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Local and Global Mild Solution for Gravitational Effects of the Time Fractional Navier–Stokes Equations

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Abstract: The gravitational effect is a physical phenomenon that explains the motion of a conductive fluid flowing under the impact of an exterior gravitational force. In this paper, we work on the Navier–Stokes equations (NSES) of the fluid flowing under the impact of an exterior gravitational force inclined at an angle of 45° with A time-fractional derivative of order $\beta \in (0,1)$. To encourage anomalous diffusion in fractal media, we apply these equations. In $H^{\delta,r}$, we prove the existence and uniqueness of local and global mild solutions. Additionally, we provide moderate local solutions in J_r. Additionally, we establish the regularity and existence of classical solutions to these equations in J_r.

Keywords: Navier–Stokes equations; Caputo fractional derivative; existence; stability; Mittag–Leffler functions; Mild solutions; regularity

MSC: 34K37; 34B15

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1. Introduction

The study of the various definitions of real number powers or complex number powers of the differentiation operator D and the integration operator J, as well as the creation of a calculus for these operators, is the subject of the mathematical analysis branch known as fractional calculus [1,2]. Furthermore, developing a calculus for such operators generalizes the classical one. The Navier–Stokes equation is a partial differential equation which is used to model the flow of incompressible fluids in fluid mechanics [3]. A model is a generalized form of the one created in the 18th century by the Swiss mathematician Leonhard Euler to describe the flow of incompressible and frictionless fluids [4]. The Navier–Stokes equation for incompressible fluid is:

$$\frac{\partial w}{\partial t} + (w.\nabla)u - \mu\nabla^2 w = -\frac{1}{\rho}\nabla P + f.$$

The fractional calculus has numerous and varied applications in the domains of engineering and science, including optics, data processing, viscoelasticity, fluid mechanics, electrochemistry, biological population models, and electromagnetics [5,6]. To better understand engineering and physical processes, fractional differential equations have been applied to simulate them. The systems that require the correct modeling of damping are modeled using fractional derivative models. In recent years, a variety of analytical and numerical approaches have been suggested in these disciplines, along with their applications to new challenges. The purpose of this Special Issue on "Fractional Calculus and its Applications in Applied Mathematics and Other Sciences" is to examine the most recent research in the areas of fractional calculus conducted by the top scholars.

The purpose of this Special Issue is to bring together top academics from a variety of engineering disciplines, including applied mathematicians, and give them a forum to present their creative research. The primary focus of the article includes analytical and numerical methods with cutting-edge mathematical modeling and new advancements in differential and integral equations of arbitrary order originating in physical systems.

In [7], the authors introduced the concept of a viscous fluid equation of motion. At this point, it might be beneficial to investigate the nature of this equation a little more. The Navier–Stokes equation, which was independently developed by two of the greatest applied mathematicians of the nineteenth century, is known when the forces operating in or on the fluid are viscosity, gravity, and pressure. To derive the spatial distributions of velocities, pressures, and shear stresses in a particular flow issue, the Navier–Stokes equation must be solved. The main causes are that the viscous-force phrase contains second derivatives, which are the derivatives of derivatives, and that the acceleration term is nonlinear, which means that it comprises products of partial derivatives. The Navier–Stokes equation can only be analytically solved in a few rare cases where one or both of these terms can be omitted or simplified. However, with today's powerful computers, numerical solutions to the entire Navier–Stokes equation are now possible for a considerably larger variety of flow situations. The Navier–Stokes equation for incompressible fluid is

$$\begin{cases} \dfrac{\partial w}{\partial t} + (w.\nabla)u - \mu\nabla^2 w = -\dfrac{1}{\rho}\nabla P + f, \\ \nabla.w = 0. \end{cases}$$

A gravity current, also known as a density current, is a largely horizontal flow in a gravitational field that is caused by a difference in the densities of one or more fluids and is confined to horizontal flow, such as a ceiling. The irresistible momentum assumptions made about the Cauchy stress tensor lead to the Navier–Stokes equation: the stress is Galilean invariant, meaning that it only depends on the spatial derivatives of the flow velocity rather than the flow velocity itself. In a purely mathematical sense, the Navier–Stokes equations are also extremely interesting. It has not yet been established whether smooth solutions always exist in three dimensions, i.e., whether they are infinitely differentiable (or even just limited) at all locations in the domain. This is despite the fact that they have a wide range of practical applications [8]. The Navier–Stokes existence and smoothness problem refer to this. Niazi et al. [9], Shafqat et al. [10], Alnahdi [11], Khan [12] and Abuasbeh et al. [13] investigated the existence and uniqueness the fractional evolution equations. The main effort on time-fractional Navier–Stokes equations has been dedicated into attempting to derive numerical solutions and analytical solutions; see Ganji et al. [14] and Momani and Zaid [15]. However, to the best of our knowledge, there are very few results on the existence and regularity of mild solutions for time-fractional Navier–Stokes equations. Recently, Carvalho-Neto [16] dealt with the existence and uniqueness of global and local mild solutions for the time-fractional Navier–Stokes equations. In this paper, inspired by the prior explanation, we analyze the time-fractional Navier–Stokes equations having a smooth boundary $\Omega = \mathcal{K}$ in $\mathbb{R}^n$ for $(n \geq 3)$:

$$\begin{cases} \partial_t^\beta w - \nu\Delta w + (w.\nabla)w = -\nabla p + g, \ t > 0, \\ \nabla.w = 0, \\ w|_{\partial\mathcal{K}} = 0, \\ w(t,x) = axcos\beta + btsin\beta, \end{cases}$$

where the Caputo fractional derivative of order $\beta \in (0,1)$ is denoted by ∂_t^β. Motion is gravitational, where the velocity field at a point $x \in \mathcal{K}$ and time $t > 0$ is denoted by $w = (w_1(t,x), w_2(t,x), w_3(t,x), \ldots, w_n(t,x))$, the pressure term is denoted by $p = p(t,x)$, the kinematic viscosity shows by symbol ν, t represents the time, and $a = a(x)$ signifies the starting velocity. The smooth boundary is $\mathcal{K}$. This model was extended by substituting a fractional derivative of order β, and $0 < \beta \leq 1$ for the first-time derivative. Since the flow of fluid due to gravity is tilted at an angle of $45°$,

$$w(0,x) = axcos45° = \frac{ax}{\sqrt{2}}$$

$$\begin{cases} \partial_t^\beta w - \nu\Delta w + (w.\nabla)w = -\nabla P + g, & t > 0, \\ \nabla.w = 0, \\ w|_{\partial\mathcal{K}} = 0, \\ w(0,x) = \dfrac{ax}{\sqrt{2}}. \end{cases} \tag{1}$$

Here are some properties, before applying the Leray projector on (1),

$$\begin{aligned} Aw &= -\nu P_L(\Delta w), \\ P_L(w) &= w, \\ P_L(\frac{\partial w}{\partial t}) &= \frac{dw}{dt}, \\ P_L(\nabla P) &= 0, \\ F(w) = F(w,w) &= -P_L(w.\nabla)w. \ (Bilinear\ property) \end{aligned}$$

Consider

$$\begin{cases} \partial_t^\beta w - \nu\Delta w + (w.\nabla)w = -\nabla P + g, \ \ t > 0, \\ \nabla.w = 0, \end{cases}$$

$$\begin{aligned} P_L(\partial_t^\beta w) - P_L(\nu\Delta w) &= -P_L(w.\nabla)w + P_L(-\nabla P) + P_L(g) \\ {}^C D_t^\beta w(t) - \nu P_L(\Delta u) &= F(u,\nu) + 0 + P_L(g) \\ {}^C D_t^\beta w(t) + A(w) &= F(w,w) + P_L(g) \\ {}^C D_t^\beta w(t) + A(w) &= F(w,w) + P_L(g). \end{aligned}$$

Now, (NSE) is converted into a time-fractional model using the Helmholtz–Leray projector P_L to Equation (1). The operator $-\nu P\Delta$ with Dirichlet boundary conditions is simply the same operator A, just like in the divergence-free function space under consideration. We then write (1) in its abstract sense, which is

$$\begin{cases} {}^C D_t^\beta w(t) = -Aw + F(w,w) + P_L(g), \ \ t > 0, \\ w(0) = \dfrac{ax}{\sqrt{2}}, \end{cases} \tag{2}$$

where $F(w,\nu) = -P_L(w.\nabla)\nu$. If the Stokes function A and the Helmholtz–Leray projection P_L resemble each other, then the Equation (2) has a similar solution to that of Equation (1). For convenience, we simply write P instead of P_L. The goal of this paper was to demonstrate the presence and distinction of the moderate global and local problem solutions of (2) in $H^{\delta,r}$. Additionally, we demonstrate the regularity findings, which indicate that there is just one classical solution if Pg is Hölder continuous. For Aw and ${}^C, Dt^\beta w(t)$ is Hölder continuous in J_r and w(t) has to be the such solution.

2. Preliminaries

In this section, we set the representations, definitions, and introductory information that will be used throughout the research [17]. Let $\Omega = \mathcal{K} = \{(x_1, x_2, \ldots, x_n) : x_n > 0\}$ be the open subset of $\mathbb{R}^n$, where $n \geq 3$. Let $1 < r < \infty$, then there is the Hodge projection on $(L^r(\mathcal{K}))^n$, which is a bounded projection P, whose range is the closure of

$$\mathcal{C}_\sigma^\infty(\mathcal{K}) := \{w \in (\mathcal{C}^\infty(\mathcal{K})^n) : \nabla.w = 0, \ w\ has\ compact\ subspace\ in\ \mathcal{K}\},$$

and whose null space is the closure of

$$\{w \in (\mathcal{C}^\infty(\mathcal{K}))^n : w = \nabla\eta, \eta \in \mathcal{C}^\infty(\mathcal{K})\}.$$

In order to make the notation clear, let $J_r := \overline{\mathcal{C}_\sigma^\infty(\mathcal{K})}^{|.|_r}$, which is a closed subspace of $(L^r(\mathcal{K}))^n$. Furthermore, $(Z^{m,r}(\mathcal{K}))^n$ is a Sobolev space with the norm $|.|_{m,r}$.
$A = -\nu P\Delta$ denotes the Stokes operator in J_r whose domain is $D_r(A) = D_r(\Delta) \cap J_r$; here,

$$D_r(\Delta) = \{w \in (Z^{2,r}(\mathcal{K}))^n : w|_{\partial\mathcal{K}} = 0\}.$$

The bounded analytic semigroup $\{e^{-tA}\}$ on J_r is generated by $-A$, which is a closed linear operator. To present our findings, we must first define the fractional power spaces associated with $-A$. For $\delta > 0$ and $w \in J_r$, describe

$$A^{-\delta}w = \frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta-1}e^{-tA}w dt.$$

Then, $A^{-\delta}$ is a one-to-one bounded operator on J_r. Let A^δ be the inverse of $A^{-\delta}$, and we denote the space $H^{\delta,r}$ by the range of $A^{-\delta}$ with the norm for $\delta > 0$.

$$|w|_{H^{\delta,r}} = |A^\delta w|_r.$$

Checking that e^{-tA} is extended (or restricted) to a bound analytic semigroup on $H^{\delta,r}$ is simple. Let X be a Banach space and J be a $\mathbb{R}$ interval. The set of all continuous X-valued functions is denoted by $\mathcal{C}(J, X)$. $\mathcal{C}^\vartheta(J, X)$ represents the set of all Hölder continuous functions with the exponent ϑ, for $0 < \vartheta < 1$. Let $\beta \in (0, 1]$ and $\nu : [0, \infty) \to X$. The fractional integral of order β for a function ν with a lower limit of zero is defined as

$$I_t^\beta \nu(t) = \int_0^t g_\beta(t-s)\nu(s)ds, \ t > 0.$$

Assuming that the right-hand-side is pointwise defined on $[0, \infty)$, where g_a denotes the Riemann–Liouville kernel

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \ t > 0.$$

Furthermore, as the Caputo fractional derivative operator of order β is represented by ${}^C D_t^\beta$, it is defined by

$${}^C D_t^\beta \nu(t) = \frac{d}{dt}[I_t^{1-\beta}(\nu(t) - \nu(0))] = \frac{d}{dt}\left(\int_0^t g_{1-\beta}(t-s)(\nu(t) - \nu(0))ds\right), \ t > 0.$$

More generally, for $w : [0, \infty] \times \mathbb{R}^n \to \mathbb{R}^n$, the Caputo fractional derivative concerning the time of function w can be written as

$$\partial_t^\beta w(t, x) = \partial_t\left(\int_0^t g_{1-\beta}(t-s)(w(t, x) - w(0, x))ds\right), \ t > 0.$$

Let us look into Mittag–Leffler's special functions in general:

$$\begin{aligned} E_\beta(-t^\beta A) &= \int_0^\infty \bar{M}_\beta(s)e^{-st^\beta A}ds, \\ E_{\beta,\beta}(-t^\beta A) &= \int_0^\infty \beta s\bar{M}_\beta(s)e^{-st^\beta A}ds, \end{aligned}$$

where $\bar{M}_\beta(\lambda)$ denotes the Mainardi–Wright type function, which is defined by

$$\bar{M}_\beta(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^m}{m!\Gamma(1-\beta(1+m))}.$$

Proposition 1.

