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On the Variable Order Fractional Calculus Characterization for the Hidden Variable Fractal Interpolation Function

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Abstract: In this study, the variable order fractional calculus of the hidden variable fractal interpolation function is explored. It extends the constant order fractional calculus to the case of variable order. The Riemann–Liouville and the Weyl–Marchaud variable order fractional calculus are investigated for hidden variable fractal interpolation function. Moreover, the conditions for the variable fractional order μ on a specified range are also derived. It is observed that, under certain conditions, the Riemann–Liouville and the Weyl–Marchaud variable order fractional calculus of the hidden variable fractal interpolation function are again the hidden variable fractal interpolation functions interpolating the new data set.

Keywords: hidden variable fractal interpolation function; variable order; Riemann–Liouville fractional calculus; Weyl–Marchaud fractional derivative

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1. Introduction

Traditional approximation approaches of data samples obtained in scientific and natural phenomena are influenced by Euclidean geometry, in which elementary functions such as linear functions, polynomials, trigonometric functions, and exponential functions play a vital role. Interpolation is an effective method in approximation theory, and there are traditional methods reported for interpolating unknown/known data of function, surfaces in Euclidean space that can be used to describe regular objects. However, many objects in natural phenomena such as clouds; forest horizons; the surface of the sea; rocks; and porous objects cannot be easily described using only regular functions in Euclidean geometry. In order to overcome this problem, Mandelbrot invented the fractal theory beyond the standard Euclidean geometric to describe irregular objects. Despite the number of ways available to define the fractals, the effective and simple way is an iterated function system in which fractal can be constructed as a unique attractor. Based on a Hutchinson theorem, Barnsley [1] provides a method for constructing fractal-interpolation function (FIF) using iterated function system theory (IFs). For more information, read [2–7]. The fractal-interpolation function becomes a powerful tool in applied science and engineering for modeling irregular phenomena in the last 30 years. The graph of the fractal-interpolation function is produced as a unique attractor of a certain unique type of iterated function system. In the evolution of FIF theory, many researchers have developed different types of FIFs, one of which is the hidden variable fractal interpolation function. A hidden variable FIF (HVFIF) is a very different, impressive, and irregular than FIF, which was developed by Barnsley et al. [8]. The purpose is to use the FIF approach on an extended data set in $\mathbb{R}^3$ such that the associated FIF's graph projected onto $\mathbb{R}^2$ provide the appropriate interpolation function for the data set $\{(x_k, y_k) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$. Working with $\mathbb{R}^3$ provides us with an extra degree of freedom, resulting in hidden variables and the HVFIF. The HVFIF are useful for modifying the fractal dimension and the shape of the interpolation functions. However, HVFIFs are not suitable for simulating curves that are half self-affine and half non-affine because they can only be used to generate non-self-affine or non-self-similar functions. Chand and Kapoor [9] developed a non-diagonal IFS that devised both self-affine and non-self-affine FIS at the same time based on free and constrained variables on a large set of interpolation data. A CHFIS is the attractor of such an IFS, and it is the preferable choice for investigating highly irregular surfaces such as clouds, sea surfaces, rock surfaces, tsunami waves, and so on. In $\mathbb{R}^2$, Bouboulis and Dalla [10] created a hidden variable vector valued FIFs on random grids. In [11], it is examined that how to build a repetitive FIS and how to calculate its box-counting dimension. The coalescence hidden variable fractal interpolation function (CHVFIF) was introduced by Chand and Kapoor in [12] to approximate both self-affine and non-self-affine functions at the same time. According to a Lipschitz exponent of the associated CHFIS, Kapoor and Srijanani [13] investigate the smoothness of such surfaces.

In general, fractal-interpolation functions are nowhere differentiable though they are everywhere continuous. Hence, researchers have recognized the significance of applying fractional calculus to analyses of fractal functions. Furthermore, the review of the literature demonstrates that fractional calculus significantly influences fractal-interpolation functions and their fractal dimension. Recent years have witnessed a number of proposals for fractal-interpolation functions with regard to constant-order fractional calculus developed in the literature, connecting the fractal-interpolation function to different fractional calculus theories. Srijanai investigated the Riemann–Liouville (**RL**) fractional integral of order v for a CHVFIF in [14]. The Weyl–Marchaud (**WM**) fractional derivative for HVFIF is discussed in [15]. The fractional calculus of the cubic spline hidden variable recurrent fractal-interpolation function was discussed by Ri and Mi-Gyong et al. [16]. For more information, read [17–28]. In [29], it is discussed that differentiating a function $f(x)$ with different integers n for different points x may lead to a discontinuous function, and this can be addressed with the choice of continuously varying order $\mu(x)$, where $\mu(x)$ must take fractional values. Further, since the value changes from point to point, the variable order fractional calculus generalizes the constant-order fractional calculus method. With this advantage, in the present paper, the order for both the fractional calculus methods is chosen as a variable rather than a constant since practical systems are changing with respect to time, and their behavior changes over time because nature always likes uncertainty . Hence, there is interest in fractional calculus with constant order rapidly transferred to variable order fractional calculus (see [29–32]). Needless to say that the research gap, the change of fractal functions when applying the variable order fractional calculus, motivated us to discuss the variable order fractional calculus of HVFIF.

The framework of this paper is follows: the construction of HVFIF is briefly explained in Section 2. The definitions of variable order **RL** fractional calculus and the variable order **WM** fractional derivative are given at the start of Section 3. Further, variable order fractional calculus is applied on HVFIF. Finally, Section 4 offers concluding remarks of the present work.

2. Brief of the Hidden Variable Fractal Interpolation Function

In order to extend the flexibility of the fractal interpolation method, a vector valued fractal function called the hidden variable interpolation function is constructed in [8]. The advantages of HVFIFs are not only limited to flexibility; they also offer greater diversity since their values are continuously dependent on hidden variables. By introducing the idea of constrained free variables and hidden variables, the HVFIF is constructed as the projection of graphs of the vector-valued function. Using the HVFIFs, both self-affine and non-self-affine functions can be approximated simultaneously. For more details of HVFIF, the reader can refer to [8–10,12–17]. In the this section, the construction of the hidden variable fractal interpolation function is recollected as the projection of an attractor of the IFS defined on $\mathbb{R}^3$.

Let a data set $\{(x_k, y_k) \in I \times \mathbb{R} : k = 0, 1, \ldots, N\}$ be given. The given data set is generalized as $\{(x_k, y_k, z_k) \in I \times \mathbb{R}^2 : k = 0, 1, \ldots, N\}$; this notation is used in the construction of an HVFIF, as follows: let $N > 1$ be a positive integer and $I := [x_0, x_N] \subset \mathbb{R}$ such that $x_0 < x_1 < \cdots < x_N$. Set $I_k = [x_{k-1}, x_k]$, and for all $k = 1, 2, \ldots, N$, let $L_k : I \to I_k$ be a contractive homeomorphism that satisfies the following end point condition:

$$L_k(x_0) = x_{k-1}, \quad L_k(x_N) = x_k.$$

Let $F_k : D \to \mathbb{R}^2$ be the N continuous mappings for a compact set D of $I \times \mathbb{R}^2$, which satisfies the following condition:

$$F_k(x_0, y_0, z_0) = (y_{k-1}, z_{k-1}),\ F_k(x_N, y_N, z_N) = (y_k, z_k)$$

$$\rho(F_k(x, y_1, z_1) - F_k(x, y_2, z_2)) \leq t\rho_E((y_1, z_1), (y_2, z_2)) \tag{1}$$

where $(x, y_1, z_1), (y_2, z_2) \in I \times \mathbb{R}^2$, $0 \leq t < 1$, ρ is the sup norm on $I \times \mathbb{R}^2$, and ρ_E is the Euclidean metric on $\mathbb{R}^2$. To define HVFIF, functions L_k and F_k have taken the following:

$$L_k(x) = a_k x + b_k, \tag{2}$$

$$F_k(x, y, z) = \mathcal{A}_k(y, z)^T + (p_k(x), q_k(x))^T, \tag{3}$$

where $\mathcal{A}_k$ is an upper triangular matrix $\begin{pmatrix} \alpha_k & \beta_k \\ 0 & \gamma_k \end{pmatrix}$ and $p_k(x), q_k(x)$ are continuous functions with two arbitrary constants. Choose α_k and γ_k as free variables such that $|\alpha_k| < 1$ and $|\gamma_k| < 1$, β_k as constrained free variables such that $|\beta_k| + |\gamma_k| < 1$. The generalized IFS required for the generating of the HVFIF corresponding to data $\{(x_k, y_k, z_k) : k = 0, 1, \ldots, N\}$ is described as

$$\{\mathbb{R}^2; w_k(x, y, z) = L_k(x), F_k(x, y, z) : k = 1, 2, \ldots, N\}. \tag{4}$$

