text{in } (0,T]\times\Omega, \\ \varphi(t,x) = 0 & \text{on } (0,T]\times\partial\Omega, \\ \varphi(0,x) = g(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ with sufficiently smooth boundary $\partial\Omega$, and g is the initial function. Recall from [30] the fractional operator

$$\left(t^{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\beta}\varphi(t) := (1-\alpha)^{\beta}t^{(\alpha-1)\beta}\frac{1-\alpha}{\Gamma(-\beta)}t^{(\alpha-1)\beta}\int_0^t \left(t^{1-\alpha}-s^{1-\alpha}\right)^{\beta-1}s^{-\alpha}\varphi(s)\mathrm{d}s, \tag{2}$$

where $\alpha < 1, \beta \in (0,1)$, Γ is the Gamma function and ∇ is the usual gradient operator. The notation ${}_{C}\mathbb{D}_t^{\alpha,\beta}$ stands for the Caputo-like counterpart of the hyper-Bessel operator with parameters $\alpha < 1$ of order $\beta \in (0,1)$ and can be defined as follows:

$${}_{C}\mathbb{D}_t^{\alpha,\beta}\varphi(t) := \left(t^{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\beta}\varphi(t) - \varphi(0)(1-\alpha)^{\beta}\frac{t^{(\alpha-1)\beta}}{\Gamma(1-\alpha)}, \tag{3}$$

provided that the right-hand side of the above equality makes sense. Since first introduced in [31] by Dimovski, the fractional hyper-Bessel operator has been shown to have applications in Brownian motion, fractional relaxation, and fractional diffusion models [30,32,33]. The regularized Caputo-like counterpart operator ${}_C\mathbb{D}_t^{\alpha,\beta}$ was introduced in [34] by Al-Musalhi et al., where the authors considered a direct problem and a inverse problem for a linear diffusion equation with the Caputo-like counterpart of the hyper-Bessel derivative. To provide an overview of topics related to Problem (1), we mention [35], where Au et al. investigated the Cauchy problem for the following equation:

$$ {}_C\mathbb{D}_t^{\alpha,\beta} u + \mathbb{L}u(t,x) = F(u), \tag{4}$$

where $\mathbb{L}$ is a generalization of $-\Delta$ and F is a nonlinearity of logarithm type, and the authors established the existence and uniqueness of a mild solution. In addition, they studied the blowing-up behavior of this solution. Tuan et al. [29] considered a terminal value problem for (4) where F is given in a linear form, and they showed that the backward problem is ill-posed and then applied a regularized Tikhonov regularization method to construct an approximating solution. In [36], Baleanu et al. investigated mild solutions to Equation (4) where F satisfies an exponential growth, and they showed the local well-posedness of mild solutions.

The first equation of Problem (1) is a modification of the classical diffusion equation. In the classical problem, Newton's derivative describes the velocity of a particle or slope of a tangent, whereas the general conformable derivative in (1) can be regarded as a special velocity and its direction and strength rely on a particular function [37]. The main goal of this work is to study the theory of existence and uniqueness of mild solutions, by which we can find an efficient numerical approach to investigate (1). In comparison with the above studies, our work possesses some new features. First, our source function is a gradient nonlinearity of Hamilton–Jacobi type. The presence of this function requires us to use different methods and, motivated by Souplet [38], we use the Picard iteration method to establish the existence and uniqueness of mild solutions. However, to deduce our results, we balance the linear and nonlinear parts of Problem (1), and to do this, we apply properties of Mittag–Leffler functions in an efficient way. Additionally, some Sobolev embeddings between Hilbert scales spaces and Lebesgue space are required to find an appropriate estimate to deal with the gradient source.

The outline of the work is as follows. Section 2 provides some preliminaries, and the main result concerning Problem (1) is given in Section 3.

