fractal and fractional

Article

Subordination Properties of Certain Operators Concerning Fractional Integral and Libera Integral Operator

Georgia Irina Oros [1,*], Gheorghe Oros [1] and Shigeyoshi Owa [2]

[1] Department of Mathematics and Computer Science, Faculty of Informatics and Sciences, University of Oradea, 410087 Oradea, Romania
[2] 1 Decembrie 1918 University, 510009 Alba Iulia, Romania
* Correspondence: georgia_oros_ro@yahoo.co.uk

Abstract: The results contained in this paper are the result of a study regarding fractional calculus combined with the classical theory of differential subordination established for analytic complex valued functions. A new operator is introduced by applying the Libera integral operator and fractional integral of order λ for analytic functions. Many subordination properties are obtained for this newly defined operator by using famous lemmas proved by important scientists concerned with geometric function theory, such as Eenigenburg, Hallenbeck, Miller, Mocanu, Nunokawa, Reade, Ruscheweyh and Suffridge. Results regarding strong starlikeness and convexity of order α are also discussed, and an example shows how the outcome of the research can be applied.

Keywords: analytic function; libera integral operator; fractional integral of order λ; differential subordination; strongly of order α

MSC: 30C45

check for updates

Citation: Oros, G.I.; Oros, G.; Owa, S. Subordination Properties of Certain Operators Concerning Fractional Integral and Libera Integral Operator. *Fractal Fract.* **2023**, *7*, 42. https://doi.org/10.3390/fractalfract7010042

Academic Editor: Natália Martins

Received: 5 December 2022
Revised: 23 December 2022
Accepted: 27 December 2022
Published: 30 December 2022

1. Introduction

Ever since the theory of differential subordination was initiated by Miller and Mocanu in the work published in 1978 [1] and 1981 [2], it was intensely used since it proves useful at re-obtaining known results in easier manners and also for providing interesting results when associated to studies involving analytic functions. A line of research which developed nicely in the context of differential subordination theory resulted after incorporating different types of operators into the study. Integral operators are an important tool when such investigations are considered as a recent survey paper shows [3]. The research started with the integral operator introduced by Alexander in 1915 [4]. A widely investigated integral operator is the Libera integral operator, introduced in 1965 [5]. Due to its properties of preserving starlikeness and convexity, it has been associated with many studies (see for example, references [6–10]) and still provides important new outcomes if combined with differential operators, such as in [11], with a confluent hypergeometric function, such as in [12], or with a generalized distribution, such as in [13]. Generalizations of the Libera operator are also considered for recent studies in papers, such as [14–17].

In the present investigation, the Libera integral operator is extended and combined with the fractional integral of order λ for introducing a new fractional calculus operator. The idea was inspired by recent publications where the fractional integral is associated with the Mittag–Leffler confluent hypergeometric function [18–20], with the confluent hypergeometric function [21,22], with the Ruscheweyh and Sălăgean Operators [23], with the convolution product of the multiplier transformation and the Ruscheweyh derivative [24], with the convolution product of Sălăgean operator and Ruscheweyh derivative [25] or with other operators [26,27].

Consider the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

denoted by A and called analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

For $f(z) \in A$, Libera [5] introduced the integral operator $L_1(f(z))$ defined as

$$\begin{aligned} L_1(f(z)) &= \frac{2}{z}\int_0^z f(t)dt \\ &= z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)a_k z^k. \end{aligned}$$

Consider the following extension for the operator $L_1(f(z))$.

$$\begin{aligned} L_2(f(z)) &= L_1(L_1(f(z))) \\ &= z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^2 a_k z^k \end{aligned}$$

and

$$\begin{aligned} L_n(f(z)) &= L_1(L_{n-1}(f(z))) \\ &= z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^n a_k z^k, \end{aligned}$$

where $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, and $L_0(f(z)) = f(z)$.

For $f(z) \in A$, the extension called fractional integral of order λ is used in [28,29] as:

$$I_z^{\lambda}(f(z)) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(t)}{(z-t)^{1-\lambda}}dt \quad (\lambda > 0),$$

where the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$, and $\Gamma(z)$ is the gamma function.

The following form in easily deduced:

$$I_z^{\lambda}(f(z)) = \frac{1}{\Gamma(2+\lambda)}z^{1+\lambda} + \sum_{k=2}^{\infty}\frac{k!}{\Gamma(k+1+\lambda)}a_k z^{k+\lambda}.$$

Using $I_z^{\lambda}(f(z))$, we consider

$$\begin{aligned} L_{\lambda}(f(z)) &= \frac{\Gamma(2+\lambda)}{z^{\lambda}}I_z^{\lambda}(f(z)) \\ &= z + \sum_{k=2}^{\infty}\frac{k!\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)}a_k z^k \quad (\lambda > 0). \end{aligned} \tag{1}$$

It follows from the above that

$$L_0(f(z)) = \lim_{\lambda \to 0} L_{\lambda}(f(z)) = f(z)$$

and

$$L_1(f(z)) = z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)a_k z^k.$$

Definition 1. *Using the operator $L_\lambda(f(z))$ given by (1), we introduce*

$$L_{n+\lambda}(f(z)) = L_n(L_\lambda(f(z))) = z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^n \frac{k!\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)}a_k z^k$$

and

$$L_{\lambda+n}(f(z)) = L_\lambda(L_n(f(z))) = z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^n \frac{k!\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)}a_k z^k$$

for $n = 0, 1, 2, \ldots$ and $0 < \lambda \leq 1$. Considering the expressions above, we have:

$$L_{n+\lambda}(f(z)) = L_{\lambda+n}(f(z)).$$

For $f(z) \in A$, $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, and such that $f(z) = g(w(z))$. If $g(z)$ is univalent in U, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ ([30,31]).