With respect to the metric $d_M\big((x, y, z), (x^*, y^*, z^*)\big) = |x - x^*| + \theta d\big((y, z), (y^*, z^*)\big)$, where $\theta = \frac{1-a}{2t}$, $a = \max\{\frac{x_k - x_{k-1}}{x_N - x_0} : k = 1, 2, \ldots, N\}$, which is equivalent to the Euclidean metric on $\mathbb{R}$; the IFS (4) is now a hyperbolic. As a consequence, there is a unique nonempty compact set $G \subseteq \mathbb{R}^3$ called the attractor. The attractor G is the graph of the continuous vector-valued function $f : I \to \mathbb{R}^2$ such that $f(x_k) = (y_k, z_k)$ for all $k = 0, 1, \ldots, N$. Letting $f = (f_1, f_2)$, it follows that the continuous function $f_1 : I \to \mathbb{R}$ interpolating the given data (x_k, y_k) is called the *hidden variable fractal interpolation function* (HVFIF). Similarly, the fractal function f_2 interpolating the data set $\{(x_k, z_k) : k = 0, 1, \ldots, N\}$ is the projection [8]. The continuous function f, which can be written as $f(x) = (f_1(x), f_2(x))$, is the graph of the attractor of the IFS provided in (4) if and only if the fixed point of the Read–Bajraktarević (RB) operator T, say the space of continuous vector-valued functions $f : I \to \mathbb{R}^2$, satisfies

$$Tf(x) = f(x) = F_k(L_k^{-1}(x), f(L_k^{-1}(x))) \tag{5}$$

for all $k = 1, 2, \ldots, N$, which can be equivalently written as

$$(f_1(x), f_2(x)) = F_k(L_k^{-1}(x), f_1(L_k^{-1}(x), f_2(L_k^{-1}(x))). \tag{6}$$

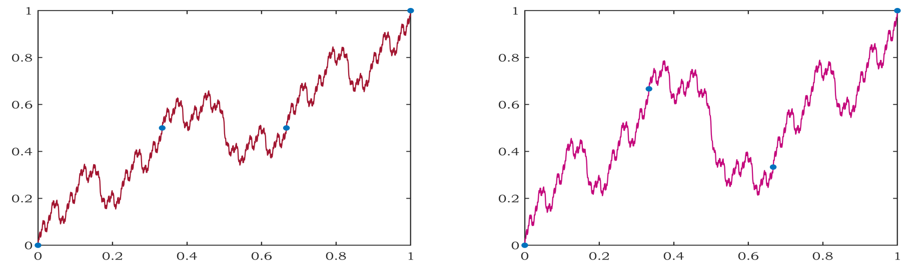
Component-wise, the image Tf of the vector valued function f can be represented as (Tf_1, Tf_2); then, for all $x \in I$, HVFIF $f_1(x)$ satisfies

$$Tf_1(L_k(x)) = f_1(L_k(x)) = F_{1k}(x, f_1(x), f_2(x)) = \alpha_k f_1(x) + \beta_k f_2(x) + p_k(x) \tag{7}$$

similarly, fractal function $f_2(x)$ satisfies

$$Tf_2(L_k(x)) = f_2(L_k(x)) = F_{2k}(x, f_2(x)) = \gamma_k f_2(x) + q_k(x). \tag{8}$$

Example 1. *Consider a data set* $(0, 0, 0), (1/3, 1/2, 2/3), (2/3, 1/2, 1/3), (1, 1, 1)$ *with the scale vectors* $\alpha = (0.3, 0.3, 0.3)$, $\beta = (0.2, 0.2, 0.2)$, *and* $\gamma = (0.5, 0.5, 0.5)$. *Figure 1a,b depict* *the graphical representation of the non-self-affine fractal function f_1 and the self-affine FIF f_2, respectively. The influence of the free variable α_k and the constrained free variable β_k can be visualized in Figure 1a, whereas Figure 1b clearly depicts the influence of the variable γ_k. In both the figures, the variables are chosen in such a way that the constraints $|\alpha_k| < 1$ and $|\beta_k| + |\gamma_k| < 1$ are satisfied.*



(**a**) (**b**)

Figure 1. Hidden variable fractal interpolation function: (**a**) non-self-affine f_1, (**b**) self-affine f_2.

3. Main Results

This section narrates the **RL** variable order fractional calculus and the **WM** variable order fractional derivative of the HVFIF. The **RL** fractional integral of variable order $\mu(s)$ where $\mu : [a, b] \to (0, 1)$ for a continuous function f is defined by

$$\mathcal{I}_a^{\mu(s)} f(x) = \frac{1}{\Gamma(\mu(s))} \int_a^x (x-t)^{\mu(s)-1} f(t) dt. \tag{9}$$

Suppose that f is an HVFIF decided by the IFS (4); then, the **RL** fractional integral of f with order $\mu(s)$ is defined as follows:

$$\mathcal{I}_a^{\mu(s)} f_1(x) = \frac{1}{\Gamma(\mu(s))} \int_a^x (x-t)^{\mu(s)-1} f_1(t) dt, \tag{10}$$

$$\mathcal{I}_a^{\mu(s)} f_2(x) = \frac{1}{\Gamma(\mu(s))} \int_a^x (x-t)^{\mu(s)-1} f_2(t) dt. \tag{11}$$

Theorem 1. *Let f be the HVFIF given in Equations (2) and (3). If $\|\mathcal{A}_k'\| < 1$ where $\mathcal{A}_k' = a_k^{\mu(s)} \begin{bmatrix} \alpha_k & \beta_k \\ 0 & \gamma_k \end{bmatrix}$, for all $k = 1, 2, \ldots, N$, then $(\mathcal{I}_{x_0}^{\mu(s)} f)(x) = ((\mathcal{I}_{x_0}^{\mu(s)} f_1)(x), (\mathcal{I}_{x_0}^{\mu(s)} f_2)(x))$ is also an HVFIF. Further, $(\mathcal{I}_{x_0}^{\mu(s)} f_1)(x)$ and $(\mathcal{I}_{x_0}^{\mu(s)} f_2)(x)$ are determined by the IFSs $\left\{L_k(x), \hat{F}_{1(k,\mu(s))}(x, \hat{y}, \hat{z})\right\}_{k=1}^N$ and $\left\{L_k(x), \hat{F}_{2(k,\mu(s))}(x, \hat{z})\right\}_{k=1}^N$, respectively, where $\hat{F}_{1(k,\mu(s))}(x, \hat{y}, \hat{z}) = a_k^{\mu(s)} \alpha_k y + a_k^{\mu(s)} \beta_k z + \hat{p}_k(x)$ and $\hat{F}_{2(k,\mu(s))} = a_k^{\mu(s)} \gamma_k z + \hat{q}_k(x)$ with $\sum_{k=1}^N a_k^{\mu(s)} \alpha_k \neq 1$, $\sum_{k=1}^N a_k^{\mu(s)} \gamma_k \neq 1$, for $k = 1, 2, \ldots, N$.*

Proof. The **RL** variable order fractional integral of f_1 is given by,

$$\begin{aligned} \mathcal{I}_{x_0}^{\mu(s)} f_1(L_k(x)) = &\frac{1}{\Gamma(\mu(s))} \int_{x_0}^{x_{k-1}} (x_{k-1}-t)^{\mu(s)-1} f_1(t) dt \\ &+ \frac{1}{\Gamma(\mu(s))} \int_{x_0}^{x_{k-1}} \Big((L_k(x)-t)^{\mu(s)-1} - (x_{k-1}-t)^{\mu(s)-1} \Big) f_1(t) dt \\ &+ \frac{1}{\Gamma(\mu(s))} \int_{x_{k-1}}^{L_k(x)} (L_k(x)-t)^{\mu(s)-1} f_1(t) dt. \end{aligned}$$

In the third term, replace $t = L_k(u)$,

$$\mathcal{I}_{x_0}^{\mu(s)} f_1(L_k(x)) = \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + \frac{a_k}{\Gamma(\mu(s))} \int_{x_0}^{x} (L_k(x) - L_k(u))^{\mu(s)-1} f_1(L_k(u)) du.$$

