fractal and fractional

Article

On Averaging Principle for Caputo–Hadamard Fractional Stochastic Differential Pantograph Equation

Mounia Mouy [1], Hamid Boulares [2], Saleh Alshammari [3], Mohammad Alshammari [3], Yamina Laskri [4] and Wael W. Mohammed [3,5,*]

1 Department of Mathematics, Faculty of Sciences, University Badji Mokhtar Annaba, P.O. Box 12, Annaba 23000, Algeria
2 Laboratory of Analysis and Control of Differential Equations "ACED", Faculty MISM, Department of Mathematics, University of Guelma, Guelma 24000, Algeria
3 Department of Mathematics, Collage of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
4 Higher School of Industrial Technologies, Annaba 23000, Algeria
5 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
* Correspondence: wael.mohammed@mans.edu.eg

Abstract: In this paper, we studied an averaging principle for Caputo–Hadamard fractional stochastic differential pantograph equation (FSDPEs) driven by Brownian motion. In light of some suggestions, the solutions to FSDPEs can be approximated by solutions to averaged stochastic systems in the sense of mean square. We expand the classical Khasminskii approach to Caputo–Hadamard fractional stochastic equations by analyzing systems solutions before and after applying averaging principle. We provided an applied example that explains the desired results to us.

Keywords: averaging principle; Caputo–Hadamard fractional derivative; pantograph equations; Khasminskii approach

MSC: 34K20; 34K30; 34K40

check for updates

Citation: Mouy, M.; Boulares, H.; Alshammari, S.; Alshammari, M.; Laskri, Y.; Mohammed, W.W. On Averaging Principle for Caputo–Hadamard Fractional Stochastic Differential Pantograph Equation. *Fractal Fract.* **2023**, *7*, 31. https://doi.org/10.3390/fractalfract7010031

Academic Editor: Carlo Cattani

Received: 8 December 2022
Revised: 23 December 2022
Accepted: 26 December 2022
Published: 28 December 2022

1. Introduction

The nature of solutions for fractional stochastic differential pantograph equations (FSDPEs) in Euclidean space n-dimensional $\mathbb{R}^n$ [1,2], is particularly interesting in practical applications. In general, the systems take the form

$$\begin{cases} \mathfrak{D}^{\alpha}_{\varsigma}\mathcal{X}(\varsigma) = b(\varsigma, \mathcal{X}(\varsigma), \mathcal{X}(1+\eta\varsigma)) + \sigma_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{X}(1+\eta\varsigma))\dfrac{d\mathfrak{B}(\varsigma)}{d\varsigma} \\ \mathcal{X}(1) = \mathcal{X}_0, \end{cases} \quad (1)$$

where $\eta \in \left(0, \frac{T-1}{T}\right)$, $\mathfrak{D}^{\alpha}_{\varsigma}$ is the Caputo–Hadamard fractional derivative (CHFD), $\alpha \in (\frac{1}{2}, 1)$, for each $\varsigma \geq 1$, $b : [1, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma_1 : [1, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable continuous functions (CF), $\mathfrak{B}(\varsigma)$ is a m-dimensional standard Brownian motion on $\{\Omega, \mathfrak{F}, P\}$ probability space. The initial value $\mathcal{X}_0$ is an $\mathfrak{F}_0$-measurable $\mathbb{R}^n$-value random variable, satisfying $E|\mathcal{X}_0|^2 < \infty$.

Solutions of non-linear FSDPEs are almost impossible to solve and very difficult. For this reason we used symmetrical methods and techniques in the widest field. It plays very important in modernity of partial calculus [3,4].

In [5], Khasminiskii was interested in studying the convergence of idle systems on the drag time scale $\varepsilon \to 0$, in resolving intermediate arguments. He concluded that averaging principle lay in the study of equations lost in terms of the relevant average. So, we have an easy way to solve these equations, as it is known that such equations have been applied to many numerical algorithms to different models, including FSDEs see [6,7].

The generalized pantograph equation has a variety of applications. Only applications in number theory are mentioned [8], in electrodynamics [9] and in the absorption of energy by the pantograph of an electronic locomotive [10–13].

We rely on this article, which aims to expand Khasminskii's classic argument into random fractional differential equations with CHFD. For our goal, with the help of rigorous mathematical deduction, which here accurately illustrates the fractional averaging principle mean square that has been reached. This means that an easy and effective way has been given to solve the FSDPEs (1) accurately. We have arranged the organization of this article as follows. We present in the second section some basic ideas, definitions, lemmas and arguments. In section 3, we explain an averaging principle obtained first, and complete with a main result. To explain this, we give a specific illustrative example.

