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New Results for Homoclinic Fractional Hamiltonian Systems of Order $\alpha \in (1/2, 1]$

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Abstract: In this manuscript, we are interested in studying the homoclinic solutions of fractional Hamiltonian system of the form $-{}_{\varsigma}\mathfrak{D}^{\alpha}_{\infty}({}_{-\infty}\mathfrak{D}^{\alpha}_{\varsigma}Z(\varsigma)) - \mathcal{A}(\varsigma)Z(\varsigma) + \nabla\varpi(\varsigma, Z(\varsigma)) = 0$, where $\alpha \in (\frac{1}{2}, 1]$, $Z \in H^{\alpha}(\mathbb{R}, \mathbb{R}^N)$ and $\varpi \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ are not periodic in ς. The characteristics of the critical point theory are used to illustrate the primary findings. Our results substantially improve and generalize the most recent results of the proposed system. We conclude our study by providing an example to highlight the significance of the theoretical results.

Keywords: fractional Hamiltonian systems; Mountain Pass Theorem; genus properties critical point

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1. Introduction

In physics, mechanics, control theory, biology, bioengineering, and economics, processes are frequently simulated using fractional ordinary and partial differential equations. The theory of fractional differential equations has consequently attracted a lot of attention in recent years. For instance, existence and stability are addressed in [1–3], and several resolution strategies are in [4–6]. The monographs [7,8] are exceptional sources for numerous techniques that are thought to be extensions of various differential equations. Recent discussions have focused in particular on equations that have both left and right fractional derivatives. With regard to their numerous applications, these kinds of equations are significant and are considered as a novel subject in the theory of fractional differential equations. Using nonlinear analytic techniques such as fixed point theory, there have appeared many results dealing with the existence and multiplicity of solutions to nonlinear fractional differential equations in this field. For instance, we name here Leray–Schauder nonlinear alternative [9], topological degree theory [10], and the comparison method, which includes upper and lower solutions and monotone iterative method [11,12], and so on. On the other hand, it has been demonstrated that the critical point theory and variational techniques are crucial for assessing whether or not differential equations have solutions. With the help of this theory, one can search for solutions to a specific boundary value problem by locating the critical points of an appropriate energy functional defined on a suitable function space. In light of this, the critical point theory has developed into a potent tool for investigating the existence of solutions to differential equations with variational forms (see [13,14] and the references therein).

Adopting the aforementioned classic research, Zhou and Lu [15] implemented the critical point theory to tackle the existence of solutions for the following fractional BVP

$$\begin{cases} {}_{\varsigma}\mathfrak{D}_{T}^{\alpha}({}_{0}\mathfrak{D}_{\varsigma}^{\alpha}Z(\varsigma)) = \nabla\varpi(\varsigma, Z(\varsigma)), \quad \text{a.e. } \varsigma \in [0, T], \\ Z(0) = Z(T), \end{cases} \tag{1}$$

where α in $(\frac{1}{2}, 1)$, $Z \in \mathbb{R}^N$, $\varpi \in C^1([0, T] \times \mathbb{R}^N, \mathbb{R})$ and $\nabla\varpi(\varsigma, Z)$ is the gradient of ϖ at Z. It is significant to note that many of the premises made in order to arrive at the conclusions in [15] weaken the fundamental theorems. Inspired by their work, Torres [16] studied the following fractional Hamiltonian systems

$$\begin{cases} -{}_{\varsigma}\mathfrak{D}_{\infty}^{\alpha}({}_{-\infty}\mathfrak{D}_{\varsigma}^{\alpha}Z(\varsigma)) - \mathcal{A}(\varsigma)Z(\varsigma) + \nabla\varpi(\varsigma, Z(\varsigma)) = 0, \\ Z \in H^{\alpha}(\mathbb{R}, \mathbb{R}^N), \end{cases} \tag{2}$$

where ${}_{-\infty}\mathfrak{D}_{\varsigma}^{\alpha}$ and ${}_{\varsigma}\mathfrak{D}_{\infty}^{\alpha}$ are left and right Liouville–Weyl fractional derivatives of order α and $\mathcal{A}(\varsigma) \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is symmetric and positive definite matrix for all $\varsigma \in \mathbb{R}$. The Mountain Pass Theorem was used in [16] to show that equations accept at least one nontrivial solution as long as $\mathcal{A}$ and ϖ can validate the following four hypotheses:

(Υ_0) $\mathcal{A}(\varsigma)$ is symmetric and positive definite matrix $\forall\, \varsigma \in \mathbb{R}$, and there exists functional $l \in C(\mathbb{R}, (0, \infty))$ while $l(\varsigma) \to \infty$ as $|\varsigma| \to \infty$ and $(\mathcal{A}(\varsigma)Z, Z) \geq l(\varsigma)|Z|^2$, for any $\varsigma \in \mathbb{R}$ and $Z \in \mathbb{R}^N$;

($\mathcal{F}_1$) $|\nabla\varpi(\varsigma, Z)| = o(|Z|)$ as $|Z| \to 0$ uniformly in $\varsigma \in \mathbb{R}$;

($\mathcal{F}_2$) There exists $\overline{\varpi} \in C(\mathbb{R}^N, \mathbb{R})$ such that $|\varpi(\varsigma, Z)| + |\nabla\varpi(\varsigma, Z)| \leq |\overline{\varpi}(Z)|$ for all $(\varsigma, Z) \in \mathbb{R} \times \mathbb{R}^N$;

($\mathcal{F}_3$) There exists some constant $\mu > 2$ such as $0 < \mu\varpi(\varsigma, Z) \leq (\nabla\varpi(\varsigma, Z), Z)$, for any $\varsigma \in \mathbb{R}$ and $Z \in \mathbb{R}^N \backslash \{0\}$.

