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Existence Results for Nonlinear Fractional Differential Inclusions via q-ROF Fixed Point

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Abstract: Fractional Differential inclusions, the multivalued version of fractional differential equations, yellow play a vital role in various fields of applied sciences. In the present article, a class of q-rung orthopair fuzzy (q-ROF) set valued mappings along with q-ROF upper/lower semi-continuity have been introduced. Based on these ideas, existence theorems for a numerical solution of a distinct class of fractional differential inclusions have been achieved with the help of Schaefer type and Banach contraction fixed point theorems. A physical example is also provided to validate the hypothesis of the main results. The notion of q-rung orthopair fuzzy mappings along with the use of fixed point techniques and a new-fangled Caputo type fractional derivative are the principal novelty of this article.

Keywords: orthopair fuzzy sets; q-rung orthopair fuzzy mapping; fuzzy differential inclusion

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1. Introduction

From the last five decades, a lot of development has been observed in the field of fuzzy set theory and its connected branches. Many scientists have initiated various ideas and applications of fuzzy sets towards decision-making, game theory, control systems engineering, robotics, image processing and optimization theory, etc. Fuzzy set has been generalized in different directions such as L-fuzzy set [1], by defining membership on lattice and intuitionistic fuzzy set [2], by considering membership and nonmembership grades, Pythagorean fuzzy set [3], having membership grades in the form of orthopairs and q-rung orthopair fuzzy set [4], have q-rung orthopair membership grades. The main idea behind all this development is to allow more space to the membership grades. In all the above mentioned papers, the authors highlighted some important characteristics of the fuzzy family. The notion of fuzzy mappings was initiated by Weiss [5] and Butnariu [6]. Consequently, Heilpern [7] proved a fixed point result for fuzzy contractive mappings to generalize Nadler's result [8]. Afterwards, many mathematicians extended the idea of fuzzy mappings in various directions (for example, see [9–13]). Fractional calculus plays an important role in the modelling of physical problems in a more efficient way. Many fractional derivatives were defined such as Riemman–Liouville, Hadamard, Caputo and Grunwald–Letnikov [14–16]. The nature of the problem decides which fractional derivative is useful. In [17], authors define a new type of fractional derivative called a Caputo–Fabrizio derivative with a non-singular kernel, it was applied in many real world problems [18–21]. The Caputo–Fabrizio derivative was unable to satisfy some required properties; for example, it cannot produce the original function if the order approaches to 1. To overcome these issues, an other derivative with a nonsingular kernel using the Mittag–Leffler function was introduced [22,23]. Many investigations and applications have been made using this derivative [24–27]. Differential inclusions are of great importance in the modelling of optimization problems and game theory. An interesting way to generalize the modelling with differential inclusions is with fuzzy differential inclusions. In 2000, Zhou et al. presented a way to generalize the differential inclusions via fuzzy mappings. An extension of this idea was made by Min et al. in [28] and proved a result for the existence of a solution of system of fuzzy differential inclusions using a continuous selection theorem. This idea was extended for fuzzy partial differential inclusions in [29], and the system of fuzzy partial differential inclusions in [30]. In this article, we introduce the notions of q-rung orthopair fuzzy (q-ROF for short) convex, concave functions, q-ROF upper/lower semicontinuous mappings and q-ROF numbers. Using these notions, we have presented existence results for fractional differential inclusions of ABC type via q-rung orthopair fuzzy mappings. We consider two problems related to open and closed q-rung orthopair fuzzy cuts of q-rung orthopair fuzzy mappings. We use well known selection theorems to prove these existence results. Finally, the existence results for obtained fractional differential equations of ABC types are proved using Schaefer type and the Banach contraction principle.

2. Preliminaries

The following are defined in [2].

Definition 1. *Consider X a non-empty set. A pair $A = \langle \mu_A, \nu_A \rangle$ where $\mu_A, \nu_A : X \to [0,1]$ are functions satisfying $\mu_A(x) + \nu_A(x) \leq 1 \forall\ x \in X$ is known as an intuitionistic fuzzy set (IF-set for short). The functions μ_A, ν_A are its membership and non-membership functions.*

Remark 1. *Every fuzzy set having membership function μ can be considered an IF-set $< \mu, 1 - \mu >$. The support of an IF-set A is the crisp set:*

$$supp(A) = \{A \in X : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\} \tag{1}$$

In 2016, Yager introduced the notion of q-rung orthopair fuzzy sets to generalize the basic idea of fuzzy sets. In the following, we include some basic definitions from [4].

Definition 2. *A q-ROF subset A of X, for short a q-ROF set, is an orthopair, $A = \langle \zeta_A, \eta_A \rangle_q$ satisfying:*

(i) $q \geq 1$,
(ii) $\zeta_A(x) \in [0,1]$ *and* $\eta_A(x) \in [0,1]$,
(iii) $(\zeta_A(x))^q + (\eta_A(x))^q \leq 1$,

where $\zeta_A, \eta_A : X \to [0,1]$ indicates the membership and nonmembership of elements in A, respectively.

The negation of a $q - ROF$ set is given by $C(a) = (1 - a^q)^{\frac{1}{q}}$.