2. Basic Settings

We begin this section with a convention that $a \lesssim b$ means a positive constant C exists such that $a \leqslant Cb$. Let $(B, \|\|_B)$ be a Banach space. We define the following space:

$$ L^\infty(0,T;B) := \Big\{ u : (0,T) \to B \;\Big|\; u \text{ is bounded almost everywhere on } (0,T) \Big\}. \tag{5}$$

Next, we recall that in $L^2(\Omega)$, the negative Laplace operator subject to Dirichlet conditions satisfies the following spectral problem:

$$ \begin{cases} -\Delta\Theta_l(x) = \lambda_l \Theta_l(x), & x \in \Omega, \\ \Theta_l(x) = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

where $\{\Theta_l\}_{l\in\mathbb{N}}$ is a set of eigenvectors which is also a orthonormal basic of $L^2(\Omega)$ and $\{\lambda_l\}_{l\in\mathbb{N}}$ is the corresponding increasing set of positive eigenvalues such that $\lambda_l \to \infty$ as $l \to \infty$. Then, for any $\sigma \geqslant 0$, we define the fractional Laplacian $(-\Delta)^\sigma$ by

$$ (-\Delta)^\sigma u := \sum_{l\in\mathbb{N}} \lambda_l^\sigma u_l \Theta_l, \tag{7}$$

where $u_l := \int_\Omega u(x)\Theta_l(x)\mathrm{d}x$ and u belongs to the following space

$$\mathbb{D}^\sigma(\Omega) := \left\{ u \in L^2(\Omega) \,\middle|\, \sum_{l\in\mathbb{N}} \lambda_l^{2\sigma} u_l^2 < \infty \right\}. \tag{8}$$

We note that $\mathbb{D}^\sigma(\Omega)$ is a Hilbert space and possesses the following norm:

$$\|u\|_{\mathbb{D}^\sigma(\Omega)} := \left\|(-\Delta)^\sigma u\right\|_{L^2(\Omega)} = \left(\sum_{l\in\mathbb{N}} \lambda_l^{2\sigma} u_l^2 \right)^{\frac{1}{2}}. \tag{9}$$

We define the Hilbert scale space with negative orders $\mathbb{D}^{-\sigma}(\Omega)$ as the dual space of $\mathbb{D}^\sigma(\Omega)$. Denote by $\langle \cdot, \cdot \rangle_*$ the dual product between $\mathbb{D}^{-\sigma}(\Omega)$ and $\mathbb{D}^\sigma(\Omega)$, and $\mathbb{D}^{-\sigma}(\Omega)$ is a Hilbert space equipped with the norm

$$\|u\|_{\mathbb{D}^{-\sigma}(\Omega)} := \left(\sum_{l\in\mathbb{N}} \lambda_l^{-2\sigma} \langle u, \Theta_l \rangle_*^2 \right)^{\frac{1}{2}}, \quad u \in \mathbb{D}^{-\sigma}(\Omega). \tag{10}$$

Remark 1 (Chapter 5 [39])**.** *For any $u \in L^2(\Omega)$ and $v \in \mathbb{D}^\sigma(\Omega)$, we have the following equality:*

$$\langle u, v \rangle_* = \int_\Omega u(x)v(x)\mathrm{d}x. \tag{11}$$

Proposition 1 (Lemma 4.7 [35])**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^N$. The following embeddings are satisfied:*

$$L^q(\Omega) \hookrightarrow \mathbb{D}^\nu(\Omega) \quad \text{if } \frac{-N}{4} < \nu \leqslant 0, \text{ and } q \geqslant \frac{2N}{N-4\nu}, \tag{12}$$

$$L^q(\Omega) \hookleftarrow \mathbb{D}^\nu(\Omega) \quad \text{if } 0 \leqslant \nu < \frac{N}{4}, \text{ and } q \leqslant \frac{2N}{N-4\nu}. \tag{13}$$

Next, we derive the mild formula for solutions of Problem (1). First, we introduce the definition of Mittag–Leffler functions, which play an important role in investigating time-fractional differential equations.