2. Preliminaries

In this section, we introduce some basic techniques, definitions, lemmas and theorems (see [14–19]).

Definition 1 ([2,19])**.** *The Riemann–Liouville fractional integral (RLFI) of order $\alpha > 0$ for a function $x : [0, +\infty) \to \mathbb{R}$ is defined as*

$$I^{\alpha}x(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma} (\varsigma - s)^{\alpha-1} x(s)ds,$$

where Γ is the Euler gamma function and it is defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\varsigma}\varsigma^{\alpha-1}d\varsigma.$$

Definition 2 ([2,19])**.** *The Hadamard fractional integral of order $\alpha > 0$ for a CF $x : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{I}_1^{\alpha}x(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_1^{\varsigma} \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}.$$

Definition 3 ([2,19])**.** *The Riemann–Liouville fractional derivative (RLFD) of order $\alpha > 0$ for a CF $x : [0, +\infty) \to \mathbb{R}$ is defined as*

$$D^{\alpha}x(\varsigma) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\varsigma} (\varsigma - s)^{n-\alpha-1} x^{(n)}(s)ds, \ n-1 < \alpha < n, \ n \in \mathbb{N}.$$

Definition 4 ([2,19])**.** *The CHFD of order $\alpha > 0$ for a CF $x : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{D}_1^{\alpha}x(\varsigma) = \frac{1}{\Gamma(n-\alpha)} \int_1^{\varsigma} \left(\log \frac{\varsigma}{s}\right)^{n-\alpha-1} \delta^n x(s)\frac{ds}{s}, \ n-1 < \alpha < n,$$

where $\delta^n = \left(\varsigma\frac{d}{d\varsigma}\right)^n, n \in \mathbb{N}$.

Lemma 1 ([2,19])**.** *Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. The equality $\left(\mathfrak{I}_1^{\alpha}\mathfrak{D}_1^{\alpha}x\right)(\varsigma) = 0$ is true if and only if*

$$x(\varsigma) = \sum_{k=1}^{n} c_k(\log\varsigma)^{\alpha-k} \text{ for each } \varsigma \in [1, \infty),$$

where $c_k \in \mathbb{R}$, $k = 1, \ldots, n$ are arbitrary constants.

Lemma 2 ([2,19])**.** *Let $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and $x \in C^{n-1}[1, \infty)$. Then*

$$\mathfrak{I}_1^{\alpha}[\mathfrak{D}_1^{\alpha}x(\varsigma)] = x(\varsigma) - \sum_{k=0}^{m-1} \frac{\left(\delta^k x\right)(1)}{\Gamma(k+1)}(\log\varsigma)^k.$$

Lemma 3 ([2,19])**.** *For all $\mu > 0$ and $\nu > -1$,*

$$\frac{1}{\Gamma(\mu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{s}\right)^{\mu-1}(\log s)^{\nu}\frac{ds}{s} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log\varsigma)^{\mu+\nu}.$$

Lemma 4 ([2,19])**.** *Let $x(\varsigma) = (\log\varsigma)^{\mu}$, where $\mu \geq 0$ and let $m-1 < \alpha \leq m$, $m \in \mathbb{N}$. Then*

$$\mathfrak{D}_1^{\alpha}x(\varsigma) = \begin{cases} 0 & \text{if } \mu \in \{0,1,\ldots,m-1\}, \\ \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log\varsigma)^{\mu-\nu} & \text{if } \mu \in \mathbb{N},\ \mu \geq m \text{ or } \mu \notin \mathbb{N},\ \mu > m-1. \end{cases}$$

Here we put some conditions on coefficient functions, to study the qualitative properties of solving Equation (1), which will help us solve it.

($\Lambda 1$) For every $x, y, z, w \in \mathbb{R}^n$ and $\varsigma \in [1, T]$, there exist three constants C_1, C_2 and C_3 are positive, so that

$$|b(\varsigma,x,y)|^2 \vee |\sigma_1(\varsigma,x,y)|^2 \leq C_1^2\left(1+|x|^2+|y|^2\right)$$
$$|b(\varsigma,x,y) - b(\varsigma,w,z)| \vee |\sigma_1(\varsigma,x,y) - \sigma_1(\varsigma,w,z)| \leq C_2|x-w| + C_3|y-z|$$

where $|.|$ is the norm of $\mathbb{R}^n$, $x_1 \vee x_2 = \max\{x_1, x_2\}$.