For $\alpha = 1$, Equation (2) is downloaded to the following standard second–order Hamiltonian system

$$\ddot{Z}(\varsigma) - \mathcal{A}(\varsigma)Z(\varsigma) + \nabla\varpi(\varsigma, Z(\varsigma)) = 0. \tag{3}$$

Several papers including [17–24] investigated the existence of homoclinic solutions for the Hamiltonian system (3) when $\mathcal{A}(\varsigma)$ and $\varpi(\varsigma, Z)$ are either independent of or periodic in ς.

In this work, we impose new standards based on the critical point theory to demonstrate the existence of infinitely many homoclinic solutions of fractional Hamiltonian system (2) where $\varpi(\varsigma, Z)$ is sub-quadratic as $|Z| \to +\infty$. In addition to condition (Υ_0), we assume that $\varpi(\varsigma, Z)$ fulfills the following three conditions:

(Λ_1) $\varpi \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and there exists γ_1, γ_2 satisfying $1 < \gamma_1 < \gamma_2 < 2$ and two functional a_1, a_2 in $\mathcal{L}^{\frac{2}{2-\gamma_1}}(\mathbb{R}, \mathbb{R}+)$ such that

$$|\varpi(\varsigma, Z)| \leq a_1(\varsigma)|Z|^{\gamma_1}, \text{ for all } (\varsigma, Z) \text{ in } \mathbb{R} \times \mathbb{R}^N, \ |Z| \leq 1,$$

and

$$|\varpi(\varsigma, Z)| \leq a_2(\varsigma)|Z|^{\gamma_2}, \text{ for all } (\varsigma, Z) \text{ in } \mathbb{R} \times \mathbb{R}^N, \ |Z| \geq 1.$$

(Λ_2) There exists b in $\mathcal{L}^{\frac{2}{2-\gamma_1}}(\mathbb{R}, \mathbb{R}^+)$ and φ in $C([0, +\infty), [0, +\infty))$ such that

$$|\nabla\varpi(\varsigma, Z)| \leq b(\varsigma)\varphi(|Z|), \text{ for all } (\varsigma, Z) \text{ in } \mathbb{R} \times \mathbb{R}^N,$$

and $\varphi(s) = O(s^{\gamma_1 - 1})$ as $s \to 0^+$.

(Λ_3) There exists an open set $J \subset \mathbb{R}$ and two constants $\gamma_3 \in (1, 2)$, $\eta > 0$ such that

$$\varpi(\varsigma, Z) \geq \eta|Z|^{\gamma_3}, \ \forall\, (\varsigma, Z) \in J \times \mathbb{R}^N, \ |Z| \leq 1.$$

It is worthy mentioning here that the results given in [14] were obtained under the condition $(\mathcal{F}_3)$, which is known as the global Ambrosetti–Rabinowitz condition. That is, $\varpi(\varsigma, Z)$ is super–quadratic when $|Z| \to \infty$. Moreover, it was assumed that Z and ϖ are periodic in ς. In this paper, however, the main results are proved under less restrictive condition where $\mathcal{A}$ is coercive at infinity, ϖ is sub-quadratic growth as $|Z| \to \infty$ ($\frac{\varpi}{|Z|^2} = 0$ if $|Z| \to \infty$) and Z and ϖ are not periodic in ς. Our results supply substantial generalizations to the recent results existing in the literature.

Significant findings of our paper are described in the following two theorems.

Theorem 1. *If conditions* (Υ_0), (Λ_1), (Λ_2), *and* (Λ_3) *hold. So,* (2) *accepts one nontrivial homoclinic solution.*

Theorem 2. *Assuming that* (Υ_0), (Λ_1), (Λ_2) *and* (Λ_3) *hold. In addition, assume that* $\varpi(\varsigma, Z)$ *is even in* Z*. Then,* (2) *has infinitely many nontrivial homoclinic solutions* $(Z_k)_{k\in\mathbb{N}}$ *such that, as* $k \to \infty$,

$$\int_{\mathbb{R}} \left[\frac{1}{2}|{}_{-\infty}\mathfrak{D}^{\alpha}_{\varsigma} Z_k(\varsigma)|^2 + \frac{1}{2}(\mathcal{A}(\varsigma)Z_k(\varsigma), Z_k(\varsigma)) - \varpi(\varsigma, Z_k(\varsigma)\right] d\varsigma \to 0^-. \tag{4}$$

The proofs of Theorems 1 and 2 are given in Section 3.

2. Essential Preliminaries

This section is devoted to stating and demonstrating some fundamental definitions and lemmas that are required in the work that follows.

Definition 1. *The left and right Liouville–Weyl fractional integrals of order* α *on* $\mathbb{R}$, $(0 < \alpha < 1)$ *are, respectively, given by*

$${}_{-\infty}I^{\alpha}_{\varkappa} Z(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\varkappa} (\varkappa - \xi)^{\alpha-1} Z(\xi)\, d\xi, \quad \varkappa \in \mathbb{R}, \tag{5}$$

and

$${}_{\varkappa}I^{\alpha}_{\infty} Z(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\infty} (\xi - \varkappa)^{\alpha-1} Z(\xi)\, d\xi, \quad \varkappa \in \mathbb{R}.$$

Definition 2. *The left and the right Liouville–Weyl fractional derivatives of order* α *on* $\mathbb{R}$, $(0 < \alpha < 1)$ *are, respectively, given by*

$${}_{-\infty}\mathfrak{D}^{\alpha}_{\varkappa} Z(\varkappa) = \frac{d}{d\varkappa}\, {}_{-\infty}I^{1-\alpha}_{\varkappa} Z(\varkappa),\ \varkappa \in \mathbb{R} \tag{6}$$

and

$${}_{\varkappa}\mathfrak{D}^{\alpha}_{\infty} Z(\varkappa) = -\frac{d}{d\varkappa}\, {}_{\varkappa}I^{1-\alpha}_{\infty} Z(\varkappa),\ \varkappa \in \mathbb{R}.$$

Remark 1. *The operators* (5) *and* (6) *can be written in the form*

$${}_{-\infty}\mathfrak{D}^{\alpha}_{\varkappa} Z(\varkappa) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{Z(\varkappa) - Z(\varkappa - \xi)}{\xi^{\alpha+1}}\, d\xi,$$

and

$${}_{\varkappa}\mathfrak{D}^{\alpha}_{\infty} Z(\varkappa) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{Z(\varkappa) - Z(\varkappa + \xi)}{\xi^{\alpha+1}}\, d\xi\,.$$

Definition 3. *A solution* x *of* (2) *is called homoclinic (to 0) if* $x \in C^2(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$, $x \neq 0$, $x(t) \to 0$ *and* $\dot{x}(t) \to 0$ *as* $|t| \to \infty$*. A function* φ *is said to be coercive if* $\varphi(t) \to \infty$ *as* $|t| \to \infty$.