The following equation is obtained using the functional equation in Equation (7):

$$\begin{aligned}\mathcal{I}_{x_0}^{\mu(s)} f_1(L_k(x)) &= \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + \frac{a_k^{\mu(s)}}{\Gamma(\mu(s))} \int_{x_0}^{x} (x-u)^{\mu(s)-1} (\alpha_k f_1(u) + \beta_k f_2(u) + p_k(x)) du \\ &= \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + a_k^{\mu(s)} \alpha_k (\mathcal{I}_{x_0}^{\mu(s)} f_1)(x) + a_k^{\mu(s)} \beta_k (\mathcal{I}_{x_0}^{\mu(s)} f_2)(x) + a_k^{\mu(s)} (\mathcal{I}_{x_0}^{\mu(s)} p_k)(x) \\ &= a_k^{\mu(s)} \alpha_k (\mathcal{I}_{x_0}^{\mu(s)} f_1)(x) + a_k^{\mu(s)} \beta_k (\mathcal{I}_{x_0}^{\mu(s)} f_2)(x) + \hat{p}_k(x).\end{aligned}$$

using the notation $\hat{p}_k(x) = \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + a_k^{\mu(s)} (\mathcal{I}_{x_0}^{\mu(s)} p_k)(x)$, $\hat{y}_{k-1} = \frac{1}{\Gamma(\mu(s))} \int_{x_0}^{x_{k-1}}$ $(x_{k-1} - t)^{\mu(s)-1} f_1(t) dt$ and $f_{1(k,\mu(s))}(x) = \frac{1}{\Gamma(\mu(s))} \int_{x_0}^{x_{k-1}} \Big((L_k(x) - t)^{\mu(s)-1} - (x_{k-1} - t)^{\mu(s)-1}$ $\Big) f_1(t) dt$. Now consider the **RL** variable order fractional integral of f_2 of order $\mu(s)$

$$\begin{aligned}\mathcal{I}_{x_0}^{\mu(s)} f_2(L_k(x)) = &\frac{1}{\Gamma(\mu(s))} \int_{x_0}^{x_{k-1}} (x_{k-1} - t)^{\mu(s)-1} f_2(t) dt \\ &+ \frac{1}{\Gamma(\mu(s))} \int_{x_0}^{x_{k-1}} \Big((L_k(x) - t)^{\mu(s)-1} - (x_{k-1} - t)^{\mu(s)-1} \Big) f_2(t) dt \\ &+ \frac{1}{\Gamma(\mu(s))} \int_{x_{k-1}}^{L_k(x)} (L_k(x) - t)^{\mu(s)-1} f_2(t) dt.\end{aligned}$$

In third term, replace $t = L_k(u)$,

$$\mathcal{I}_{x_0}^{\mu(s)} f_2(L_k(x)) = \hat{y}_{k-1} + f_{2(k,\mu(s))}(x) + \frac{a_k}{\Gamma(\mu(s))} \int_{x_0}^{x} (L_k(x) - L_k(u))^{\mu(s)-1} f_2(L_k(u)) du.$$

Using the functional equation in Equation (8), the following equation is obtained:

$$\begin{aligned}\mathcal{I}_{x_0}^{\mu(s)} f_2(L_k(x)) &= \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + \frac{a_k^{\mu(s)}}{\Gamma(\mu(s))} \int_{x_0}^{x} (x-u)^{\mu(s)-1} (\gamma_k f_2(u) + q_k(x)) du \\ &= \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + a_k^{\mu(s)} \gamma_k (\mathcal{I}_{x_0}^{\mu(s)} f_2)(x) + a_k^{\mu(s)} (\mathcal{I}_{x_0}^{\mu(s)} q_k)(x) \\ &= a_k^{\mu(s)} \gamma_k (\mathcal{I}_{x_0}^{\mu(s)} f_1)(x) + \hat{q}_k(x).\end{aligned}$$

Apply the notation $\hat{q}_k(x) = \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + a_k^{\mu(s)} (\mathcal{I}_{x_0}^{\mu(s)} q_k)(x)$, $\hat{z}_{k-1} = \frac{1}{\Gamma(\mu(s))} \int_{x_0}^{x_{k-1}}$ $(x_{k-1} - t)^{\mu(s)-1} f_2(t) dt$ and $f_{2(k,\mu(s))}(x) = \frac{1}{\Gamma(\mu(s))} \int_{x_0}^{x_{k-1}} \Big((L_k(x) - t)^{\mu(s)-1} - (x_{k-1} - t)^{\mu(s)-1}$ $\Big) f_2(t) dt$. Hence, the **RL** variable order fractional integral of HVFIF is also HVFIF. Consider the following equation for calculating new data points:

$$\begin{aligned}\mathcal{I}_{x_0}^{\mu(s)} f_1(L_k(x)) = &\hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + a_k^{\mu(s)} \alpha_k (\mathcal{I}_{x_0}^{\mu(s)} f_1)(x) + a_k^{\mu(s)} \beta_k (\mathcal{I}_{x_0}^{\mu(s)} f_2)(x) \\ &+ a_k^{\mu(s)} (\mathcal{I}_{x_0}^{\mu(s)} p_k)(x).\end{aligned}$$

Let $x = x_N$ and $L_k(x) = x_k$. Then, the above equation provides,

$$\begin{aligned}\hat{y}_k - \hat{y}_{k-1} = &f_{1(k,\mu(s))}(x_N) + a_k^{\mu(s)} \alpha_k \hat{y}_N + a_k^{\mu(s)} \beta_k (\mathcal{I}_{x_0}^{\mu(s)} f_2)(x_N) \\ &+ a_k^{\mu(s)} (\mathcal{I}_{x_0}^{\mu(s)} p_k)(x_N).\end{aligned}$$

The equation above is as follows when considered form the perspective of the equation $\hat{y}_k = \hat{y}_0 + \sum_{n=1}^{k}(\hat{y}_n - \hat{y}_{n-1})$:

$$\hat{y}_k = \sum_{n=1}^{k}\left(f_{1(n,\mu(s))}(x_N) + a_n^{\mu(s)}\alpha_n\hat{y}_N + a_n^{\mu(s)}\beta_n(\mathcal{I}_{x_0}^{\mu(s)}f_2)(x_N) + a_n^{\mu(s)}(\mathcal{I}_{x_0}^{\mu(s)}p_n)(x_N)\right). \quad (12)$$

By changing k to N in Equation (12), $\hat{y}_N$ is obtained as,

$$\hat{y}_N = \sum_{n=1}^{N}\left(f_{1(n,\mu(s))}(x_N) + a_n^{\mu(s)}\beta_n(\mathcal{I}_{x_0}^{\mu(s)}f_2)(x_N) + a_n^{\mu(s)}(\mathcal{I}_{x_0}^{\mu(s)}p_n)(x_N)\right)\Bigg/1 - \sum_{n=1}^{N}a_n^{\mu(s)}\alpha_n.$$

Consider the following equation to determine the new data set $\{(x_k, \hat{z}_k) : K = 0, 1, \ldots, N\}$.

$$\mathcal{I}_{x_0}^{\mu(s)}f_2(L_k(x)) = \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + a_k^{\mu(s)}\gamma_k(\mathcal{I}_{x_0}^{\mu(s)}f_2)(x) + a_k^{\mu(s)}(\mathcal{I}_{x_0}^{\mu(s)}q_k)(x).$$

Let $x = x_N$ and $L_k(x) = x_k$. The above equation then yields,

$$\hat{z}_k - \hat{z}_{k-1} = f_{2(k,\mu(s))}(x_N) + a_k^{\mu(s)}\gamma_k\hat{z}_N + a_k^{\mu(s)}(\mathcal{I}_{x_0}^{\mu(s)}q_k)(x_N).$$

The equation above is as follows when considered from the perspective of the equation $\hat{z}_k = \hat{z}_0 + \sum_{n=1}^{k}(\hat{z}_n - \hat{z}_{n-1})$:

$$\hat{z}_k = \sum_{n=1}^{k}\left(f_{2(n,\mu(s))}(x_N) + a_n^{\mu(s)}\gamma_n\hat{z}_N + a_n^{\mu(s)}(\mathcal{I}_{x_0}^{\mu(s)}q_n)(x_N)\right). \quad (13)$$

By changing k to N in Equation (13), $\hat{z}_N$ is obtained as

$$\hat{z}_N = \sum_{n=1}^{N}\left(f_{2(n,\mu(s))}(x_N) + a_n^{\mu(s)}(\mathcal{I}_{x_0}^{\mu(s)}q_n)(x_N)\right)\Bigg/1 - \sum_{n=1}^{N}a_n^{\mu(s)}\gamma_n.$$