In coordination with pivotal research of Zone [20], Zhang and Agarwal [21], as we recognize that by proposal ($\Lambda 1$), FSDPEs (1) has a unique solution

$$\begin{aligned} \mathcal{X}(\varsigma) = \mathcal{X}_0 &+ \frac{1}{\Gamma(\alpha)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{s}\right)^{\alpha-1} b(s,\mathcal{X}(s),\mathcal{X}(1+\eta s))\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{s}\right)^{\alpha-1}\sigma_1(s,\mathcal{X}(s),\mathcal{X}(1+\eta s))\frac{d\mathfrak{B}(s)}{s}, \end{aligned} \tag{2}$$

$\mathcal{X}(\varsigma)$ is $\mathfrak{F}(\varsigma)$-adapted and $E\left(\int_1^T|\mathcal{X}(\varsigma)|^2 d\varsigma\right) < \infty$.

3. An Averaging Principle

In this part we investigated the averaging principle for FSDPEs, combining the results of existence and uniqueness. Let us consider the standard form of Equation (1):

$$\begin{aligned} \mathcal{X}_{\epsilon}(\varsigma) = \mathcal{X}_0 &+ \frac{\epsilon}{\Gamma(\alpha)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{s}\right)^{\alpha-1} b(s,\mathcal{X}_{\epsilon}(s),\mathcal{X}_{\epsilon}(1+\eta s))\frac{ds}{s} \\ &+ \frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{s}\right)^{\alpha-1}\sigma_1(s,\mathcal{X}_{\epsilon}(s),\mathcal{X}_{\epsilon}(1+\eta s))\frac{d\mathfrak{B}(s)}{s}, \end{aligned} \tag{3}$$

where the initial value $\mathcal{X}_0$, coefficients b and σ_1 it has the same meaning as in Equation (1). We also denote by ϵ_0 a fixed number, and $\epsilon \in [0, \epsilon_0]$ is a positive small parameter.

Before we continue with the averaging principle, we impose some measurable coefficients, $\overline{b} : \mathbb{R}^n \to \mathbb{R}^n, \overline{\sigma} : \mathbb{R}^n \to \mathbb{R}^n$, satisfying ($\Lambda 1$) and the additional inequalities:

($\Lambda 2$) For any $T_1 \in [1, T]$, $x, y \in \mathbb{R}^n$, there exist two positive bounded functions $\Psi_i(T_1), i = 1, 2$ such that

$$\frac{1}{\log T_1}\int_1^{T_1}\left|b(s,x,y) - \overline{b}(x,y)\right|\frac{ds}{s} \leq \Psi_1(T_1)(1+|x|+|y|),$$
$$\frac{1}{\log T_1}\int_1^{T_1}|\sigma_1(s,x,y) - \overline{\sigma}_1(x,y)|^2\frac{ds}{s} \leq \Psi_2(T_1)(1+|x|^2+|y|^2),$$

where $\lim\limits_{T_1\to\infty}\Psi_i(T_1) = 0$.

With sufficient help above, we will explain that the exact solution $\mathcal{X}_{\epsilon}(\varsigma)$ converges, as $\epsilon \to 0$, tend to $Z_{\epsilon}(\varsigma)$ of the averaged system

$$Z_\epsilon(\varsigma) = \mathcal{X}_0 + \frac{\epsilon}{\Gamma(\alpha)} \int_1^\varsigma (\log \frac{\varsigma}{s})^{\alpha-1} \overline{b}(Z_\epsilon(s), Z_\epsilon(1+\eta s)) \frac{ds}{s}$$
$$+ \frac{\sqrt{\epsilon}}{\Gamma(\alpha)} \int_1^\varsigma (\log \frac{\varsigma}{s})^{\alpha-1} \overline{\sigma}_1(Z_\epsilon(s), Z_\epsilon(1+\eta s)) \frac{d\mathfrak{B}(s)}{s}. \quad (4)$$

We come now and present the main result of this research.