We recall that the Fourier transform of $Z(\cdot)$ is

$$\widehat{Z}(w) = \int_{-\infty}^{\infty} e^{-i\varkappa.w} Z(\varkappa)\, d\varkappa.$$

The semi–norm is given by

$$|Z|_{I^{\alpha}_{-\infty}} := \|_{-\infty}\mathfrak{D}^{\alpha}_{\varkappa} Z\|_{\mathbb{L}^2},\ \alpha > 0,$$

while the norm is

$$\|Z\|_{I^{\alpha}_{-\infty}} := \left(\|Z\|^2_{\mathbb{L}^2} + |Z|^2_{I^{\alpha}_{-\infty}}\right)^{1/2}.$$

We denote by $I^{\alpha}_{-\infty}(\mathbb{R})$ the completion of $C_0^{\infty}(\mathbb{R})$ coupled with the norm $\|\cdot\|_{I^{\alpha}_{-\infty}}$, that is

$$I^{\alpha}_{-\infty}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{I^{\alpha}_{-\infty}}}.$$

Further, we define the semi-norm by

$$|Z|_{\alpha} = \||w|^{\alpha}\widehat{Z}\|_{\mathbb{L}^2},\ 0 < \alpha < 1,$$

and the norm by

$$\|Z\|_{\alpha} = (\|Z\|^2_{\mathbb{L}^2} + |Z|^2_{\alpha})^{1/2}.$$

We define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ in terms of the Fourier transform as follows:

$$H^{\alpha}(\mathbb{R}) := \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{\alpha}}.$$

Noting that $Z \in \mathbb{L}^2(\mathbb{R})$ is an element of $I^{\alpha}_{-\infty}(\mathbb{R})$ if and only if

$$|w|^{\alpha}\widehat{Z} \in \mathbb{L}^2(\mathbb{R}).$$

In particular, we obtain

$$|Z|_{I^{\alpha}_{-\infty}} = \||w|^{\alpha}\widehat{Z}\|_{\mathbb{L}^2(\mathbb{R})}.$$

Therefore, if the semi-norm and the norm are equivalent, then $H^{\alpha}(\mathbb{R})$ and $I^{\alpha}_{-\infty}(\mathbb{R})$ are also equivalent [16].

Similar to $I^{\alpha}_{-\infty}(\mathbb{R})$, we define $I^{\alpha}_{\infty}(\mathbb{R})$. Thus, the semi-norm $|Z|_{I^{\alpha}_{\infty}}$ and the norm $|Z|_{I^{\alpha}_{\infty}}$ of Z are, respectively, given by

$$|Z|_{I^{\alpha}_{\infty}} := \|_{\varkappa}\mathfrak{D}^{\alpha}_{\infty}\|_{\mathbb{L}^2(\mathbb{R})},$$

and

$$\|Z\|_{I^{\alpha}_{\infty}} := (\|Z\|^2_{\mathbb{L}^2} + |Z|^2_{I^{\alpha}_{\infty}})^{1/2}.$$

Letting

$$I^{\alpha}_{-\infty}(\mathbb{R}) := \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{I^{\alpha}_{-\infty}}}.$$

Additionally, if the semi-norm and the norm are equivalent, then $I^{\alpha}_{\infty}(\mathbb{R})$ and $I^{\alpha}_{-\infty}(\mathbb{R})$ are equivalent.

Lemma 1 ([16])**.** *If $\alpha > \frac{1}{2}$, then $H^{\alpha}(\mathbb{R})$ is included in the continuous real functions space $C(\mathbb{R})$, and there exists a constant C_{α} (noted by C) such that*

$$\|Z\|_{\mathbb{L}^{\infty}} = \sup_{Z\in\mathbb{R}} |Z(\varkappa)| \le C\|Z\|_{\alpha}. \tag{7}$$

Remark 2. *If $Z \in H^{\alpha}(\mathbb{R})$, then $Z \in \mathbb{L}^{q}(\mathbb{R})$ for any q in $[2, \infty]$, as*

$$\int_{\mathbb{R}} |Z(\varkappa)|^{q} d\varkappa \leq \|Z\|_{\mathbb{L}^{\infty}}^{q-2} \|Z\|_{\mathbb{L}^{2}}^{2}.$$

Next, we define the fractional space and construct the variational framework of the fractional Hamiltonian systems (2). To this end, letting

$$E = X^{\alpha} = \left\{ Z \text{ in } H^{\alpha}(\mathbb{R}, \mathbb{R}^{n}) : \int_{\mathbb{R}} |{}_{-\infty}\mathfrak{D}_{\varsigma}^{\alpha} Z(\varsigma)|^{2} + (\mathcal{A}(\varsigma)Z(\varsigma), Z(\varsigma))\, d\varsigma < \infty \right\}. \quad (8)$$