Remark 2. *It is clear that IF sets are $q - ROF$ sets with $q = 1$ and Pythagorean fuzzy sets are $q - ROF$ sets with $q = 2$.*

Theorem 1. *If A is $q_1 - ROF$ set on X and if $q_2 > q_1$, then A is also $q_2 - ROF$ set on X.*

Definition 3. *For a $q - ROF$ set $A = \langle \zeta_A, \eta_A \rangle_q$, the strength of commitment at rung q is defined as*

$$S(A^q(x)) = (\zeta_A(x))^q + (\eta_A(x))^q)^{\frac{1}{q}},$$

which is actually a Minkowski metric.

The hesitancy of a *q-ROF* set membership grade is defined as

$$Hes_{A^q}(x) = (1 - (S_{A^q}(x))^q)^{\frac{1}{q}}$$
$$= (1 - ((\zeta_A(x))^q + (\eta_A(x))^q)^{\frac{1}{q}}.$$

Furthermore, if $p > q$, then $S_{A^p}(x) \leq S_{A^q}(x)$ and correspondingly $Hes_{A^p}(x) \geq Hes_{A^q}(x)$.
Let $F^q(X)$ denote the family of all *q-ROF* sets defined on set X. Yager et al. in 2016 defined some basic set operations on *q-ROF* sets.

(i) For $A_1, A_2 \in F^q(X)$ with membership grades $A_1 = \langle \zeta_{\wp_1}, \eta_{A_1} \rangle_q$ and $A_2 = \langle \zeta_{A_2}, \eta_{A_2} \rangle_q$, then $\zeta^q_{A_1} + \eta^q_{A_1} = \left(S_{A^q_1}\right)^q \leq 1$ and $\zeta^q_{A_2} + \eta^q_{A_2} = \left(S_{A^q_2}\right)^q \leq 1$.

(ii) Let $D = A_1 \cap A_2$, and the intersection of *q-ROF* sets is defined as $D = \langle \zeta_D, \eta_D \rangle_q$ where $\zeta_D = \min\{\zeta_{A_1}, \zeta_{A_2}\}$ and $\eta_D = \max\{\eta_{A_1}, \eta_{A_2}\}$; note that, since $\zeta^q_D + \eta^q_D \leq 1$, it ensures that $D \in F^q(X)$.

(iii) Let $D = A_1 \cap A_2$, intersection of *q-ROF* sets is defined as $D = \langle \zeta_D, \eta_D \rangle_q$ where $\zeta_D = \min\{\zeta_{A_1}, \zeta_{A_2}\}$ and $\eta_D = \max\{\eta_{A_1}, \eta_{A_2}\}$, note that, since $\zeta^q_D + \eta^q_D \leq 1$, it ensures that $D \in F^q(X)$.

(iv) For $A, B \in F^q(X)$, $A \subset B$ if $A(x) \leqslant B(x)$ for each $x \in X$, that is, $\zeta_A(x) \leq \zeta_B(x)$ and $\eta_A(x) \geq \eta_B(x)$.

(v) For a set $A \in F^q(X)$, the complement $\overline{A}$ is defined as $\overline{A} = \langle \eta_A, \zeta_A \rangle_q$.

Banach [31] in 1922 presented the famous Banach contraction principle.

Theorem 2. *Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping such that:*

$$d(Tx, Ty) \leq ad(x, y),$$

for all $x, y \in X, a \in (0, 1)$.

Schaefer Theorem [32] is as follows.

Theorem 3. *Let$(X, \|.\|)$ be a norm space. H be a continuous mapping of X into X, which is compact on each bounded subset D of X. Then, either*

(i) *$x = \lambda Hx$ has a solution in x, or*

(ii) *the set of all such solutions $0 \leq \lambda \leq 1$ is unbounded.*

The following is defined in [33].

Definition 4. *Associate with each real number s a positive measure μ_s on R^n by setting,*

$$d\mu_s(y) = (1 + |y|^2)^s dm_n(y)$$

if $f \in L^2(\mu_s)$ (where L^k is the space of Lebesgue integrable functions), that is, if $\int |f|^2 d\mu_s < \infty$, then f is a tempered distribution. Hence, f is the Fourier Transform of a tempered distribution u. The vector space of all u so obtained will be denoted by H^s, equipped with the norm,

$$\|u\|_s = \left(\int_{R_n} |\widehat{u}|^2 d\mu_s\right)^{1/2}$$

These spaces H^s are called Sobolev spaces.

It is known that the Mittag–Leffler function is the solution of the following fractional differential equation:

$$\frac{D^t f}{Dx^t} = af, \text{ for } 0 < t < 1.$$

Now, we recall recently a used fractional derivative and integral as given in [22,23].

Definition 5. *Considering $f \in H^1(a,b)$ and $t \in [0,1]$, then Atangana–Baleanu–Caputo fractional derivative (ABC fractional derivative) of order t is given by*

$$ {}_b^{ABC}D_\ell^t(f(\ell)) = \frac{B(t)}{1-t}\int_b^\ell f'(x)E_t\left(-t\frac{(\ell-x)^t}{1-t}\right)dx, \tag{2}$$

where $B(t)$ denotes a normalization function satisfying $B(0) = B(1) = 1$. The associated integral is defined by

$$ {}_a^{AB}I_\ell^t\{f(\ell)\} = \frac{1-t}{B(t)}f(\ell) + \frac{t}{B(t)\Gamma(t)}\int_a^\ell f(s)(\ell-s)^{t-1}ds. \tag{3}$$

The following concepts are defined in [34].