We note that $f(z) \in A$ belongs to the class of starlike functions of order α in U if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in U)$$

for $0 \leq \alpha < 1$ and that $f(z) \in A$ belongs to the class of convex functions of order α in U if

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in U)$$

for $0 \leq \alpha < 1$.

In addition, the analytic function $p(z)$, $z \in U$, satisfies the condition

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (z \in U)$$

for certain real values $\alpha > 0$ if

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha \quad (z \in U).$$

In Section 2 of the paper, a series of properties are proved for the newly introduced operator $L_\lambda(f(z))$ given by (1) considering the theory of differential subordination and a well-known lemma from Miller and Mocanu [30,32]. The study on operator $L_\lambda(f(z))$ is continued in Section 3 with results obtained by using lemmas from Suffridge [33] and its improved form obtained by Hallenbeck and Ruscheweyh [34]. Results related to the Briot–Bouquet differential subordination involving the operator $L_\lambda(f(z))$ are also obtained in Section 3 by using a lemma from Eenigenburg, Miller, Mocanu and Reade [35]. The study considering the operator $L_\lambda(f(z))$ and known lemmas is concluded in Section 3 with two theorems that use a result proved by Nunokawa [36,37] for obtaining certain univalence conditions for the operator $L_\lambda(f(z))$. The necessary lemmas cited above are listed in every section before each new result that is obtained as application. In Section 4, strong starlikeness and convexity of order α are investigated regarding the operator $L_\lambda(f(z))$, and an example is also presented as an application for the new results.

2. Subordination Results Regarding $L_{n+\lambda}(F(Z))$

To consider some properties of $L_{n+\lambda}(f(z))$, the following result proved by Miller and Mocanu ([30,32]) (also from Jack [38]) will be considered in the study.

Lemma 1 ([30,32,38])**.** *Let $w(z)$ be analytic in U with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then we have*

$$z_0 w'(z_0) = m w(z_0)$$

and

$$\mathrm{Re}\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq m,$$

where $m \geq 1$.

Using the lemma presented above, the following theorem can be stated and proved:

Theorem 1. *Consider the function $f(z) \in A$ satisfying the subordination*

$$\frac{L_{n+\lambda}(f(z))}{z} \prec \frac{\alpha(1+z)}{\alpha + (2-\alpha)z} \quad (z \in U) \tag{2}$$

for certain real values $\alpha > 1$. The subordination (2) gives:

$$\left|\frac{L_{n+\lambda}(f(z))}{z} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \quad (z \in U). \tag{3}$$

Proof. With condition (2), there exists an analytic function $w(z)$ satisfying the properties needed for the definition of subordination and

$$\frac{L_{n+\lambda}(f(z))}{z} = \frac{\alpha(1+w(z))}{\alpha + (2-\alpha)w(z)} \quad (z \in U). \tag{4}$$

Using relation (4) have that

$$|w(z)| = \left|\frac{\alpha\left(\frac{L_{n+\lambda}(f(z))}{z} - 1\right)}{\alpha - (2-\alpha)\frac{L_{n+\lambda}(f(z))}{z}}\right| < 1 \quad (z \in U),$$

and that

$$2\left|\frac{L_{n+\lambda}(f(z))}{z}\right|^2 - \alpha\left(\frac{L_{n+\lambda}(f(z))}{z} + \overline{\left(\frac{L_{n+\lambda}(f(z))}{z}\right)}\right) < 0 \tag{5}$$

for $z \in U$. Hence, inequality (3) holds. □

Remark 1. *The result (3) in Theorem 1 shows us that*

$$0 < \mathrm{Re}\left(\frac{L_{n+\lambda}(f(z))}{z}\right) < \alpha \quad (z \in U)$$

for $\alpha > 1$.

Let us consider the analytic function $f(z)$ such that

$$L_{n-1+\lambda}(f(z)) = \frac{z(4+5z-2z^2)}{(2-z)^2} \quad (z \in U).$$

Then, we see that

$$\begin{aligned} L_{n+\lambda}(f(z)) &= \frac{2}{z}\int_0^z L_{n-1+\lambda}(f(t))dt \\ &= \frac{2}{z}\int_0^z \frac{t(4+5t-2t^2)}{(2-t)^2}dt \\ &= \frac{2z(1+z)}{2-z}. \end{aligned} \tag{6}$$

The function obtained in (6) can be used in subordination (2) and satisfies the inequality (3) for $\alpha = 4$.

For an analytic function $f(z)$, the following result can be proved.

Theorem 2. *If* $f(z) \in A$ *satisfies*

$$\mathrm{Re}\left(\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))} - 1\right) < \frac{1}{4(\alpha-1)} \quad (z \in U) \tag{7}$$

for $1 < \alpha \leq 2$ *or*

$$\mathrm{Re}\left(\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))} - 1\right) < \frac{\alpha-1}{4} \quad (z \in U) \tag{8}$$

for $\alpha > 2$, *then*

$$\left|\frac{L_{n+\lambda}(f(z))}{z} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \quad (z \in U).$$

Proof. Consider an analytic function $w(z)$ that satisfies relation (4). We know that $w(0) = 0$, and we obtain from (4) that

$$\frac{z(L_{n+\lambda}(f(z)))'}{L_{n+\lambda}(f(z))} - 1 = \frac{zw'(z)}{w(z)}\left(\frac{w(z)}{1-w(z)} - \frac{(2-\alpha)w(z)}{\alpha+(2-\alpha)w(z)}\right). \tag{9}$$

Since

$$z(L_{n+\lambda}(f(z)))' = 2L_{n-1+\lambda}(f(z)) - L_{n+\lambda}(f(z)),$$

Equation (9) becomes

$$\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))} - 1 = \frac{zw'(z)}{2w(z)}\left(\frac{w(z)}{1-w(z)} - \frac{(2-\alpha)w(z)}{\alpha+(2-\alpha)w(z)}\right).$$

For the considered function $w(z)$, assume that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1.$$