The space X^{α} is a reflexive and separable Hilbert space under the inner product

$$(Z, v)_{X^{\alpha}} = \int_{\mathbb{R}} ({}_{-\infty}\mathfrak{D}_{\varsigma}^{\alpha} Z(\varsigma) . {}_{-\infty}\mathfrak{D}_{\varsigma}^{\alpha} v(\varsigma)) + (\mathcal{A}(\varsigma)Z(\varsigma), v(\varsigma)) d\varsigma,$$

with the norm

$$\|Z\|^{2} = (Z, Z)_{X^{\alpha}}.$$

Lemma 2. *If $\mathcal{A}$ satisfies (Y_0), then X^{α} is continuously embedded in $H^{\alpha}(\mathbb{R}, \mathbb{R}^{n})$.*

Proof. Since $l \in C(\mathbb{R}, (0, \infty))$ and l is coercive, then $l_{*} := \min_{\varsigma \in \mathbb{R}} l(\varsigma)$ exists. So, we obtain

$$(\mathcal{A}(\varsigma)Z(\varsigma), Z(\varsigma)) \geq l(\varsigma)|\varsigma|^{2} \geq l_{*}|\varsigma|^{2}, \text{ for any real } \varsigma.$$

Thus,

$$\begin{aligned} \|Z\|_{\alpha}^{2} &= \int_{\mathbb{R}} (|{}_{-\infty}\mathfrak{D}_{\varsigma}^{\alpha} Z(\varsigma)|^{2} + (\mathcal{A}(\varsigma)Z(\varsigma), Z(\varsigma))) d\varsigma \\ &\leq \int_{\mathbb{R}} |{}_{-\infty}\mathfrak{D}_{\varsigma}^{\alpha} Z(\varsigma)|^{2} d\varsigma + \frac{1}{l_{*}} \int_{\mathbb{R}} (\mathcal{A}(\varsigma)Z(\varsigma), Z(\varsigma)) d\varsigma. \end{aligned}$$

Therefore,

$$\|Z\|_{\alpha}^{2} \leq K\|Z\|^{2}, \quad (9)$$

where $K := \max\left(1, \frac{1}{l_{*}}\right)$. □

It is difficult to demonstrate that there are infinitely many solutions to the Hamiltonian systems (2) because the Sobolev embedding is not compact under the assumptions of Theorems 1 and 2. We will utilize the following lemma to ensure that the task is made simple:

Lemma 3. *If $\mathcal{A}$ satisfies the condition (Y_0), then the embedding of X^{α} in $\mathbb{L}^{2}(\mathbb{R})$ is compact.*

Proof. Form Lemma 2 and Remark 2, we obtain the continuity of $X^{\alpha} \hookrightarrow \mathbb{L}^{2}(\mathbb{R})$. Let $(Z_k) \in X^{\alpha}$ be a sequence such that $Z_k \rightharpoonup Z$ in X^{α}. We will prove that $Z_k \to Z$ in $\mathbb{L}^{2}(\mathbb{R})$ functional. The Banach–Steinhauss theorem implies that

$$A := \sup_{k \in \mathbb{N}} \|Z_k - Z\| < \infty.$$

Let $\epsilon > 0$. Since $\lim_{|\varsigma| \to \infty} l(\varsigma) = \infty$, there exits a real $T_0 > 0$ such that

$$\frac{1}{l(\varsigma)} \leq \epsilon, \text{ for all } |\varsigma| \geq T_0.$$

Therefore,

$$\int_{|\varsigma|\geq T_0} |Z_k(\varsigma) - Z(\varsigma)|^2 d\varsigma \leq \epsilon \int_{|\varsigma|\geq T_0} l(\varsigma)|Z_k(\varsigma) - Z(\varsigma)|^2 d\varsigma$$
$$\leq \epsilon \|Z_k - Z\|^2 \leq \epsilon A^2. \tag{10}$$

Moreover, Sobolev's theorem ([13]) implies that $Z_k \to Z$ uniformly on $[-T_0, T_0]$. Thus, there is $k_0 \in \mathbb{N}$ such that

$$\int_{|\varsigma|\leq T_0} |Z_k(\varsigma) - Z(\varsigma)|^2 d\varsigma \leq \epsilon, \text{ for all } k \geq k_0. \tag{11}$$

By combining (10) and (11), we obtain that $Z_k \to Z$ in $\mathbb{L}^2(\mathbb{R})$. □

Remark 3. *We note that Remark 2 and Lemma 3 assure the embedding of X^α in $L^q(\mathbb{R})$. For $q \in (2, \infty)$, the operator X^α is also continuous and compact. Consequently, by the Lemma 1, there exists a constant C_α satisfies*

$$\|Z_q\| \leq C_q \|Z\| \text{ for any } q \in [2, \infty]. \tag{12}$$

Lemma 4. *Under the condition of Theorem 1, if $Z_k \rightharpoonup Z$ in X^α, then $\nabla\varpi(\varsigma, Z_k) \to \nabla\varpi(\varsigma, Z)$ in $\mathbb{L}^2(\mathbb{R})$.*

Proof. Assuming $Z_k \rightharpoonup Z$ in X^α. Consequently, by using the Banach–Steinhauss theorem, there exists $M > 0$ such that

$$\sup_{k\in\mathbb{N}} \|Z_k\| \leq M \text{ and } \|Z\| \leq M. \tag{13}$$

By (Λ_2), there exists $M_1 > 0$ such as

$$\varphi(|Z|) \leq M_1 |Z|^{\gamma_1 - 1}, \text{ for all } |Z| \leq M. \tag{14}$$