(i) $E_{\beta,\beta}(-t^\beta A) = \frac{1}{2\pi i}\int_{\Gamma\theta} E_{\beta,\beta}(-t^\beta \mu)(\mu I + A)^{-1} d\mu;$

(ii) $A^\alpha E_{\beta,\beta}(-t^\beta A) = \frac{1}{2\pi i}\int_{\Gamma\theta} \mu^\alpha E_{\beta,\beta}(-t^\beta \mu)(\mu I + A)^{-1} d\mu.$

Proof.

(i) In view of $\int_0^\infty \beta s \bar{M}_\beta(s) e^{-st} ds = E_{\beta,\beta}(-t)$ and Fubini theorem, we have

$$\begin{aligned} E_{\beta,\beta}(-t^\beta A) &= \int_0^\infty \beta s \bar{M}_\beta(s) e^{-st^\beta A} ds \\ &= \frac{1}{2\pi i}\int_0^\infty \beta s \bar{M}_\beta(s) \int_{\Gamma\theta} e^{-\mu s t^\beta}(\mu I + A)^{-1} d\mu ds \\ &= \frac{1}{2\pi i}\int_{\Gamma\theta} E_{\beta,\beta}(-t^\beta \mu)(\mu I + A)^{-1} d\mu, \end{aligned}$$

here, Γ_θ is the appropriate integral route.

(ii) Similarly

$$\begin{aligned} A^\alpha E_{\beta,\beta}(-t^\beta A) &= \int_0^\infty \beta s \bar{M}_\beta(s) A^\alpha e^{-st^\beta A} ds \\ &= \frac{1}{2\pi i}\int_0^\infty \beta s \bar{M}_\beta(s) \int_{\Gamma\theta} \mu^\alpha e^{-\mu s t^\beta}(\mu I + A)^{-1} d\mu ds \\ &= \frac{1}{2\pi i}\int_{\Gamma\theta} \mu^\alpha E_{\beta,\beta}(-\mu t^\beta)(\mu I + A)^{-1} d\mu. \end{aligned}$$

□

The outcomes that follow are similar.

Lemma 1 ([18])**.** *The operators $E_\beta(-t^\beta A)$ and $E_{\beta,\beta}(-t^\beta A)$ are continuous for $t > 0$, in a uniform operator topology. Moreover, the uniformly continuous on $[r, \infty)$ for $r > 0$.*

Lemma 2 ([19])**.** *Take $0 < \beta < 1$ which implies:*

(i) $\forall\, w \in X$, $\lim_{t\to 0^+} E_\beta(-t^\beta A)w = w;$

(ii) $\forall\, w \in D(A)$ *and* $t > 0$, ${}^C D_t^\beta E_\beta(-t^\beta A)w = -AE_\beta(-t^\beta A)w;$

(iii) $\forall\, w \in X$, $E'_\beta(-t^\beta A)w = -t^{\beta-1} A E_{\beta,\beta}(-t^\beta A)w;$

(iv) $\forall\, t > 0$, $E_\beta(-t^\beta A)w = I_t^{1-\beta}\left(t^{\beta-1} E_{\beta,\beta}(-t^\beta A)w\right).$

We discuss the lemma below for the function $h : [0, \infty) \to X$, before discussing the notion of a mild solution to the issue (2). For details, we refer the reader to [20].

Lemma 3. *If x(t) is solution of Equation (2) for $w(0) = \frac{ax}{\sqrt{2}}$, then w(t) is given as*

$$w(t) = \frac{at}{\sqrt{2}} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\big(-Aw(s) + F(w(s), w(s)) + Pg\big) ds, \ for\ t \geq 0, \quad (3)$$

holds, then

$$\begin{aligned} w(t) &= \frac{at}{\sqrt{2}} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(-Aw(s))ds \\ &+ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F(w(s),w(s))ds + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}Pgds. \end{aligned}$$

Taking the Laplace on both sides

$$w(\lambda) = \frac{a}{\sqrt{2}\lambda^2} + \frac{1}{\lambda^\beta}[-Aw(\lambda)] + \frac{1}{\lambda^\beta}F(w,w(\lambda)) + \frac{1}{\lambda^\beta}Pg(\lambda).$$

Multiplying by λ^β on both sides

$$\begin{aligned} \lambda^\beta w(\lambda) &= \frac{a}{\sqrt{2}}\lambda^{\beta-2} + [-Aw(\lambda)] + F(w,w(\lambda)) + Pg(\lambda) \\ (\lambda^\beta + A)w(\lambda) &= \frac{a}{\sqrt{2}}\lambda^{\beta-2} + F(w,w(\lambda)) + Pg(\lambda) \\ w(\lambda) &= (\lambda^\beta + A)^{-1}\frac{a}{\sqrt{2}}\lambda^{\beta-2} + (\lambda^\beta + A)^{-1}F(w,w(\lambda)) + (\lambda^\beta + A)^{-1}Pg(\lambda). \end{aligned}$$

Taking the Laplace inverse on both sides

$$\begin{aligned} w(t) &= \frac{a}{\sqrt{2}}\int_0^t E_\beta(-(t-s)^\beta A)ds + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)F(w(s),w(s))ds \\ &+ \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)Pg(s)ds. \end{aligned}$$

Definition 1. *A function $w : [0,\infty) \to H^{\delta,r} or (J_r)$ is termed a global mild solution of problem (2) in $H^{\delta,r}$, if $w \in C([0,\infty), H^{\delta,r})$ and $t \in [0,\infty)$,*

$$\begin{aligned} w(t) &= \frac{a}{\sqrt{2}}\int_0^t E_\beta(-(t-s)^\beta A)ds + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)F(w(s),w(s))ds \\ &+ \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)Pg(s)ds. \end{aligned} \tag{4}$$

Definition 2. *Let $0 < \tilde{T} < \infty$. If $w \in C([0,\tilde{T}], H^{\delta,r})$ or $C([0,\tilde{T}], H^{\delta,r})$ and w and satisfy (4) for $t \in [0,\tilde{T}]$. A function $w : [0,\tilde{T}] \to H^{\beta,r}$ or (J_r) is called a local mild solution of problem (2) in $H^{\beta,r}$ or (J_r).*

We define three operators for case ξ, η, ϕ,

$$\begin{aligned} \xi(t) &= \frac{a}{\sqrt{2}}\int_0^t E_\beta(-(t-s)^\beta A)ds, \\ \eta(t) &= \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)Pg(s)ds, \\ \phi(w,\nu)(t) &= \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)F(w(s),\nu(s)))ds. \end{aligned}$$

Definition 3. *A non-negative measurable function f is defined on measurable set $\mathcal{E}$ is said to be integrable if, $\int_{\mathcal{E}} f < \infty$.*

Definition 4. *Let g be integrable over $\mathcal{E}$ and let $< f_n >$ be a sequence of measurable function such that $|f_n| \le g$ on $\mathcal{E}$ and $\lim_{n\to\infty} f_n = f$ i.e.,*

$$\lim_{n\to\infty}\int_{\mathcal{E}} f_n = \int_{\mathcal{E}} f.$$

Clearly, every f_n is integrable on $\mathcal{E}$ and furthermore, it follows that $\lim_{n\to\infty} f_n = f$ on $\mathcal{E}$ and $|f_n| \leq g$ on $\mathcal{E}$ that $|f| \leq g$ and hence f is integrable on $\mathcal{E}$.

Lemma 4. *Let $(X, ||.||_X)$ denote the Banach space, and let a bilinear operator which is defined as $G : X \times X \to X$ and a positive real number L such that*

$$||G(u,v)||_X \leq L||u||_X||v||_X, \quad \forall u, v \in X.$$

Then, for any $u_0 \in X$ with $||u_0||_X < \frac{1}{4L}$, there is just unique solution $u \in X$ to the equation $u = u_0 + G(u,u)$.

3. Global and Local Existence in $H^{\delta,r}$

For the existence and unique property of a mild solution to the situation (2) in $H^{\delta,r}$, we provide adequate conditions. For this, we suppose that:
(e) For $t > 0$, Pg is continuous and $|pg(t)|_r = 0(t^{-\beta(1-\delta})$ for $0 < \delta < 1$ as $t \to 0$.

Lemma 5. *Let $1 < r < \infty$ and $\delta_1 \leq \delta_2$. Then, the existence of the constant is $\mathcal{C} = \mathcal{C}(\delta_1, \delta_2)$ such that*

$$|e^{-tA}v|_{H^{\delta_2,r}} \leq \mathcal{C}t^{-(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}}, \ t > 0,$$

for $v \in H^{\delta_1,r}$. Furthermore,

$$\lim_{t\to 0} t^{(\delta_2-\delta_1)}|e^{-tA}v|_{H^{\delta_2,r}} = 0.$$

We will now examine a fundamental lemma that will enable us to demonstrate the final major theorems of this section.

Lemma 6. *Let $1 < r < \infty$ and also $\delta_1 \leq \delta_2$. Then, for any $\tilde{T} > 0$, there is a constant $\mathcal{C}_1 = \mathcal{C}_1(\delta_1, \delta_2) > 0$ such that*

$$|E_\beta(-t^\beta A)|_{H^{\delta_2,r}} \leq \mathcal{C}_1 t^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}}, \quad \text{and} \quad |E_{\beta,\beta}(-t^\beta A)|_{H^{\delta_2,r}} \leq \mathcal{C}_1 t^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}}$$

for all $v \in H^{\delta_1,r}$ and $t \in (0, \tilde{T}]$. Furthermore, $\lim_{t\to 0} t^{\beta(\delta_2-\delta_1)}|E_\beta(-t^\beta A)v|_{H^{\delta_2,r}} = 0$.

Proof. Let $v \in H^{\delta_1,r}$. By the previous Lemma 5, we find that

$$\begin{aligned} |E_\beta(-t^\beta A)v|_{H^{\delta_2,r}} &\leq \int_0^\infty \bar{M}_\beta(s)|e^{-st^\beta A}v|_{H^{\delta_2,r}} ds \\ &\leq \left(\mathcal{C}\int_0^\infty \bar{M}_\beta(s)s^{-(\delta_2-\delta_1)}ds\right)t^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}} \\ &\leq \mathcal{C}_1 t^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}}. \end{aligned}$$

Additionally, the dominated convergence theorem of Lebesgue demonstrates that

$$\lim_{t\to 0} t^{\beta(\delta_2-\delta_1)}|E_\beta(-t^\beta A)v|_{H^{\delta_2,r}} \leq \int_0^\infty \bar{M}_\beta(s) \lim_{t\to 0} t^{\beta(\delta_2-\delta_1)}|e^{-st^\beta A}v|_{H^{\beta_2,r}} ds = 0.$$

Similarly,

$$\begin{aligned} |E_{\beta,\beta}(-t^\beta A)v|_{H^{\delta_2,r}} &\leq \int_0^\infty \beta s\bar{M}_\beta(s)|e^{-st^\beta A}v|_{H^{\delta_2,r}} ds \\ &\leq \left(\beta\mathcal{C}\int_0^\infty \bar{M}_\beta(s)s^{1-(\delta_2-\delta_1)}ds\right)t^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}} \\ &\leq \mathcal{C}_1 t^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}}, \end{aligned}$$

where constant $\mathcal{C}_1 = \mathcal{C}_1(\beta, \delta_1, \delta_2)$ is

$$\mathcal{C}_1 \geq \mathcal{C}\ max\left\{\frac{\Gamma(1-\delta_2+\delta_1)}{\Gamma(1+\beta(\delta_1-\delta_2))}, \frac{\beta\Gamma(2-\delta_2+\delta_1)}{\Gamma(1+\beta(1+\delta_1-\delta_2))}\right\}.$$

□

4. Global Existence in $H^{\delta,r}$

No portion of the above section of this article deals with the existence of a global mild solution to the problem (2) in $H^{\delta,r}$. We let

$$\begin{aligned}
\mathcal{M}(t) &= \sup_{s\in(0,t]} \{s^{\beta(1-\delta)}|pg(s)|_r\}, \\
\mathcal{B}_1 &= \mathcal{C}_1\ max\{\mathcal{B}(\beta(1-\delta)), 1-(\beta(1-\delta)), \mathcal{B}(\beta(1-\alpha), 1-\beta(1-\delta))\}, \\
L &\geq \mathcal{M}\mathcal{C}_1\ max\left\{\mathcal{B}(\beta(1-\delta)), 1-2\beta(\alpha-\delta)), \mathcal{B}(\beta(1-\alpha), 1-2\beta(\alpha-\delta))\right\}.
\end{aligned}$$

We assume that $\mathcal{M}$ is provided afterward.