In this situation, we write $w(z_0) = e^{i\theta}$ $(0 \leq \theta < 2\pi)$ and

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1).$$

Using the properties seen above, we have

$$\begin{aligned} \mathrm{Re}\left(\frac{L_{n-1+\lambda}(f(z_0))}{L_{n+\lambda}(f(z_0))} - 1\right) &= \frac{k}{2}\mathrm{Re}\left(\frac{w(z_0)}{1-w(z_0)} - \frac{(2-\alpha)w(z_0)}{\alpha+(2-\alpha)w(z_0)}\right) \\ &= \frac{k}{2}\left(\frac{1}{2} - \frac{(2-\alpha)(2-\alpha+\alpha\cos\theta)}{\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)\cos\theta}\right). \end{aligned}$$

Considering a function $g(t)$ given by

$$g(t) = \frac{2-\alpha+\alpha t}{\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)t} \quad (t=\cos\theta),$$

we have

$$g'(t) = \frac{4\alpha(\alpha-1)}{(\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)t)^2} > 0.$$

Since $g(t)$ is increasing for $t=\cos\theta$, we obtain for $1<\alpha\leq 2$ that

$$\text{Re}\left(\frac{L_{n-1+\lambda}(f(z_0))}{L_{n+\lambda}(f(z_0))}-1\right) \geq \frac{k}{4(\alpha-1)} \geq \frac{1}{4(\alpha-1)} \tag{10}$$

and

$$\text{Re}\left(\frac{L_{n-1+\lambda}(f(z_0))}{L_{n+\lambda}(f(z_0))}-1\right) \geq \frac{(\alpha-1)k}{4} \geq \frac{\alpha-1}{4} \quad (\alpha>2). \tag{11}$$

Since (10) contradicts (7) and (11) contradicts (8), we say that there is no $w(z)$ such that $w(0)=0$ and $|w(z_0)|=1$ for $z_0\in U$. This implies that

$$|w(z)| = \left|\frac{\alpha\left(\frac{L_{n+\lambda}(f(z))}{z}-1\right)}{\alpha-(2-\alpha)\frac{L_{n+\lambda}(f(z))}{z}}\right| < 1 \quad (z\in U),$$

that is the inequality (5). □

Next, our result is

Theorem 3. *Consider the analytic function $f(z)$ satisfying the conditions*

$$\text{Re}\left(\frac{L_{n+\lambda}(f(z))}{L_{n+1+\lambda}(f(z))}-\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))}-1\right) < \frac{1}{4(\alpha-1)} \quad (z\in U),$$

for $1<\alpha\leq 2$ or

$$\text{Re}\left(\frac{L_{n+\lambda}(f(z))}{L_{n+1+\lambda}(f(z))}-\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))}-1\right) < \frac{\alpha-1}{4} \quad (z\in U),$$

for $\alpha>2$.

Then,

$$\left|\frac{L_{n+1+\lambda}(f(z))}{L_{n+\lambda}(f(z))}-\frac{\alpha}{2}\right| < \frac{\alpha}{2} \quad (z\in U).$$

Proof. Consider a function $w(z)$ satisfying

$$\frac{L_{n+1+\lambda}(f(z))}{L_{n+\gamma}(f(z))} = \frac{\alpha(1+w(z))}{\alpha+(2-\alpha)w(z)} \quad (z\in U).$$

This shows that $w(0)=0$.
Using

$$z(L_{n+1+\lambda}(f(z)))' = 2L_{n+\lambda}(f(z)) - L_{n+1+\lambda}(f(z))$$

and

$$z(L_{n+\lambda}(f(z)))' = 2L_{n-1+\lambda}(f(z)) - L_{n+\lambda}(f(z)),$$

we obtain that

$$\frac{L_{n+\lambda}(f(z))}{L_{n+1+\lambda}(f(z))}-\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))}-1 = \frac{zw'(z)}{2w(z)}\left(\frac{w(z)}{1-w(z)}-\frac{(2-\alpha)w(z)}{\alpha+(2-\alpha)w(z)}\right).$$

From this point on, the proof of this theorem is completed by following the same steps as for the proof of Theorem 2. □

3. Applications of Subordinations by Suffridge

We first introduce the following lemma proved by Suffridge [27].

Lemma 2 ([27])**.** *If a function $p(z)$ is analytic in U with $p(0) = 1$ and satisfies*

$$zp'(z) \prec h(z) \quad (z \in U)$$

for some starlike function $h(z)$, then

$$p(z) \prec \int_0^z \frac{h(t)}{t} \quad (z \in U).$$

Applying the above lemma, we have

Theorem 4. *Consider the analytic function $f(z)$ satisfying the following subordination*

$$\frac{L_{n-1+\lambda}(f(z)) - L_{n+\lambda}(f(z))}{z} \prec \frac{1 + (1-2\alpha)z}{2(1-z)} \quad (z \in U),$$

for certain real values α $(0 \le \alpha < 1)$.

Then

$$\frac{L_{n+\lambda}(f(z))}{z} \prec \log\left(\frac{\sqrt{z}}{(1-z)^{1-\alpha}}\right) \quad (z \in U).$$

Proof. Consider the analytic function $p(z)$ with $p(0) = 1$ given by:

$$p(z) = \frac{L_{n+\lambda}(f(z))}{z}.$$

In addition, consider the starlike function of order α $h(z)$ given by

$$h(z) = \frac{1 + (1-2\alpha)z}{1-z} \quad (z \in U),$$

for $0 \le \alpha < 1$.