Further, by (8), for any $Z \in X^\alpha$, there exists $T > 0$ such that

$$|Z(\varsigma)| \leq M, \text{ for all } |\varsigma| \geq T. \tag{15}$$

Therefore, from the inequalities (12), (13), (14) and (15), and by using Hölder inequality, we obtain

$$\begin{aligned}
&\int_{|\varsigma|\geq T} |\nabla\varpi(\varsigma, Z_k(\varsigma)) - \nabla\varpi(\varsigma, Z(\varsigma))|^2 d\varsigma \\
&\leq 2\int_{|\varsigma|\geq T} \left(|\nabla\varpi(\varsigma, Z_k(\varsigma))|^2 + |\nabla\varpi(\varsigma, Z(\varsigma))|^2\right) d\varsigma \\
&\leq 2M_1^2 \int_{|\varsigma|\geq T} |b(\varsigma)|^2 \left(|Z_k(\varsigma)|^{2(\gamma_1-1)} + |Z(\varsigma)|^{2(\gamma_1-1)}\right) d\varsigma \\
&\leq 2M_1^2 \left(\int_{|\varsigma|\geq T} |b(\varsigma)|^{\frac{2}{2-\gamma_1}} d\varsigma\right)^{2-\gamma_1} \left(\int_{|\varsigma|\geq T} |Z_k(\varsigma)|^2 d\varsigma\right)^{\gamma_1-1} \\
&+ 2M_1^2 \left(\int_{|\varsigma|\geq T} |b(\varsigma)|^{\frac{2}{2-\gamma_1}} d\varsigma\right)^{2-\gamma_1} \left(\int_{|\varsigma|\geq T} |Z(\varsigma)|^2 d\varsigma\right)^{\gamma_1-1} \\
&\leq 2M_1^2 \|b\|^2_{\frac{2}{2-\gamma_1}} \left(\|Z_k\|_2^{2(\gamma_1-1)} + \|Z\|_2^{2(\gamma_1-1)}\right) \\
&\leq 4M_1^2 M^{2(\gamma_1-1)} C_2^{2(\gamma_1-1)} \|b\|^2_{\frac{2}{2-\gamma_1}}.
\end{aligned} \tag{16}$$

Moreover, since $\nabla\varpi(\varsigma, Z)$ is continuous, there is a constant $d > 0$ such that

$$\int_{|\varsigma|\leq T} |\nabla\varpi(\varsigma, Z_k(\varsigma)) - \nabla\varpi(\varsigma, Z(\varsigma))|^2 d\varsigma \leq d. \tag{17}$$

Thus, by combining (16) and (17), we obtain

$$\int_{\mathbb{R}} |\nabla\varpi(\varsigma, Z_k(\varsigma)) - \nabla\varpi(\varsigma, Z(\varsigma))|^2 d\varsigma \leq d + 4M_1^2 M^{2(\gamma_1-1)} C_2^{2(\gamma_1-1)} \|b\|^2_{\frac{2}{2-\gamma_1}}. \tag{18}$$

However, by Lemma 3, the fact $Z_k \rightharpoonup Z$ implies the existence of a subsequence $(Z_k)_{k\in\mathbb{N}}$ such that $Z_k \to Z \in \mathbb{L}^2(\mathbb{R})$, which yields $Z_k(\varsigma) \to Z(\varsigma)$ for almost every $\varsigma \in \mathbb{R}$. Thus, the proof is completed by applying the Lebesgue's convergence Theorem. □

Lemma 5 ([13]). *Let $I \in C^1(B, \mathbb{R})$ satisfying the Palais–Smale condition (PS) and bounded below. Then, $c = \inf_B I$ is a critical value of I.*

To find solutions of (2) under the conditions of Theorem 2, we use the genus properties. For this, we recall some definitions and results from [14]. Denote by B the real Banach space. For $I \in C^1(B, \mathbb{R})$ and $c \in \mathbb{R}$, let us define the following sets:

$$\Sigma := \{A \subset B\backslash\{0\} \text{ such that } A \text{ symmetric with respect to 0 and closed in } B\},$$

$$K_c := \{Z \in B : I(Z) = c, I'(Z) = 0\},$$

and

$$I^c := \{Z \in B : I(Z) \leq c\}.$$

Definition 4. *For $A \in \Sigma$, we call the genus of A is j (denoted by $\Gamma(A) = j$) if there is an odd map ψ in $C(A, \mathbb{R}^j\backslash\{0\})$, where j is the smallest integer satisfy this property.*

Lemma 6 ([14]). *Let $I \in C^1$ be an even functional on B that satisfies the Palais–Smale (PS) condition. Further, for every $j \in \mathbb{N}$, let $\Sigma_j = \{A \in \Sigma : \Gamma(A) \geq j\}$ and $c_j = \inf_{A\in\Sigma_j} \sup_{Z\in A} I(Z)$.*

(i) If $\Sigma_j \neq \emptyset$ and $c_j \in \mathbb{R}$, then c_j is a critical value of I.
(ii) If there exists a natural number r such that $c_j = c_{j+1} = \ldots = c_{j+r} = c \in \mathbb{R}$, and $c \neq I(0)$, then $\Gamma(K_c) \geq r + 1$.

Remark 4 ([14]). *If K_c belongs to Σ and $\Gamma(K_c) > 1$, then K_c has infinitely many distinct points. Thus, I contains infinitely many distinct critical points in B.*

3. Proofs of Main Results

First, we construct the variational framework to prove the existence of solutions for (2). We define $I : X^\alpha \to \mathbb{R}$, by

$$\begin{aligned} I(Z) &= \int_{\mathbb{R}} \left[\frac{1}{2}|{}_{-\infty}\mathfrak{D}^\alpha_\varsigma Z(\varsigma)|^2 + \frac{1}{2}(\mathcal{A}(\varsigma)Z(\varsigma), Z(\varsigma)) - \varpi(\varsigma, Z(\varsigma)\right] d\varsigma \\ &= \frac{1}{2}\|Z\|^2 - \int_{\mathbb{R}} \varpi(\varsigma, Z(\varsigma)) d\varsigma. \end{aligned} \tag{19}$$

Under the assumptions of Theorem 1, we obtain

$$I'(Z)v = \int_{\mathbb{R}} \left[\left({}_{-\infty}\mathfrak{D}^\alpha_\varsigma Z(\varsigma), {}_{-\infty}\mathfrak{D}^\alpha_\varsigma v(\varsigma)\right) + (\mathcal{A}(\varsigma)Z(\varsigma), v(\varsigma)) - (\nabla\varpi(\varsigma, Z(\varsigma)), v(\varsigma))\right] d\varsigma \tag{20}$$

for any $Z,v \in X^\alpha$. This implies that

$$I'(v)v = \|v\|^2 - \int_{\mathbb{R}} (\nabla\varpi(\varsigma, Z(\varsigma)), v(\varsigma))d\varsigma. \tag{21}$$

Furthermore, I is defined on X^α and continuously Fréchet-differentiable functional; that is $I \in C^1(X^\alpha, \mathbb{R})$.