Theorem 1. *Suggest that* $(\Lambda 1) - (\Lambda 2)$ *are satisfied. For* $\delta_1 > 0$ *there exists* $L > 1, \epsilon_1 \in (0, \epsilon_0]$ *and* $\beta \in (0,1)$ *us such for every* $\epsilon \in (0, \epsilon_1]$,

$$\mathrm{E}\left(\sup_{\varsigma \epsilon \left[1, L^{\epsilon - \beta}\right]} |\mathcal{X}_\epsilon(\varsigma) - Z_\epsilon(\varsigma)|^2 \right) \leq \delta_1. \quad (5)$$

Proof. For any $\varsigma \in [1, u] \subset [1, T]$,

$$\begin{aligned} &\mathcal{X}_\epsilon(\varsigma) - Z_\epsilon(\varsigma) \\ &= \frac{\epsilon}{\Gamma(\alpha)} \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} \left[b(s, \mathcal{X}_\epsilon(s), \mathcal{X}_\epsilon(1+\eta s)) - \overline{b}(Z_\epsilon(s), Z_\epsilon(1+\eta s)) \right] \frac{ds}{s} \\ &+ \frac{\sqrt{\epsilon}}{\Gamma(\alpha)} \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} [\sigma_1(s, \mathcal{X}_\epsilon(s), \mathcal{X}_\epsilon(1+\eta s)) - \overline{\sigma}_1(Z_\epsilon(s), Z_\epsilon(1+\eta s))] \frac{d\mathfrak{B}(s)}{s}. \quad (6) \end{aligned}$$

Using the elementary inequality

$$|x_1 + x_2|^2 \leq 2(|x_1|^2 + |x_2|^2), \quad (7)$$

we have

$$\begin{aligned} &E\left(\sup_{1 \leq \varsigma \leq u} |\mathcal{X}_\epsilon(\varsigma) - Z_\epsilon(\varsigma)|^2 \right) \\ &\leq \frac{2\epsilon^2}{\Gamma(\alpha)^2} E \sup_{1 \leq \varsigma \leq u} \left| \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} [b(s, \mathcal{X}_\epsilon(s), \mathcal{X}_\epsilon(1+\eta s)) \right. \\ &\left. - \overline{b}(Z_\epsilon(s), Z_\epsilon(1+\eta s)) \frac{ds}{s} \right] \Bigg|^2 \\ &+ \frac{2\epsilon}{\Gamma(\alpha)^2} E \sup_{1 \leq \varsigma \leq u} \left| \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} [\sigma_1(s, \mathcal{X}_\epsilon(s), \mathcal{X}_\epsilon(1+\eta s)) \right. \\ &\left. - \overline{\sigma}_1(Z_\epsilon(s), Z_\epsilon(1+\eta s)) \frac{d\mathfrak{B}(s)}{s} \right] \Bigg|^2 \\ &= I_1 + I_2. \quad (8) \end{aligned}$$

Recalling inequality (7), we obtain

$$\begin{aligned} I_1 &\leq \frac{4\epsilon^2}{\Gamma(\alpha)^2} E \sup_{1 \leq \varsigma \leq u} \left| \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} [b(s, \mathcal{X}_\epsilon(s), \mathcal{X}_\epsilon(1+\eta s)) \right. \\ &\left. - b(Z_\epsilon(s), Z_\epsilon(1+\eta s))] \frac{ds}{s} \right|^2 \\ &+ \frac{4\epsilon}{\Gamma(\alpha)^2} E \sup_{1 \leq \varsigma \leq u} \left| \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} [b(s, Z_\epsilon(s), Z_\epsilon(1+\eta s)) \right. \\ &\left. - \overline{b}(Z_\epsilon(s), Z_\epsilon(1+\eta s)) \right] \frac{ds}{s} \Bigg|^2 \\ &= I_{11} + I_{12}. \quad (9) \end{aligned}$$

Using the Cauchy–Schwarz inequality and condition $(\Lambda 1)$, we obtain

$$I_{11} \leq K_{11}\epsilon^2 \log u \int_1^u (\log \frac{u}{s})^{2\alpha-2} E\left(\sup_{1\leq s_1 \leq s} |\mathcal{X}_\epsilon(s_1) - Z_\epsilon(s_1)|^2 \frac{ds}{s} \right), \tag{10}$$

where $K_{11} = \dfrac{8\left(C_2^2 + C_3^2\right)}{\Gamma(\alpha)^2}$. By the definition of variable upper limit integration,