Definition 6. *Let $A \in F^q(X)$ and $x \in X$, then, respectively, the q-rung closed and open α-level cuts, for $\alpha \in (0,1]$ of A are defined by*

$$[A]_\alpha^q = \{x \in X : (\zeta_A(x))^q \geq \alpha \text{ and } (\eta_A(x))^q \leq 1-\alpha\}$$

and

$$[A]_\alpha^q = \{x \in X : (\zeta_A(x))^q > \alpha \text{ and } (\eta_A(x))^q < 1-\alpha\}.$$

Definition 7. *Let $\alpha, \beta \in (0,1]$ and $\alpha + \beta \leq 1$, then, respectively, the q-rung closed and open (α,β)-cuts of A are defined by*

$$[A]_{(\alpha,\beta)}^q = \{x \in X : (\zeta_A(x))^q \geq \alpha \text{ and } (\eta_A(x))^q \leq \beta\}$$

and

$$A_{(\alpha,\beta)}^q = \{x \in X : (\zeta_A(x))^q > \alpha \text{ and } (\eta_A(x))^q < \beta\}.$$

Definition 8. *Consider X be a non-empty set, Y be metric space. A mapping $S : X \to F^q(Y)$ is called q-ROF mapping.*

Definition 9. *A point $x^* \in X$ is called q-ROF fixed point of a q-ROF mapping $S : X \to F^q(X)$ if there exists $\alpha, \beta \in (0,1]$ such that $x^* \in [Sx^*]_{(\alpha,\beta)}^q$.*

3. Existence Results

Definition 10. *Let ∇_n^q represent the set of all q-ROF sets defined on set $\mathbb{R}^n$ such that, for $A \in \nabla_n^q$, the following properties are satisfied:*
(i) If there exist $x, y \in \mathbb{R}^n$ such that $(\zeta_A(x))^q = 1$ and $(\eta_A(y))^q = 0$, then A is normal;
(ii) A is q-ROF convex, that is, ζ_A is q-rung fuzzy convex, and η_A is q-rung fuzzy concave, that is, for $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$,

$$(\zeta_A(\lambda x + (1-\lambda)y))^q \geq \min\{(\zeta_A(x))^q, (\zeta_A(y))^q\},$$
$$(\eta_A(\lambda x + (1-\lambda)y))^q \leq \max\{(\eta_A(x))^q, (\eta_A(y))^q\};$$

(iii) A is upper semicontinuous that is for any $\alpha, \beta \in [0,1]$, $[A]^q_{(\alpha,\beta)}$ *is a closed subset of* $\mathbb{R}^n$;
(iv) the closure of $[A]^q_{(0,1)}$ *is compact.*

In fact ∇^q_n denotes the set of all *q-ROF* numbers. For $A, B \in \nabla^q_n$, define

$$H\Big([A]^q_{(\alpha,\beta)}, [B]^q_{(\alpha,\beta)}\Big) = \max\left\{ \sup_{b\in[B]^q_{(\alpha,\beta)}} d\Big(b, [A]^q_{(\alpha,\beta)}\Big), \sup_{a\in[A]^q_{(\alpha,\beta)}} d\Big(a, [B]^q_{(\alpha,\beta)}\Big) \right\},$$

the Hausdorff distance, clearly $[A]^q_{(\alpha,\beta)}, [B]^q_{(\alpha,\beta)}$ are compact subsets of $\mathbb{R}^n$. Again, for $A, B \in \nabla^q_n$, we define

$$H(A,B) = \sup_{\alpha,\beta}\Big\{ H\Big([A]^q_{(\alpha,\beta)}, [B]^q_{(\alpha,\beta)}\Big)\Big\}.$$

Definition 11. *Let* (X, d_X) *and* (Y, d_Y) *be any two metric spaces. A multivalued map* $\mathcal{S} : X \to 2^Y$ *is upper semicontinuous at* $\varepsilon_0 \in X$ *if and only if, for any neighborhood M of* $\mathcal{S}(\varepsilon_0)$*, there exists a neighborhood N of* ε_0 *such that, for each* $\varepsilon > 0$ *in N,* $\mathcal{S}(\varepsilon) \subset M$. $\mathcal{S}$ *is called upper semicontinuous on X if it is upper semicontinuous at any point* $\varepsilon_0 \in X$.

Remark 3. *A q-ROF valued map* $S : X \to \nabla^q_n$ *can generate a vector valued function* $\widetilde{S} : X \times \mathbb{R}^n \to [0,1]^2$*, where, for any* $x \in X$, $y \in \mathbb{R}^n$, $\widetilde{S}(x,y) = S(x)(y)$.

Throughout this paper, an open subset in $\mathbb{R} \times \mathbb{R}^n$ is denoted by Φ with $(\ell_0, \varepsilon_0) \in \Phi$.