Since

$$\begin{aligned} zp'(z) &= \frac{z(L_{n+\lambda}(f(z)))' - L_{n+\lambda}(f(z))}{z} \\ &= \frac{2(L_{n-1+\lambda}(f(z)) - L_{n+\gamma}(f(z)))}{z} \end{aligned}$$

and

$$\begin{aligned} \int_0^z \frac{h(t)}{t} dt &= \int_0^z \left(\frac{1}{t} - \frac{2(1-\alpha)}{1-t}\right) dt \\ &= \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right), \end{aligned}$$

by applying Lemma 2, we obtain that

$$\frac{L_{n-1+\lambda}(f(z)) - L_{n+\lambda}(f(z))}{z} \prec \frac{1}{2} \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right) \quad (z \in U).$$

Hence, the proof is completed. □

Taking $\alpha = \frac{1}{2}$ in Theorem 4, the following corollary emerges:

Corollary 1. *If $f(z) \in A$ satisfies*

$$\frac{L_{n-1+\lambda}(f(z)) - L_{n+\lambda}(f(z))}{z} \prec \frac{1}{1-z} \quad (z \in U),$$

then

$$\frac{L_{n+\lambda}(f(z))}{z} \prec \frac{1}{2}\log\left(\frac{z}{1-z}\right) \quad (z \in U).$$

Hallenbeck and Ruscheweyh [28] obtained the following form for Lemma 2 given by Suffridge:

Lemma 3 ([28]). *If a function $p(z)$ is analytic in U with $p(0) = 1$ and satisfies*

$$p(z) + zp'(z) \prec h(z) \quad (z \in U)$$

for some convex function $h(z)$, then

$$p(z) \prec \frac{1}{z}\int_0^z h(t)dt \quad (z \in U).$$

Now, we prove the following result.

Theorem 5. *Consider the analytic function $f(z)$ satisfying the subordination*

$$2\frac{L_{n-1+\lambda}(f(z))}{z} - \frac{L_{n+\lambda}(f(z))}{z} \prec \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right) \quad (z \in U),$$

for certain real values of α $(0 \le \alpha < 1)$.

Then,

$$\frac{L_{n+\lambda}(f(z))}{z} \prec \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right) + \frac{2(1-\alpha)}{z}\log(1-z) + (1-2\alpha) \quad (z \in U).$$

Proof. Consider the analytic function $p(z)$, $z \in U$, with $p(0) = 1$, given by

$$p(z) = \frac{L_{n+\lambda}(f(z))}{z}.$$

Using it, we can write:

$$p(z) + zp'(z) = 2\frac{L_{n-1+\lambda}(f(z))}{z} - \frac{L_{n+\lambda}(f(z))}{z}.$$

Further, we know that a function $h(z)$ given by

$$h(z) = \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right) \quad (z \in U)$$

satisfies

$$zh'(z) = \frac{1+(1-2\alpha)z}{1-z}.$$

Thus, $h(z)$ is convex in U because $zh'(z)$ is starlike of order α in U. Applying Lemma 3, we obtain

$$\begin{aligned}\frac{L_{n+\lambda}(f(z))}{z} &\prec \frac{1}{z}\int_0^z \left(\log\left(\frac{t}{(1-t)^{2(1-\alpha)}}\right)\right)dt \\ &= \frac{1}{z}\int_0^z (\log t - 2(1-\alpha)\log(1-t))dt \\ &= \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right) + \frac{2(1-\alpha)}{z}\log(1-z) + (1-2\alpha) \quad (z \in U).\end{aligned}$$

□

Choosing $\alpha = \frac{1}{2}$ in Theorem 5, we obtain the following corollary.

Corollary 2. *Consider the analytic function $f(z)$ satisfying the following subordination:*

$$2\frac{L_{n-1+\lambda}(f(z))}{z} - \frac{L_{n+\lambda}(f(z))}{z} \prec \log\left(\frac{z}{1-z}\right) \quad (z \in U)$$

Then,

$$\frac{L_{n+\lambda}(f(z))}{z} \prec \log\left(\frac{z}{1-z}\right) + \frac{1}{z}\log(1-z) \quad (z \in U).$$

Theorem 6. *Consider the analytic function $f(z)$ satisfying the following subordination:*

$$2\frac{L_{n-1+\lambda}(f(z))}{z} - \frac{L_{n+\lambda}(f(z))}{z} \prec \frac{1+z}{1-z} \quad (z \in U).$$

Then,

$$\frac{L_{n+\lambda}(f(z))}{z} \prec \frac{2}{z}\log\left(\frac{1}{1-z}\right) - 1 \quad (z \in U). \tag{12}$$

Proof. Letting

$$p(z) = \frac{L_{n+\lambda}(f(z))}{z} \text{ and } h(z) = \frac{1+z}{1-z},$$

we have that $p(z)$ is analytic in U with $p(0) = 1$, and $h(z)$ is convex in U. Since

$$\begin{aligned}\frac{1}{z}\int_0^z h(t)dt &= \frac{1}{z}\int_0^z \left(\frac{1+t}{1-t}\right)dt \\ &= \frac{2}{z}\log\left(\frac{1}{1-z}\right) - 1,\end{aligned}$$

we have the subordination (12). □

Next, the lemma given below, proved by Eenigenburg, Miller, Mocanu and Reade [29], is used for obtaining a new result.

Lemma 4 ([29]). *Let $h(z)$ be convex in U with* $\mathrm{Re}(\beta h(z) + \gamma) > 0$ $(\beta \neq 0)$. *If $p(z)$ is analytic in U with $p(0) = h(0)$, then the subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in U)$$

satisfies

$$p(z) \prec h(z) \quad (z \in U).$$

With this lemma, we have

Theorem 7. *Consider the analytic function $f(z)$ satisfying the following subordination:*

$$\frac{L_{n+1+\lambda}(f(z)) - 2L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))} + \frac{2L_{n+\lambda}(f(z))}{L_{n+1+\lambda}(f(z))} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in U),$$

for $0 \le \alpha < 1$.

Then,

$$\frac{L_{n+1+\lambda}(f(z))}{L_{n+\lambda}(f(z))} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in U).$$

Proof. Consider the analytic function $p(z)$, $z \in U$, with $p(0) = 1$, given by

$$p(z) = \frac{L_{n+1+\lambda}(f(z))}{L_{n+\lambda}(f(z))}.$$

In addition, consider the convex function of order α given by

$$h(z) = \frac{1+(1-2\alpha)z}{1-z}, \quad 0 \le \alpha < 1,$$

$h(0) = 1$.