$$\begin{aligned} I_{12} \leq & \frac{4\epsilon^2}{\Gamma(\alpha)^2} E \sup_{1\leq \varsigma \leq u} \left| \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} d\left[\int_1^s b(\tau, Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau)) \right.\right. \\ & \left.\left. - \overline{b}(Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau)) \frac{d\tau}{\tau} \right] \right|^2, \end{aligned} \tag{11}$$

integration by parts is used,

$$\begin{aligned} I_{12} \leq & \frac{4\epsilon^2(\alpha-1)^2}{\Gamma(\alpha)^2} E \sup_{1\leq \varsigma \leq u} \left| \int_1^\varsigma \left(\int_1^s b(\tau, Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau)) \right.\right. \\ & \left.\left. - \overline{b}(Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau)) \frac{d\tau}{\tau} \right) (\log \frac{\varsigma}{s})^{\alpha-2} \frac{ds}{s} \right|^2, \end{aligned} \tag{12}$$

then together with the hypothesis $(\Lambda 2)$ and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} I_{12} \leq & \frac{4\epsilon^2(\alpha-1)^2(\log u)^{2\alpha-3}}{(2\alpha-3)\Gamma(\alpha)^2} \\ & \times E \int_1^u \left| \int_1^s b(\tau, Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau)) - \overline{b}(Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau)) \frac{d\tau}{\tau} \right|^2 \frac{ds}{s} \\ & \leq K_{12}\epsilon^2(\log u)^{2\alpha}, \end{aligned} \tag{13}$$

in which

$$\begin{aligned} K_{12} = & \frac{4(\alpha-1)^2}{(2\alpha-3)\Gamma(\alpha)^2} \sup_{1\leq \varsigma \leq u} \Psi_1(\varsigma)^2 \left[1 + E\left(\sup_{1\leq \tau \leq u} |Z_\epsilon(\tau)|^2 \right) \right. \\ & \left. + E\left(\sup_{1\leq \tau \leq u} |Z_\epsilon(1+\eta\tau)|^2 \right) \right]. \end{aligned} \tag{14}$$

With the same technique we look forward to the second term,

$$\begin{aligned} I_2 \leq & \frac{4\epsilon^2}{\Gamma(\alpha)^2} E \sup_{1\leq \varsigma \leq u} \left| \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} [\sigma_1(s, \mathcal{X}_\epsilon(s), \mathcal{X}_\epsilon(1+\eta s)) \right. \\ & \left. - \sigma_1(s, Z_\epsilon(s), Z_\epsilon(1+\eta s))] \frac{d\mathfrak{B}(s)}{s} \right|^2 \\ & + \frac{4\epsilon}{\Gamma(\alpha)^2} E \sup_{1\leq \varsigma \leq u} \left| \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{\alpha-1} [\sigma_1(s, Z_\epsilon(s), Z_\epsilon(1+\eta s)) \right. \\ & \left. - \overline{\sigma}_1(Z_\epsilon(s), Z_\epsilon(1+\eta s))] \frac{d\mathfrak{B}(s)}{s} \right|^2 \\ & = I_{21} + I_{22}. \end{aligned} \tag{15}$$

By applying Doob's martingale inequality, Itô's formula and condition $(\Lambda 1)$,

$$\begin{aligned} I_{21} \leq & \frac{4\epsilon}{\Gamma(\alpha)^2} E \int_1^u \left(\log \frac{u}{s}\right)^{2\alpha-2} |\sigma_1(s, \mathcal{X}_\epsilon(s), \mathcal{X}_\epsilon(1+\eta s)) \\ & - \sigma_1(s, Z_\epsilon(s), Z_\epsilon(1+\eta s))|^2 \frac{ds}{s} \\ & \leq K_{21}\epsilon \int_1^u \left(\log \frac{u}{s}\right)^{2\alpha-2} E\left(\sup_{1\leq s_1\leq s} |\mathcal{X}_\epsilon(s_1) - Z_\epsilon(s_1)|^2\right) \frac{ds}{s}, \end{aligned} \tag{16}$$

where $K_{21} = \dfrac{8(C_2^2 + C_3^2)}{\Gamma(\alpha)^2}$. Applying Doob's martingale inequality and Itô's formula again,

$$\begin{aligned} I_{22} \leq & \frac{4\epsilon}{\Gamma(\alpha)^2} E \int_1^u \left(\log \frac{u}{s}\right)^{2\alpha-2} |\sigma_1(s, \mathcal{X}_\epsilon(s), \mathcal{X}_\epsilon(1+\eta s)) \\ & - \overline{\sigma}_1(Z_\epsilon(s), Z_\epsilon(1+\eta s))|^2 \frac{ds}{s}. \end{aligned} \tag{17}$$