Lemma 1. *Consider* Γ *a paracompact Hausdorff topological space, E a topological vector space,* $S : \Gamma \to 2^E$ *a multifunction having nonempty convex values. If S possesses open lower sections, that is, for any* $e \in E$, $S^{-1}(e) = \{x \in \Gamma : e \in S(x)\}$ *is open in* Γ*, then a function* $f : \Gamma \to E$ *exists, which is continuous and* $f(x) \in S(x)$ *for any* $x \in \Gamma$.

Definition 12. *Let* $S : \Phi \to \nabla^q_n$ *be a q-ROF map. The mapping S is called lower open if* $(S(\ell, \varepsilon))(x)$ *is lower semicontinuous at* $(\ell, \varepsilon) \in \Phi$.

The following lemma is from [28].

Lemma 2. *Let* $\Phi \subset \mathbb{R} \times \mathbb{R}^n$ *be an open set,* $(\ell_0, \varepsilon_0) \in \Phi$ *and* $S : \Phi \to \Gamma(\mathbb{R}^n)$ *an upper semicontinuous mutivalued operator. Then, there exists an interval* $L = [\ell_0 - \epsilon, \ell_0 + \epsilon] \subset \mathbb{R}$ *(for* $\epsilon > 0$*) and* $J > 0$ *such that*
(i) $L \times B_{\mathbb{R}^n}(\varepsilon_0, \epsilon J) \subset \Phi$;
(ii) $\|S(\ell, \varepsilon)\| \leq J$ *on* $L \times B_{\mathbb{R}^n}(\varepsilon_0, \epsilon J)$;
where $\Gamma(\mathbb{R}^n)$ *represents the set of all compact and convex subsets of* $\mathbb{R}^n$.

Lemma 3. *Consider X a Banach space and L a measurable space. Assume that* $A, B : L \longrightarrow \Gamma(X)$ *is any two multivalued measurable compact operators. For any measurable selection* $a(\ell) \in A(\ell)$*, a measurable selection* $b(\ell) \in B(\ell)$ *exists such that*

$$\|a(\ell) - b(\ell)\| \leq H(A(\ell), B(\ell)),$$

for all $\ell \in L$.

Theorem 4 assures the existence of solution of the following fractional differential inclusion. Let $\mathbb{R}^n$ represent a Banach space having norm $\|.\|_\infty$, and Φ represents an open subset of $[\ell_0 - \epsilon, \ell_0 + \epsilon] \times \mathbb{R}^n$. Suppose $S : \Phi \to \nabla^q_n$ be a *q*-ROF map and $\alpha, \beta : \mathbb{R}^n \to [0,1]$ are upper and lower semicontinuous functions. Assume the *q*-rung fuzzy differential

inclusions:
for a q-ROF map $S:\Phi\to\nabla_n^q$, find $\varepsilon\in C(I,\mathbb{R}^n)$ such that

$$\Big(\zeta_{S(\ell,\varepsilon)}\big({}^{ABC}D_\ell^t(\varepsilon(\ell))\big)\Big)^q>\alpha \text{ and } \Big(\eta_{S(\ell,\varepsilon)}\big({}^{ABC}D_\ell^t(\varepsilon(\ell))\big)\Big)^q<\beta,$$
$$\text{with initial condition } \varepsilon(\ell_0)=\varepsilon_0,$$

in other words,

$$\begin{cases} {}^{ABC}D_\ell^t(\varepsilon(\ell))\in\Big(S_{(\ell,\varepsilon(\ell))}\Big)^q_{(\alpha,\beta)},\\ \varepsilon(\ell_0)=\varepsilon_0.\end{cases}$$

Theorem 4. *Let $S:\Phi\to\nabla_n^q$ be a q-ROF map, which is also a bounded and lower open fuzzy surjection. Suppose ζ_S is q-rung fuzzy convex, η_S is q-rung fuzzy concave, and $\alpha,\beta:\mathbb{R}^n\to[0,1]$ are upper and lower semicontinuous functions, respectively, satisfying $(S(\ell,\varepsilon(\ell)))^q_{(\alpha,\beta)}$ is nonempty for every (ℓ,ε) in Φ. Then, there exists continuous selection $\widetilde{f}:\Phi\to\mathbb{R}^n$ with*

$$\Big(\zeta_{\widetilde{f}}\Big({}^{ABC}D_\ell^t(\varepsilon(\ell))\Big)\Big)^q>\alpha,\ \Big(\eta_{\widetilde{f}}\Big({}^{ABC}D_\ell^t(\varepsilon(\ell))\Big)\Big)^q<\beta$$

and ${}^{ABC}D_\ell^t(\varepsilon(\ell))\in(S(\ell,\varepsilon(\ell)))^q_{(\alpha,\beta)}$ for all (ℓ,ε) in Φ.