□

Remark 1. *Theorem 1 shows the variable order **RL** fractional integral of HVFIF with the predefined condition* $(\mathcal{I}_{x_0}^{\mu(s)}f_1)(x_0) = (\mathcal{I}_{x_0}^{\mu(s)}f_2)(x_0) = 0$*. Likewise, the the **RL** variable order fractional integral f is defined with the upper limit x_N as*

$$\mathcal{I}_{x_N}^{\mu(s)}f_1(L_k(x)) = -\frac{1}{\Gamma(\mu(s))}\int_x^{x_N}(L_k(x) - t)^{\mu(s)-1}f_1(t)dt$$

$$\mathcal{I}_{x_N}^{\mu(s)}f_2(L_k(x)) = -\frac{1}{\Gamma(\mu(s))}\int_x^{x_N}(L_k(x) - t)^{\mu(s)-1}f_2(t)dt$$

Then, one can obtain that the function $(\mathcal{I}_{x_N}^{\mu(s)}f(L_k(x)))$ is again HVFIF with variable order generated by the IFS $\{L_k(x), \hat{F}_{1(k,\mu(s))}(x, \hat{y}, \hat{z})\}_{i=1}^{N}$ and $\{L_k(x), \hat{F}_{2(k,\mu(s))}(x, \hat{z})\}_{i=1}^{N}$ where

$$\hat{f}_1(x) = \hat{y}_k - f_{1(k,\mu(s))}(x) + a_k^{\mu(s)}\alpha_k(\mathcal{I}_{x_N}^{\mu(s)} f_1)(x) + a_k^{\mu(s)}\beta_k(\mathcal{I}_{x_N}^{\mu(s)} f_2)(x) + a_k^{\mu(s)}(\mathcal{I}_{x_N}^{\mu(s)} p_k)(x)$$

and $\hat{f}_2(x) = \hat{z}_k - f_{2(k,\mu(s))}(x) + a_k^{\mu(s)}\gamma_k(\mathcal{I}_{x_N}^{\mu(s)} f_2)(x) + a_k^{\mu(s)}(\mathcal{I}_{x_N}^{\mu(s)} q_k)(x)$

$$\begin{aligned}
\hat{p}_k(x) &= \hat{y}_k - f_{1(k,\mu(s))}(x) + a_k^{\mu(s)}(\mathcal{I}_{x_N}^{\mu(s)} p_k)(x),\\
\hat{y}_{k-1} &= -\sum_{n=1}^{k}\Big(f_{1(n,\mu(s))}(x_0) - a_n^{\mu(s)}\alpha_n\hat{y}_0 - a_n^{\mu(s)}\beta_n(\mathcal{I}_{x_N}^{\mu(s)} f_2)(x_0)\\
&\quad + a_n^{\mu(s)}(\mathcal{I}_{x_N}^{\mu(s)} p_n)(x_0)\Big),\\
\hat{y}_0 &= -\sum_{n=1}^{N}\Big(f_{1(n,\mu(s))}(x_0) - a_n^{\mu(s)}\beta_n(\mathcal{I}_{x_N}^{\mu(s)} f_2)(x_0)\\
&\quad + a_n^{\mu(s)}(\mathcal{I}_{x_N}^{\mu(s)} p_n)(x_0)\Big)\Big/ 1 - \sum_{n=1}^{N} a_n^{\mu(s)}\alpha_n,\\
\hat{q}_k(x) &= \hat{z}_k - f_{2(k,\mu(s))}(x) + a_k^{\mu(s)}(\mathcal{I}_{x_N}^{\mu(s)} q_k)(x),\\
\hat{z}_{k-1} &= -\sum_{n=1}^{k}\Big(f_{2(n,\mu(s))}(x_0) - a_n^{\mu(s)}\gamma_n\hat{z}_0 - a_n^{\mu(s)}(\mathcal{I}_{x_N}^{\mu(s)} q_n)(x_0)\Big),\\
\hat{z}_0 &= -\sum_{n=1}^{N}\Big(f_{2(n,\mu(s))}(x_0) - a_n^{\mu(s)}(\mathcal{I}_{x_N}^{\mu(s)} q_n)(x_0)\Big)\Big/ 1 - \sum_{n=1}^{N} a_n^{\mu(s)}\gamma_n.
\end{aligned}$$

If the variable order $\mu(s)$ is taken as a constant function, the above result coincides with the result presented by Srijanani in [14]. Additionally, if we take $\mu = 1$, then the above result coincides with the result presented by Chand [9]. The above remark gives the **RL** variable order fractional integral of HVFIF with the predefined condition $(\mathcal{I}_{x_N}^{\mu(s)} f)(x_N) = 0$.

Example 2. *Observe the data points in Example 1 with the same scale vectors. Figure 2a presents the* **RL** *variable order fractional integral of f_1 with order $\mu(s) = 0.5$. Similarly, the graphical representation of the* **RL** *variable order fractional integral of f_2 with order $\mu(s) = 0.5$ is displayed in Figure 2b. In both Figure 2a,b, the fractional order $\mu(s)$ is chosen to be the same $\mu(s) = 0.5$ for clear comparison.*

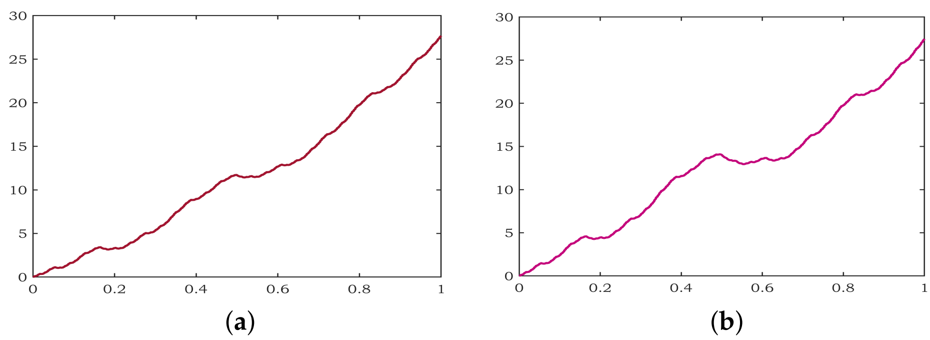


Figure 2. The **RL** fractional integral: (**a**) $I^{\mu(s)} f_1$ with variable order $\mu(s) = 0.5$, (**b**) $I^{\mu(s)} f_2$ with order $\mu(s) = 0.5$.

The following theorem explains the **RL** variable order fractional derivative of HVFIF with the preset initial condition $(\mathcal{D}_{x_0}^{\mu(s)} f)(x_0) = 0$. The **RL** fractional derivative of variable order $\mu(s)$ where $\mu : [a, b] \to (0, 1)$, for a continuous function f is defined by the equation

$$\mathcal{D}_a^{\mu(s)} f(x) = \frac{1}{1 - \Gamma(\mu(s))}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\mu(s)}}dt. \tag{14}$$

Suppose f is an HVFIF decisived by IFS (4); then, the **RL** fractional derivative of f of order $\mu(s)$ is defined as follows:

$$\mathcal{D}_a^{\mu(s)} f_1(x) = \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_a^x \frac{f_1(t)}{(x-t)^{\mu(s)}} dt, \tag{15}$$

$$\mathcal{D}_a^{\mu(s)} f_2(x) = \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_a^x \frac{f_2(t)}{(x-t)^{\mu(s)}} dt. \tag{16}$$

Theorem 2. *Let f be the HVFIF given in Equations (2) and (3). If $\|\mathcal{A}_k'\| < 1$ where $\mathcal{A}_k' = a_k^{1-\mu(s)} \begin{bmatrix} \alpha_k & \beta_k \\ 0 & \gamma_k \end{bmatrix}$, for all $k = 1, 2, \ldots, N$, then $(\mathcal{D}_{x_0}^{\mu(s)} f)(x) = (\mathcal{D}_{x_0}^{\mu(s)} f_1)(x), (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x)$ is also an HVFIF. Further, $(\mathcal{D}_{x_0}^{\mu(s)} f_1)(x)$ and $(\mathcal{D}_{x_0}^{\mu(s)} f_2)(x)$ are determined by the IFSs $\left\{L_k(x), \hat{F}_{1(k,\mu(s))}(x, \hat{y}, \hat{z})\right\}_{k=1}^N$ and $\left\{L_k(x), \hat{F}_{2(k,\mu(s))}(x, \hat{z})\right\}_{k=1}^N$, respectively, where $\hat{F}_{1(k,\mu(s))}(x, \hat{y}, \hat{z}) = a_k^{1-\mu(s)} \alpha_k y + a_k^{1-\mu(s)} \beta_k z + \hat{p}_k(x)$ and $\hat{F}_{2(k,\mu(s))} = a_k^{1-\mu(s)} \gamma_k z + \hat{q}_k(x)$ with $\sum_{k=1}^N a_k^{1-\mu(s)} \alpha_k \neq 1$, $\sum_{k=1}^N a_k^{1-\mu(s)} \gamma_k \neq 1$, for $k = 1, 2, \ldots, N$.*