Definition 1. *For $\beta_1 \in \mathbb{R}^+, \beta_2 \in \mathbb{R}$ and $z \in \mathbb{C}$, the Mittag-Leffler function is defined as follows*

$$E_{\beta_1,\beta_2}(z) := \sum_{n\in\mathbb{N}} \frac{z^n}{\Gamma(n\beta_1 + \beta_2)}. \tag{14}$$

Suppose that $\varphi \in L^\infty(0,\infty; L^2(\Omega))$, and we find from the first equation of Problem (1) that

$$_C\mathbb{D}_t^{\alpha,\beta} \varphi_l(t) + \lambda_l^\sigma \varphi_l(t) = \left|\nabla\varphi(t)\right|_l^p, \quad t > 0, \tag{15}$$

here, we recall that $\varphi_l = \int_\Omega \varphi(x)\Theta_l(x)\mathrm{d}x$, $|\nabla\varphi|$ is the module of the gradient of φ and $|\nabla\varphi(t)|_l^p = \int_\Omega |\nabla\varphi(t,x)|^p \Theta_l(x)\mathrm{d}x$.

In order to solve this equation, we recall the following theorem from ([34]) (Section 2):

Theorem 1. *Let $\alpha < (-\infty, 1)$, $\lambda > 0$ and $\beta \in (0,1)$. For any $t > 0$, solutions of the following fractional differential equation*

$$_C\mathbb{D}_t^{\alpha,\beta} u(t) + \lambda u(t) = f(t) \tag{16}$$

are represented by the formula below:

$$
\begin{aligned}
u(t) =&E_{\beta,1}\left(\frac{-\lambda t^{(1-\alpha)\beta}}{(1-\alpha)^{\beta}}\right)u(0) \\
&+\frac{1}{(1-\alpha)^{\beta}\Gamma(\beta)}\int_0^t\left(t^{1-\alpha}-s^{1-\alpha}\right)^{\beta-1}f(s)\mathrm{d}(s^{1-\alpha}) \\
&-\frac{\lambda}{(1-\alpha)^{2\beta}}\int_0^t\left(t^{1-\alpha}-s^{1-\alpha}\right)^{2\beta-1}E_{\beta,2\beta}\left(-\frac{\lambda\left(t^{1-\alpha}-s^{1-\alpha}\right)^{\beta}}{(1-\alpha)^{\beta}}\right)f(s)\mathrm{d}(s^{1-\alpha}).
\end{aligned}
\tag{17}
$$

Based on the above theorem and some calculations, we derive the following equivalent equation of the (15):

$$
\begin{aligned}
\varphi_l(t) =&E_{\beta,1}\left(\frac{-\lambda_l^{\sigma}t^{(1-\alpha)\beta}}{(1-\alpha)^{\beta}}\right)g_l \\
&+\int_0^t\frac{\left(t^{1-\alpha}-s^{1-\alpha}\right)^{\beta-1}}{(1-\alpha)^{\beta}}E_{\beta,\beta}\left(-\frac{\lambda_l^{\sigma}\left(t^{1-\alpha}-s^{1-\alpha}\right)^{\beta}}{(1-\alpha)^{\beta}}\right)\left|\nabla\varphi(s)\right|_l^p\mathrm{d}(s^{1-\alpha}).
\end{aligned}
\tag{18}
$$

Recall that, for any $u \in L^2(\Omega)$, we have the Fourier expansion $u(x) = \sum_{l\in\mathbb{N}} u_l\Theta_l(x)$. Based on (18), we obtain the formula of the Fourier coefficient $\varphi_l(t)$ at $t \in (0,T)$ of a mild solution $\varphi \in L^\infty(0,T;\mathbb{D}^\nu(\Omega))$ of Problem (1). In summary, the solution $\varphi \in L^\infty(0,T;\mathbb{D}^\nu(\Omega))$ can be studied via the following equivalent integral equation:

$$
\varphi(t,x) = R_{1,\sigma}(t^{1-\alpha})g(x) + \int_0^t R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left|\nabla\varphi(s,x)\right|^p\mathrm{d}(s^{1-\alpha}), \tag{19}
$$

where

$$
R_{1,\sigma}(t)u(x) := \sum_{l\in\mathbb{N}} E_{\beta,1}\left(\frac{-\lambda_l^{\sigma}t^{\beta}}{(1-\alpha)^{\beta}}\right)u_l\Theta_l(x), \tag{20}
$$

$$
R_{2,\sigma}(t)u(x) := \sum_{l\in\mathbb{N}} \frac{t^{\beta-1}}{(1-\alpha)^{\beta}}E_{\beta,\beta}\left(-\frac{\lambda_l^{\sigma}t^{\beta}}{(1-\alpha)^{\beta}}\right)u_l\Theta_l(x). \tag{21}
$$