Proof. Define a set-valued function $\widetilde{S}:\Phi\to2^{\mathbb{R}^n}$ as

$$\widetilde{S}(\ell,\varepsilon(\ell))=\Big(S_{(\ell,\varepsilon(\ell))}\Big)^q_{(\alpha,\beta)}$$

where

$$\Big(S_{(\ell,\varepsilon(\ell))}\Big)^q_{(\alpha,\beta)}=\{x\in C(I,\mathbb{R}^n):(\zeta_S(x(\ell)))^q>\alpha \text{ and } (\eta_S(x(\ell)))^q<\beta\}$$

for each $(\ell,\varepsilon)\in\Phi$. Obviously $\widetilde{S}(\ell,\varepsilon(\ell))$ is nonempty, for each $(\ell,\varepsilon)\in\Phi$. Suppose for $x,y\in\widetilde{S}(\ell,\varepsilon(\ell))$ and $\lambda\in[0,1]$:

$$\Big(\zeta_{S_{(\ell,\varepsilon)}}(\lambda x+(1-\lambda)y)\Big)^q\geq\min\Big\{\Big(\zeta_{S_{(\ell,\varepsilon)}}(x)\Big)^q,\Big(\zeta_{S_{(\ell,\varepsilon)}}(y)\Big)^q\Big\}>\alpha,$$
$$\Big(\eta_{S_{(\ell,\varepsilon)}}(\lambda x+(1-\lambda)y)\Big)^q\leq\max\Big\{\Big(\eta_{S_{(\ell,\varepsilon)}}(x)\Big)^q,\Big(\eta_{S_{(\ell,\varepsilon)}}(y)\Big)^q\Big\}<\beta.$$

Then, the q-rung convexity of ζ_S and the q-rung concavity of η_S ensure $\lambda x+(1-\lambda)y\in\widetilde{S}(\ell,\varepsilon(\ell))$. Thus, $(S(\ell,\varepsilon(\ell)))^q_{(\alpha,\beta)}$ is a convex set on Φ. Next, it is shown that $\widetilde{S}$ has open lower sections. Consider for any $v\in\mathbb{R}^n$

$$\widetilde{S}^{-1}(v)=\Big\{(\ell,\varepsilon)\in\Phi:v\in\widetilde{S}(\ell,\varepsilon(\ell))\Big\},$$
$$=\Big\{(\ell,\varepsilon)\in\Phi:\big(\zeta_{\widetilde{S}}(v)\big)^q>\alpha \text{ and } \big(\eta_{\widetilde{S}}(v)\big)^q<\beta\Big\}.$$

In order to do so, it will be enough to prove that the complement of $\widetilde{S}^{-1}(v)$ that is the set $\Big\{(\ell,\varepsilon)\in\Phi:\big(\zeta_{\widetilde{S}}(v)\big)^q\leq\alpha \text{ and } \big(\eta_{\widetilde{S}}(v)\big)^q\geq\beta\Big\}$ is closed. For this, suppose $\{(\ell_n,\varepsilon_n)\}_{n\in\mathbb{N}}$ a sequence in $\Big(\widetilde{S}^{-1}(v)\Big)^c$ such that $(\ell_n,\varepsilon_n)\to(\ell,\varepsilon)$. Since S is lower open, α is upper semicontinuous and β is lower semicontinuous, we have

$$\Big(\zeta_{\widetilde{S_{(\ell_n,\varepsilon_n)}}}(v)\Big)^q\leq\alpha(\ell_n,\varepsilon_n),\Big(\eta_{\widetilde{S(\ell_n,\varepsilon_n)}}(v)\Big)^q\geq\beta(\ell_n,\varepsilon_n)$$

which implies

$$\begin{aligned}\left(\zeta_{\widetilde{S_{(\ell,\varepsilon)}}}(v)\right)^q &\leq \liminf_{n\to\infty}\left(\zeta_{\widetilde{S_{(\ell_n,\varepsilon_n)}}}(v)\right)^q \\ &\leq \limsup_{n\to\infty\infty}\alpha(\ell_n,\varepsilon_n) \\ &\leq \alpha(\ell,\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\left(\eta_{\widetilde{S_{(\ell,\varepsilon)}}}(v)\right)^q &\geq \limsup_{n\to\infty\infty}\left(\eta_{\widetilde{S(\ell_n,\varepsilon_n)}}(v)\right)^q \\ &\geq \liminf_{n\to\infty}\beta(\ell_n,\varepsilon_n) \\ &\geq \beta(\ell,\varepsilon).\end{aligned}$$

Hence, $(\ell,\varepsilon) \in \left(\widetilde{S}^{-1}(v)\right)^c$. Thus, $\widetilde{S}$ contains open lower sections. Now, with the help of a Proposition, a continuous selection $\widetilde{f} : \Phi \to \mathbb{R}^n$ exists so that $\widetilde{f}(\ell,\varepsilon) \in \left(S_{(\ell,\varepsilon(\ell))}\right)^q_{(\alpha,\beta)}$ for each $(\ell,\varepsilon) \in \Phi$. Since S is surjection and $\widetilde{S}(\ell,\varepsilon(\ell))$ is bounded, therefore ${}^{ABC}D^t_\ell(\varepsilon(\ell)) = \widetilde{f}(\ell,\varepsilon)$ for each $(\ell,\varepsilon) \in \Phi$. □

Theorem 5 assures the existence of solution of the following fractional differential inclusion.