Integrating by parts, produces

$$\begin{aligned} I_{22} \leq & \frac{4\epsilon}{\Gamma(\alpha)^2} E \int_1^u \left(\log \frac{u}{s}\right)^{2\alpha-2} d\left[\int_1^s |\sigma_1(\tau, Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau)) \right. \\ & \left. - \overline{\sigma}_1(Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau))|^2 \frac{d\tau}{\tau}\right] \\ \leq & \frac{4\epsilon(2\alpha-2)}{\Gamma(\alpha)^2} E \int_1^u \left(\int_1^s |\sigma_1(\tau, Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau)) \right. \\ & \left. - \overline{\sigma}_1(Z_\epsilon(\tau), Z_\epsilon(1+\eta\tau))|^2 \frac{d\tau}{\tau}\right) \left(\log \frac{u}{s}\right)^{2\alpha-3} \frac{ds}{s}, \end{aligned} \tag{18}$$

thanks to the hypothesis $(\Lambda 2)$, we can conclude

$$\begin{aligned} I_{22} \leq & \frac{4\epsilon(2\alpha-2)}{\Gamma(\alpha)^2} E \int_1^u \left(\sup_{1\leq s_1\leq s} \Psi_2(s_1) \left[1 + E\left(\sup_{1\leq\tau\leq s} |Z_\epsilon(\tau)|^2\right) \right. \right. \\ & \left. \left. + E\left(\sup_{1\leq\tau\leq s} |Z_\epsilon(1+\eta\tau)|^2\right)\right]\right) (\log s) \left(\log \frac{u}{s}\right)^{2\alpha-3} \frac{ds}{s} \\ \leq & K_{22}\epsilon(\log u)^{2\alpha-1}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} K_{22} = & \frac{3(2\alpha-2)}{\alpha(2\alpha-1)\Gamma(\alpha)^2} \sup_{1\leq\varsigma\leq u} \Psi_2(\varsigma) \left[1 + E\left(\sup_{1\leq\varsigma\leq u} |Z_\epsilon(\tau)|^2\right) \right. \\ & \left. + E\left(\sup_{1\leq\varsigma\leq u} |Z_\epsilon(1+\eta\tau)|^2\right)\right]. \end{aligned} \tag{20}$$

Now, substituting Equations (10)–(19) into (8), for any $u \in [1, T]$, we find

$$\begin{aligned} & E\left(\sup_{1\leq\varsigma\leq u} |\mathcal{X}_\epsilon(\varsigma)|^2\right) \\ & \leq K_{12}\epsilon^2 u^{2\alpha} + K_{22}\epsilon u^{2\alpha-1} \\ & + \left(K_{11}\epsilon^2 u + K_{21}\epsilon\right) \int_1^u \left(\log \frac{u}{s}\right)^{(2\alpha-1)-1} E\left(\sup_{1\leq s_1\leq s} |\mathcal{X}_\epsilon(s_1) - Z_\epsilon(s_1)|^2\right) \frac{ds}{s}, \end{aligned} \tag{21}$$

depending on the Gronwall–Bellman inequality [22], we find

$$
\begin{aligned}
&E\left(\sup_{1\le\varsigma\le u}|\mathcal{X}_\epsilon(\varsigma)-Z_\epsilon(\varsigma)|^2\right)\\
&\le\left(K_{12}\epsilon^2(\log u)^{2\alpha}+K_{22}\epsilon(\log u)^{2\alpha-1}\right)\\
&\times\sum_{k=0}^{\infty}\frac{\left(\left(K_{11}\epsilon^2(\log u)^{2\alpha}+K_{21}\epsilon(\log u)^{2\alpha-1}\right)\Gamma(2\alpha-1)\right)^k}{\Gamma(k(2\alpha-1)+1)}.
\end{aligned}
\tag{22}
$$