Remark 2. *The function φ in (19) is actually described by the limit (in $L^\infty(0,T;\mathbb{D}^\nu(\Omega))$) of the sequence $\{\varphi_j\}_{j\in\mathbb{N}}$, which is defined by*

$$
\varphi_1(t,x) := R_{1,\sigma}(t^{1-\alpha})g(x) \tag{22}
$$

and

$$
\varphi_{j+1}(t,x) := \varphi_1(t,x) + \int_0^t R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left|\nabla\varphi_j(s,x)\right|^p\mathrm{d}(s^{1-\alpha}). \tag{23}
$$

3. Existence and Uniqueness

This section begins with some linear estimates for $R_{1,\sigma}$ and $R_{2,\sigma}$, which are derived via the Fourier series of L^2 functions and Parseval's equality.

Lemma 1 ([8] Theorem 1.6)**.** *Let $\beta_1 \in (0,1)$ and $\beta_2 \in \mathbb{R}$ and $\varsigma \in (\frac{\pi\beta_1}{2},\pi)$. Then, for any $z \in \mathbb{C}$ such that*

$$
\varsigma \leqslant |\arg z| \leqslant \pi, \tag{24}
$$

the following estimate is satisfied:

$$\left|E_{\beta_1,\beta_2}(z)\right| \lesssim \frac{1}{1+|z|}. \tag{25}$$

Lemma 2 (Linear estimates)**.** *Let $\nu \geqslant 0$ and $0 < \sigma \leqslant 1$. The following estimates hold:*

1. *For any $u \in \mathbb{D}^\nu(\Omega)$,*

$$\left\|R_{1,\sigma}(t)u\right\|_{\mathbb{D}^\nu(\Omega)} \lesssim \|u\|_{\mathbb{D}^\nu(\Omega)}, \quad t > 0. \tag{26}$$

2. *For any $\theta \in [0,1]$ and $u \in \mathbb{D}^\nu(\Omega)$,*

$$\left\|R_{2,\sigma}(t)u\right\|_{\mathbb{D}^\nu(\Omega)} \lesssim t^{\beta-\theta\beta-1}\|u\|_{\mathbb{D}^{\nu-\theta\sigma}(\Omega)}, \quad t > 0. \tag{27}$$

Proof.

1. Suppose that $u \in \mathbb{D}^\nu(\Omega)$. The definition of $\mathbb{D}^\nu(\Omega)$ and Parseval's equality show that

$$\begin{aligned}\left\|R_{1,\sigma}(t)u\right\|^2_{\mathbb{D}^\nu(\Omega)} &= \left\|(-\Delta)^\nu R_{1,\sigma}(t)u\right\|_{L^2(\Omega)} \\ &= \sum_{l\in\mathbb{N}} \lambda_l^{2\nu}\left[E_{\beta,1}\left(\frac{-\lambda_l^\sigma t^\beta}{(1-\alpha)^\beta}\right)\right]^2 u_l^2. \end{aligned} \tag{28}$$

Applying Lemma 1, we find that

$$\left|E_{\beta,1}\left(\frac{-\lambda_l^\sigma t^\beta}{(1-\alpha)^\beta}\right)\right| \lesssim \frac{(1-\alpha)^\beta}{(1-\alpha)^\beta+\lambda_l^\sigma t^\beta}. \tag{29}$$

Combining (28) and (29) yields

$$\left\|R_{1,\sigma}(t)u\right\|_{\mathbb{D}^\nu(\Omega)} \lesssim \|u\|_{\mathbb{D}^\nu(\Omega)}. \tag{30}$$