Proof. The **RL** fractional derivative of f_1 is given by,

$$\begin{aligned}
\mathcal{D}_{x_0}^{\mu(s)} f_1(L_k(x)) &= \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x_{k-1}} \frac{f_1(t)}{(L_k(x)-t)^{\mu(s)}} dt \\
&+ \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x_{k-1}} \left(\frac{f_1(t)}{(x_{k-1}-t)^{\mu(s)}} - \frac{f_1(t)}{(x_{k-1}-t)^{\mu(s)}} \right) dt \\
&+ \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_{k-1}}^{L_k(x)} \frac{f_1(t)}{(L_k(x)-t)^{\mu(s)}} du.
\end{aligned}$$

Replace the variable $t = L_k(u)$ in third term,

$$\mathcal{D}_{x_0}^{\mu(s)} f_1(L_k(x)) = \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + \frac{a_k}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x} \frac{f_1(L_k(u))}{(L_k(x)-L_k(u))^{\mu(s)}} du.$$

Using the functional equation in Equation (7), we obtained the following equation:

$$\begin{aligned}
(\mathcal{D}_{x_0}^{\mu(s)} f_1)(L_k(x)) &= \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + \frac{a_k}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x} \frac{(\alpha_k f_1(u) + \beta_k f_2(u) + p_k(x))}{a_k^{\mu(s)} (x-u)^{\mu(s)}} \\
&= \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + a_k^{1-\mu(s)} \alpha_k (\mathcal{D}_{x_0}^{\mu(s)} f_1)(x) + a_k^{1-\mu(s)} \beta_k (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x) \\
&+ a_k^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} p_k)(x) \\
&= a_k^{1-\mu(s)} \alpha_k (\mathcal{D}_{x_0}^{\mu(s)} f_1)(x) + a_k^{1-\mu(s)} \beta_k (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x) + \hat{p}_k(x).
\end{aligned}$$

Here, take $\hat{p}_k(x) = \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + a_k^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} p_k)(x)$, $\hat{y}_{k-1} = \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x_{k-1}} \frac{f_1(t)}{(x_{k-1}-t)^{\mu(s)}} dt$ and $f_{1(k,\mu(s))}(x) = \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x_{k-1}} (\frac{f_1(t)}{(L_k(x)-t)^{\mu(s)}} - \frac{f_1(t)}{(x_{k-1}-t)^{\mu(s)}}) dt$. Now, consider the **RL** fractional derivative of f_2 of order $\mu(s)$

$$\begin{aligned}
\mathcal{D}_{x_0}^{\mu(s)} f_2(L_k(x)) &= \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x_{k-1}} \frac{f_2(t)}{(L_k(x)-t)^{\mu(s)}} dt \\
&+ \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x_{k-1}} \left(\frac{f_2(t)}{(x_{k-1}-t)^{\mu(s)}} - \frac{f_2(t)}{(x_{k-1}-t)^{\mu(s)}} \right) dt \\
&+ \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_{k-1}}^{L_k(x)} \frac{f_2(t)}{(L_k(x)-t)^{\mu(s)}} dt.
\end{aligned}$$

In the third term, replace $t = L_k(u)$

$$\mathcal{D}_{x_0}^{\mu(s)} f_1(L_k(x)) = \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + \frac{a_k}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x} \frac{f_2(L_k(u))}{(L_k(x) - L_k(u))^{\mu(s)}} du.$$

The following equation is obtained using the functional equation in Equation (8):

$$\begin{aligned}(\mathcal{D}_{x_0}^{\mu(s)} f_2)(L_k(x)) &= \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + \frac{a_k}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x} \frac{(\gamma_k f_2(u) + q_k(x))}{a_k^{\mu(s)} (x-u)^{\mu(s)}} \\ &= \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + a_k^{1-\mu(s)} \gamma_k (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x) + a_k^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} q_k)(x) \\ &= a_k^{1-\mu(s)} \gamma_k (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x) + \hat{q}_k(x).\end{aligned}$$

Here, $\hat{q}_k(x) = \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + a_k^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} q_k)(x)$, $\hat{z}_{k-1} = \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x_{k-1}} \frac{f_2(t)}{(x_{k-1}-t)^{\mu(s)}} dt$ and $f_{2(k,\mu(s))}(x) = \frac{1}{1-\Gamma(\mu(s))} \frac{d}{dx} \int_{x_0}^{x_{k-1}} (\frac{f_2(t)}{(L_k(x)-t)^{\mu(s)}} - \frac{f_2(t)}{(x_{k-1}-t)^{\mu(s)}}) dt$. Hence, the **RL** variable order fractional derivative of HVFIF is also HVFIF. Consider the following:

$$\begin{aligned}\mathcal{D}_{x_0}^{\mu(s)} f_1(L_k(x)) = \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + a_k^{1-\mu(s)} \alpha_k (\mathcal{D}_{x_0}^{\mu(s)} f_1)(x) + a_k^{1-\mu(s)} \beta_k (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x) \\ + a_k^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} p_k)(x).\end{aligned}$$

Let $x = x_N$ and $L_k(x) = x_k$. Then, the above equation provides,

$$\begin{aligned}\hat{y}_k - \hat{y}_{k-1} = f_{1(k,\mu(s))}(x_N) + a_k^{1-\mu(s)} \hat{y}_N + a_k^{1-\mu(s)} \beta_k (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x_N) \\ + a_k^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} p_k)(x_N).\end{aligned}$$

We know $\hat{y}_k = \hat{y}_0 + \sum_{n=1}^{k} (\hat{y}_n - \hat{y}_{n-1})$, the aforementioned equation is as follows:

$$\begin{aligned}\hat{y}_k = \sum_{n=1}^{k} \Big(f_{1(n,\mu(s))}(x_N) + a_n^{1-\mu(s)} \alpha_n (\mathcal{D}_{x_0}^{\mu(s)} f_1)(x_N) + a_n^{1-\mu(s)} \beta_n (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x_N) \\ + a_n^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} p_n)(x_N) \Big).\end{aligned} \tag{17}$$

In Equation (17), $\hat{y}_N$ is obtained by converting k to N,

$$\begin{aligned}\hat{y}_N = \sum_{n=1}^{N} \Big(f_{1(n,\mu(s))}(x_N) + a_n^{1-\mu(s)} \beta_n (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x_N) \\ + a_n^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} p_n)(x_N) \Big) \Big/ 1 - \sum_{n=1}^{N} a_n^{1-\mu(s)} \alpha_n.\end{aligned}$$

Consider the following equation to determine the new data set $\{(x_k, \hat{z}_k) : K = 0, 1, \ldots, N\}$

$$\mathcal{D}_{x_0}^{\mu(s)} f_2(L_k(x)) = \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + a_k^{1-\mu(s)} \gamma_k (\mathcal{D}_{x_0}^{\mu(s)} f_2)(x) + a_k^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} q_k)(x).$$

Let $x = x_N$ and $L_k(x) = x_k$

$$\hat{z}_k - \hat{z}_{k-1} = f_{2(k,\mu(s))}(x_N) + a_k^{1-\mu(s)} \gamma_k \hat{z}_N + a_k^{1-\mu(s)} (\mathcal{D}_{x_0}^{\mu(s)} q_k)(x_N).$$

We know $\hat{z}_k = \hat{z}_0 + \sum_{n=1}^{k}(\hat{z}_n - \hat{z}_{n-1})$; the aforementioned equation is as follows:

$$\hat{z}_k = \sum_{n=1}^{k}\left(f_{2(n,\mu(s))}(x_N) + a_n^{1-\mu(s)}\gamma_n\hat{z}_N + a_n^{1-\mu(s)}(\mathcal{D}_{x_0}^{\mu(s)}q_n)(x_N)\right). \tag{18}$$

In the Equation (18), $\hat{z}_N$ is obtained by converting k to N

$$\hat{z}_N = \sum_{n=1}^{N}\left(f_{2(n,\mu(s))}(x_N) + a_n^{1-\mu(s)}(\mathcal{D}_{x_0}^{\mu(s)}q_n)(x_N)\right)\Big/ 1 - \sum_{n=1}^{N}a_n^{1-\mu(s)}\gamma_n.$$