This implies that we can select $\beta\in(0,1)$ and $L>1$, such that for every $\varsigma\in\left[1,L^{\epsilon^{-\beta}}\right]\subseteq[1,T]$ having

$$
E\left(\sup_{1\le\varsigma\le L^{\epsilon^{-\beta}}}|\mathcal{X}_\epsilon(\varsigma)-Z_\epsilon(\varsigma)|^2\right)\le C\epsilon^{1-\beta},
\tag{23}
$$

where

$$
\begin{aligned}
C=&\left(K_{12}(\log L)^{2\alpha}\epsilon^{1+\beta-2\alpha\beta}+K_{22}(\log L)^{2\alpha-1}\epsilon^{2\beta(1-\alpha)}\right)\\
&\times\sum_{k=0}^{\infty}\frac{\left(\left(K_{11}(\log L)^{2\alpha}\epsilon^{2(1-\alpha\beta)}+K_{21}(\log L)^{2\alpha-1}\epsilon^{1+\beta(1-2\alpha)}\right)\Gamma(2\alpha-1)\right)^k}{\Gamma(k(2\alpha-1)+1)},
\end{aligned}
\tag{24}
$$

is a constant. Hence, for any given number $\delta_{1,}$there exists $\epsilon_1\in(0,\epsilon_0]$ such that for each $\epsilon\in(0,\epsilon_1]$ and $\varsigma\in\left[1,L^{\epsilon^{-\beta}}\right]$ having

$$
E\left(\sup_{1\le\varsigma\le L^{\epsilon^{-\beta}}}|\mathcal{X}_\epsilon(\varsigma)-Z_\epsilon(\varsigma)|^2\right)\le\delta_1.
\tag{25}
$$

finished the proof. □

4. Example

We present the following equation FSDPEs

$$
\begin{cases}
\mathfrak{D}_1^\alpha\mathcal{X}_\epsilon(\varsigma)=3\epsilon(\mathcal{X}_\epsilon(\varsigma)+\mathcal{X}_\epsilon(1+\eta\varsigma))\log^2(\varsigma)+\sqrt{\epsilon}\dfrac{d\mathfrak{B}(\varsigma)}{d\varsigma},\\
\mathcal{X}(1)=0,
\end{cases}
\tag{26}
$$

where $\eta\in\left(0,\frac{\pi-1}{\pi}\right),\alpha\in\left(\frac{1}{2},1\right)$. The coefficients $b(\varsigma,\mathcal{X}_\epsilon,Y_\epsilon)=3(\mathcal{X}_\epsilon+Y_\epsilon)\log^2(\varsigma)$ and $\sigma_1(\varsigma,\mathcal{X}_\epsilon,Y_\epsilon)=1$ verify the conditions $(\Lambda1)$, so there has a unique solution to FSDPEs (26).

Define

$$
\overline{b}(\mathcal{X}_\epsilon,Y_\epsilon)=\frac{1}{\log\pi}\int_1^\pi b(\varsigma,\mathcal{X}_\epsilon,Y_\epsilon)\frac{d\varsigma}{\varsigma}=(\mathcal{X}_\epsilon+Y_\epsilon)\log^2(\pi),\ \overline{\sigma}_1(\mathcal{X}_\epsilon,Y_\epsilon)=1,
$$

it is easily seen $(\Lambda2)$ holds, so the averaging form of (26) is

$$
\mathfrak{D}_1^\alpha Z_\epsilon(\varsigma)=\epsilon(Z_\epsilon(\varsigma)+Z_\epsilon(1+\eta\varsigma))\log^2(\pi)+\sqrt{\epsilon}\frac{d\mathfrak{B}(\varsigma)}{d\varsigma},\ Z_\epsilon(1)=\mathcal{X}_0.
\tag{27}
$$

Depending to Theorem 1, as $\epsilon\to0$, the solution $\mathcal{X}_\epsilon(\varsigma)$ and $Z_\epsilon(\varsigma)$ to Equations (26) and (27) are equivalent in the sense of mean square.

5. Conclusions

Previously, many researchers studied the averaging principle for Caputo fractional stochastic differential equations approximated by solutions to averaged stochastic systems in the sense of mean square. The new idea in our research in (1) is a discussion of a special kind of Caputo–Hadamard fractional stochastic differential pantograph equations driven by Brownian motion. We have also made two commitments, the solutions to FSDPEs can be approximated by solutions to averaged stochastic systems in the sense of mean square. Moreover, we extend the classical Khasminskii approach to Caputo–Hadamard fractional stochastic differential pantograph equations.