Consider the q-rung fuzzy differential inclusions:
for a q-ROF map $S : \Phi \to \nabla^q_n$ evaluating $\varepsilon \in C(I,\mathbb{R}^n)$ such that

$$\begin{cases} \left(\zeta_{S(\ell,\varepsilon)}\left({}^{ABC}D^t_\ell(\varepsilon(\ell))\right)\right)^q \geq \alpha \text{ and } \left(\eta_{S(\ell,\varepsilon)}\left({}^{ABC}D^t_\ell(\varepsilon(\ell))\right)\right)^q \leq \beta, \\ \varepsilon(\ell_0) = \varepsilon_0, \end{cases}$$

in other words,

$$\begin{cases} {}^{ABC}D^t_\ell(\varepsilon(\ell)) \in \left(S_{(\ell,\varepsilon(\ell))}\right)^q_{(\alpha,\beta)}, \\ \varepsilon(\ell_0) = \varepsilon_0. \end{cases}$$

Theorem 5. *Let $S : \Phi \to \nabla^q_n$ be a uniformly continuous and q-rung fuzzy integrably bounded mapping and $\alpha,\beta : \mathbb{R}^n \to [0,1]$ are uniformly continuous. If, for every (ℓ,ε) and $(\ell,\bar{\varepsilon})$ in Φ,:*

$$H(S(\ell,\varepsilon), S(\ell,\bar{\varepsilon})) \leq \|\varepsilon - \bar{\varepsilon}\|$$

and

$$\left[\frac{1-t}{B(t)} + a\frac{t}{B(t)}\right] < 1$$

are satisfied, where $a \in \mathbb{R}$. Then, a solution of the above fractional differential inclusion exists.

Proof. Define a set-valued function $\widetilde{S} : \Phi \to 2^{\mathbb{R}^n}$ as

$$\widetilde{S}(\ell,\varepsilon(\ell)) = \left[S_{(\ell,\varepsilon(\ell))}\right]^q_{(\alpha,\beta)},$$

where

$$\left[S_{(\ell,\varepsilon(\ell))}\right]^q_{(\alpha,\beta)} = \left\{\varsigma \in C(I,\mathbb{R}^n) : (\zeta_S(x(\ell)))^q \geq \alpha \text{ and } (\eta_S(x(\ell)))^q \leq \beta\right\}$$

for each $(\ell,\varepsilon) \in \Phi$. Now, we show that the mapping $\widetilde{S}$ is upper semicontinuous. For this, consider the neighborhood of $\widetilde{S}\left(\widetilde{\ell},\widetilde{\varepsilon}\right)$, for $\left(\widetilde{\ell},\widetilde{\varepsilon}\right) \in \Phi$, as follows:

$$\mathcal{N}_r^{\widetilde{S}\left(\widetilde{\ell},\widetilde{\varepsilon}\right)} = \left\{\varepsilon \in \mathbb{R}^n : d\left(\varepsilon, \widetilde{S}\left(\widetilde{\ell},\widetilde{\varepsilon}\right)\right) < r\right\}.$$

For$(\ell,\varepsilon) \in \Phi$ and $v \in \widetilde{S}(\ell,\varepsilon)$, we have

$$\begin{aligned} d\left(v,\widetilde{S}\left(\widetilde{\ell},\widetilde{\varepsilon}\right)\right) &\leq H\left(\widetilde{S}(\ell,\varepsilon),\widetilde{S}\left(\widetilde{\ell},\widetilde{\varepsilon}\right)\right) \\ &= H\left(\left[S_{(\ell,\varepsilon)}\right]^q_{(\alpha,\beta)},\left[S_{\left(\widetilde{\ell},\widetilde{\varepsilon}\right)}\right]^q_{\left(\widetilde{\alpha},\widetilde{\beta}\right)}\right) \\ &\leq H\left(\left[S_{\left(\widetilde{\ell},\widetilde{\varepsilon}\right)}\right]^q_{\left(\widetilde{\alpha},\widetilde{\beta}\right)},\left[S_{(\ell,\varepsilon)}\right]^q_{\left(\widetilde{\alpha},\widetilde{\beta}\right)}\right) + H\left(\left[S_{(\ell,\varepsilon)}\right]^q_{\left(\widetilde{\alpha},\widetilde{\beta}\right)},\left[S_{(\ell,\varepsilon)}\right]^q_{(\alpha,\beta)}\right) \\ &\leq H\left(\widetilde{S}\left(\widetilde{\ell},\widetilde{\varepsilon}\right),\widetilde{S}(\ell,\varepsilon)\right) + H\left(\left[S_{(\ell,\varepsilon)}\right]^q_{\left(\widetilde{\alpha},\widetilde{\beta}\right)},\left[S_{(\ell,\varepsilon)}\right]^q_{(\alpha,\beta)}\right). \end{aligned}$$

Since S and α are uniformly continuous, utilizing the above inequality, a small enough neighborhood $\aleph$ of $\left(\widetilde{\ell},\widetilde{\varepsilon}\right)$ in Φ can be found, satisfying for all $(\ell,\varepsilon) \in \aleph$ and $\delta \in \widetilde{S}(\ell,\varepsilon)$

$$d\left(\delta,\widetilde{S}\left(\widetilde{\ell},\widetilde{\varepsilon}\right)\right) < r,$$

thus

$$\widetilde{S}(\aleph) \subset \mathcal{N}_r^{\widetilde{S}\left(\widetilde{\ell},\widetilde{\varepsilon}\right)},$$

which shows $\widetilde{S}$ is upper semicontinuous. Thus, utilizing lemma 3, a real constant $\kappa > 0$ exists such that

$$\max_{(\ell,\varepsilon)}\left\|\widetilde{S}(\ell,\varepsilon)\right\| \leq \kappa.$$