Taking $\beta = 1$ and $\gamma = 0$ in Lemma 4, we say that

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in U)$$

implies

$$p(z) \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in U).$$

Since

$$\frac{zp'(z)}{p(z)} = 2\left(\frac{L_{n+\lambda}(f(z))}{L_{n+1+\lambda}(f(z))} - \frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))}\right),$$

we prove the theorem with Lemma 4. □

Next, we consider the following lemma proved by Nunokawa ([30,31]).

Lemma 5 ([30,31])**.** *Let a function $p(z)$ be analytic in U with $p(0) = 1$. If there exists a point z_0 $(|z_0| < 1)$ such that*

$$|\arg(p(z))| < \frac{\pi}{2}\beta \quad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\beta$$

for some real $\beta > 0$, then

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg(p(z_0))}{\pi}$$

for some real k such that

$$k \ge \frac{1}{2}\left(a + \frac{1}{a}\right) > 1,$$

where

$$(p(z_0))^{\frac{1}{\beta}} = \pm ia \quad (a > 0).$$

Now, we derive

Theorem 8. *Let $f(z) \in A$ and*

$$F(z) = \frac{L_{n+\lambda}(f(z)) - \alpha z}{(1-\alpha)z} + 2\frac{L_{n-1+\lambda}(f(z)) - \alpha z}{L_{n+\lambda}(f(z)) - \alpha z} - 2 \tag{13}$$

for $0 \le \alpha < 1$. If $f(z)$ satisfies

$$F(z)^2 - 1 \prec \frac{16z}{(1-z)^2} \quad (z \in U), \tag{14}$$

then,

$$\text{Re}\left(\frac{L_{n+\lambda}(f(z))}{z}\right) > \alpha \quad (z \in U).$$

Proof. Consider the analytic function $p(z)$, $z \in U$, with $p(0) = 1$, given by:

$$p(z) = \frac{L_{n+\lambda}(f(z))}{z}.$$

For such $p(z)$, assume that there exists a point z_0 ($|z_0| < 1$) such that

$$\text{Re}\left(\frac{p(z) - \alpha}{1-\alpha}\right) > 0 \quad (|z| < |z_0| < 1)$$

and

$$\text{Re}\left(\frac{p(z_0) - \alpha}{1-\alpha}\right) = 0.$$

If

$$\frac{p(z_0) - \alpha}{1-\alpha} \neq 0,$$

Lemma 5 gives us that

$$\begin{aligned} \frac{z_0 p'(z_0)}{p(z_0) - \alpha} &= \frac{2ik}{\pi} \arg\left(\frac{p(z_0) - \alpha}{1-\alpha}\right) \\ &= \frac{2ik}{\pi} \arg(p(z_0) - \alpha) \end{aligned}$$

for some real k such that $k \ge \frac{1}{2}\left(a + \frac{1}{a}\right) > 1$ with

$$\left(\frac{p(z_0) - \alpha}{1-\alpha}\right)^{\frac{1}{\beta}} = \pm ia \quad (a > 0).$$

It follows from the above that

$$\begin{aligned} \left\{\frac{p(z_0) - \alpha}{1-\alpha} + \frac{z_0 p'(z_0)}{p(z_0) - \alpha}\right\} - 1 &= F(z_0)^2 - 1 \\ &= (\pm ia \pm ik)^2 - 1 \\ &\le -\left(a + \frac{a^2+1}{2a}\right)^2 - 1. \end{aligned}$$

Let us consider a function $h(a)$ given by

$$h(a) = a + \frac{a^2+1}{2a} \quad (a > 0).$$

Then, $h(a)$ satisfies

$$h(a) \geq h\left(\sqrt{\frac{1}{3}}\right) = \sqrt{3}.$$

This gives us that

$$\left\{\frac{p(z_0)-\alpha}{1-\alpha} + \frac{z_0 p'(z_0)}{p(z_0)-\alpha}\right\}^2 - 1 = F(z_0)^2 - 1 \leq -4.$$

Here, we define a function $g(z)$ by

$$g(z) = \frac{16z}{(1-z)^2} \quad (z \in U).$$

Then, $g(z)$ maps U onto the domain with the slit $(-\infty, -4)$. This contradicts our condition (14).

Having the contradiction, we conclude that $p(z)$ satisfies the condition

$$\mathrm{Re}\left(\frac{p(z)-\alpha}{1-\alpha}\right) = \mathrm{Re}\left(\frac{\frac{L_{n+\lambda}(f(z))}{z} - \alpha}{1-\alpha}\right) > 0,$$

for all $z \in U$. Hence, the proof of the theorem is completed. □

Next, our theorem is

Theorem 9. *Consider a function $F(z)$ given by (13) where $f(z)$ is analytic in U and $0 \leq \alpha < 1$. If $F(z)$ satisfies*

$$F(z) \prec \frac{1+z}{1-z} \quad (z \in U), \tag{15}$$

then

$$\mathrm{Re}\left(\frac{L_{n+\lambda}(f(z))}{z}\right) > \alpha \quad (z \in U).$$

Proof. Consider the analytic function $p(z)$ given by

$$p(z) = \frac{L_{n+\lambda}(f(z))}{z}.$$

Then, there exists a point z_0 $(|z_0| < 1)$ such that

$$\mathrm{Re}\left(\frac{p(z)-\alpha}{1-\alpha}\right) > 0 \quad (|z| < |z_0| < 1) \tag{16}$$

and

$$\mathrm{Re}\left(\frac{p(z_0)-\alpha}{1-\alpha}\right) = 0. \tag{17}$$

If

$$\frac{p(z_0)-\alpha}{1-\alpha} = 0,$$

by Lemma 5, we have

$$\frac{z_0 p'(z_0)}{p(z_0)-\alpha} = \frac{2ik}{\pi} \arg(p(z_0) - \alpha)$$

for some real $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) > 1$ with

$$\frac{p(z_0) - \alpha}{1 - \alpha} = \pm ia \quad (a > 0).$$

With the properties obtained so far, we can write

$$\frac{p(z_0) - \alpha}{1 - \alpha} + \frac{z_0 p'(z_0)}{p(z_0) - \alpha} = F(z_0) = \pm i(a + k).$$