□

Remark 2. *Theorem 2 describes the* ***RL*** *variable order fractional derivative of HVFIF with the predefined condition* $(\mathcal{D}_{x_0}^{\mu(s)}f_1)(x_0) = (\mathcal{D}_{x_0}^{\mu(s)}f_2)(x_0) = 0$. *Additionally, the* ***RL*** *variable order fractional derivative* f *is defined with the upper limit* x_N *as*

$$\mathcal{D}_{x_N}^{\mu(s)}f_1(L_k(x)) = -\frac{1}{1-\Gamma(\mu(s))}\frac{d}{dx}\int_x^{x_N}\frac{f_1(t)}{(L_k(x)-t)^{\mu(s)}}dt$$

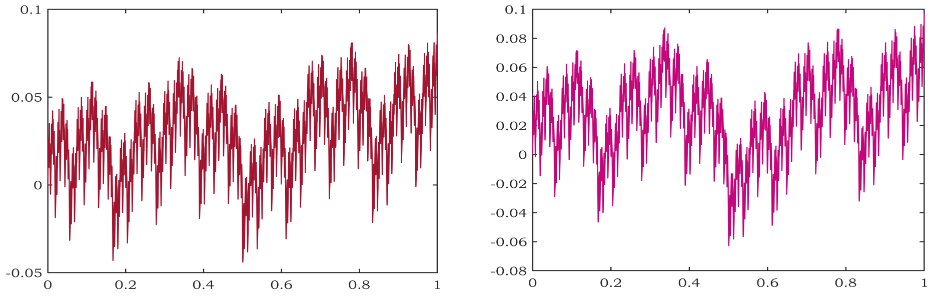
$$\mathcal{D}_{x_N}^{\mu(s)}f_2(L_k(x)) = -\frac{1}{1-\Gamma(\mu(s))}\frac{d}{dx}\int_x^{x_N}\frac{f_2(t)}{(L_k(x)-t)^{\mu(s)}}dt$$

then one can obtain that function $(\mathcal{D}_{x_N}^{\mu(s)}f(L_k(x)))$ *is again HVFIF with the variable order generated by the IFS* $\{L_k(x), \hat{F}_{1(k,\mu(s))}(x,\hat{y},\hat{z})\}_{i=1}^{N}$ *and* $\{L_k(x), \hat{F}_{2(k,\mu(s))}(x,\hat{z})\}_{i=1}^{N}$ *where* $\hat{f}_1(x) = \hat{y}_k - f_{1(k,\mu(s))}(x) + a_k^{1-\mu(s)}\alpha_k(\mathcal{D}_{x_N}^{\mu(s)}f_1)(x) + a_k^{1-\mu(s)}\beta_k(\mathcal{D}_{x_N}^{\mu(s)}f_2)(x) + a_k^{1-\mu(s)}(\mathcal{D}_{x_N}^{\mu(s)}p_k)(x)$ *and* $\hat{f}_2(x) = \hat{z}_k - f_{2(k,\mu(s))}(x) + a_k^{1-\mu(s)}\gamma_k(\mathcal{D}_{x_N}^{\mu(s)}f_2)(x) + a_k^{1-\mu(s)}(\mathcal{D}_{x_N}^{\mu(s)}q_k)(x)$

$$\begin{aligned}
\hat{p}_k(x) &= \hat{y}_k - f_{1(k,\mu(s))}(x) + a_k^{1-\mu(s)}(\mathcal{D}_{x_N}^{\mu(s)}p_k)(x),\\
\hat{y}_{k-1} &= -\sum_{n=1}^{k}\Big(f_{1(n,\mu(s))}(x_0) - a_n^{1-\mu(s)}\alpha_n\hat{y}_0 - a_n^{1-\mu(s)}\beta_n(\mathcal{D}_{x_N}^{\mu(s)}f_2)(x_0)\\
&\quad - a_n^{1-\mu(s)}(\mathcal{D}_{x_N}^{\mu(s)}p_n)(x_0)\Big),\\
\hat{y}_0 &= -\sum_{n=1}^{N}\Big(f_{1(n,\mu(s))}(x_0) - a_n^{1-\mu(s)}\beta_n(\mathcal{D}_{x_N}^{\mu(s)}f_2)(x_0)\\
&\quad - a_n^{1-\mu(s)}(\mathcal{D}_{x_N}^{\mu(s)}p_n)(x_0)\Big)\Big/ 1 - \sum_{n=1}^{N}a_n^{1-\mu(s)}\alpha_n,\\
\hat{q}_k(x) &= \hat{z}_k - f_{2(k,\mu(s))}(x) + a_k^{1-\mu(s)}(\mathcal{D}_{x_N}^{\mu(s)}q_k)(x),\\
\hat{z}_{k-1} &= -\sum_{n=1}^{k}\left(f_{2(n,\mu(s))}(x_0) - a_n^{1-\mu(s)}\gamma_n\hat{z}_0 - a_n^{1-\mu(s)}(\mathcal{D}_{x_N}^{\mu(s)}q_n)(x_0)\right),\\
\hat{z}_0 &= -\sum_{n=1}^{N}\left(f_{2(n,\mu(s))}(x_0) - a_n^{1-\mu(s)}(\mathcal{D}_{x_N}^{\mu(s)}q_n)(x_0)\right)\Big/ 1 - \sum_{n=1}^{N}a_n^{1-\mu(s)}\gamma_n.
\end{aligned}$$

If the variable order $\mu(s)$ is considered as a constant function, the above conclusion gives the Srijanani's result [14]. The **RL** variable order fractional derivative of HVFIF with the predefined condition $(\mathcal{D}_{x_N}^{\mu(s)}f)(x_N) = 0$ is stated by the above remark.

Example 3. *Consider the same data points and scale factors in Example 1 to obtain the* ***RL*** *variable order fractional derivative of* f_1 *and the* ***RL*** *variable order fractional derivative of* f_2. *Figure 3a shows that* $\mathcal{D}_{0.5}f_1$ *and Figure 3b depict graphical representations of* $\mathcal{D}_{0.5}f_2$. *The same choice of* *the fractional order $\mu(s)$ helps to observe the non-self affine and self-affine nature of $\mathcal{D}^{\mu(s)}f_1$ and $\mathcal{D}^{\mu(s)}f_2$ in Figure 3a,b respectively.*



(**a**) (**b**)

Figure 3. The **RL** fractional derivative: (**a**) $\mathcal{D}^{\mu(s)}f_1$ with variable order $\mu(s) = 0.5$; (**b**) $\mathcal{D}^{\mu(s)}f_2$ with order $\mu(s) = 0.5$.

The **WM** variable order fractional derivative of HVFIF with a predetermined initial condition $(\mathscr{D}^{\mu(s)}_{x_0}f)(x_0) = 0$ is demonstrated in the following theorem.

Let variable order $\mu(s)$, where $\mu : [a,b] \to (0,1)$ of the **WM** fractional derivative of continuous function f, be defined by the equation

$$(\mathscr{D}^{\mu(s)}_{x_0}f)(x) = \frac{\mu(s)}{\Gamma(1-\mu(s))}\int_a^x \frac{f(x)-f(x-t)}{t^{1+\mu(s)}}dt. \tag{19}$$

Suppose f is an HVFIF decisived by the IFS (4); then, the **WM** fractional derivative of f of order $\mu(s)$ is defined as follows:

$$(\mathscr{D}^{\mu(s)}_{x_0}f_1)(x) = \frac{\mu(s)}{\Gamma(1-\mu(s))}\int_a^x \frac{f_1(x)-f_1(x-t)}{t^{1+\mu(s)}}dt, \tag{20}$$

$$(\mathscr{D}^{\mu(s)}_{x_0}f_2)(x) = \frac{\mu(s)}{\Gamma(1-\mu(s))}\int_a^x \frac{f_2(x)-f_2(x-t)}{t^{1+\mu(s)}}dt. \tag{21}$$

Theorem 3. *Let f be the HVFIF given in Equations (2) and (3). If $\|\mathcal{A}'_k\| < 1$ where $\mathcal{A}'_k = a_k^{-\mu(s)}\begin{bmatrix}\alpha_k & \beta_k \\ 0 & \gamma_k\end{bmatrix}$, for all $k = 1,2,\ldots,N$, then $(\mathscr{D}^{\mu(s)}_{x_0}f)(x) = (\mathscr{D}^{\mu(s)}_{x_0}f_1)(x), (\mathscr{D}^{\mu(s)}_{x_0}f_2)(x)$ is also an HVFIF. Further, $(\mathscr{D}^{\mu(s)}_{x_0}f_1)(x)$ and $(\mathscr{D}^{\mu(s)}_{x_0}f_2)(x)$ are determined by the IFSs $\left\{L_k(x), \hat{F}_{1(k,\mu(s))}(x,\hat{y},\hat{z})\right\}_{k=1}^N$ and $\left\{L_k(x), \hat{F}_{2(k,\mu(s))}(x,\hat{z})\right\}_{k=1}^N$, respectively, where $\hat{F}_{1(k,\mu(s))}(x,\hat{y},\hat{z}) = a_k^{-\mu(s)}\alpha_k y + a_k^{-\mu(s)}\beta_k z + \hat{p}_k(x)$ and $\hat{F}_{2(k,\mu(s))} = a_k^{-\mu(s)}\gamma_k z + \hat{q}_k(x)$ with $\sum_{k=1}^N a_k^{-\mu(s)}\alpha_k \neq 1, \sum_{k=1}^N a_k^{-\mu(s)}\gamma_k \neq 1$, for $k = 1,2,\ldots,N$.*