Let

$$W = \left\{\begin{array}{c} v \in C(I,\mathbb{R}^n) : \|v(\ell) - v_0\| \leq \kappa_1, \\ \text{for all } \ell \in I = [0,a], \\ \text{where } v(\ell_0) = v_0 \end{array}\right\}$$

with a metric $d_W : W \times W \to \mathbb{R} \cup \{+\infty\}$ defined by;

$$d_W(v_1,v_2) = \sup_{\ell \in I}\{\|v_1(\ell) - v_2(\ell)\|\}.$$

Then, (W,d_W) become a complete generalized metric space [28].
Now, define a multivalued mapping $T : W \longrightarrow 2^W$ by

$$T(v) = \left\{\overline{v} : \overline{v}(\ell) \in v_0 + \frac{1-t}{B(t)}\left[S_{(\tau,v(\tau))}\right]^q_{(\alpha,\beta)} + \frac{t}{B(t)}\int_0^\ell \left[S_{(\tau,v(\tau))}\right]^q_{(\alpha,\beta)} d\tau \quad \textit{almost everywhere in} \;\; I\right\},$$

provided that $\left[S_{(\tau,v(\tau))}\right]^q_{(\alpha,\beta)} = \{0\}$ at $\tau = \ell_0$, where $\int_0^\ell \left[S_{(\tau,v(\tau))}\right]^q_{(\alpha,\beta)} d\tau$ is the multivalued integral by Aumann [35].
Now, we show that $T(v)$ is nonempty for all $v \in W$. Since the multivalued operator $\widetilde{S}(\ell,\varepsilon(\ell)) = \left[S_{(\ell,\varepsilon(\ell))}\right]^q_{(\alpha,\beta)}$ is upper semicontinuous with compact values, therefore according to Kuratowski–Ryll–Nardzewski selection, Theorem [36], $\widetilde{S}(\ell,\varepsilon)$ contains a measurable

selection $f(\ell,\varepsilon) \in \widetilde{S}(\ell,\varepsilon)$ for all $\ell \in I$ and, by a given condition $f(\ell,\varepsilon(\ell))$, is also Lebesgue integrable. Let

$$\widehat{v}(\ell) = v_0 + \frac{1-t}{B(t)} f(\tau, v(\tau)) + \frac{t}{B(t)} \int_0^\ell f(\tau, v(\tau)) d\tau,$$

then $\widehat{v} \in T(v)$, implying $T(v) \neq \emptyset$.
Next, it is claimed that $T(v)$ is closed for each $v \in W$. Assume that (v_n) is a sequence in $T(v)$ which is convergent to $v^\circ \in W$. Since we know that

$$v_n(\ell) \in v_0 + \frac{1-t}{B(t)} \widetilde{S}(\tau, v(\tau)) + \frac{t}{B(t)} \int_0^\ell \widetilde{S}(\tau, v(\tau)) d\tau \ \ \textit{almost everywhere in } I,$$

and the set $v_0 + \frac{1-t}{B(t)} \widetilde{S}(\tau, v(\tau)) + \frac{t}{B(\varrho)} \int_0^\ell \widetilde{S}(\tau, v(\tau)) d\tau$ is closed [37], it follows that $v^\circ \in T(v)$.
Now, we prove that T is a multivalued contraction. For this purpose, choose $v_2 \in T(v_1)$, which implies the existence of $f(\ell, v_1) \in \widetilde{S}(\ell, v_1)$ such that

$$v_2(\ell) = v_0 + \frac{1-t}{B(t)} f(\tau, v_1(\tau)) + \frac{t}{B(\varrho)} \int_0^\ell f(\tau, v_1(\tau)) d\tau.$$

Then, by Lemma 4, a measurable selection $f(\ell, v_2) \in \widetilde{S}(\ell, v_2)$ exists such that

$$\begin{aligned} \|f(\ell, v_2) - f(\ell, v_1)\| &\leq H\Big(\widetilde{S}(\ell, v_2), \widetilde{S}(\ell, v_1)\Big) \\ &= H\bigg(\Big[S_{(\ell, v_2)}\Big]^q_{(\alpha,\beta)}, \Big[S_{(\ell, v_1)}\Big]^q_{(\alpha,\beta)}\bigg) \\ &\leq H(S(\ell, v_2), S(\ell, v_1)). \end{aligned}$$

Let $v_3 \in T(v_2)$, then

$$v_3(\ell) = v_0 + \frac{1-t}{B(t)} f(\tau, v_2(\tau)) + \frac{t}{B(t)} \int_0^\ell f(\tau, v_2(\tau)) d\tau.$$