Since

$$\operatorname{Re}\left(\frac{1+z}{1-z}\right) > 0 \quad (z \in U),$$

we say that

$$F(z) \not\prec \frac{1+z}{1-z} \quad (z \in U).$$

This means that there is no z_0 ($|z_0| < 1$) such that (16) and (17) are satisfied. Hence, we obtain the stated conclusion of the theorem.

$$\operatorname{Re}(p(z) - \alpha) = \operatorname{Re}\left(\frac{L_{n+\lambda}(f(z))}{z} - \alpha\right) > 0 \quad (z \in U).$$

□

Remark 2. *Considering $f(z)$ an analytic function in U given by*

$$\frac{L_{n+\lambda}(f(z))}{z} = \frac{1}{1-z}$$

and $\alpha = 0$ in Theorem 9, we have that

$$F(z) = \frac{1+z}{1-z} \quad (z \in U).$$

Therefore, $f(z)$ satisfies the subordination (15) for $\alpha = 0$. For such $f(z)$, we know that

$$\operatorname{Re}\left(\frac{L_{n+\lambda}(f(z))}{z}\right) > \frac{1}{2} > 0 \quad (z \in U).$$

4. Results Regarding Strong Properties of Order α

Let $f(z) \in A$ and $L_{n+\lambda}(f(z))$ be defined by (1) for $n = 0, 1, 2, \ldots$ and $0 \leq \lambda \leq 1$. For $f(z) \in A$ satisfying

$$\operatorname{Re}\left(\frac{L_{n+\lambda}(f(z))}{z}\right) > \alpha \quad (z \in U),$$

$f(z)$ is said to be strongly of order α in U if $f(z)$ satisfies

$$\left|\arg\left(\frac{L_{n+\lambda}(f(z))}{z}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U),$$

where $0 \leq \alpha < 1$.

If $f(z) \in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U)$$

for $0 \leq \alpha < 1$, then $f(z)$ is said to be strongly starlike of order α in U. In addition, if $f(z) \in A$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U)$$

for $0 \leq \alpha < 1$, then we say that $f(z)$ is strongly convex of order α in U.

Let us consider a function $w(z)$ defined by

$$w(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in U)$$

for $0 \leq \alpha < 1$, then we see that

$$\arg w(z) = \alpha \arg\left(\frac{1+z}{1-z}\right) = \frac{\pi}{2}\alpha \quad (z \in U).$$

Thus, a function $f(z)$ given by

$$f(z) = \exp\left(\int_0^z \left(\frac{1+t}{1-t}\right)^{\alpha} dt\right)$$

is strongly starlike of order α in U and a function $f(z) \in A$ given by

$$f'(z) = \frac{1}{z}\exp\left(\int_0^z \left(\frac{1+t}{1-t}\right)^{\alpha} dt\right)$$

is strongly convex of order α in U.

Now, we derive

Theorem 10. *If $f(z) \in A$ satisfies*

$$\left|\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))} - 1\right| < \frac{\alpha}{4}\mathrm{Re}\left(\frac{1+\beta z}{1-z}\right) \quad (z \in U)$$

for some real α $(0 \leq \alpha < 1)$ and some real β $(\beta \neq -1)$, then

$$\left|\arg\left(\frac{L_{n+\lambda}(f(z))}{z}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U).$$

Proof. Define a function $p(z)$ by

$$p(z) = \frac{L_{n+\lambda}(f(z))}{z}.$$

Then, $p(z)$ is analytic in U, and $p(0) = 1$. This function $p(z)$ satisfies

$$\frac{zp'(z)}{p(z)} = 2\left(\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))} - 1\right).$$

It follows from the above that

$$
\begin{aligned}
\left|\arg\left(\frac{L_{n+\lambda}(f(z))}{z}\right)\right| &= |\arg(p(z))| = |\mathrm{Im}(\log(p(z)))| = \left|\mathrm{Im}\int_0^z (\log(p(t)))' dt\right| \\
&= \left|\mathrm{Im}\int_0^z \frac{p'(t)}{p(t)} dt\right| = \left|\mathrm{Im}\int_0^z \frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})} e^{i\theta} d\rho\right| \\
&\leq \int_0^r \left|\mathrm{Im}\left(\frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})} e^{i\theta}\right)\right| d\rho \leq \int_{-r}^r \left|\frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})}\right| d\rho \\
&\leq \frac{r}{2}\int_0^{2\pi} \left|\frac{p'(re^{i\theta})}{p(re^{i\theta})}\right| d\theta = \frac{1}{2}\int_0^{2\pi} \left|\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}\right| d\theta \\
&= \int_0^{2\pi} \left|\frac{L_{n-1+\lambda}(f(re^{i\theta}))}{L_{n+\lambda}(f(re^{i\theta}))} - 1\right| dt < \frac{\alpha}{4}\int_0^{2\pi} \mathrm{Re}\left(\frac{1+\beta re^{i\theta}}{1-re^{i\theta}}\right) d\theta \\
&= \frac{\alpha}{4}\int_0^{2\pi} \left\{\frac{1-\beta}{2} + \left(\frac{1+\beta}{2}\right)\frac{1-r^2}{1+r^2-2r\cos\theta}\right\} d\theta = \frac{\pi}{2}\alpha,
\end{aligned}
$$

because by Poisson integral

$$
\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos\theta} d\theta = 1.
$$