Theorem 1. *Let $1 < r < \infty$, $0 < \delta < 1$ and (e) hold. For all $\dfrac{at}{\sqrt{2}} \in H^{\delta,r}$. Suppose that*

$$\mathcal{C}_1\left|\frac{at}{\sqrt{2}}\right|_{H^{\delta,r}} + \mathcal{B}_1\mathcal{M}_\infty < \frac{1}{4L}. \tag{5}$$

The above mention $\mathcal{M}_\infty$ is defined as $\mathcal{M}_\infty := \sup_{s\in(0,\infty)} \{s^{\beta(1-\delta)}Pg(s)\}$. If $\frac{n}{2r} - \frac{1}{2} < \delta$, then there is a $\alpha > max\{\delta, \frac{1}{2}\}$ and a function which is unique $w : [0,\infty) \to H^{\delta,r}$ helps to satisfy:

(a) *The function $w : [0,\infty) \to H^{\delta,r}$ is continuous and $w(0) = \frac{at}{\sqrt{2}}$;*
(b) *The function $w : (0,\infty) \to H^{\alpha,r}$ is continuous and $lim_{t\to 0} t^{\beta(\alpha-\delta)}|w(t)|_{H^{\alpha,r}} = 0$;*
(c) *w satisfy (4) for $t \in [0,\infty)$.*

Proof. Take $\alpha = \frac{1+\delta}{2}$. Here, X_∞ is a space containing all the well-defined curves $u : (0,\infty) \to H^{\delta,r}$, also $X_\infty = X[\infty]$ and the X_∞ term is a complete as well as non-empty metric space:

(i) The continuous and bounded function is $w : [0,\infty) \to H^{\delta,r}$;
(ii) Furthermore, the function is also continuous and bounded $w : (0,\infty) \to H^{\alpha,r}$, moreover,

$$\lim_{t\to 0} t^{\beta(\alpha-\delta)}|w(t)|_{H^{\alpha,r}} = 0;$$

having a fundamental norm

$$||w||_{X_\infty} = max\left\{\sup_{t\geq 0}|w(t)|_{H^{\delta,r}}, \sup_{t\geq 0} t^{\beta(\alpha-\delta)}|w(t)|_{H^{\alpha,r}}\right\}.$$

Since it is clear that the mapping $\mathcal{F} : H^{\alpha,r} \times H^{\alpha,r} \to J_r$ is a well defined, bounded as well as bilinear mapping because of a Weissler argument, hence ∃ $\mathcal{M}$ in such a way that for $w, \nu \in H^{\alpha,r}$,

$$\begin{aligned}
|\mathcal{F}(w,\nu)|_r &\leq \mathcal{M}|w|_{H^{\alpha,t}}|\nu|_{H^{\alpha,r}}, \\
|\mathcal{F}(w,w) - \mathcal{F}(\nu,\nu)|_r &\leq \mathcal{M}(|w|_{H^{\alpha,r}} + |\nu|_{H^{\alpha,r}})|w-\nu|_{H^{\alpha,r}}.
\end{aligned} \tag{6}$$

Step I
Let us suppose that $w, \nu \in X_\infty$. Here, the operator $\phi(w(t), \nu(t)) \in \mathcal{C}([0,\infty), H^{\delta,r})$ as well

as the operator $\phi(w(t), \nu(t)) \in \mathcal{C}((0,\infty), H^{\alpha,r})$. Consider $t < t_0$ as completely arbitrary $t_0 \geq 0$ is fixed and $\varepsilon > 0$ is very small (the situation it follows is related). There are

$$\begin{aligned}
&\left|\phi(w(t), \nu(t)) - \phi(w(t_0), \nu(t_0))\right|_{H^{\delta,r}} \\
\leq\; & \int_{t_0}^{t} (t-s)^{\beta-1} |E_{\beta,\beta}(-(t-s)^{\beta} A) F(w(s), \nu(s))|_{H^{\delta,r}} ds \\
+\; & \int_{0}^{t_0} |((t-s)^{\beta-1} - (t_0-s)^{\beta-1}) E_{\beta,\beta}(-(t-s)^{\beta} A) F(w(s), \nu(s)|_{H^{\delta,r}} ds \\
+\; & \int_{0}^{t_0-\varepsilon} (t_0-s)^{\beta-1} |\big(E_{\beta,\beta}(-(t-s)^{\beta} A) - E_{\beta,\beta}(-(t_0-s)^{\beta} A)\big) F(w(s), \nu(s)|_{H^{\delta,r}} ds \\
+\; & \int_{t_0-\varepsilon}^{t_0} (t_0-s)^{\beta-1} |E_{\beta,\beta}(-(t-s)^{\beta} A) - E_{\beta,\beta}(-(t_0-s)^{\beta} A) F(w(s), \nu(s)|_{H^{\delta,r}} ds \\
:=\; & G_{11}(t) + G_{22}(t) + G_{33}(t) + G_{44}(t).
\end{aligned}$$

Each of these four terms is estimated independently. For $G_{11}(t)$, in light of the above Lemma 6, we find

$$\begin{aligned}
G_{11}(t) &\leq \mathcal{C}_1 \int_{t_0}^{t} (t-s)^{\beta(1-\delta)-1} |F(w(s), \nu(s)|_r ds \\
&\leq \mathcal{M}\mathcal{C}_1 \int_{t_0}^{t} (t-s)^{\beta(1-\delta)-1} |w(s)|_{H^{\alpha,r}} |\nu(s)|_{H^{\alpha,r}} ds \\
&\leq \mathcal{M}\mathcal{C}_1 \int_{t_0}^{t} (t-s)^{\beta(1-\delta)-1} s^{-2\beta(\alpha-\delta)} ds \sup_{s\in[0,t]} \left\{ s^{2\beta(\alpha-\delta)} |w(s)|_{H^{\alpha,r}} |\nu(s)|_{H^{\alpha,r}} \right\} \\
&= \mathcal{M}\mathcal{C}_1 \int_{t_0/t}^{1} (1-s)^{\beta(1-\delta)-1} s^{-2\beta(\alpha-\delta)} ds \sup_{s\in[0,t]} \left\{ s^{2\beta(\alpha-\delta)} |w(s)|_{H^{\alpha,r}} |\nu(s)|_{H^{\alpha,r}} \right\}.
\end{aligned}$$

There exists $\tilde{\delta} > 0$ and from the definition of the Beta function, and $\tilde{\delta}$ is very small for $0 < t - t_0 < \tilde{\delta}$,

$$\int_{t_0/t}^{1} (1-s)^{\beta(1-\delta)-1} s^{-2\beta(\alpha-\delta)} ds \to 0.$$

Consequently, $G_{11}(t)$ approaches to 0 as $t - t_0$ approaches to 0. For $G_{12}(t)$,

$$\begin{aligned}
G_{12}(t) &\leq \mathcal{C}_1 \int_{0}^{t_0} \big((t_0-s)^{\beta-1} - (t-s)^{\beta-1}\big) (t-s)^{-\beta\delta} |F(w(s), \nu(s)|_r ds \\
&\leq \mathcal{M}\mathcal{C}_1 \int_{0}^{t_0} \big((t_0-s)^{\beta-1} - (t-s)^{\beta-1}\big) (t-s)^{-\beta\delta} s^{-2\beta(\alpha-\delta)} ds \sup_{s\in[0,t_0]} \left\{ s^{2\beta(\alpha-\delta)} |w(s)|_{H^{\alpha,r}} |\nu(s)|_{H^{\alpha,r}} \right\}.
\end{aligned}$$

It is interesting to note that

$$\begin{aligned}
&\int_{0}^{t_0} |(t_0-s)^{\beta-1} - (t-s)^{\beta-1}| (t-s)^{-\beta\delta} s^{-2\beta(\alpha-\delta)} ds \\
&\leq \int_{0}^{t_0} (t-s)^{\beta-1} (t-s)^{-\beta\delta} s^{-2\beta(\alpha-\delta)} ds + \int_{0}^{t_0} (t_0-s)^{\beta-1} (t-s)^{-\beta\delta} s^{-2\beta(\alpha-\delta)} ds \\
&\leq 2 \int_{0}^{t_0} (t_0-s)^{\beta(1-\delta)-1} s^{-2\beta(\alpha-\delta)} ds \\
&= 2\mathcal{B}(\beta(1-\delta), 1 - 2\beta(\alpha-\delta)).
\end{aligned}$$

Thus, by theorem (LDC), there are

$$\int_{0}^{t_0} (t_0-s)^{\beta-1} - (t-s)^{\beta-1} (t-s)^{-\beta\delta} s^{-2\beta(\alpha-\delta)} ds \to 0 \; as \; t \to t_0.$$

We conclude that a limiting value of $G_{12}(t)$ is equal to zero as t approaches t_0. Now, we move towards $G_{13}(t)$,

$$\begin{aligned} G_{13}(t) &\leq \int_0^{t_0-\varepsilon}(t_0-s)^{\beta-1}|(E_{\beta,\beta}(-(t-s)^{\beta}A)+E_{\beta,\beta}(-(t_0-s)^{\beta}A)F(w(s),\nu(s))|_{H^{\delta,r}}ds \\ &\leq \int_0^{t_0-\varepsilon}(t_0-s)^{\beta-1}((t-s)^{-\alpha\beta}+(t_0-s)^{-\alpha\beta})|F(w(s),\nu(s)|_{H^{\delta,r}}ds \\ &\leq 2\mathcal{M}\mathcal{C}_1\int_0^{t_0-\varepsilon}(t_0-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds \sup_{s\in[0,t_0]}\{s^{2\beta(\alpha-\delta)}|w(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\alpha,r}}\}. \end{aligned}$$

Using the (LDC) theorem one more time, the operator $E_{\beta,\beta}(-t^{\beta}A)$ is uniform continuous by Lemma 1, which shows

$$\begin{aligned} \lim_{t\to t_0}G_{13}(t) &= \int_0^{t_0-\varepsilon}(t_0-s)^{\beta-1}\lim_{t\to t_0}|E_{\beta,\beta}(-(t-s)^{\beta}A)-E_{\beta,\beta}(-(t_0-s)^{\beta}A)F(w(s),\nu(s))|_{H^{\delta,r}}ds \\ &= 0. \end{aligned}$$

For $G_{14}(t)$, from calculations, we find conclusions that

$$\begin{aligned} G_{14}(t) &\leq \int_{t_0-\varepsilon}^{t_0}(t_0-s)^{\beta-1}((t-s)^{-\beta\delta}+(t_0-s)^{-\beta\delta})|F(w(s),\nu(s)|_r ds \\ &\leq 2\mathcal{M}\mathcal{C}_1\int_{t_0-\varepsilon}^{t_0}(t_0-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)} \sup_{s\in[t_0-\varepsilon,t_0]}\{s^{2\beta(\alpha-\delta)}|w(s)|_{H^{\delta,r}}|\nu(s)|_{H^{\delta,r}}\} \\ &\to 0,\ as\ \varepsilon\to 0. \end{aligned}$$

From the characteristics of β-function, we find that

$$\left|\phi(w(t),\nu(t))-\phi(w(t_0),\nu(t_0))\right|_{H^{\delta,r}} \to 0,\ as\ t\to\ t_0.$$

The continuous operator $\phi(w,\nu)$ calculated in $\mathcal{C}((0,\infty),H^{\alpha,r})$ follows the same conversation as before. Thus, we skip the explanation.
Step II
This must prove that $\phi\ :\ X_\infty\times X_\infty\ \to\ X_\infty$ is bilinear as well as a continuous operator. From Lemma 6, it gives

$$\begin{aligned} |\phi(w(t),\nu(t))|_{H^{\delta,r}} &\leq \left|\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^{\beta}A)F(w(s),\nu(s)ds\right|_{H^{\delta,r}} \\ &\leq \mathcal{C}_1\int_0^t(t-s)^{\beta(1-\delta)-1}|F(w(s),\nu(s)|_r ds \\ &\leq \mathcal{M}\mathcal{C}_1\int_0^t(t-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds \sup_{s\in[0,t]}\{s^{2\beta(\alpha-\delta)}|w(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\alpha,r}}\} \\ &= \mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\delta),1-2\beta(\alpha-\delta))||w||_{X_\infty}||\nu||_{X_\infty}, \end{aligned}$$

and

$$\begin{aligned} |\phi(w(t),\nu(t))|_{H^{\alpha,r}} &\leq \left|\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^{\beta}A)F(w(s),\nu(s))ds\right|_{H^{\alpha,r}} \\ &\leq \mathcal{C}_1\int_0^t(t-s)^{\beta(1-\alpha)-1}|F(w(s),\nu(s)|_r\ ds \\ &\leq \mathcal{M}\mathcal{C}_1\int_0^t(t-s)^{\beta(1-\alpha)-1}s^{-2\beta(\alpha-\delta)}ds \sup_{s\in[0,t]}\{s^{2\beta(\alpha-\delta)}|w(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\alpha,r}}\} \\ &= \mathcal{M}\mathcal{C}_1 t^{-\beta(\alpha-\delta)}\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))||w||_{X_\infty}||\nu||_{X_\infty}. \end{aligned}$$