Proof. The **WM** fractional derivative Equation (20) gives,

$$\begin{aligned}(\mathscr{D}^{\mu(s)}_{x_0}f_1)(L_k(x)) = {} & \frac{\mu(s)}{\Gamma(1-\mu(s))}\int_{x_0}^{x_{k-1}} \frac{f_1(x_{k-1})-f_1(t)}{(x_{k-1}-t)^{1+\mu(s)}}dt \\ & + \frac{\mu(s)}{\Gamma(1-\mu(s))}\int_{x_0}^{x_{k-1}} \left(\frac{f_1(L_k(x))-f_1(t)}{(L_k(x)-t)^{1+\mu(s)}} - \frac{f_1(x_{k-1})-f_1(t)}{(x_{k-1}-t)^{1+\mu(s)}}\right)dt \\ & + \frac{\mu(s)}{\Gamma(1-\mu(s))}\int_{x_{k-1}}^{L_k(x)} \frac{f_1(L_k(x))-f_1(t)}{(L_k(x)-t)^{1+\mu(s)}}dt.\end{aligned}$$

Replace $t = L_k(u)$ in the third term,

$$(\mathscr{D}_{x_0}^{\mu(s)} f_1)(L_k(x)) = \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + \frac{a_k \mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x} \frac{f_1(L_k(x)) - f_1(L_k(u))}{(L_k(x) - L_k(u))^{1+\mu(s)}} du.$$

The following equation is obtained using the functional equation Equation (7):

$$\begin{aligned}
(\mathscr{D}_{x_0}^{\mu(s)} f_1)(L_k(x)) &= \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + \frac{a_k^{-\mu(s)} \mu(s) \alpha_k}{\Gamma(1-\mu(s))} \int_{x_0}^{x} \frac{f_1(x) - f_1(u)}{(x-u)^{1+\mu(s)}} du \\
&+ \frac{a_k^{-\mu(s)} \mu(s) \beta_k}{\Gamma(1-\mu(s))} \int_{x_0}^{x} \frac{f_2(x) - f_2(u)}{(x-u)^{1+\mu(s)}} du + \frac{a_k^{-\mu(s)} \mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x} \frac{p_k(x) - p_k(u)}{(x-u)^{1+\mu(s)}} du \\
&= \hat{y}_{n-1} + f_{1(k,\mu(s))}(x) + a_k^{-\mu(s)} \alpha_k (\mathscr{D}_{x_0}^{\mu(s)} f_1)(x) + a_k^{-\mu(s)} \beta_k (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x) \\
&+ a_k^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} p_k)(x) \\
&= a_k^{-\mu(s)} \alpha_k (\mathscr{D}_{x_0}^{\mu(s)} f_1)(x) + a_k^{-\mu(s)} \beta_k (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x) + \hat{p}_{k,\mu(s)}(x).
\end{aligned}$$

Here, $\hat{p}_{k,\mu(s)}(x) = \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + a_k^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} p_k)(x)$, $\hat{y}_{k-1} = \frac{\mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x_{k-1}} \frac{f_1(x_{k-1}) - f_1(t)}{(x_{k-1}-t)^{1+\mu(s)}} dt$ and $f_{1(k,\mu(s))}(x) = \frac{\mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x_{k-1}} \left(\frac{f_1(L_k(x)) - f_1(t)}{(L_k(x)-t)^{1+\mu(s)}} - \frac{f_1(x_{k-1}) - f_1(t)}{(x_{k-1}-t)^{1+\mu(s)}} \right) dt$. Now, consider the **WM** fractional derivative of f_2 of order $\mu(s)$,

$$\begin{aligned}
(\mathscr{D}_{x_0}^{\mu(s)} f_2)(L_k(x)) &= \frac{\mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x_{k-1}} \frac{f_2(x_{k-1}) - f_2(t)}{(x_{k-1}-t)^{1+\mu(s)}} dt \\
&+ \frac{\mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x_{k-1}} \left(\frac{f_2(L_k(x)) - f_2(t)}{(L_k(x)-t)^{1+\mu(s)}} - \frac{f_2(x_{k-1}) - f_2(t)}{(x_{k-1}-t)^{1+\mu(s)}} \right) dt \\
&+ \frac{\mu(s)}{\Gamma(1-\mu(s))} \int_{x_{k-1}}^{L_k(x)} \frac{f_2(L_k(x)) - f_2(t)}{(L_k(x)-t)^{1+\mu(s)}} dt.
\end{aligned}$$

Replace $t = L_k(u)$ in the third term,

$$(\mathscr{D}_{x_0}^{\mu(s)} f_2)(L_k(x)) = \hat{y}_{k-1} + f_{2(k,\mu(s))}(x) + \frac{a_k \mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x} \frac{f_2(L_k(x)) - f_2(L_k(u))}{(L_k(x) - L_k(u))^{1+\mu(s)}} du.$$

The following equation is obtained using the functional equation Equation (8):

$$\begin{aligned}
(\mathscr{D}_{x_0}^{\mu(s)} f_2)(L_k(x)) &= \hat{z}_{k-1} + f_{1(k,\mu(s))}(x) + \frac{a_k^{-\mu(s)} \mu(s) \gamma_k}{\Gamma(1-\mu(s))} \int_{x_0}^{x} \frac{f_2(x) - f_2(u)}{(x-u)^{1+\mu(s)}} du \\
&+ \frac{a_k^{-\mu(s)} \mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x} \frac{q_k(x) - q_k(u)}{(x-u)^{1+\mu(s)}} du \\
&= \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + a_k^{-\mu(s)} \gamma_k (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x) + a_k^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} q_k)(x) \\
&= a_k^{-\mu(s)} \gamma_k (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x) + \hat{q}_{k,\mu(s)}(x).
\end{aligned}$$

Here, $\hat{q}_{k,\mu(s)}(x) = \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + a_k^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} q_k)(x)$, $\hat{z}_{k-1} = \frac{\mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x_{k-1}} \frac{f_2(x_{k-1}) - f_2(t)}{(x_{k-1}-t)^{1+\mu(s)}} dt$ and $f_{1(k,\mu(s))}(x) = \frac{\mu(s)}{\Gamma(1-\mu(s))} \int_{x_0}^{x_{k-1}} \left(\frac{f_1(L_k(x)) - f_1(t)}{(L_k(x)-t)^{1+\mu(s)}} - \frac{f_1(x_{k-1}) - f_1(t)}{(x_{k-1}-t)^{1+\mu(s)}} \right) dt$. Hence, the **WM** variable order fractional derivative of HVFIF is also HVFIF. Consider the following equation for calculating new data points:

$$(\mathscr{D}_{x_0}^{\mu(s)} f_1)(L_k(x)) = \hat{y}_{k-1} + f_{1(k,\mu(s))}(x) + a_k^{-\mu(s)} \alpha_k (\mathscr{D}_{x_0}^{\mu(s)} f_1)(x) + a_k^{-\mu(s)} \beta_k (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x) + a_k^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} p_k)(x).$$

Let $x = x_N$ and $L_k(x_N) = x_k$. Then, the above equation gives

$$\hat{y}_k - \hat{y}_{k-1} = f_{1(k,\mu(s))}(x_N) + a_k^{-\mu(s)} \alpha_k \hat{y}_N + a_k^{-\mu(s)} \beta_k (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x_N) + a_k^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} p_k)(x).$$

The aforementioned equation is as follows when considered from the perspective of the equation $\hat{y}_k = \hat{y}_0 + \sum_{n=1}^{k} (\hat{y}_n - \hat{y}_{n-1})$:

$$\hat{y}_k = \sum_{n=1}^{k} \left(f_{1(n,\mu(s))}(x_N) + a_n^{-\mu(s)} \alpha_n \hat{y}_N + a_n^{-\mu(s)} \beta_n (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x_N) + a_n^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} p_n)(x_N) \right). \tag{22}$$