This completes the proof of the theorem. □

Example 1. *Consider a function $f(z) \in A$ given by*

$$
L_{n+\lambda}(f(z)) = z\left(\frac{2}{2-z}\right)^{3\alpha} \quad (z \in U),
$$

with $0 \leq \alpha < 1$. Note that a function

$$
w(z) = \frac{2}{2-z}
$$

satisfies

$$
\left|w(z) - \frac{4}{3}\right| < \frac{2}{3} \quad (z \in U)
$$

and

$$
|\arg w(z)| < \frac{\pi}{6} \quad (z \in U).
$$

This gives us that

$$
\left|\arg\left(\frac{L_{n+\lambda}(f(z))}{z}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U).
$$

For such $f(z)$, we have

$$
\left|\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))} - 1\right| = \frac{3}{2}\alpha\left|\frac{z}{2-z}\right| < \frac{3}{2}\alpha \quad (z \in U).
$$

Thus, if we consider a real β such that $\beta \leq -11$, then $f(z)$ satisfies

$$
\left|\frac{L_{n-1+\lambda}(f(z))}{L_{n+\lambda}(f(z))} - 1\right| < \frac{3}{2}\alpha \leq \frac{\alpha(1-\beta)}{8} < \frac{\alpha}{4}\mathrm{Re}\left(\frac{1+\beta z}{1-z}\right),
$$

for $z \in U$.

5. Conclusions

The outcome of this paper falls within the research topic which concerns incorporating fractional calculus in geometric function theory by defining new fractional operators and conducting studies involving the theory of differential subordination. The operator used for the investigation denoted by $L_\lambda(f(z))$ is introduced in Definition 1 using fractional integral of order λ defined in [28,29] and the Libera integral operator [5]. The necessary known definitions regarding the analytic functions are shown in the Introduction. Section 2 contains three theorems that show the results of the study conducted on the operator $L_\lambda(f(z))$ by applying a famous lemma from Miller and Mocanu [30,32] which is presented at the beginning of this section. Section 3 starts with recalling the lemma from Suffridge [33] which is used for obtaining the new results regarding the operator $L_\lambda(f(z))$ contained in Theorem 4 and Corollary 1. This lemma was modified by Hallenbeck and Ruscheweyh [34]. Their resulting lemma is first listed, and then Theorems 5 and 6 and Corollary 2 present the new results obtained by applying it to the operator $L_\lambda(f(z))$. A lemma from Eenigenburg, Miller, Mocanu and Reade [35] is next stated and used for obtaining the new result involving the operator $L_\lambda(f(z))$ presented in Theorem 7. A lemma proved by Nunokawa [36,37] is next listed and applied to the operator $L_\lambda(f(z))$ for the new outcome presented in Theorems 8 and 9. In Section 4, the basic definition regarding strong starlikeness and strong convexity of order α are recalled, and a new result concerning the strong starlikeness of order α of the operator $L_\lambda(f(z))$ is proved. An example is also provided in order to show a certain application of the theoretical result presented in Theorem 10.

As future uses of the results presented here, the operator $L_\lambda(f(z))$ given by (1) can be applied for defining new subclasses of analytic functions with certain geometric properties given by the characteristics of this operator already proven in this paper. The classes could be further investigated considering the strong starlikeness of order α of the operator $L_\lambda(f(z))$ having as inspiration recent studies such as [39].

Author Contributions: Conceptualisation, S.O.; methodology, G.I.O., G.O. and S.O.; software, G.I.O.; validation, G.I.O., G.O. and S.O.; formal analysis, G.I.O., G.O. and S.O.; investigation, G.I.O., G.O. and S.O.; resources, G.I.O. and S.O.; data curation, G.I.O. and S.O.; writing—original draft preparation, S.O.; writing—review and editing, G.I.O., G.O. and S.O.; visualisation, G.I.O.; supervision, S.O.; project administration, G.I.O.; funding acquisition, G.I.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable

Informed Consent Statement: Not applicable

Data Availability Statement: Not applicable

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Miller, S.S.; Mocanu, P.T. Second order differential inequalities in the complex plane. *J. Math. Anal. Appl.* **1978**, *65*, 289–305. [CrossRef]
2. Miller, S.S.; Mocanu, P.T. Differential subordinations and univalent functions. *Mich. Math. J.* **1981**, *28*, 157–172. [CrossRef]
3. Ahuja, O.P.; Çetinkaya, A. A Survey on the theory of integral and related operators in Geometric Function Theory. In *Mathematical Analysis and Computing*; Mohapatra, R.N., Yugesh, S., Kalpana, G., Kalaivani, C., Eds.; ICMAC 2019. Springer Proceedings in Mathematics & Statistics; Springer: Singapore, 2021; Volume 344. [CrossRef]
4. Alexander, J.W. Functions which map the interior of the unit circle upon simple regions. *Ann. Math.* **1915**, *17*, 12–22. [CrossRef]
5. Libera, R.J. Some classes of regular univalent functions. *Proc. Am. Math. Soc.* **1965**, *16*, 755–758. [CrossRef]
6. Owa, S.; Srivastava, H.M. Some applications of the generalized Libera integral operator. *Proc. Jpn. Acad. Ser. A Math. Sci.* **1986**, *62*, 125–128. [CrossRef]
7. Nunokawa, M. On starlikeness of Libera transformation. *Complex Var. Elliptic Equ.* **1991**, *17*, 79–83. [CrossRef]
8. Acu, M. A preserving property of a generalized Libera integral operator. *Gen. Math.* **2004**, *12*, 41–45.
9. Oros, G.; Oros, G.I. Convexity condition for the Libera integral operator. *Complex Var. Elliptic Equ.* **2006**, *51*, 69–76. [CrossRef]
10. Szász, R. A sharp criterion for the univalence of Libera operator. *Creat. Math. Inf.* **2008**, *17*, 65–71.