3.1. Proof of Theorem 1

First, we prove that I is bounded below. From the hypothesis (Λ_1) and Hölder inequality, we obtain

$$\begin{aligned}
I(Z) &\geq \frac{1}{2}\|Z\|^2 - \int_{\mathbb{R}(|Z(\varsigma)|\leq 1)} a_1(\varsigma)|Z(\varsigma)|^{\gamma_1} d\varsigma - \int_{\mathbb{R}(|Z(\varsigma)|\geq 1)} a_2(\varsigma)|Z(\varsigma)|^{\gamma_2} d\varsigma \\
&\geq \frac{1}{2}\|Z\|^2 - \left(\int_{\mathbb{R}(|Z(\varsigma)|\leq 1)} |a_1(\varsigma)|^{\frac{2}{2-\gamma_1}} d\varsigma\right)^{\frac{2-\gamma_1}{2}} \|Z(\varsigma)\|_2^{\gamma_1} \\
&- \left(\int_{\mathbb{R}(|Z(\varsigma)|\leq 1)} |a_2(\varsigma)|^{\frac{2}{2-\gamma_2}} d\varsigma\right)^{\frac{2-\gamma_2}{2}} \left(\int_{\mathbb{R}(|Z(\varsigma)|\geq 1)} |Z(\varsigma)|^{\frac{2\gamma_2}{\gamma_1}} d\varsigma\right)^{\frac{\gamma_1}{2}} \\
&\geq \frac{1}{2}\|Z\|^2 - C_2^{\gamma_1}\|a_1\|_{\frac{2}{2-\gamma_1}}\|Z\|^{\gamma_1} - C_2^{\gamma_1}\|a_2\|_{\frac{2}{2-\gamma_1}}\|Z\|_\infty^{\gamma_2-\gamma_1}\|Z\|^{\gamma_2} \\
&\geq \frac{1}{2}\|Z\|^2 - C_2^{\gamma_1}\|a_1\|_{\frac{2}{2-\gamma_1}}\|Z\|^{\gamma_1} - C_2^{\gamma_1}C_\infty^{\gamma_2-\gamma_1}\|a_2\|_{\frac{2}{2-\gamma_1}}\|Z\|^{\gamma_2}.
\end{aligned} \tag{22}$$

Since $1 < \gamma_1 < \gamma_2$, from (22), we conclude

$$I(Z) \to \infty \text{ as } \|Z\| \to \infty.$$

Thus, I is bounded below.

Now, we show that I satisfies the (PS) condition. To this end, let $(Z_k)_{k\in\mathbb{N}}$ be a sequence in X^α such that $(I(Z_k))$ is bounded and $I'(Z_k) \to 0$ as $k \to \infty$. So, by (19) and (22), it follows that there exists a positive real constant A such that

$$\|Z_k\| \leq A, \quad \text{for all} \quad k \in \mathbb{N}. \tag{23}$$

It follows from (21) that

$$(I'(Z_k) - I'(Z))(Z_k - Z) = \|Z_k - Z\|^2 - \int_{\mathbb{R}} (\nabla\varpi(\varsigma, Z_k(\varsigma)) - \nabla\varpi(\varsigma, Z(\varsigma)), Z_k - Z(\varsigma))d\varsigma.$$

Since $(I'(Z_k) - I'(Z))(Z_k - Z) \to 0$ as $k \to \infty$, by the Lemma 4, we deduce that

$$\|Z_k - Z\|^2 \to 0 \text{ as } \ k \to \infty.$$

Consequently, I validates the Palais–Smale condition (PS) as desired.

Now, by Lemma 5, it follows that $c = \inf_{X^\alpha} I(Z)$ is a critical value of I. Thus, there exists a critical point $Z_* \in X^\alpha$ such that $I(Z_*) = c$.

It is remaining to show that $Z_* \neq 0$. Let $Z_0 \in (\mathcal{W}_0^{1,2}(J) \cap X^\alpha)\backslash\{0\}$ and $\|Z_0\|_\infty \leq 1$. Then, by (Λ_1), (Λ_3) and (19), we obtain

$$\begin{aligned}
I(sZ_0) &= \frac{s^2}{2}\|Z_0\|^2 - \int_{\mathbb{R}} \varpi(\varsigma, sZ_0(\varsigma))d\varsigma = \frac{s^2}{2}\|Z_0\|^2 - \int_J \varpi(\varsigma, sZ_0(\varsigma))d\varsigma \\
&\leq \frac{s^2}{2}\|Z_0\|^2 - \eta s^{\gamma_3}\int_J |Z_0|^{\gamma_3} d\varsigma, \ 0 < s < 1.
\end{aligned} \tag{24}$$

Since $1 < \gamma_3 < 2$, it follows from (24) that $I(sZ_0) < 0$ for $s > 0$ small enough. Hence $I(Z_*) = c < 0$, and thus Z_* is nontrivial critical point of I. Therefore, $Z_* = Z(\varsigma)$ is nontrivial solution of (2).