Hence,

$$\sup_{t\in[0,\infty)} t^{\beta(\alpha-\delta)}|\phi(w(t),\nu(t))|_{H^{\alpha,r}} \leq \mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))||w||_{X_\infty}||\nu||_{X_\infty}.$$

To be more accurate,

$$\lim_{t\to t_0} t^{\beta(\alpha-\delta)}|\phi(w(t),\nu(t))|_{H^{\alpha,r}} = 0.$$

Thus, $\phi(w,\nu)$ *belongs to* X_∞ and $||\phi(w(t),\nu(t))||_{X_\infty} \leq L||w||_{X_\infty}||\nu||_{X_\infty}$.
Step III
Let $0 \leq t \leq t_0$. Since

$$\begin{aligned}
& |\eta(t)-\eta(t_0)|_{H^{\delta,r}} \\
\leq\ & \int_{t_0}^{t}(t-s)^{\beta-1}|E_{\beta,\beta}(-(t-s)^{\beta}A)Pg(s)|_{H^{\delta,r}}ds \\
+\ & \int_0^{t_0}((t_0-s)^{\beta-1}-(t-s))^{\beta-1}|E_{\beta,\beta}(-(t-s)^{\beta}A)Pg(s)|_{H^{\delta,r}}ds \\
+\ & \int_0^{t_0-\varepsilon}(t_0-s)^{\beta-1}|E_{\beta,\beta}(-(t-s)^{\beta}A)-E_{\beta,\beta}(-(t_0-s)^{\beta}A)Pg(s)|_{H^{\delta,r}}ds \\
+\ & \int_{t_0-\varepsilon}^{t_0}(t_0-s)^{\beta-1}|E_{\beta,\beta}(-(t-s)^{\beta}A)-E_{\beta,\beta}(-(t_0-s)^{\beta}A)Pg(s)|_{H^{\delta,r}}ds \\
\leq\ & \mathcal{C}_1\int_{t_0}^{t}(t-s)^{\beta(1-\delta)-1}|Pg(s)|_r ds \\
+\ & \mathcal{C}_1\int_0^{t_0}\big((t_0-s)^{\beta-1}-(t-s)^{\beta-1}\big)(t-s)^{-\beta\delta}|Pg(s)|_r ds \\
+\ & \mathcal{C}_1\int_0^{t_0-\varepsilon}(t_0-s)^{\beta-1}|E_{\beta,\beta}(-(t-s)^{\beta}A)-E_{\beta,\beta}(-(t_0-s)^{\beta}A)Pg(s)|_{H^{\delta,r}}ds \\
+\ & 2\mathcal{C}_1\int_{t_0-\varepsilon}^{t_0}(t_0-s)^{\beta(1-\delta)-1}|Pg(s)|_r ds
\end{aligned}$$

$$\begin{aligned}
\leq\ & \mathcal{M}(t)\mathcal{C}_1\int_{t_0}^{t}(t-s)^{\beta(1-\delta)-1}s^{-\beta(1-\delta)}ds \\
+\ & \mathcal{M}(t)\mathcal{C}_1\int_0^{t_0}((t_0-s)^{\beta-1}-(t-s)^{\beta-1})(t-s)^{-\beta\delta}s^{-\beta(1-\delta)}ds \\
+\ & \mathcal{M}(t)\mathcal{C}_1\int_0^{t_0-\varepsilon}(t_0-s)^{\beta-1}|E_{\beta,\beta}(-(t-s)^{\beta}A)-E_{\beta,\beta}(-(t_0-s)^{\beta}A)s^{-\beta(1-\delta)}ds \\
+\ & 2\mathcal{M}(t)\mathcal{C}_1\int_{t_0-\varepsilon}^{t_0}(t_0-s)^{\beta(1-\delta)-1}s^{-\beta(1-\delta)}ds.
\end{aligned}$$

From the result of Lemma 1 as well as the property of β function, the first, second, and third and final integral approaches to 0 as $t \to t_0$ as ε *tends* 0, which suggests

$$|\eta(t)-\eta(t_0)|_{H^{\delta,r}} \ \to\ 0 \ \ as \ \ t\to t_0.$$

We evaluated that $\eta(t)$ is continuous in $H^{\alpha,r}$ which implies a similarly prior explanation:

$$\begin{aligned}
|\eta(t)|_{H^{\delta,r}} &\leq \left|\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^{\beta}A)Pg(s)ds\right|_{H^{\delta,r}} \\
&\leq \mathcal{C}_1\int_0^t(t-s)^{\beta(1-\delta)-1}\big|Pg(s)\big|_r ds \\
&\leq \mathcal{M}(t)\mathcal{C}_1\int_0^t(t-s)^{\beta(1-\delta)-1}s^{-\beta(1-\delta)}ds \\
&= \mathcal{M}(t)\mathcal{C}_1\mathcal{B}(\beta(1-\delta),1-\beta(1-\delta)),
\end{aligned} \tag{7}$$

as well as

$$\begin{aligned}
|\eta(t)|_{H^{\alpha,r}} &\leq \left| \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A) Pg(s) ds \right|_{H^{\alpha,r}} \\
&\leq \mathcal{C}_1 \int_0^t (t-s)^{\beta(1-\alpha)-1} \left| Pg(s) \right|_r ds \\
&\leq \mathcal{M}(t) \mathcal{C}_1 \int_0^t (t-s)^{\beta(1-\alpha)-1} s^{-\beta(1-\delta)} ds \\
&= t^{-\beta(\alpha-\delta)} \mathcal{M}(t) \mathcal{C}_1 \mathcal{B}(\beta(1-\alpha), 1-\beta(1-\delta)).
\end{aligned}$$

Specifically,

$$t^{\beta(\alpha-\delta)} |\eta(t)|_{H^{\alpha,r}} \leq \mathcal{M}(t) \mathcal{C}_1 \mathcal{B}(\beta(1-\alpha), 1-\beta(1-\delta)) \to 0, \ as\ t \to 0.$$

As we know that, if $t \to 0$, then $\mathcal{M}(t) \to 0$. As a result of an assumption (e), it is guaranteed that $\eta(t) \in X_\infty$ and
$||\eta(t)||_\infty \leq \mathcal{B}_1 \mathcal{M}_\infty$.
For $\frac{at}{\sqrt{2}} \in H^{\delta,r}$, and from the statement of Lemma 1, it is simple to see that

$$\begin{aligned}
E_\beta(-t^\beta A) \frac{a}{\sqrt{2}} &\in \mathcal{C}([0,\infty), H^{\delta,r}) \\
E_\beta(-t^\beta A) \frac{a}{\sqrt{2}} &\in \mathcal{C}((0,\infty), H^{\alpha,r}).
\end{aligned}$$

Therefore, also

$$\begin{aligned}
\int_0^t E_\beta(-(t-s)^\beta A) \frac{a}{\sqrt{2}} ds &\in \mathcal{C}([0,\infty), H^{\delta,r}) \\
\int_0^t E_\beta(-(t-s)^\beta A) \frac{a}{\sqrt{2}} ds &\in \mathcal{C}((0,\infty), H^{\alpha,r}).
\end{aligned}$$

From Lemma 6, which suggests that $\forall t \in (0, \tilde{T}]$,

$$\begin{aligned}
\int_0^t E_\beta(-(t-s)^\beta A) \frac{a}{\sqrt{2}} ds &\in X_\infty, \\
t^{\beta(\alpha-\delta)} \int_0^t E_\beta(-(t-s)^\beta A) \frac{a}{\sqrt{2}} ds &\in \mathcal{C}((0,\infty), H^{\alpha,r}), \\
\left|\left| \int_0^t E_\beta(-(t-s)^\beta A) \frac{a}{\sqrt{2}} ds \right|\right|_{X_\infty} &\leq \int_0^t \left|\left| E_\beta(-(t-s)^\beta A) \frac{a}{\sqrt{2}} ds \right|\right|_{X_\infty}, \\
&\leq \int_0^t \mathcal{C}_1 \left| \frac{a}{\sqrt{2}} \right|_{H^{\delta,r}} ds, \\
&= \mathcal{C}_1 \left| \frac{at}{\sqrt{2}} \right|_{H^{\delta,r}}.
\end{aligned}$$

With the help of Equation (5), the inequality gives us

$$\begin{aligned}
\left|\left| \int_0^t E_\beta(-(t-s)^\beta A) \frac{a}{\sqrt{2}} ds + \eta(t) \right|\right|_{X_\infty} &\leq \left|\left| \int_0^t E_\beta(-(t-s)^\beta A) \frac{a}{\sqrt{2}} ds \right|\right|_{X_\infty} + \left|\left| \eta(t) \right|\right|_{X_\infty} \\
&\leq \mathcal{C}_1 \left| \frac{at}{\sqrt{2}} \right|_{H^{\delta,r}} + \mathcal{B}_1 \mathcal{M}_\infty \leq \frac{1}{4L}
\end{aligned}$$

which shows the result that $\mathcal{F}$ has a unique and special fixed point.
Step IV

For the purpose of demonstrating that $w(t) \to \frac{at}{\sqrt{2}}$ in $H^{\delta,r}$, by assigning $t \to 0$. We must demonstrate that:

$$\begin{aligned}
\lim_{t\to 0}\xi(t) &= \lim_{t\to 0}\frac{a}{\sqrt{2}}\int_0^t E_\beta(-t^\beta A)ds = 0,\\
\lim_{t\to 0}\eta(t) &= \lim_{t\to 0}\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)Pg(s)ds = 0,\\
\lim_{t\to 0}\phi(w,v)(t) &= \lim_{t\to 0}\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)F(w(s),w(s))ds = 0
\end{aligned}$$

in $H^{\delta,r}$. It is understood that $\lim_{t\to t_0}\eta(t) = 0$ and $\lim_{t\to t_0}\mathcal{M}(t) = 0$ with (7). Additionally,

$$\begin{aligned}
&\left|\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)F(w(s),w(s))ds\right|_{H^{\delta,r}}\\
&\le \mathcal{C}_1\int_0^t (t-s)^{\beta(1-\delta)-1}\left|F(w(s),w(s))\right|_r ds
\end{aligned}$$

$$\begin{aligned}
&\le \mathcal{MC}_1\int_0^t (t-s)^{\beta(1-\delta)-1}|w(s)|^2_{H^{\alpha,r}}ds\\
&\le \mathcal{MC}_1\int_0^t (t-s)^{\beta(1-\delta)-1}|w(s)|^2_{H^{\alpha,r}}ds\\
&\le \mathcal{MC}_1\int_0^t (t-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds \sup_{s\in(0,t]}\left\{s^{2\beta(\alpha-\delta)}|(w(s)|^2_{H^{\alpha,r}}\right\}\\
&= \mathcal{MC}_1\mathcal{B}(\beta(1-\delta),1-2\beta(\alpha-\delta)) \sup_{s\in(0,t]}\left\{s^{2\beta(\alpha-\delta)}|w(s)|^2_{H^{\alpha,r}}\right\} \to 0 \ \ as \ \ t\to 0.
\end{aligned}$$

□

5. Local Existence in $H^{\delta,r}$

This part is further separated into the local mild solution to the problem (2) in $H^{\delta,r}$.

Theorem 2. *Let $1 < r < \infty$, $0 < \delta < 1$ and (e) exist. Let us suppose that*

$$\frac{n}{2r} - \frac{1}{2} < \delta. \tag{8}$$

The existence of a function $\alpha > max\{\delta, \frac{1}{2}\}$ is such that for all $\frac{at}{\sqrt{2}} \in H^{\delta,r}$ $\exists$ $\tilde{T}_* > 0$ and a distinctive continuous function $u : [0, \tilde{T}_*] \to H^{\delta,r}$ which is constant:

(i) A continuous mapping in $H^{\delta,r}$ is defined as $w : [0, \tilde{T}_*] \to H^{\delta,r}$ with $w(0) = \frac{at}{\sqrt{2}}$;
(ii) A continuous mapping in $H^{\alpha,r}$ is defined as $w : (0, \tilde{T}_*] \to H^{\alpha,r}$ with a limiting value of function $\lim_{t\to 0} t^{\beta(\alpha-\delta)}|w(t)|_{H^{\alpha,r}} = 0$;
(iii) w holds (4) for $t \in [0, \tilde{T}_*]$.