In Equation (22), $\hat{y}_N$ is obtained by varying k to N

$$\hat{y}_N = \sum_{n=1}^{N} \left(f_{1(n,\mu(s))}(x_N) + a_n^{-\mu(s)} \beta_n (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x_N) + a_n^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} p_n)(x_N) \right) \Big/ 1 - \sum_{n=1}^{N} a_n^{-\mu(s)} d_n.$$

Consider the following equation to determine the new data set $\{(x_k, \hat{z}_k) : K = 0, 1, \ldots, N\}$

$$\mathscr{D}_{x_0}^{\mu(s)} f_2(L_k(x)) = \hat{z}_{k-1} + f_{2(k,\mu(s))}(x) + a_k^{-\mu(s)} \gamma_k (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x) + a_k^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} q_k)(x).$$

Let $x = x_N$ and $L_k(x) = x_k$. Then, the above equation yields,

$$\hat{z}_k - \hat{z}_{k-1} = f_{2(k,\mu(s))}(x_N) + a_k^{-\mu(s)} \gamma_k \hat{z}_N + a_k^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} q_k)(x_N).$$

The aforementioned equation is as follows when considered from the perspective of the equation $\hat{z}_k = \hat{z}_0 + \sum_{n=1}^{k} (\hat{z}_n - \hat{z}_{n-1})$:

$$\hat{z}_k = \sum_{n=1}^{k} \left(f_{2(n,\mu(s))}(x_N) + a_n^{-\mu(s)} \gamma_n \hat{z}_N + a_n^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} q_n)(x_N) \right). \tag{23}$$

In Equation (23), $\hat{z}_N$ is obtained by varying k to N

$$\hat{z}_N = \sum_{n=1}^{N} \left(f_{2(n,\mu(s))}(x_N) + a_n^{-\mu(s)} (\mathscr{D}_{x_0}^{\mu(s)} q_n)(x_N) \right) \Big/ 1 - \sum_{n=1}^{N} a_n^{-\mu(s)} \gamma_n.$$

$\square$

Remark 3. *The* ***WM*** *fractional derivative with the variable order of HVFIF is demonstrated in Theorem 3 with a predefined condition* $(\mathscr{D}_{x_0}^{\mu(s)} f_1)(x_0) = (\mathscr{D}_{x_0}^{\mu(s)} f_2)(x_0) = 0$. *Similarly, the* ***WM*** *variable order fractional derivative f is defined with the upper limit x_N as*

$$(\mathscr{D}_{x_N}^{\mu(s)} f_1)(L_k(x)) = -\frac{\mu(s)}{\Gamma(1-\mu(s))} \int_x^{x_N} \frac{f_1(L_k(x)) - f_1(t)}{(L_k(x) - t)^{1+\mu(s)}} dt$$

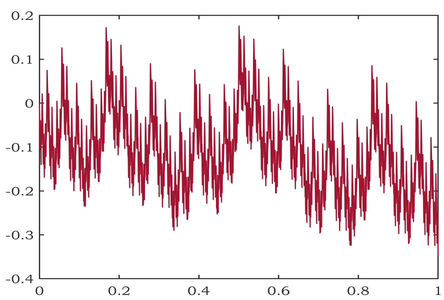
$$(\mathscr{D}_{x_N}^{\mu(s)} f_2)(L_k(x)) = -\frac{\mu(s)}{\Gamma(1-\mu(s))} \int_x^{x_N} \frac{f_2(L_k(x)) - f_2(t)}{(L_k(x)-t)^{1+\mu(s)}} dt$$

then, one can obtain that function $(\mathscr{D}_{x_N}^{\mu(s)} f(L_k(x))$ *is again HVFIF with the variable order generated by the IFS* $\{L_k(x), \hat{F}_{1(k,\mu(s))}(x,\hat{y},\hat{z})\}_{i=1}^N$ *and* $\{L_k(x), \hat{F}_{2(k,\mu(s))}(x,\hat{z})\}_{i=1}^N$ *where* $\hat{f}_1(x) = \hat{y}_k - f_{1(k,\mu(s))}(x) + a_k^{-\mu(s)} \alpha_k (\mathscr{D}_{x_N}^{\mu(s)} f_1)(x) + a_k^{-\mu(s)} \beta_k (\mathscr{D}_{x_N}^{\mu(s)} f_2)(x) + a_k^{-\mu(s)} (\mathscr{D}_{x_N}^{\mu(s)} p_k)(x)$ *and* $\hat{f}_2(x) = \hat{z}_k - f_{2(k,\mu(s))}(x) + a_k^{-\mu(s)} \gamma_k (\mathscr{D}_{x_N}^{\mu(s)} f_2)(x) + a_k^{-\mu(s)} (\mathscr{D}_{x_N}^{\mu(s)} q_k)(x)$

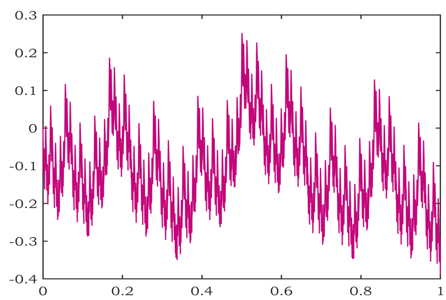
$$\begin{aligned}
\hat{p}_k(x) &= \hat{y}_k - f_{1(k,\mu(s))}(x) + a_k^{-\mu(s)} (\mathscr{D}_{x_N}^{\mu(s)} p_k)(x), \\
\hat{y}_{k-1} &= -\sum_{n=1}^{k} \Big(f_{1(n,\mu(s))}(x_0) - a_n^{-\mu(s)} \alpha_n \hat{y}_0 - a_n^{-\mu(s)} \beta_n (\mathscr{D}_{x_N}^{\mu(s)} f_2)(x_0) \\
&\quad - a_n^{-\mu(s)} (\mathscr{D}_{x_N}^{\mu(s)} p_n)(x_0) \Big), \\
\hat{y}_0 &= -\sum_{n=1}^{N} \Big(f_{1(n,\mu(s))}(x_0) - a_n^{-\mu(s)} \beta_n (\mathscr{D}_{x_N}^{-\mu(s)} f_2)(x_0) \\
&\quad - a_n^{-\mu(s)} (\mathscr{D}_{x_N}^{\mu(s)} p_n)(x_0) \Big) \Big/ 1 - \sum_{n=1}^{N} a_n^{-\mu(s)} \alpha_n, \\
\hat{q}_k(x) &= \hat{z}_k - f_{2(k,\mu(s))}(x) + a_k^{-\mu(s)} (\mathscr{D}_{x_N}^{\mu(s)} q_k)(x), \\
\hat{z}_{k-1} &= -\sum_{n=1}^{k} \Big(f_{2(n,\mu(s))}(x_0) - a_n^{-\mu(s)} \gamma_n \hat{z}_0 - a_n^{-\mu(s)} (\mathscr{D}_{x_N}^{\mu(s)} q_n)(x_0) \Big), \\
\hat{z}_0 &= -\sum_{n=1}^{N} \Big(f_{2(n,\mu(s))}(x_0) - a_n^{-\mu(s)} (\mathscr{D}_{x_N}^{\mu(s)} q_n)(x_0) \Big) \Big/ 1 - \sum_{n=1}^{N} a_n^{-\mu(s)} \gamma_n.
\end{aligned}$$

When the variable order $\mu(s)$ *is considered as a constant function, the above conclusion provides Priyanka's [27] result.*

Example 4. *The* ***WM*** *variable order fractional derivative of* f_1 *and the* ***WM*** *variable order fractional derivative of* f_2 *are calculated using the same data points and scale factors as in Example 1.* $\mathscr{D}^{0.5} f_1$ *is shown in Figure 4a, while a graphical representation of* $\mathscr{D}^{0.5} f_2$ *is given in Figure 4b.*



(a)



(b)

Figure 4. The **WM** fractional derivative: (**a**) $\mathscr{D}^{\mu(s)} f_1$ with variable order $\mu(s) = 0.5$, (**b**) $\mathscr{D}^{\mu(s)} f_2$ with order $\mu(s) = 0.5$.

4. Concluding Remarks

This work introduces a novel approach to investigate variable order fractional integration. The evaluation of fractional integration and differentiation with continuously varying-order $\mu(s)$ for each $s \in [x_0, x_N]$ is associated with the change of constant fractional order μ as a function of s. The **RL** variable order fractional calculus and the **WM** variable order fractional derivative of the HVFIF are explored with predefined initial conditions. Further, the resultant functions are demonstrated as the HVFIF.

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