2. Similarly, Lemma 1 implies

$$\begin{aligned}\left|E_{\beta,\beta}\left(-\frac{\lambda_l^\sigma t^\beta}{(1-\alpha)^\beta}\right)\right| &\lesssim \left[\frac{(1-\alpha)^\beta}{(1-\alpha)^\beta+\lambda_l^\sigma t^\beta}\right]^{1-\theta}\left[\frac{(1-\alpha)^\beta}{(1-\alpha)^\beta+\lambda_l^\sigma t^\beta}\right]^{\theta} \\ &\lesssim \lambda_l^{-\sigma\theta}t^{-\theta\beta}, \end{aligned} \tag{31}$$

for any $\theta \in [0,1]$. For any $u \in \mathbb{D}^{\nu-\theta\sigma}(\Omega)$, one has

$$\left\|R_{2,\sigma}(t)u\right\|^2_{\mathbb{D}^\nu(\Omega)} = \sum_{l\in\mathbb{N}} \lambda_l^{2\nu}\left[\frac{t^{\beta-1}}{(1-\alpha)^\beta}E_{\beta,\beta}\left(\frac{-\lambda_l^\sigma t^\beta}{(1-\alpha)^\beta}\right)\right]^2 u_l^2. \tag{32}$$

Based on estimate (31), we deduce

$$\begin{aligned}\left\|R_{2,\sigma}(t)u\right\|_{\mathbb{D}^\nu(\Omega)} &\lesssim t^{\beta-\theta\beta-1}\left(\sum_{l\in\mathbb{N}}\lambda_l^{2\nu-2\theta\sigma}u_l^2\right)^{\frac{1}{2}} \\ &= t^{\beta-\theta\beta-1}\|u\|_{\mathbb{D}^{\nu-\theta\sigma}(\Omega)}. \end{aligned} \tag{33}$$

The proof is completed. □

Next, we provide a lemma about the nonlinear estimate that helps us to completely define the source function $|\nabla u|^p$ and find an appropriate way to deal with it.

Lemma 3 (Nonlinear estimates)**.** *Let $N \geqslant 1$ and ν, γ, p be constants such that*

$$\nu < \gamma \leqslant \frac{N}{4} + \nu, \tag{34}$$

$$\frac{1}{2} \leqslant \nu < \frac{N}{4} + \frac{1}{2}, \tag{35}$$

$$\max\left\{1, \frac{2N}{N-4(\nu-\gamma)}\right\} p \leqslant \frac{2N}{N-4(\nu-\frac{1}{2})}. \tag{36}$$

Then, for any $u, v \in \mathbb{D}^{\nu}(\Omega)$, we have the following nonlinear estimate:

$$\left\| |\nabla u|^p - |\nabla v|^p \right\|_{\mathbb{D}^{\nu-\gamma}(\Omega)} \lesssim \left(\|u\|^{p-1}_{\mathbb{D}^{\nu}(\Omega)} + \|u\|^{p-1}_{\mathbb{D}^{\nu}(\Omega)} \right) \left\| u - v \right\|_{\mathbb{D}^{\nu}(\Omega)}. \tag{37}$$

Proof. We first note that there exists a positive constant q such that

$$\max\left\{1, \frac{2N}{N-4(\nu-\gamma)}\right\} p \leqslant q \leqslant \frac{2N}{N-4(\nu-\frac{1}{2})}. \tag{38}$$

Hölder's inequality thus helps us to derive

$$\left\| |\nabla u|^p - |\nabla v|^p \right\|_{L^{q/p}(\Omega)} \lesssim \left(\|\nabla u\|^{p-1}_{L^{q}(\Omega)} + \|\nabla u\|^{p-1}_{L^{q}(\Omega)} \right) \left\| \nabla u - \nabla v \right\|_{L^{q}(\Omega)}. \tag{39}$$

Then, we apply the inclusion $\mathbb{D}^{\nu-\frac{1}{2}}(\Omega) \hookrightarrow L^{q}(\Omega)$ and deduce

$$\left\| |\nabla u|^p - |\nabla v|^p \right\|_{L^{q/p}(\Omega)} \lesssim \left(\|\nabla u\|^{p-1}_{\mathbb{D}^{\nu-\frac{1}{2}}(\Omega)} + \|\nabla u\|^{p-1}_{\mathbb{D}^{\nu-\frac{1}{2}}(\Omega)} \right) \left\| \nabla u - \nabla v \right\|_{\mathbb{D}^{\nu-\frac{1}{2}}(\Omega)}. \tag{40}$$