Proof. Take $\alpha = \frac{1+\delta}{2}$ and a value $\frac{at}{\sqrt{2}} \in H^{\delta,r}$. Let us define $X_{\tilde{T}}$ as the space of curves in such a way that $w : (0, \tilde{T}] \to H^{\delta,r}$, moreover $X_{\tilde{T}} = X[\tilde{T}]$:

(a*) A continuous mapping is defined as $w : [0, \tilde{T}] \to H^{\delta,r}$;
(b*) A continuous mapping with limiting value of function is defined as $w : (0, \tilde{T}] \to H^{\alpha,r}$, with $lim_{t\to 0}t^{\beta(\alpha-\delta)}|w(t)|_{H^{\alpha,r}} = 0$;

with a norm defined by

$$||w||_X = \sup_{t\in[0,\tilde{T}]} \left\{ t^{\beta(\alpha-\delta)} |w(t)|_{H^{\alpha,r}} \right\}.$$

From the proof of Theorem 1, we notice that the operator ϕ is continuous and shows a linear map $\phi : X \times X \to X$ and $\eta(t) \in X$. From Lemma 1, we can claim $\forall\, t \in (0, \tilde{T}]$,

$$\begin{aligned} E_\beta(-t^\beta A)\frac{a}{\sqrt{2}} &\in \mathcal{C}([0,\tilde{T}], H^{\delta,r}), \\ E_\beta(-t^\beta A)\frac{a}{\sqrt{2}} &\in \mathcal{C}([0,\tilde{T}], H^{\alpha,r}), \\ \int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds &\in \mathcal{C}([0,\tilde{T}, H^{\delta,r}), \\ \int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds &\in \mathcal{C}((0,\tilde{T}, H^{\alpha,r}). \end{aligned}$$

Hence, from the previous Lemma 6, this yields that

$$\begin{aligned} \int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds &\in X, \\ t^{\beta(\alpha-\delta)} \int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds &\in \mathcal{C}([0,\tilde{T}], H^{\alpha,r}). \end{aligned}$$

$$\begin{aligned} \left|\left| \int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds \right|\right|_X &\leq \int_0^t \left|\left| E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds \right|\right|_X, \\ &\leq \int_0^t \mathcal{C}_1 \left|\frac{a}{\sqrt{2}}\right|_{H^{\delta,r}} ds, \\ &= \mathcal{C}_1 \left|\frac{at}{\sqrt{2}}\right|_{H^{\delta,r}}. \end{aligned}$$

With the help of Equation (5), the inequality gives us

$$\begin{aligned} \left|\left| \int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds + \eta(t) \right|\right|_{X[\tilde{T}_*]} &\leq \left|\left| \int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds \right|\right|_{X[\tilde{T}_*]} + \left|\left| \eta(t) \right|\right|_{X[\tilde{T}_*]} \\ &\leq \mathcal{C}_1 \left|\frac{at}{\sqrt{2}}\right|_{H^{\delta,r}} + \mathcal{B}_1\mathcal{M}_\infty \leq \frac{1}{4L}. \end{aligned}$$

This gives a result that $\mathcal{F}$ has a unique fixed point due to Lemma 4. □

6. Existence Locally in J_r

We are using the iteration method in this part intended for the thought of the local existence of a mild solution to the problem (2) in J_r. Let $\alpha = \frac{1+\delta}{2}$.

Theorem 3. *Suppose $1 < r < \infty$, $0 < \delta < 1$ and (e) holds. Let us take the value in $H^{\delta,r}$*

$$\frac{at}{\sqrt{2}} \in H^{\delta,r} \ \text{with} \ \frac{n}{2r} - \frac{1}{2} < \delta.$$

Thus, the mild solution of (2) is unique answer for $\frac{at}{\sqrt{2}} \in H^{\delta,r}$ in J_r. Additionally, $t^{\beta(\alpha-\delta)} A^\alpha w(t)$ is bounded as $t \to 0$, Moreover, w and $A^\alpha u$ are both continuous functions in $[0, \tilde{T}]$ and $(0, \tilde{T}]$, respectively.

Proof. Step I Set

$$\tilde{\kappa}(t) := \sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|A^{\alpha}w(s)|_r$$

also

$$\tilde{\psi}(t) := \phi(w,w)(t) = \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^{\beta}A)F(w(s),w(s))ds.$$

The instant results from Step II in Theorem 1 reveal that, A continuous functions $\tilde{\psi}(t)$ and $A^{\alpha}\tilde{\psi}(t)$ exist in $[0,\tilde{T}]$ and $(0,\tilde{T}]$, respectively, with the value

$$\begin{aligned}
|A^{\alpha}\tilde{\psi}(t)|_r &\leq \left|\int_0^t A^{\alpha}(t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^{\beta}A)F(w(s),w(s))ds\right|_r \\
&\leq \mathcal{C}_1\int_0^t A^{\alpha}(t-s)^{\beta(1-\alpha)-1}|F(u(s),u(s))|_r\, ds \\
&\leq \mathcal{M}\mathcal{C}_1\int_0^t A^{\alpha}(t-s)^{\beta(1-\alpha)-1}s^{-2\beta(\alpha-\delta)}\sup_{s\in(0,t]} s^{2\beta(\alpha-\delta)}|w(s)|_r|w(s)|_r\, ds \quad (9)\\
&= \mathcal{M}\mathcal{C}_1\int_0^t (t-s)^{\beta(1-\alpha)-1}s^{-2\beta(\alpha-\delta)}ds\{\sup_{s\in(0,t]} A^{\alpha}s^{2\beta(\alpha-\delta)}|w(s)|^2\} \\
|A^{\alpha}\tilde{\psi}(t)|_r &\leq \mathcal{M}\mathcal{C}_1 t^{-\beta(\alpha-\delta)}\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}^2(t).
\end{aligned}$$

We also take into account the integral $\eta(t)$. Given that (e) is true,

$$\left|Pg(s)\right|_r \leq \mathcal{M}(t)s^{\beta(1-\delta)}.$$

The above inequality with a continuous function $\mathcal{M}(t)$ holds. Theorem 1's third stage reveals that $A^{\alpha}\eta(t)$ is continuous in $(0,\tilde{T}]$ with the results

$$\begin{aligned}
|A^{\alpha}\eta(t)|_r &= \left|\int_0^t (t-s)^{\beta-1}A^{\alpha}E_{\beta,\beta}(-(t-s)^{\beta}A)Pg(s))\right|_{H^{\alpha,r}} ds \\
&\leq \mathcal{C}_1\int_0^t (t-s)^{\beta(1-\alpha)-1}\left|A^{\alpha}Pg(s)\right|_r ds \\
&\leq \mathcal{C}_1\mathcal{M}(t)\int_0^t (t-s)^{\beta(1-\alpha)-1}s^{-\beta(1-\delta)}ds \quad (10)\\
&= t^{-\beta(\alpha-\delta)}\mathcal{C}_1\mathcal{M}(t)\mathcal{B}(\beta(1-\delta),1-\beta(1-\delta)) \\
|A^{\alpha}\eta(t)|_r &\leq t^{-\beta(\alpha-\delta)}\mathcal{C}_1\mathcal{M}(t)\mathcal{B}(\beta(1-\delta),1-\beta(1-\delta)).
\end{aligned}$$

For $|Pg(t)|_r = 0(t^{-\beta(1-\delta)})$, t tends towards zero and $\mathcal{M}(t) = 0$. The above Equation (10) concludes that $|A^{\alpha}\eta(t)|_r = 0(t^{-\beta(\alpha-\delta)})$ as t approaches 0. We establish the continuity of η in J_r. In actuality, we take$0 < t_0 < t < \tilde{T}$, resulting in

$$
\begin{aligned}
|\eta(t)-\eta(t_0)|_r \leq\ & \mathcal{C}_3\int_{t_0}^{t}(t-s)^{\beta-1}\big|Pg(s)\big|_r ds+\mathcal{C}_3\int_0^{t_0}((t_0-s)^{\beta-1}-(t-s))^{\beta-1}\big|Pg(s)\big|_r ds \\
+\ & \mathcal{C}_3\int_0^{t_0-\varepsilon}(t_0-s)^{\beta-1}||E_{\beta,\beta}(-(t-s)^{\beta}A)-E_{\beta,\beta}(-(t_0-s)^{\beta}A)||\ |Pg(s)|_r ds \\
+\ & 2\mathcal{C}_3\int_{t_0-\varepsilon}^{t_0}(t_0-s)^{\beta-1}|Pg(s)|_r ds \\
\leq\ & \mathcal{C}_3 M(t)\int_{t_0}^{t}(t-s)^{\beta-1}s^{-\beta(1-\delta)}ds \\
+\ & \mathcal{C}_3\mathcal{M}(t)\int_0^{t_0}((t-s)^{\beta-1}-(t_0-s)^{\beta-1})\, s^{-\beta(1-\delta)}ds \\
+\ & \mathcal{C}_3\mathcal{M}(t)\int_0^{t_0-\varepsilon}(t_0-s)^{\beta-1}s^{-\beta(1-\delta)}ds \sup_{s\in[0,t-\varepsilon]}||E_{\beta,\beta}(-(t-s)^{\beta}A)-E_{\beta,\beta}(-(t_0-s)^{\beta}A)|| \\
+\ & 2\mathcal{C}_3\mathcal{M}(t)\int_{t_0-\varepsilon}^{t_0}(t_0-s)^{\beta-1}s^{-\beta(1-\delta)}ds \ \to\ 0,\ \ as\ t\ \to\ t_0.
\end{aligned}
$$

Additionally, we think about the function $\int_0^t E_\beta(-(t-s)^\beta A)\dfrac{a}{\sqrt{2}}\,ds$. It is clear from the Lemma 6 that

$$
\begin{aligned}
\left|\int_0^t A^{\alpha}E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds\right|_r \leq\ & \int_0^t\left|A^{\alpha}E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}\right|_r ds \\
\leq\ & C_1 t^{-\beta(\alpha-\delta)}A^{\delta}\Big|\int_0^t\frac{a}{\sqrt{2}}ds\Big|_r \\
=\ & C_1 t^{-\beta(\alpha-\delta)}\Big|\frac{a}{\sqrt{2}}t\Big|_{H^{\delta,r}}\lim_{t\to 0}t^{\beta(\alpha-\delta)}\left|\int_0^t A^{\alpha}E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds\right|_r, \\
=\ & \lim_{t\to 0}t^{\beta(\alpha-\delta)}\left|\int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds\right|_{H^{\alpha,r}} \\
=\ & 0.
\end{aligned}
$$

Step II
Now, using the successive approximation method, we create the solution:

$$
\begin{aligned}
w_0(t) &= \textstyle\int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}ds+\eta(t), \\
w_{n+1}(t) &= w_0(t)+\phi(w_n,w_n)(t),\ n=0,1,2,\ldots.
\end{aligned} \tag{11}
$$

Using the above results, we have $\tilde{\kappa}_n(t) := \sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|A^{\alpha}w_n(s)|_r$ which are continuous and increasing in $[0,\tilde{T}]$ with a value of $\tilde{\kappa}_n(0)=0$. However, given (9) and (10) and that the inequality is satisfied by $\tilde{\kappa}_n(t)$,

$$\tilde{\kappa}_{n+1}(t)\leq\tilde{\kappa}_0(t)+\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_n^2(t). \tag{12}$$

For $\tilde{\kappa}_0(0)=0$, select $\tilde{T}>0$ in such a way that

$$4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_0(\tilde{T})<1. \tag{13}$$

The sequence $\tilde{\kappa}_n(\tilde{T})$ is hence guaranteed to be bounded by fundamental consideration (12), i.e.,

$$\tilde{\kappa}_n(\tilde{T})\leq\tilde{\rho}(\tilde{T}),\ n=0,1,2,\ldots.$$

There are
$$\tilde{\rho}(t) = \frac{1-\sqrt{1-4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_0(t)}}{2\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))}.$$
It is true that $\tilde{\kappa}n(t) \le \tilde{\rho}(t)$ holds for any value of $t \in (0,\tilde{T}]$. Similarly, we say $\tilde{\rho}(t) \le 2\tilde{\kappa}_0(t)$. Let us think about equality

$$z_{n+1}(t) = \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)[F(w_{n+1}(s),w_{n+1}(s)-F(w_n(s),w_n(s))]ds,$$

for $t \in (0,\tilde{T}]$ and $z_n = u_{n+1}-u_n,\ n = 0,1,2,\ldots$. By writing,

$$Z_n(t) := \sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|A^\alpha z_n(s)|_r.$$