3.2. Proof of Theorem 2

By Lemma 5 and the proof of Theorem 1, $I \in C^1(X^\alpha, \mathbb{R})$ is bounded below and satisfies the (PS) condition. It is clear that I is even and $I(0) = 0$. In order to apply the Lemma 6, we show that

$$\forall\, n \in \mathbb{N}\, \exists\, \epsilon > 0 \text{ such that } \gamma(I^{-\epsilon}) \geq n. \tag{25}$$

For any natural n, take n disjoint open sets J_i such that $\bigcup_{i=1}^{n} J_i \subset J$. For $i = 1, 2, \ldots, n$, choose $Z_i \in \left(\mathcal{W}_0^{1,2}(J_i) \cap X^\alpha\right) \backslash \{0\}$ such that $\|Z_i\| = 1$. Letting

$$E_n := span\{Z_1, Z_2, \ldots, Z_n\} \text{ and } S_n := \{Z \in E_n : \|Z\| = 1\}.$$

For each $Z \in E_n$, there exist $\lambda_i \in \mathbb{R}, i = 1, 2, \ldots, n$ such that

$$Z(\varsigma) = \sum_{i=1}^{n} \lambda_i Z_i(\varsigma) \text{ for } \varsigma \in \mathbb{R}. \tag{26}$$

Hence,

$$\|Z\|_{\gamma_3} = \left(\int_{\mathbb{R}} |Z(\varsigma)|^{\gamma_3} d\varsigma\right)^{\frac{1}{\gamma_3}} = \left(\sum_{i=1}^{n} |\lambda_i|^{\gamma_3} \int_{J_i} |Z_i(\varsigma)|^{\gamma_3} d\varsigma\right)^{\frac{1}{\gamma_3}}, \tag{27}$$

and hence

$$\begin{aligned} \|Z\|^2 &= \int_{\mathbb{R}} (|{}_{-\infty}\mathfrak{D}_\varsigma^\alpha Z(\varsigma)|^2 + (\mathcal{A}(\varsigma)Z(\varsigma), Z(\varsigma)))d\varsigma \\ &= \sum_{i=1}^{n} \lambda_i^2 \int_{\mathbb{R}} (|{}_{-\infty}\mathfrak{D}_\varsigma^\alpha Z_i(\varsigma)|^2 + (\mathcal{A}(\varsigma)Z_i(\varsigma), Z_i(\varsigma)))d\varsigma = \sum_{i=1}^{n} \lambda_i^2. \end{aligned} \tag{28}$$

There exists a constant $c > 0$ such that all norms of a finite dimensional normed space are similar

$$c\|Z\| \leq \|Z\|_{\gamma_3} \text{ for } Z \in E_n. \tag{29}$$

So, by (Λ_1), (Λ_3), (27)–(29), we have

$$\begin{aligned} I(sZ) &= \frac{s^2}{2}\|Z\|^2 - \int_{\mathbb{R}} \varpi(\varsigma, sZ(\varsigma))d\varsigma = \frac{s^2}{2}\|Z\|^2 - \sum_{i=1}^{n} \int_{J_i} \varpi(\varsigma, s\lambda_i Z_i(\varsigma))d\varsigma \\ &\leq \frac{s^2}{2}\|Z\|^2 - \eta s^{\gamma_3} \sum_{i=1}^{n} |\lambda_i|^{\gamma_3} \int_{J_i} |Z_i(\varsigma)|^{\gamma_3} d\varsigma \\ &= \frac{s^2}{2}\|Z\|^2 - \eta s^{\gamma_3} \|Z\|_{\gamma_3}^{\gamma_3} \leq \frac{s^2}{2}\|Z\|^2 - \eta(cs)^{\gamma_3}\|Z\|^{\gamma_3} \\ &= \frac{s^2}{2} - \eta(cs)^{\gamma_3}, \text{ for all } Z \in S_n, \text{ with } 0 < s \leq 1. \end{aligned} \tag{30}$$

From (30), it follows that there exists $\epsilon > 0$ and $\sigma > 0$ such as

$$I(\sigma Z) < -\epsilon \text{ for } Z \in S_n. \tag{31}$$

Letting

$$S_n^\sigma := \{\sigma Z : Z \in S_n\} \text{ and } \Omega := \left\{(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i^2 < \sigma^2\right\}.$$

Thus, from (31), is results that $I(Z) < -\epsilon$ for $Z \in S_n^\sigma$. In addition, we have $I \in C^1(X^\alpha, \mathbb{R})$ and even. This implies that

$$S_n^\sigma \subset I^{-\epsilon} \in \Sigma. \tag{32}$$

From (26) and (28) we deduce that there exists $\psi \in C(S_n^\sigma, \partial\Omega)$ an odd homeomorphism mapping ([14]), we obtain

$$\Gamma(I^{-\epsilon}) \geq \Gamma(S_n^\sigma) = n, \tag{33}$$

Let $c_n = \inf_{A\in\Sigma_n} \sup_{Z\in A} I(Z)$. Since I is bounded below on E, from (33) we obtain $-\infty < c_n \leq -\epsilon < 0$, and so $c_n \in \mathbb{R}_+$. We know that I has infinitely many nontrivial critical points (by using Lemma 3). Thus, the system 2 possesses infinitely many non trivial solutions.

Next, we show that $c_n \to 0^-$ as $n \to +\infty$. Define

$$X_n := span\{e_n\},\ Z_n = \overline{\bigoplus_{k=n}^{\infty} X_k},$$

where $\{e_n\}_{n=1}^{\infty}$ the standard orthogonal basis of X^α, and let

$$\beta_n = \sup_{Z\in Z_n, \|Z\|=1} \|Z\|_{\mathbb{L}^2}. \tag{34}$$

We claim that $\beta_n \to 0$ as $n \to +\infty$. Indeed, $0 < \beta_{n+1} \leq \beta_n$, and so $\beta_n \to \beta \geq 0$ as $n \to +\infty$. Now, for all $n \geq 1$, there exists $Z_n \in Z_n$ as such $\|Z_n\| = 1$ and $\|Z_n\| \geq \frac{\beta_n}{2}$. By definition of Z_n, it follows that $Z_n \to 0$ in X^α. Thus, by Lemma 3, we obtain $Z_n \to 0$ in $\mathbb{L}^2(\mathbb{R})$, and so $\beta = 0$. This proves our claim. Moreover, we have

$$I(Z) \geq \frac{1}{2}\|Z\|^2 - C_2^{\gamma_1}\|a_1\|_{\frac{2}{2-\gamma_1}}\|Z\|^{\gamma_1} - C_2^{\gamma_1}C_\infty^{\gamma_2-\gamma_1}\|a_2\|_{\frac{2}{2-\gamma_1}}\|Z\|^{\gamma_2}.$$