It immediately follows that

$$\left\| |\nabla u|^p - |\nabla v|^p \right\|_{L^{q/p}(\Omega)} \lesssim \left(\|u\|^{p-1}_{\mathbb{D}^{\nu}(\Omega)} + \|u\|^{p-1}_{\mathbb{D}^{\nu}(\Omega)} \right) \left\| u - v \right\|_{\mathbb{D}^{\nu}(\Omega)}. \tag{41}$$

This result together with the embedding $L^{q/p}(\Omega) \hookrightarrow \mathbb{D}^{\nu-\gamma}(\Omega)$ yield the desired estimate, provided that $q/p \geqslant \frac{2N}{N-4(\nu-\gamma)}$. The proof is completed. □

Theorem 2. *Suppose that $N \geqslant 1$ and ν, σ, θ, p satisfy the following assumptions:*

$$0 < \theta < 1, 0 < \sigma \leqslant 1 \tag{42}$$

$$\frac{1}{2} \leqslant \nu < \frac{N}{4} + \frac{1}{2}, \tag{43}$$

$$\nu < \theta\sigma \leqslant \frac{N}{4} + \nu, \tag{44}$$

$$\max\left\{1, \frac{2N}{N-4(\nu-\theta\sigma)}\right\} p \leqslant \frac{2N}{N-4(\nu-\frac{1}{2})}. \tag{45}$$

In addition, assume that $g \in \mathbb{D}^{\nu}(\Omega)$. Then, there exists a positive constant $T > 0$ such that Problem (1) *has a unique mild solution $\varphi \in L^{\infty}(0, T; \mathbb{D}^{\nu}(\Omega))$.*

Proof. First, for any $T > 0$, we denote by $\mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$ a closed ball in $L^\infty(0,T;\mathbb{D}^\nu(\Omega))$ centered at zero with radius $R > 0$. Next, we consider a sequence of functions $\{\varphi_j\}_{j\in\mathbb{N}}$ defined in Remark 2. By induction, we show that if $g \in \mathbb{D}^\nu(\Omega)$,$\{\varphi_j\}_{j\in\mathbb{N}}$ is a subset of $\mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$ for some appropriate constants $R > 0$ and $T > 0$. Indeed, for $g \in \mathbb{D}^\nu(\Omega)$, we can apply Lemma 2 and deduce

$$\begin{aligned} \|\varphi_1(t)\|_{\mathbb{D}^\nu(\Omega)} &= \left\|R_{1,\sigma}(t^{1-\alpha})g\right\|_{\mathbb{D}^\nu(\Omega)} \\ &\lesssim \|g\|_{\mathbb{D}^\nu(\Omega)} \\ &< \frac{1}{2}R, \qquad\qquad t > 0. \end{aligned} \tag{46}$$

Thus, $\varphi_1 \in \mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$. Next, for $j \geqslant 2$, we suppose that $\varphi_j \in \mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$. For $t > 0$, the triangle inequality yields

$$\|\varphi_{j+1}(t)\|_{\mathbb{D}^\nu(\Omega)} \leqslant \|\varphi_1(t)\|_{\mathbb{D}^\nu(\Omega)} + \int_0^t \left\|R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left|\nabla\varphi_j(s,x)\right|^p\right\|_{\mathbb{D}^\nu(\Omega)} \mathrm{d}(s^{1-\alpha}). \tag{47}$$

According to Lemma 2, the following estimate holds:

$$\left\|R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left|\nabla\varphi_j(s,x)\right|^p\right\|_{\mathbb{D}^\nu(\Omega)} \lesssim (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\left\|\left|\nabla\varphi_j(s)\right|^p\right\|_{\mathbb{D}^{\nu-\theta\sigma}(\Omega)}. \tag{48}$$

Assumptions of ν, σ, θ enable us to use Lemma 3 and derive

$$\left\|R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left|\nabla\varphi_j(s,x)\right|^p\right\|_{\mathbb{D}^\nu(\Omega)} \lesssim (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\|\varphi_j(s)\|^p_{\mathbb{D}^\nu(\Omega)}, \tag{49}$$

where we chose $u = \varphi_j$ and $v = 0$. Therefore, for any $t > 0$, we find that

$$\begin{aligned} &\int_0^t \left\|R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left|\nabla\varphi_j(s,x)\right|^p\right\|_{\mathbb{D}^\nu(\Omega)} \mathrm{d}(s^{1-\alpha}) \\ &\quad\lesssim \int_0^t (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\|\varphi_j(s)\|^p_{\mathbb{D}^\nu(\Omega)} \mathrm{d}(s^{1-\alpha}). \end{aligned} \tag{50}$$