11. Oros, G.I. New differential subordination obtained by using a differential-integral Ruscheweyh-Libera operator. *Miskolc Math. Notes* **2020**, *21*, 303–317. [CrossRef]
12. Oros, G.I. Study on new integral operators defined using confluent hypergeometric function. *Adv. Differ. Equ.* **2021**, *2021*, 342. [CrossRef]
13. Hamzat, J.O.; Oladipo, A.T.; Oros, G.I. Application of a Multiplier Transformation to Libera Integral Operator Associated with Generalized Distribution. *Symmetry* **2022**, *14*, 1934. [CrossRef]
14. Guney, H.O.; Owa, S. New extension of Alexander and Libera integral operators. *Turkish J. Math.* **2022**, *46*, 17. [CrossRef]
15. Chandralekha, S. Inclusion properties for subclasses of multivalent regular functions defined on the unit disk. *Malaya J. Mat.* **2021**, *9*, 684–689. [CrossRef]
16. Aouf, M.K.; Mostafa, A.O.; Bulboacă, T. Properties of a certain class of multivalent functions. *Bol. Soc. Parana Mat.* **2022**, *40*, 1–9. [CrossRef]
17. Kanwal, B.; Hussain, S.; Abdeljawad, T. On certain inclusion relations of functions with bounded rotations associated with Mittag-Leffler functions. *AIMS Math.* **2022**, *7*, 7866–7887. [CrossRef]
18. Ghanim, F.; Al-Janaby, H.F. An analytical study on Mittag-Leffler-confluent hypergeometric functions with fractional integral operator. *Math. Meth. Appl. Sci.* **2021**, *44*, 3605–3614. [CrossRef]
19. Ghanim, F.; Al-Janaby, H.F.; Bazighifan, O. Some New Extensions on Fractional Differential and Integral Properties for Mittag-Leffler Confluent Hypergeometric Function. *Fractal Fract.* **2021**, *5*, 143. [CrossRef]
20. Ghanim, F.; Bendak, S.; Al Hawarneh, A. Certain implementations in fractional calculus operators involving Mittag-Leffler-confluent hypergeometric functions. *Proc. R. Soc. A* **2022**, *478*, 20210839. [CrossRef]
21. Alb Lupaş, A. New Applications of the Fractional Integral on Analytic Functions. *Symmetry* **2021**, *13*, 423. [CrossRef]
22. Acu, M.; Oros, G.; Rus, A.M. Fractional Integral of the Confluent Hypergeometric Function Related to Fuzzy Differential Subordination Theory. *Fractal Fract.* **2022**, *6*, 413. [CrossRef]
23. Alb Lupaş, A. On Special Fuzzy Differential Subordinations Obtained for Riemann-Liouville Fractional Integral of Ruscheweyh and Sălăgean Operators. *Axioms* **2022**, *11*, 428. [CrossRef]
24. Alb Lupaş, A. New Applications of Fractional Integral for Introducing Subclasses of Analytic Functions. *Symmetry* **2022**, *14*, 419. [CrossRef]
25. Alb Lupaş, A. Subordination results for a fractional integral operator. *Probl. Anal. Issues Anal.* **2022**, *11*, 20–31. [CrossRef]
26. Wanas, A.K.; Hammadi, N.J. Applications of Fractional Calculus on a Certain Class of Univalent Functions Associated with Wanas Operator. *Earthline J. Math. Sci.* **2022**, *9*, 117–129. [CrossRef]
27. Srivastava, H.M.; Kashuri, A.; Mohammed, P.O.; Nonlaopon, K. Certain Inequalities Pertaining to Some New Generalized Fractional Integral Operators. *Fractal Fract.* **2021**, *5*, 160. [CrossRef]
28. Owa, S. On the distortion theorems I. *Kyungpook Math. J.* **1978**, *18*, 53–59.
29. Owa, S.; Srivastava, H.M. Univalent and starlike generalized hypergeometric functions. *Can. J. Math.* **1987**, *39*, 1057–1077. [CrossRef]
30. Miller, S.S.; Mocanu, P.T. *Differential Subordinations, Theory and Applications*; Marcel Dekker Inc.: New York, NY, USA, 2000.
31. Pommerenke, C. *Univalent Functions*; Vanderhoeck and Ruprecht: Göttingen, Germany, 1975.
32. Miller, S.S.; Mocanu, P.T. Briot-Bouquet differential equations and differential subordinations. *Complex Var.* **1997**, *33*, 217–237. [CrossRef]
33. Suffridge, T.J. Some remarks on convex maps on the unit disc. *Duke Math. J.* **1970**, *37*, 775–777. [CrossRef]
34. Hallenbeck, D.J.; Ruscheweyh, S. Subordination by convex functions. *Proc. Am. Math. Soc.* **1975**, *52*, 191–195. [CrossRef]
35. Eenigenburg, P.; Miller, S.S.; Mocanu, P.T.; Reade, M.O. On a Briot-Bouquet differential subordination. In *General Inequalities 3*; I.S.N.M. Birkhäuser Verlag: Basel, Switzerland, 1983; Volume 64, pp. 339–348.
36. Nunokawa, M. On properties of non-Carathéodory functions. *Proc. Jpn. Acad.* **1992**, *68*, 152–153. [CrossRef]
37. Nunokawa, M. On the order of strongly starlikeness of strongly convex functions. *Proc. Jpn. Acad.* **1993**, *69*, 234–237. [CrossRef]
38. Jack, I.S. Functions starlike and convex of order alpha. *J. Lond. Math. Soc.* **1971**, *3*, 469–471. [CrossRef]
39. Sümer Eker, S.; Şeker, B.; Çekiç, B.; Acu, M. Sharp Bounds for the Second Hankel Determinant of Logarithmic Coefficients for Strongly Starlike and Strongly Convex Functions. *Axioms* **2022**, *11*, 369. [CrossRef]