In light of (5), there are

$$\begin{aligned}
& \left|F(w_{n+1}(s),w_{n+1}(s)-F(w_n(s),w_n(s))\right|_r \\
\le\ & \mathcal{M}\left(|w_{n+1}|_{H^{\alpha,r}}+|w_n|_{H^{\alpha,r}}\right)|w_{n+1}-w_n|_{H^{\alpha,r}} \\
=\ & \mathcal{M}\left(|w_{n+1}|_{H^{\alpha,r}}+|w_n|_{H^{\alpha,r}}\right)A^\alpha z_n \sup_{s\in(0,t]} s^{-\beta(\alpha-\delta)}s^{\beta(\alpha-\delta)} \\
=\ & \mathcal{M}\left(|A^\alpha w_{n+1}|_r+|A^\alpha w_n|_r\right)z_n \sup_{s\in(0,t]} s^{-\beta(\alpha-\delta)}s^{\beta(\alpha-\delta)} \\
=\ & \mathcal{M}\Bigg(\sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|A^\alpha w_{n+1}|_r \\
+\ & \sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|A^\alpha w_n|_r\Bigg)z_n s^{-\beta(\alpha-\delta)} \\
\le\ & \mathcal{M}\left(\tilde{\kappa}_{n+1}+\tilde{\kappa}_n\right)z_n s^{-\beta(\alpha-\delta)} \sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}s^{-\beta(\alpha-\delta)}A^\alpha \\
\le\ & \mathcal{M}\left(\tilde{\kappa}_{n+1}+\tilde{\kappa}_n\right) \sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|A^\alpha z_n(s)|_r s^{-2\beta(\alpha-\delta)} \\
\left|F(w_{n+1}(s),w_{n+1}(s)-F(w_n(s),w_n(s))\right|_r \le\ & \mathcal{M}\left(\tilde{\kappa}_{n+1}+\tilde{\kappa}_n\right)Z_n(s)s^{-2\beta(\alpha-\delta)}.
\end{aligned}$$

This is implied by Step II in Theorem 1 that

$$t^{\beta(\alpha-\delta)}|A^\alpha z_{n+1}(t)|_r \le 2\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-\beta(1-\delta))\tilde{\rho}(\tilde{T})Z_n(t).$$

Such inequality results in

$$\begin{aligned}
Z_{n+1}(\tilde{T}) &\le 2\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\rho}(\tilde{T})Z_n(\tilde{T}) \\
&\le 4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_0(\tilde{T})Z_n(\tilde{T}).
\end{aligned} \tag{14}$$

As per (13) with (14), it is indeed obvious that

$$\lim_{n\to 0}\frac{Z_{n+1}(\tilde{T})}{Z_n(\tilde{T})} < 4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_0(\tilde{T}) < 1.$$

Thus, the convergence of the series $\Sigma_{n=0}^\infty Z_n(\tilde{T})$ implies the uniform convergence of the series $\Sigma_{n=0}^\infty t^{\beta(\alpha-\delta)}A^\alpha z_n(t)$ for $t \in (0,\tilde{T}]$; hence, the uniform convergence of the sequence $\{t^{\beta(\alpha-\delta)}A^\alpha w_n(t)\}$ holds in $(0,\tilde{T}]$. Thus,

$$\lim_{n\to\infty} w_n(t) = w(t) \in D(A^\alpha)$$

and

$$\lim_{n\to\infty} t^{\beta(\alpha-\delta)} A^{\alpha} w_n(t) = t^{\beta(\alpha-\delta)} A^{\alpha} w(t) \ uniformly.$$

According to the boundedness theorem, "A function f continuous on a bounded and closed interval is necessarily a bounded function". Therefore, both $A^{-\alpha}$ and A^{α} are bound and closed, respectively. Accordingly, the function $\tilde{\kappa}(t) = \sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|A^{\alpha}w(s)|_r$ also satisfies

$$\tilde{\kappa}(t) \le \tilde{\rho}(t) \le 2\tilde{\kappa}_0(t),\ t \in (0,t], \tag{15}$$

as well as

$$\begin{aligned} \varsigma_n &= \sup_{s\in(0,\tilde{T}]} s^{2\beta(\alpha-\delta)}|F(w_n(s), w_n(s) - F(w(s), w(s))|_r \\ &\le \mathcal{M}(\tilde{\kappa}_n(\tilde{T}) + \tilde{\kappa}(\tilde{T})) \sup_{s\in(0,\tilde{T}]} s^{\beta(\alpha-\delta)}|A^{\alpha}(w_n(s) - w(s))|_r \ \to\ 0,\ as\ n \to\ \infty. \end{aligned}$$

To conclude, it is necessary to confirm that w is a mild solution to the problem (2) in the domain of $[0, \tilde{T}]$,

$$|\phi(w_n, w_n)(t) - \phi(w, w)(t)|_r \le \int_0^t (t-s)^{\beta-1}\varsigma_n s^{-2\beta(\alpha-\delta)} ds = t^{\beta\delta}\varsigma_n\ \to\ 0,\ (n\to\infty).$$

In other words, we obtain $\phi(w_n, w_n)(t) - \phi(w, w)(t)$ as the limits on both sides of (10) are taken, and we deduce that

$$w(t) = w_0(t) + \phi(w, w)(t). \tag{16}$$

Let $w(0) = \dfrac{at}{\sqrt{2}}$, and we learn that (16) is true for both $t \in [0, \tilde{T}]$ and $w \in \mathcal{C}([0, \tilde{T}], J_r)$. Additionally, the continuity of $A^{\alpha}w(t)$ in $(0, \tilde{T}]$ is derived from the uniform convergence of $t^{\beta(\alpha-\delta)}A^{\alpha}w_n(t)$ to $t^{\beta(\alpha-\delta)}A^{\alpha}w(t)$. We conclude that $|A^{\alpha}w(t)|_r = 0(t^{-\beta(\alpha-\delta)})$ is clear from (15) and $\tilde{\kappa}_0(t) = 0$.

Step III

We demonstrate the distinction of mild solutions. Suppose that w and ν are mild solutions to the problem (2).

Let $z = w - \nu$, and consider the inequality

$$z(t) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^{\beta}A)[F(w(s), w(s) - F(\nu(s), \nu(s))]ds.$$

Expressing the solution

$$\tilde{\kappa}(t) := max\{\sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|A^{\alpha}(w(s))|_r, \sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|A^{\alpha}(\nu(s))|_r\}.$$

From (5) and Lemma 6, we have the inequality

$$|A^{\alpha}z(t)|_r \le \mathcal{M}\mathcal{C}_1\tilde{\kappa}(t)\int_0^t (t-s)^{\beta(1-\alpha)-1}s^{-\beta(\alpha-\delta)}|A^{\alpha}z(s)|_r ds.$$

The Gronwall inequality demonstrates that for $t \in (0, \tilde{T}]$, $A^{\alpha}z(t) = 0$. This indicates that, for $t \in [0, \tilde{T}]$, $z(t) = w(t) - \nu(t) \equiv 0$. The mild solution is a special result. □

7. Regularity

The regularity of a w solution that answers the problem is examined in this section (2). In this essay, we suppose that:
(e_1) With an exponent $\vartheta \in (0, \beta(1-\alpha))$, $Pg(t)$ is Hölder continuous, that is,

$$\begin{aligned} |f(x) - f(y)| &\leq ||x-y||^{\beta}, \\ \big|Pg(t) - Pg(s)\big|_r &\leq L|t-s|^{\vartheta}, \ for \ all \ 0 < t, \ s \leq \tilde{T}. \end{aligned}$$

Definition 5. *An expression* $w : [0, \tilde{T}] \to J_r$. *If* $w \in \mathcal{C}([0, \tilde{T}], J_r)$ *with* ${}^{C}D_t^{\beta} w(t) \in \mathcal{C}((0, \tilde{T}], Jr)$, *this accepts values in* $D(A)$ *and solves (2) for every* $t \in (0, \tilde{T}]$; *then,* J_r *is referred to as the classical solution of the (2).*

Lemma 7. *Let* (e_1) *be satisfied. If*

$$\eta_1(t) := \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^{\beta} A)\big(Pg(t) - Pg(s)\big) ds, \ \forall \ t \in (0, \tilde{T}],$$

then $\eta_1(t) \in D(A)$ *also* $A\eta_1(t) \in \mathcal{C}^{\vartheta}([0, \tilde{T}], J_r)$.

Proof. To be fixed, $t \in (0, \tilde{T}]$. Let us think about

$$(t-s)^{\beta-1} |AE_{\beta,\beta}(-(t-s)^{\beta} A)\big(Pg(s) - Pg(t)\big)|_r.$$

Lemma 6 with (e_1) give us

$$\begin{aligned} (t-s)^{\beta-1}\Big|AE_{\beta,\beta}(-(t-s)^{\beta}A)\big(Pg(s) - Pg(t)\big)\Big|_r &\leq \mathcal{C}_1 (t-s)^{\beta-1}(t-s)^{-\beta}\big|Pg(s) - Pg(t)\big|_r \\ &= \mathcal{C}_1 (t-s)^{-1}\big|Pg(s) - Pg(t)\big|_r \\ (t-s)^{\beta-1}\Big|AE_{\beta,\beta}(-(t-s)^{\beta}A)\big(Pg(s) - Pg(t)\big)\Big|_r &\leq \mathcal{C}_1 L (t-s)^{\vartheta-1} \in L^1([0, \tilde{T}], J_r). \end{aligned} \tag{17}$$

Then,

$$\begin{aligned} |A\eta_1(t)|_r &\leq \int_0^t (t-s)^{\beta-1}\Big|AE_{\beta,\beta}(-(t-s)^{\beta}A)\big(pg(s) - pg(t)\big)\Big|_r ds \\ &\leq \mathcal{C}_1 L \int_0^t (t-s)^{\vartheta-1} ds \leq \frac{\mathcal{C}_1 L t^{\vartheta}}{\vartheta} < \infty. \end{aligned}$$

Since A is closed, we can write $\eta_1(t) \in D(A)$.
It is necessary to demonstrate that $A\eta_1(t)$ is *Holder* continuous. Because

$$\frac{d}{dt}\big(t^{\beta-1}E_{\beta,\beta}(-\mu t^{\beta})\big) = t^{\beta-2}E_{\beta,\beta-1}(-\mu t^{\beta}).$$

Then,

$$\begin{aligned} &\frac{d}{dt}\big(t^{\beta-1}AE_{\beta,\beta}(-t^{\beta}A)\big) \\ = \ &\frac{1}{2\pi i}\int_{\Gamma_\theta} t^{\beta-2}E_{\beta,\beta-1}(-\mu t^{\beta})A(\mu I + A)^{-1} d\mu \\ = \ &\frac{1}{2\pi i}\int_{\Gamma_\theta} t^{\beta-2}E_{\beta,\beta-1}(-\mu t^{\beta}) d\mu - \frac{1}{2\pi i}\int_{\Gamma_\theta} t^{\beta-2}\mu E_{\beta,\beta-1}(-\mu t^{\beta})(\mu I + A)^{-1} d\mu. \end{aligned}$$

$$\begin{aligned} Put - \mu t^{\beta} &= \psi \\ -t^{\beta} d\mu &= d\psi \\ d\mu &= -\frac{d\psi}{t^{\beta}} \end{aligned}$$

$$= \frac{1}{2\pi i}\int_{\Gamma'_\theta} -t^{\beta-2}E_{\beta,\beta-1}(\psi)\frac{1}{t^{\beta}}d\psi - \frac{1}{2\pi i}\int_{\Gamma'_\theta} t^{\beta-2}E_{\beta,\beta-1}(\psi)\frac{\psi}{t^{\beta}}\left(\frac{-\psi}{t^{\beta}}I + A\right)^{-1}\frac{1}{t^{\beta}}d\psi.$$

In light of

$$\begin{aligned} ||(\mu I + A)^{-1}|| &\leq \frac{\mathcal{C}}{|\mu|}, \textit{ we derive that} \\ \left|\left|\frac{d}{dt}\left(t^{\beta-1}AE_{\beta,\beta}(-t^{\beta}A)\right)\right|\right| &\leq \mathcal{C}_{\beta}t^{-2},\ 0 < t < \tilde{T}. \end{aligned}$$

From MVT, we derive that $\forall\ 0 < s < t \leq \tilde{T}$, and we obtain

$$\begin{aligned} \left|\left|t^{\beta-1}AE_{\beta,\beta}(-t^{\beta}A) - s^{\beta-1}AE_{\beta,\beta}(-s^{\beta}A)\right|\right| &= \left|\left|\int_s^t \frac{d}{d\tau}\left(\tau^{\beta-1}AE_{\beta,\beta}(-\tau^{\beta}A)\right)d\tau\right|\right| \\ &\leq \int_s^t \left|\left|\frac{d}{d\tau}\left(\tau^{\beta-1}AE_{\beta,\beta}(-\tau^{\beta}A)\right)\right|\right|d\tau \\ &\leq \mathcal{C}_{\beta}\int_s^t \tau^{-2}d\tau = \mathcal{C}_{\beta}(s^{-1} - t^{-1}). \end{aligned} \tag{18}$$

Take $h > 0$ such that $0 < t < t + h \leq \tilde{T}$, then

$$\begin{aligned} & A\eta_1(t+h) - A\eta_1(t) \\ = & \int_0^t \Big((t+h-s)^{\beta-1}AE_{\beta,\beta}(-(t+h-s)^{\beta}A) \\ - & (t-s)^{\beta-1}AE_{\beta,\beta}(-(t-s)^{\beta}A)\Big)\big(Pg(s) - Pg(t)\big)ds \\ + & \int_0^t (t+h-s)^{\beta-1}AE_{\beta,\beta}(-(t+h-s)^{\beta}A)\big(Pg(t) - Pg(t+h)\big)ds \\ + & \int_t^{t+h} (t+h-s)^{\beta-1}AE_{\beta,\beta}(-(t+h-s)^{\beta}A)\big(Pg(s) - Pg(t+h)\big)ds \\ := & I_1(t) + I_2(t) + I_3(t). \end{aligned} \tag{19}$$