Since $\varphi_j \in \mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$, one has

$$\|\varphi_j(t)\|_{\mathbb{D}^\nu(\Omega)} \leqslant R, \quad \text{for almost } t \in (0,T). \tag{51}$$

Thus, (50) is equivalent to

$$\begin{aligned} &\int_0^t \left\|R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left|\nabla\varphi_j(s,x)\right|^p\right\|_{\mathbb{D}^\nu(\Omega)} \mathrm{d}(s^{1-\alpha}) \\ &\lesssim \int_0^t (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\left[\operatorname*{ess\,sup}_{t\in(0,T)}\|\varphi_j(s)\|_{\mathbb{D}^\nu(\Omega)}\right]^p \mathrm{d}(s^{1-\alpha}) \\ &\lesssim M^p\int_0^t (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\mathrm{d}(s^{1-\alpha}). \end{aligned} \tag{52}$$

Since $\theta < 1$, the last integral is convergent. We thus can find a sufficiently small constant T such that $T^{(1-\alpha)^{\beta-\theta\beta}}R^{p-1} \leqslant \frac{1}{2}$. Therefore, one has

$$\int_0^t \left\|R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left|\nabla\varphi_j(s,x)\right|^p\right\|_{\mathbb{D}^\nu(\Omega)} \mathrm{d}(s^{1-\alpha}) \leqslant \frac{1}{2}R. \tag{53}$$

Combining (46), (47) and (53) gives us

$$\|\varphi_{j+1}(t)\|_{\mathbb{D}^\nu(\Omega)} \leqslant R. \tag{54}$$

We can now conclude that $\varphi_{j+1} \in \mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$. Thus, $\{\varphi_j\}_{j\in\mathbb{N}}$ is a subset of $\mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$.

Next, we prove that $\{\varphi_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$. Let φ_{j-1} and φ_j be two elements of $\{\varphi_j\}_{j\in\mathbb{N}} \subset \mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$. We have

$$\left\|\varphi_{j+1}(t)-\varphi_j(t)\right\|_{\mathbb{D}^\nu(\Omega)} \leqslant \int_0^t \left\| R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left[\left|\nabla\varphi_j(s)\right|^p - \left|\nabla\varphi_{j-1}(s)\right|^p\right]\right\|_{\mathbb{D}^\nu(\Omega)} \mathrm{d}(s^{1-\alpha}). \tag{55}$$

Repeated application of Lemma 2 enables us to write

$$\left\|\varphi_{j+1}(t)-\varphi_j(t)\right\|_{\mathbb{D}^\nu(\Omega)} \lesssim \int_0^t (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\left\|\left|\nabla\varphi_j(s)\right|^p - \left|\nabla\varphi_{j-1}(s)\right|^p\right\|_{\mathbb{D}^{\nu-\theta\sigma}(\Omega)}\mathrm{d}(s^{1-\alpha}). \tag{56}$$

It follows that

$$\begin{aligned}&\left\|\varphi_{j+1}(t)-\varphi_j(t)\right\|_{\mathbb{D}^\nu(\Omega)}\\ &\lesssim \int_0^t (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\left(\|\varphi_j(s)\|^{p-1}_{\mathbb{D}^\nu(\Omega)}(s) + \|\varphi_{j-1}\|^{p-1}_{\mathbb{D}^\nu(\Omega)}\right)\left\|\varphi_j(s)-\varphi_{j-1}(s)\right\|_{\mathbb{D}^\nu(\Omega)}\mathrm{d}(s^{1-\alpha}).\end{aligned} \tag{57}$$

Similar to the above arguments, since $\varphi_{j-1}, \varphi_j \in \mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$, we have

$$\begin{cases}\operatorname*{ess\,sup}_{t\in(0,T)} \|\varphi_{j-1}(t)\|_{\mathbb{D}^\nu(\Omega)} \leqslant R,\\ \operatorname*{ess\,sup}_{t\in(0,T)} \|\varphi_j(t)\|_{\mathbb{D}^\nu(\Omega)} \leqslant R.\end{cases} \tag{58}$$