This implies that $I(Z)$ is coercive and $I(Z) \to +\infty$ as $\|Z\| \to +\infty$. Hence, there exists a $\tau > 0$ such that $I(Z) \to 0$ for $\|Z\| \geq \tau$. Moreover, for any $A \in \Sigma_n$, $\Gamma(A) \geq n$, and so $A \cap Z_n \neq \emptyset$. Thus, (34), yields

$$\begin{aligned}
\sup_{Z\in A} I(Z) &\geq \inf_{Z\in Z_n, \|Z\|\leq\tau} I(Z) \\
&\geq \inf_{Z\in Z_n, \|Z\|\leq\tau} \left(\frac{1}{2}\|Z\|^2 - \beta_n^{\gamma_1}\|a_1\|_{\frac{2}{2-\gamma_1}}\|Z\|^{\gamma_1} - \beta_n^{\gamma_1}C_\infty^{\gamma_2-\gamma_1}\|a_2\|_{\frac{2}{2-\gamma_1}}\|Z\|^{\gamma_2}\right) \\
&\geq -\beta_n^{\gamma_1}\|a_1\|_{\frac{2}{2-\gamma_1}}\tau^{\gamma_1} - \beta_n^{\gamma_1}C_\infty^{\gamma_2-\gamma_1}.
\end{aligned}$$

Therefore,

$$c_n = \inf_{A\in\Sigma_n} \sup_{Z\in A} I(Z) \geq -\beta_n^{\gamma_1}\|a_1\|_{\frac{2}{2-\gamma_1}}\tau^{\gamma_1} - \beta_n^{\gamma_1}C_\infty^{\gamma_2-\gamma_1}\|a_2\|_{\frac{2}{2-\gamma_1}}\tau^{\gamma_2}.$$

Combining this with $c_n < 0$ and $\beta_n \to 0$, we obtain $c_n \to 0^-$ as $n \to +\infty$ as desired.

4. Example

Consider system (2) with $\mathcal{A}(\varsigma) = (1+\varsigma^2)I_N$, where I_N is the identity matrix of order N and

$$\varpi(\varsigma, Z) = \frac{e^{-\varsigma^2}cos(\varsigma)}{1+|\varsigma|}|Z|^{\frac{4}{3}} + \frac{e^{-\varsigma^2}sin(\varsigma)}{1+|\varsigma|}|Z|^{\frac{3}{2}}.$$

Then, we obtain

$$\nabla\varpi(\varsigma, Z) = \frac{4e^{-\varsigma^2}cos(\varsigma)}{3(1+|\varsigma|)}|Z|^{\frac{-2}{3}}Z + \frac{3e^{-\varsigma^2}sin(\varsigma)}{2(1+|\varsigma|)}|Z|^{\frac{-1}{2}}Z,$$

$$|\varpi(\varsigma, Z)| \le \frac{2e^{-\varsigma^2}}{1+|\varsigma|}|Z|^{\frac{4}{3}}, \forall\ (\varsigma, Z) \in \mathbb{R}\times\mathbb{R}^N, |Z| \le 1,$$

$$|\varpi(\varsigma, Z)| \le \frac{2e^{-\varsigma^2}}{1+|\varsigma|}|Z|^{\frac{3}{2}}, \forall\ (\varsigma, Z) \in \mathbb{R}\times\mathbb{R}^N, |Z| \ge 1,$$

$$|\nabla\varpi(\varsigma, Z)| \le \frac{2e^{-\varsigma^2}|Z|^{\frac{1}{3}} + 9|Z|^{\frac{1}{2}}}{6(1+|\varsigma|)}, \forall\ (\varsigma, Z) \in \mathbb{R}\times\mathbb{R}^N,$$

and

$$\varpi(\varsigma, Z) \ge \frac{3e^{-\frac{\pi^2}{9}}|Z|^{\frac{4}{3}}}{2(3+\pi)}, \forall\ (\varsigma, Z) \in (0, \frac{\pi}{3})\times\mathbb{R}^N, |Z| \le 1.$$

Therefore, the conditions of Theorem 2 are satisfied, where

$$\frac{4}{3} = \gamma_1 = \gamma_3 < \gamma_2 = \frac{3}{2},\ a_1(\varsigma) = a_2(\varsigma) = b(\varsigma) = \frac{2e^{-\varsigma^2}}{1+|\varsigma|},\ \varphi(s) = \frac{8s^{\frac{1}{3}} + 9s^{\frac{1}{2}}}{12}.$$

Thus, by applying Theorem 2, we conclude that the system (2) has infinitely many nontrivial solutions.

Remark 5. *In light of the above example, one can easily figure out that Z and ϖ are not periodic in ς. Moreover, ϖ is of sub-quadratic. Therefore, System (2) with the above parameters can not be commented by the results obtained in [14]. In contrast to the outcome and conditions suggested in [15], our assumptions in the present paper are more effective. The resulting example supports the validity of the proposed hypotheses.*

5. Conclusions

We investigated in this research, the existence of infinitely many homoclinic solutions for fractional Hamiltonian systems (2). The present method is different from those considered in the literature in the sense that it provides less restrictive assumptions and assumes that $\mathcal{A}$ is coercive at infinity, ϖ is of sub-quadratic growth as $|Z| \to \infty$, and that Z and ϖ are not periodic in ς. The properties of the critical point theory have been employed to prove the main results. The findings in this paper not only generalize but also improve the recent results on fractional Hamiltonian systems (2). We provide a concrete example that demonstrates the advantage of our theorems over the previous results.

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