The three major terms are discussed here one by one. We have (18) with (e_1) for $I_1(t)$,

$$\begin{aligned} |I_1(t)|_r \leq & \int_0^t \left|\left|(t+h-s)^{\beta-1}AE_{\beta,\beta}(-(t+h-s)^{\beta}A) - (t-s)^{\beta-1}AE_{\beta,\beta}(-(t-s)^{\beta}A)\right|\right| \\ & \big|Pg(s) - Pg(t)\big|_r ds \\ \leq & \mathcal{C}_{\beta}Lh\int_0^t (t+h-s)^{-1}(t-s)^{\vartheta-1}ds \\ \leq & \mathcal{C}_{\beta}Lh\int_0^t (h+s)^{-1}(t-s)^{\vartheta-1}ds \\ \leq & \mathcal{C}_{\beta}L\int_0^t \frac{h}{s+h}s^{\vartheta-1}ds + \mathcal{C}_{\beta}Lh\int_h^{\infty}\frac{s}{s+h}s^{\vartheta-1}ds \\ \leq & \mathcal{C}_{\beta}Lh^{\vartheta}. \end{aligned} \tag{20}$$

For solving $I_2(t)$, Lemma 6 and (e_1) are used here, and we have

$$\begin{aligned} |I_2(t)|_r &\leq \int_0^t (t+h-s)^{\beta-1} \Big| AE_{\beta,\beta}(-(t+h-s)^\beta A)\big(Pg(t) - Pg(t+h)\big)\Big|_r ds \\ &\leq \mathcal{C}_1 \int_0^t (t+h-s)^{-1} \Big| \big(Pg(t) - Pg(t+h)\big)\Big|_r ds \\ &\leq \mathcal{C}_1 L h^\vartheta \int_0^t (t+h-s)^{-1} ds \\ &= \mathcal{C}_1 L [ln(h) - ln(t+h)] h^\vartheta. \end{aligned} \tag{21}$$

Moreover, for solving $I_3(t)$, Lemma 6 and (e_1) are used here, and we have

$$\begin{aligned} |I_3(t)|_r &\leq \int_t^{t+h} (t+h-s)^{\beta-1} \Big| AE_{\beta,\beta}(-(t+h-s)^\beta A)\big(Pg(s) - Pg(t+h)\big)\Big|_r ds \\ &\leq \mathcal{C}_1 \int_t^{t+h} (t+h-s)^{-1} \Big| Pg(s) - Pg(t+h)\big)\Big|_r ds \\ &\leq \mathcal{C}_1 L \int_t^{t+h} (t+h-s)^{\vartheta-1} ds = \mathcal{C}_1 L \frac{h^\vartheta}{\vartheta}. \end{aligned} \tag{22}$$

By combining all the above (20), (21) with (22), we deduce that Hölder's continuity of $A\eta_1(t)$ exists. □

Theorem 4. *Suppose that Theorem 3's assumptions are satisfied. The mild solution of Equation (2) is the classical one applicable to any* $\frac{at}{\sqrt{2}} \in D(A)$ *if* (e_1) *is true.*

Proof. From Lemma 2(ii), it is guaranteed that function $w(t) = \int_0^t E_\beta(-(t-s)^\beta A)\frac{a}{\sqrt{2}}dt \ (t > 0)$ is a classical solution for the given problem, for value $\frac{at}{\sqrt{2}} \in D(A)$:

$$\begin{cases} {}^C D_t^\beta w = -Aw, \ t > 0, \\ w(0) = \dfrac{ax}{\sqrt{2}}. \end{cases}$$

Step I
By verifying,

$$\eta(t) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A) Pg(s) ds$$

is the classical solution to the following problem

$$\begin{cases} {}^C D_t^\beta w = -Aw + Pg(t), \ t > 0, \\ u(0) = 0. \end{cases}$$

Thus, it follows from Theorem 3 that $\eta \in \mathcal{C}([0, \tilde{T}], J_r)$. By rewriting $\eta(t) = \eta_1(t)_+\eta_2(t)$, then

$$\begin{aligned} \eta_1(t) &= \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A)\big(Pg(s) - Pg(t)\big) ds, \\ \eta_2(t) &= \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A) Pg(t) ds. \end{aligned}$$

From Lemma 7, we can write $\eta_1(t) \in D(A)$. To demonstrate similar results for $\eta_2(t)$. By Lemma 2(iii), we find that

$$A\eta_2(t) = Pg(t) - E_\beta(-t^\beta A) Pg(t).$$

Since (e_1) exists, therefore

$$|A\eta_2(t)|_r \leq (1+\mathcal{C}_1)|Pg(t)|_r,$$

thus

$$\eta_2(t) \in D(A) \; for \; t \in (0,\tilde{T}] \; and \; \eta_2(t) \in \mathcal{C}^{\vartheta}((0,\tilde{T}],J_r). \tag{23}$$

Furthermore, we verify for ${}^{C}D_t^{\beta}\eta \in \mathcal{C}((0,\tilde{T}],J_r)$. In light of the Lemma 2(iv) with the condition $\eta(0)=0$, there are

$${}^{C}D_t^{\beta}\eta(t) = \frac{d}{dt}(I_t^{1-\beta}\eta(t)) = \frac{d}{dt}\bigg(E_\beta(-t^\beta A) * Pg\bigg).$$

It must still be demonstrated that $E_\beta(t^\beta A) * Pg$ in J_r is continuously differentiable.
Let $0 < h \leq \tilde{T} - t$, and we derive:

$$\begin{aligned} &\frac{1}{h}\bigg(E_\beta(-(t+h)^\beta A) * Pg - E_\beta(-t^\beta A) * Pg\bigg) \\ =& \int_0^t \frac{1}{h}\bigg(E_\beta(-(t+h-s)^\beta A)Pg(s) - E_\beta(-(t-s)^\beta A)Pg(s)\bigg)ds \\ &+\frac{1}{h}\int_t^{t+h} E_\beta(-(t+h-s)^\beta A)Pg(s)ds. \end{aligned}$$

Note that

$$\begin{aligned} &\int_0^t \frac{1}{h}\bigg|E_\beta(-(t+h-s)^\beta A)Pg(s) - E_\beta(-(t-s)^\beta A)Pg(s)\bigg|_r ds \\ \leq\;& \mathcal{C}_1\frac{1}{h}\int_0^t \bigg|E_\beta(-(t+h-s)^\beta A)Pg(s)\bigg|_r ds \\ +\;& \mathcal{C}_1\frac{1}{h}\int_0^t \bigg|E_\beta(-(t-s))^\beta A)Pg(s)\bigg|_r ds \\ \leq\;& \mathcal{C}_1\mathcal{M}(t)\frac{1}{h}\int_0^t (t+h-s)^\beta s^{-\beta(1-\delta)}ds \\ +\;& \mathcal{C}_1\mathcal{M}(t)\frac{1}{h}\int_0^t (t-s)^{-\beta} s^{-\beta(1-\delta)}ds \\ \leq\;& \mathcal{C}_1\mathcal{M}(t)\frac{1}{h}\bigg((t+h)^{1-\beta} + t^{1-\beta}\bigg)\mathcal{B}(1-\beta, 1-\beta(1-\delta)). \end{aligned}$$

We find the following result by using the LDC theorem given by Lebesgue

$$\begin{aligned} &\lim_{h\to 0}\int_0^t \frac{1}{h}\bigg(E_\beta(-(t+h-s)^\beta A)Pg(s) - E_\beta(-(t-s)^\beta A)Pg(s)\bigg)ds \\ &= -\int_0^t (t-s)^{\beta-1} A E_{\beta,\beta}(-(t-s)^\beta A)Pg(s)ds \\ &= A\eta(t). \end{aligned}$$

In contrast,

$$\begin{aligned}
&\frac{1}{h}\int_t^{t+h} E_\beta(-(t+h-s)^\beta A)Pg(s)ds \\
&= \frac{1}{h}\int_0^h E_\beta(-s^\beta A)Pg(t+h-s)ds \\
&= \frac{1}{h}\int_0^h E_\beta(-s^\beta A)\Big(Pg(t+h-s) - Pg(t-s)\Big)\Big)ds \\
&+ \frac{1}{h}\int_0^h E_\beta(-s^\beta A)Pg(t-s) - Pg(t)ds \\
&+ \frac{1}{h}\int_0^h E_\beta(-s^\beta A)Pg(t)ds.
\end{aligned}$$

Lemmas 1 and 6 with property (e_1) give us

$$\begin{aligned}
\left|\frac{1}{h}\int_0^t E_\beta(-s^\beta A)Pg(t+h-s) - Pg(t-s))ds\right|_r &\leq \mathcal{C}_1 L h^\vartheta, \\
\left|\frac{1}{h}\int_0^t E_\beta(-s^\beta A)Pg(t-s)) - Pg(t))ds\right|_r &\leq \mathcal{C}_1 L \frac{h^\vartheta}{\vartheta+1}.
\end{aligned}$$

Consequently, Lemma 2(i) offers that

$$\lim_{h\to 0}\frac{1}{h}\int_0^h E_\beta((-s)^\beta A)Pg(s)ds = Pg(t).$$

Hence,

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} E_\beta((t+h-s)^\beta A)Pg(s)ds = Pg(t).$$

Our conclusion is that $E_\beta(t^\beta A) * Pg$ is differentiable at t_+ and

$$\frac{d}{dt}\big(E_\beta(t^\beta A) * Pg\big)_+ = A\eta(t) + Pg(t).$$

Similarly, $E_\beta(t^\beta A) * Pg$ is differential at t_- and

$$\frac{d}{dt}\big(E_\beta(t^\beta A) * Pg\big)_- = A\eta(t) + Pg(t).$$

We show that $A\eta = A\eta_1 + A\eta_2 \in \mathcal{C}((0,\tilde{T}], J_r)$. It is clear that the given function

$$\eta_2(t) = Pg(t) - E_\beta(t^\beta A)Pg(t).$$

In consideration of Lemma 1, it is continuous because of Lemma 2(iii). Furthermore, in addition, Lemma 7 tells us that $A\eta_1(t)$ is also continuous. Accordingly, ${}^C D_t^\beta \eta \in \mathcal{C}((0,\tilde{T}], J_r)$.
Step II
Let us suppose that w is the mild solution of (2). To demonstrate $F(w,w) \in \mathcal{C}^\vartheta((0,\tilde{T}], J_r)$, from (5), we must verify that $A^\alpha w$ is Hölder continuous in J_r. Apply $h > 0$ in a way that

$0 < t < t+h$.
Indicate $\tilde{\phi}(t) := E_\beta(-t^\beta A)\dfrac{a}{\sqrt{2}}$ through Lemma 2(iv) and (6), then

$$\begin{aligned}|A^\alpha\tilde{\phi}(t+h) - A^\alpha\tilde{\phi}(t)|_r &= \left|\int_t^{t+h} -s^{\beta-1}A^\alpha E_{\beta,\beta}(-s^\beta A)\frac{a}{\sqrt{2}}ds\right|_r \\ &\leq \int_t^{t+h} s^{\beta-1}\left|A^{\alpha-\delta}E_{\beta,\beta}(-s^\beta A)A^\delta\frac{a}{\sqrt{2}}\right|_r ds \\ &\leq \mathcal{C}_1\int_t^{t+h} s^{\beta-1}s^{-\beta(\alpha-\delta)}\left|A^\delta\frac{a}{\sqrt{2}}\right|_r ds\end{aligned}$$

$$\begin{aligned} &= \mathcal{C}_1\int_t^{t+h} s^{\beta(\delta-\alpha)+\beta-1}\left|A^\delta\frac{a}{\sqrt{2}}\right|_r ds \\ &= \mathcal{C}_1\int_t^{t+h} s^{\beta(1+\delta-\alpha)-1}ds|A^\delta\frac{a}{\sqrt{2}}|_r \\ &= \frac{\mathcal{C}_1|a|_{H^{\delta,r}}}{\beta(1+\delta-\alpha)}\big((t+h)^{\beta(1+\delta-\alpha)} - t^{\beta(1+\delta-\alpha)}\big) \\ &\leq \frac{\mathcal{C}_1|\frac{a}{\sqrt{2}}|_{H^{\delta,r}}}{\beta(1+\delta-\alpha)}h^{\beta(1+\delta-\alpha)}.\end{aligned}$$