Consider

$$\begin{aligned} \|v_3(\ell) - v_2(\ell)\| = \Bigg\| &\left(\tfrac{1-\varrho}{B(t)} f(\tau, v_2(\tau)) + \tfrac{t}{B(t)} \int_0^\ell f(\tau, v_2(\tau)) d\tau \right) \\ &- \left(\tfrac{1-t}{B(t)} f(\tau, v_1(\tau)) + \tfrac{t}{B(t)} \int_0^\ell f(\tau, v_1(\tau)) d\tau \right) \Bigg\| \end{aligned}$$

$$\leq \left\| \frac{1-t}{B(t)}[f(\tau,v_2(\tau)) - f(\tau,v_1(\tau))] + \frac{t}{B(t)}\left[\int_0^{\ell} f(\tau,v_2(\tau))d\tau - \int_0^{\ell} f(\tau,v_1(\tau))d\tau\right]\right\| d\tau$$

$$\leq \left[\frac{1-t}{B(t)} + a\frac{t}{B(t)}\right]\|f(\tau,v_2(\tau)) - f(\tau,v_1(\tau))\|$$

$$\leq \left[\frac{1-t}{B(t)} + a\frac{t}{B(t)}\right]H(S(\tau,v_2),S(\tau,v_1))d\tau$$

$$\leq \left[\frac{1-t}{B(t)} + a\frac{t}{B(t)}\right]\|v_2 - v_1\|$$

$$\leq \left[\frac{1-t}{B(t)} + a\frac{t}{B(t)}\right]d(v_1,v_2),$$

consequently, we have

$$H(T(v_{n+1}),T(v_n)) \leq \left[\frac{1-t}{B(t)} + a\frac{t}{B(t)}\right]^n d(v_1,v_2).$$

Hence, Nadler's theorem implies the existence of a fixed point $v \in T(v)$, which is the solution of the given Problem 2. □

The following example shows the solution of fuzzy fractional differential inclusion obtained by using Theorem 4.

Example 1. *Consider the following fuzzy fractional differential inclusion:*

$$^{ABC}D^t f(x,z) + cf(x,z) \in \left[\chi_{\left(\frac{1}{B(t)+(1-t)c}\left[\frac{tz^t}{\Gamma(t+1)}+B(t)\sin x\right]\right)}\right]^q_{(a(u(x,y)),\beta(u(x,y)))},$$

$$\text{where } f(x,0) = \sin x,$$

$x,y \in \mathbb{R}$. *Suppose that* $F : \Phi \to \nabla^q_n$ *is a q-rung orthopair fuzzy map and* $\alpha,\beta : \mathbb{R}^n \to [0,1]$ *are upper and lower semicontinuous functions such that*

$$\left[F_{(x,y,u(x,y))}\right]^q_{(a(u(x,y)),\beta(u(x,y)))} := \left[\chi_{\left(\frac{1}{B(t)+(1-t)c}\left[\frac{tz^t}{\Gamma(t+1)}+B(t)\sin x\right]\right)}\right]^q_{(a(u(x,y)),\beta(u(x,y)))}$$

is non-empty for every $(x,y,u) \in \Phi$ *and* χ *is the characteristic function, defined as*

$$H\left(\begin{array}{c}\left[\chi_{\left(\frac{1}{B(t)+(1-t)c}\left[\frac{tz^t}{\Gamma(t+1)}+B(t)\sin x\right]\right)}\right]^q_{(a(u(x,y)),\beta(u(x,y)))},\\ \left[\chi_{\left(\frac{1}{B(t)+(1-t)c}\left[\frac{tz^t}{\Gamma(t+1)}+B(t)\sin x\right]\right)}\right]^q_{(a(v(x,y)),\beta(v(x,y)))}\end{array}\right) = 0.$$

Now, Theorem 4 assures the existence of a continuous selection $\widetilde{f}(x,y,u) \in \left[F_{(x,y,u(x,y))}\right]^q_{(a(u(x,y)),\beta(u(x,y)))}$ *for each* $(x,y,u) \in \Phi$ *such that*

$$\left(\zeta_{\widetilde{f}}\left(\frac{1}{B(t)+(1-t)c}\left[\frac{tz^t}{\Gamma(t+1)}+B(t)\sin x\right]\right)\right)^q > \alpha,$$

and

$$\left(\eta_{\widetilde{f}}\left(\frac{1}{B(t)+(1-t)c}\left[\frac{tz^t}{\Gamma(t+1)}+B(t)\sin x\right]\right)\right)^q < \beta.$$

Hence,

$$\frac{1}{B(t)+(1-t)c}\left[\frac{tz^t}{\Gamma(t+1)}+B(t)\sin x\right] \in \left[F_{(x,y,u(x,y))}\right]^q_{(\alpha(u(x,y)),\beta(u(x,y)))}.$$

4. Conclusions

The notions of q-rung orthopair fuzzy convex, concave functions, q-rung orthopair fuzzy mappings, q-rung orthopair fuzzy upper/lower semicontinuous mappings and q-rung orthopair fuzzy numbers are introduced. On the basis of these notions and by using Schaefer type and Banach contraction principle, the existence results for special type of fractional differential inclusions have been proved. An example in support of main results is also presented. The idea can be extended further in various directions for obtaining fuzzy fixed points, fuzzy common fixed point and coincidence point results in various metric spaces under diverse contractive conditions.

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