Therefore, we obtain the following estimate:

$$\begin{aligned}&\left\|\varphi_{j+1}(t)-\varphi_j(t)\right\|_{\mathbb{D}^\nu(\Omega)} \qquad (59)\\ &\lesssim \int_0^t (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\left[\operatorname*{ess\,sup}_{s\in(0,T)}\|\varphi_j(s)\|_{\mathbb{D}^\nu(\Omega)}(s)\right]^{p-1}\left\|\varphi_j(s)-\varphi_{j-1}(s)\right\|_{\mathbb{D}^\nu(\Omega)}\mathrm{d}(s^{1-\alpha})\\ &+\int_0^t (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\left[\operatorname*{ess\,sup}_{s\in(0,T)}\|\varphi_{j-1}(s)\|_{\mathbb{D}^\nu(\Omega)}(s)\right]^{p-1}\left\|\varphi_j(s)-\varphi_{j-1}(s)\right\|_{\mathbb{D}^\nu(\Omega)}\mathrm{d}(s^{1-\alpha})\\ &\lesssim R^{p-1}\left[\int_0^t (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\mathrm{d}(s^{1-\alpha})\right]\operatorname*{ess\,sup}_{t\in(0,T)}\left\|\varphi_j(t)-\varphi_{j-1}(t)\right\|_{\mathbb{D}^\nu(\Omega)}. \qquad (60)\end{aligned}$$

From the fact that

$$\int_0^t (t^{1-\alpha}-s^{1-\alpha})^{\beta-\theta\beta-1}\mathrm{d}(s^{1-\alpha}) \lesssim T^{(1-\alpha)^{\beta-\theta\beta}}, \tag{61}$$

by a a suitable choice of T, we have

$$\left\|\varphi_{j+1}(t)-\varphi_j(t)\right\|_{\mathbb{D}^\nu(\Omega)} \leqslant \frac{1}{2}\operatorname*{ess\,sup}_{t\in(0,T)}\left\|\varphi_j(t)-\varphi_{j-1}(t)\right\|_{\mathbb{D}^\nu(\Omega)}, \quad t>0. \tag{62}$$

This is equivalent to the following result:

$$\operatorname*{ess\,sup}_{t\in(0,T)}\left\|\varphi_{j+1}(t)-\varphi_j(t)\right\|_{\mathbb{D}^\nu(\Omega)} \leqslant \frac{1}{2}\operatorname*{ess\,sup}_{t\in(0,T)}\left\|\varphi_j(t)-\varphi_{j-1}(t)\right\|_{\mathbb{D}^\nu(\Omega)}. \tag{63}$$

From the above estimate, we easily deduce that $\{\varphi_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{B}_R(0,T;\mathbb{D}^\nu(\Omega))$. The completeness of $L^\infty(0,T;\mathbb{D}^\nu(\Omega))$ ensures the unique existence of a function φ such that

$$\lim_{j\to\infty} \operatorname*{ess\,sup}_{t\in(0,T)} \left\|\varphi_j(t)-\varphi(t)\right\|_{\mathbb{D}^\nu(\Omega)} = 0. \quad (64)$$

Therefore, we find that

$$\varphi(t,x) = \lim_{j\to\infty}\varphi_j(t,x) = R_{1,\sigma}(t^{1-\alpha})g(x) + \int_0^t R_{2,\sigma}(t^{1-\alpha}-s^{1-\alpha})\left|\nabla\varphi(s,x)\right|^p \mathrm{d}(s^{1-\alpha}). \quad (65)$$

We can now conclude that Problem (1) possesses a unique mild solution $\varphi \in L^\infty(0,T;\mathbb{D}^\nu(\Omega))$. The theorem is thus proven. □

4. Conclusions

In this study, we prove the existence and uniqueness of a mild solution to an initial value problem for a fractional diffusion equation with the Caputo-like counterpart of the hyper-Bessel derivative and a gradient source function. The result hopefully can be extended in future works to global results, and indeed the blowing-up behavior of mild solutions is also an interesting open problem.

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