Thus, $A^\alpha\tilde{\phi} \in \mathcal{C}^\vartheta((0,\tilde{T}], J_r)$. Apply h in a way that $\varepsilon \leq t < t+h \leq \tilde{T}$, for all small $\varepsilon > 0$, since

$$\begin{aligned} & |A^\alpha\eta(t+h) - A^\alpha\eta(t)|_r \\ \leq & \left|\int_t^{t+h}(t+h-s)^{\beta-1}A^\alpha E_{\beta,\beta}(-(t+h-s)^\beta A)Pg(s)ds\right|_r \\ + & \left|\int_0^t A^\alpha\big((t+h-s)^{\beta-1}E_{\beta,\beta}(-(t+h-s)^\beta A) - (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)\big)Pg(s)ds\right|_r \\ = & \eta_1(t) + \eta_2(t).\end{aligned}$$

By implementing Lemma 6 and (e), the result is

$$\begin{aligned}\eta_1(t) &\leq \mathcal{C}_1\int_t^{t+h}(t+h-s)^{\beta(1-\alpha)-1}|Pg(s)|_r ds \\ &\leq \mathcal{C}_1\mathcal{M}(t)\int_t^{t+h}(t+h-s)^{\beta(1-\alpha)-1}s^{-\beta(1-\delta)}ds \\ &\leq \mathcal{M}(t)\frac{\mathcal{C}_1}{\beta(1-\alpha)}h^{\beta(1-\alpha)}t^{-\beta(1-\delta)} \\ &\leq \mathcal{M}(t)\frac{\mathcal{C}_1}{\beta(1-\alpha)}h^{\beta(1-\alpha)}\varepsilon^{-\beta(1-\delta)}.\end{aligned}$$

To calculate $\eta_2(t)$, we find the inequality

$$\begin{aligned}\frac{d}{dt}\big(t^{\beta-1}A^\alpha E_{\beta,\beta}(-t^\beta A)\big) &= \frac{1}{2\pi i}\int_\Gamma \mu^\alpha t^{\beta-2}E_{\beta,\beta-1}(-\mu t^\beta)(\mu I + A)^{-1}d\mu \\ &= \frac{1}{2\pi i}\int_{\Gamma'} -\left(-\frac{\psi}{t^\beta}\right)^\alpha t^{\beta-2}E_{\beta,\beta-1}(\psi)\left(-\frac{\psi}{t^\beta}I + A\right)^{-1}\frac{1}{t^\beta}d\psi.\end{aligned}$$

The above equation gives

$$\left\|\frac{d}{dt}(t^{\beta-1}A^\alpha E_{\beta,\beta}(-t^\beta A))\right\| \leq C_\beta t^{\beta(1-\alpha)-2}.$$

The mean value theorem gives yields

$$\left\|t^{\beta-1}A^{\alpha}E_{\beta,\beta}(-t^{\beta}A)-s^{\beta-1}A^{\alpha}E_{\beta,\beta}(-s^{\beta}A)\right\| \leq \int_s^t \left\|\frac{d}{d\tau}(\tau^{\beta-1}A^{\alpha}E_{\beta,\beta}(-\tau^{\beta}A))\right\|d\tau$$
$$\leq \mathcal{C}_{\beta}\int_s^t \tau^{\beta(1-\alpha)-2}d\tau = \mathcal{C}_{\beta}\big(s^{\beta(1-\alpha)-1}-t^{\beta(1-\alpha)-1}\big).$$

Thus,

$$\begin{aligned}
\eta_2(t) &\leq \int_0^t \bigg|A^{\alpha}\Big((t+h-s)^{\beta-1}E_{\beta,\beta}(-(t+h-s)^{\beta}A) \\
&- (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^{\beta}A\Big)Pg(s)\bigg|_r ds \\
&\leq \int_0^t \Big((t-s)^{\beta(1-\alpha)-1}-(t+h-s)^{\beta(1-\alpha)-1}\Big)\big|Pg(s)\big|_r ds \\
&\leq \mathcal{C}_{\beta}\mathcal{M}(t)\bigg(\int_0^t (t-s)^{\beta(1-\alpha)-1}s^{-\beta(1-\delta)}ds - \int_0^{t+h}(t+h-s)^{\beta(1-\alpha)-1}s^{-\beta(1-\delta)}ds\bigg) \\
&+ \mathcal{C}_{\beta}\mathcal{M}(t)\int_t^{t+h}(t+h-s)^{\beta(1-\alpha)-1}s^{-\beta(1-\delta)}ds \\
&\leq \mathcal{C}_{\beta}\mathcal{M}(t)\big(t^{\beta(\delta-\alpha)}-(t+h)^{\beta(\delta-\alpha)}\big)\mathcal{B}(\beta(1-\alpha),1-\beta(1-\delta)) + C_{\beta}M(t)h^{\beta(1-\alpha)}t^{-\beta(1-\delta)} \\
&\leq \mathcal{C}_{\beta}\mathcal{M}(t)h^{\beta(\alpha-\delta)}[\varepsilon(\varepsilon+h)]^{\beta(\delta-\alpha)} + \mathcal{C}_{\beta}\mathcal{M}(t)h^{\beta(1-\alpha)}\varepsilon^{-\beta(1-\delta)}.
\end{aligned}$$

This guarantees that $A^{\alpha}\eta \in \mathcal{C}^{\vartheta}([\varepsilon,\tilde{T}],J_r)$. Owing to random ε, $A^{\alpha}\eta \in \mathcal{C}^{\vartheta}((0,\tilde{T}],J_r)$.

$$\tilde{\psi}(t) = \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^{\beta}A)F(w(s),w(s)))ds.$$

Here, we know that $|F(w(s),w(s)))|_r \leq \mathcal{M}\tilde{\kappa}^2(t)s^{-2\beta(\alpha-\delta)}$, in which the supplied function's continuity exists but is also bounded in $(0,\tilde{T}]$, and $\tilde{\kappa}(t) := \sup_{s\in(0,t]} s^{\beta(\alpha-\delta)}|w(s)|_{H^{\alpha,r}}$. We are able to provide the Hölder continuity of $A^{\alpha}\tilde{\psi}$ in the same fashion as in $\mathcal{C}^{\vartheta}((0,\tilde{T}],J_r)$. Therefore, we can write

$$A^{\alpha}w(t) = A^{\alpha}\tilde{\varphi}(t) + A^{\alpha}\eta(t) + A^{\alpha}\tilde{\psi}(t) \ \in C^{\vartheta}((0,\tilde{T}],J_r)$$

Seeing as $F(w,w) \in \mathcal{C}^{\vartheta}((0,\tilde{T}],J_r)$ is demonstrated. From the previous Step II, these give the results that ${}^{C}D_t^{\beta}\psi \in \mathcal{C}^{\vartheta}((0,\tilde{T}],J_r)$, $A\tilde{\psi} \in \mathcal{C}((0,\tilde{T}],J_r)$ and ${}^{C}D_t^{\beta}\tilde{\psi} = -A\tilde{\psi} + F(w,w)$. Similarly to what we did, we obtained that ${}^{C}D_t^{\beta}w \in \mathcal{C}((0,\tilde{T}],J_r)$, $Aw \in \mathcal{C}((0,\tilde{T}],J_r)$ and ${}^{C}D_t^{\beta}w = -Aw + F(w,w) + Pg$.
Thus, the conclusion is that w is a classical solution as a result. □

Theorem 5. *Suppose that (e_1) is true. If w shows a classical solution of (2), then $Aw \in \mathcal{C}^{\nu}((0,\tilde{T}],J_r)$ also ${}^{C}D_t^{\beta}w \in \mathcal{C}^{\nu}((0,\tilde{T}],J_r)$.*

Proof. Unless w shows a classical solution of (2), hence we can write $w(t) = \tilde{\varphi}(t) + \eta(t) + \tilde{\psi}(t)$. The evidence is adequate to show $A\tilde{\varphi} \in \mathcal{C}^{\beta(1-\delta)}((0,\tilde{T}],J_r)$. For every $\varepsilon > 0$, it is still important to prove that

$A\tilde{\phi} \in \mathcal{C}^{\beta(1-\delta)}([\varepsilon, \tilde{T}], J_r)$. In fact, using Lemma 2(iii), h is picked in a way that $\varepsilon \le t < t+h \le \tilde{T}$:

$$\begin{aligned} |A\tilde{\phi}(t+h) - A\tilde{\phi}(t)|_r &= \left| \int_t^{t+h} -s^{\beta-1} A^2 E_{\beta,\beta}(-s^\beta A) \frac{a}{\sqrt{2}} ds \right|_r \\ &\le \mathcal{C}_1 \int_t^{t+h} s^{-\beta(1-\delta)-1} ds |\frac{a}{\sqrt{2}}|_{H^{\delta,r}} \\ &= \frac{\mathcal{C}_1 |\frac{a}{\sqrt{2}}|_{H^{\delta,r}}}{\beta} \big(t^{-\beta(1-\delta)} - (t+h)^{-\beta(1-\delta)}\big) \\ &\le \frac{\mathcal{C}_1 |\frac{a}{\sqrt{2}}|_{H^{\delta,r}}}{\beta} \frac{h^{\beta(1-\delta)}}{[\varepsilon(\varepsilon+h)]^{\beta(1-\delta)}}. \end{aligned}$$

Similarly, from Lemma 7, we can write $\eta(t)$ as

$$\begin{aligned} \eta(t) &= \eta_1(t) + \eta_2(t) \\ &= \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A)\big(Pg(s) - Pg(t)\big) ds \\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A) Pg(t) ds, \end{aligned}$$

in the domain of $t \in (0, \tilde{T}]$. Lemma 7 with (23) leads to the conclusion that $A\eta_1(t) \in \mathcal{C}^{\nu}([0, \tilde{T}], J_r)$ and $A\eta_2(t) \in \mathcal{C}^{\tilde{\Phi}}((0, \tilde{T}], J_r)$, accordingly. □

8. Application

Assuming that $X \in L^2(\pi, 2\pi)$ and $e_n(x) = 5\sqrt{5/2\pi} sin x, n = 1, 2, \ldots$. Then, $(e_n, n = 1, 2, \ldots)$ is an orthonormal base of X. We define an infinite dimensional space $U = X$ and consider the following system governed by the semi-linear heat equation:

$$\begin{cases} {}^cD_t^{6/7} Y(t,x) = {}^c D_t^{4/5} Y(t,x) + f(t, Y(t,x)) + Au(t,x), 0 < t < a, \pi < x < 2\pi, \\ Y(0,x) = Y_0(x), \pi \le x \le 2\pi, \\ Y(t,0) = Y(t, 2\pi), 0 \le t \le a, \end{cases} \quad (24)$$

where the nonlinear function f is considered as an operator satisfying hypothesis H_1 and for each $u \in L^2(0, a; U)$ of the form $\sum_{n=1}^{\infty} \hat{u}_n o(t) e_n$; here, we define

$$Au(t) = \sum_{n=1}^{\infty} \hat{u}_n o(t) e_n,$$

where

$$\hat{u}_n(t) = \begin{cases} 0, 0 \le t < a(1 - \frac{1}{n}), \\ u_n(t), a(1 - \frac{1}{n}) \le t \le a. \end{cases} \quad (25)$$

Because

$$\|Bu\|_{L^2(0,a,X)} \le \|u\|_{L^2(0,a,X)},$$

the operator B is bounded from U into $L^2(J, X)$. In fact, it is not difficult to check that $\overline{AU} \ne L^2(J, X)$. Then, let t be an arbitrary element in $L^2(o, a, X)$ and $h \in X$ be defined by

$$h = E_\beta(-a-s)^\beta Y(0)x + \int_0^a (a-s)^{\beta-1} \Im_{\frac{4}{5}}(a-s)\phi(s) ds.$$

Assume that

$$\varphi(t) = \sum_{n=1}^{\infty} f_n(t) e_n,$$

and

$$h = \sum_{n=1}^{\infty} h_n(t)e_n.$$

Then, we claim that for every given $\varphi \in L^2(0, a, X)$, there exists $u \in U$ such that

$$E_\beta(-a-s)^\beta \Upsilon(0)x + \int_0^t (a-s)^{\beta-1}\Im_{\frac{6}{7}}(a-s)Au(s)ds$$
$$= E_\beta(-a-s)^\beta \Upsilon(0)x + \int_0^t (a-s)^{\beta-1}\Im_{\frac{6}{7}}(a-s)t(s)ds,$$

which means that the condition H_2 is satisfied, as assumptions H_1 and H_2 are satisfied.

9. Conclusions

This study uses Helmholtz–Leray projection to demonstrate the existence and uniqueness of a solution for the fractional order Navier–Stokes equations. Meanwhile, we offer a local viable solution in $\mathbf{S}_\wp$. The Navier–Stokes equations (NSEs) with time-fractional derivatives of order $\gamma \in (0, 1)$ are used to simulate anomaly diffusion in fractal media. We demonstrate the existence of regular classical solutions to these equations in $\mathbf{S}_\wp$. The concept put forth in this article may be expanded upon in future work through the inclusion of observability and the generalization of other activities. Much research is being performed in this fascinating area, which may result in a wide range of applications and theories.

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