Author Contributions: Methodology, M.M., H.B., M.A. and Y.L.; Validation, S.A.; Formal analysis, M.M., H.B., S.A. and M.A.; Data curation, S.A. and M.A.; Writing—original draft, M.M., H.B., S.A., M.A. and W.W.M.; Writing—review & editing, Y.L. and W.W.M.; Supervision, Y.L. and W.W.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: All data are available in this paper.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Chen, L.; Hu, F.; Zhu, W. Stochastic dynamics and fractional optimal control of quasi integrable Hamiltonian systems with fractional derivative damping. *Fract. Calc. Appl. Anal.* **2013**, *16*, 189–225. [CrossRef]
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
3. Chen, W.; Sun, H.G.; Li, X.C. *Fractional Derivative Modeling of Mechanics and Engineering Problems*; Science Press: Beijing, China, 2010. (In Chinese)
4. Wu, A.; Zeng, Z. *Boundedness, Mittag-Leffler Stability and Asymptotical ω-Periodicity of Fractional-Order Fuzzy Neural Networks*; Elsevier Science Ltd.: Amsterdam, The Netherlands, 2016.
5. Khasminskii, R.Z. On the principle of averaging the Itô stochastic differential equations. *Kibernetika* **1968**, *4*, 260–279.
6. Pandit, S.; Mittal, R.C. A numerical algorithm based on scale-3 Haar wavelets for fractional advection dispersion equation. *Eng. Comput.* **2021**, *38*, 1706–1724. [CrossRef]
7. Mittal, R.C.; Pandit, S. A Numerical Algorithm to Capture Spin Patterns of Fractional Bloch Nuclear Magnetic Resonance Flow Models. *J. Comput. Nonlinear Dynam.* **2019**, *14*, 081001. [CrossRef]
8. Mahler, K. On a special functional equation. *J. Lond. Math. Soc.* **1940**, *15*, 115–123. [CrossRef]
9. Fox, L.; Mayers, D.F.; Ockendon, J.R.; Tayler, A.B. On a functional differential equation. *IMA J. Appl. Math.* **1971**, *8*, 271–307. [CrossRef]
10. Hale, J. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
11. Iserles, A. On the generalized pantograph functional-differential equation. *Eur. J. Appl. Math.* **1993**, *4*, 1–38. [CrossRef]
12. Kato, T. *Asymptotic Behavior of Solutions of the Functional Differential Equation $y'(x) = ay(\eta x) + by(x)$, in Delay and Functional Differential Equations and Their Applications*; Academic Press: Cambridge, MA, USA, 1972; pp. 197–217.
13. Ockendon, J.R.; Tayler, A.B. The dynamics of a current collection system for an electric locomotive. *Proc. R. Soc. Lond. A.* **1971**, *322*, 447–468.
14. Agarwal, R.P.; Zhou, Y.; He, Y. Existence of fractional functional differential equations. *Comput. Math. Appl.* **2010**, *59*, 1095–1100. [CrossRef]
15. Boulares, H.; Benchaabane, A.; Pakkaranang, N.; Shafqat, R.; Panyanak, B. Qualitative Properties of Positive Solutions of a Kind for Fractional Pantograph Problems using Technique Fixed Point Theory. *Fractal Fract.* **2022**, *6*, 593. [CrossRef]
16. Boulares, H.; Alqudah, M.A.; Abdeljawad, T. Existence of solutions for a semipositone fractional boundary value pantograph problem. *AIMS Math.* **2022**, *7*, 19510–19519. [CrossRef]
17. Hallaci, A.; Boulares, H.; Ardjouni, A. Existence and uniqueness for delay fractional dif ferential equations with mixed fractional derivatives. *Open J. Math. Anal.* **2020**, *4*, 26–31. [CrossRef]
18. Ardjouni, A.; Boulares, H.; Laskri, Y. Stability in higher-order nonlinear fractional differential equations. *Acta Comment. Tartu. Math.* **2018**, *22*, 37–47. [CrossRef]
19. Podlubny, I. *Fractional Differential Equations, Mathematics in Science and Engineering*; Academic Press: New York, NY, USA, 1999.
20. Mohammed, W.W.; Blömker, D. Fast-diffusion limit for reaction-diffusion equations with multiplicative noise. *J. Math. Anal. Appl.* **2021**, *496*, 124808. [CrossRef]

21. Zhang, X.M.; Agarwal, P.; Liu, Z.H., et al. Existence and uniqueness of solutions for stochastic differential equations of fractional-order $q > 1$ with finite delays. *Adv. Differ. Equ.* **2017**, *2017*, 123.
22. Ye, H.; Gao, J.; Ding, Y. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl*. **2007**, *328*, 1075–1081. [CrossRef]