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Threshold Results for the Existence of Global and Blow-Up Solutions to a Time Fractional Diffusion System with a Nonlinear Memory Term in a Bounded Domain

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Abstract: In this paper, we consider a time fractional diffusion system with a nonlinear memory term in a bounded domain. We mainly prove some blow-up and global existence results for this problem. Moreover, we also give the decay estimates of the global solutions. Our proof relies on the eigenfunction method combined with the asymptotic behavior of the solution of a fractional differential inequality system, the estimates of the solution operators and the asymptotic behavior of the Mittag–Leffler function. In particular, we give the critical exponents of this problem in different cases. Our results show that, in some cases, whether one of the initial values is identically equal to zero has a great influence on blow-up and global existence of the solutions for this problem, which is a remarkable property of time fractional diffusion systems because the classical diffusion systems can not admit this property.

Keywords: time fractional diffusion system; blow-up; global existence; critical exponent; nonlinear memory

MSC: 35R11; 35F55; 35B44; 35A01

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1. Introduction

In recent years, many research studies have focused on time fractional diffusion equations and systems since they are useful to model phenomena such as viscoelasticity, anomalous diffusion phenomena, quantum mechanics, etc.(see, e.g., [1–7]). For example, time fractional diffusion equations can often be used to model physical systems exhibiting anomalous diffusion (see, e.g., [1,4,7]). In many complex dynamical systems, the diffusion processes do not follow Gaussian statistics, and then the related transport behavior can not be described by the Fick second law. The mean squared displacement of a diffusive particle usually follows the power type law, i.e., $\langle x^2(t)\rangle \sim \text{const}\cdot t^\alpha$, which is linear in t in the classical diffusion process. Since the mean squared displacement describes how fast particles diffuse, the diffusion process is called the sub-diffusion process when $0<\alpha<1$ and is called the sup-diffusion process when $1<\alpha<2$, see, e.g., [1,7]. Hence, recently, there have been a lot of literature studies studying time fractional differential equations and systems, see, e.g., [1,8–34]. For instance, in [20], the authors considered the blow-up and global existence of the solution to a Cauchy problem for a time-space fractional diffusion equation, where the time derivative is taken in the sense of the Caputo–Hadamard type and the spatial derivative is taken by the fractional Laplace operator. They also verified the blow-up results by numerical simulations. In [23], the authors generalized some theorems of counting zeros for analytical functions, and obtained an algebraic test to determine the stability of fractional order systems by the matrix inequalities. In [15], an initial-boundary value problem for the Caputo time fractional diffusion equation was studied, and the equivalence of viscosity solutions and distributional solutions for this problem was proved.

The goals of this paper are to prove blow-up and global existence results and give the decay estimates of the global solutions for the following Caputo time fractional diffusion system:

$$\begin{cases} {}_0^C D_t^{\alpha} u - \triangle u = {}_0 I_t^{1-\gamma_1}(|v|^{p-1}v), & (t,x) \in (0,T)\times\Omega \\ {}_0^C D_t^{\beta} v - \triangle v = {}_0 I_t^{1-\gamma_2}(|u|^{q-1}u), & (t,x) \in (0,T)\times\Omega, \\ u(t,x) = 0, \ v(t,x) = 0, & (t,x) \in (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $0 < \alpha, \beta \le 1$, $0 \le \gamma_1, \gamma_2 < 1$, $p, q \ge 1$, $pq > 1$ and $u_0, v_0 \in C_0(\Omega)$. Here, ${}_0^C D_t^{\alpha} u = \frac{\partial}{\partial t}[{}_0 I_t^{1-\alpha}(u(t,x) - u_0(x))]$ is the Caputo derivative of u with respect to t.

Firstly, let us dwell on some known results on blow-up and global existence of the solution for time fractional diffusion systems. In [32], Zhang et al. discussed the semilinear time fractional diffusion system

$$\begin{cases} {}_0^C D_t^{\alpha} u - \triangle u = |v|^{p-1}v, & x \in \mathbb{R}^N, \ t > 0, \\ {}_0^C D_t^{\alpha} v - \triangle v = |u|^{q-1}u, & x \in \mathbb{R}^N, \ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2}$$

where $0 < \alpha < 1$, $p, q > 1$, $u_0, v_0 \not\equiv 0$ with $u_0, v_0 \in C_0(\mathbb{R}^N)$, and gave the Fujita critical exponent of (2), which is the same as that of the classical diffusion system (i.e., (2) with $\alpha = 1$). They showed that problem (2) can admit global nontrivial solutions in the critical case, whereas for a classical diffusion system (i.e., (2) with $\alpha = 1$), all positive solutions blow up in finite time in the critical case. In [8], a time fractional diffusion system on $\mathbb{R}^N$ with two different fractional powers was considered and some blow-up and global existence results were proved.

Let us now turn to the study of time fractional diffusion equations with nonlinear memory terms on both $\mathbb{R}^N$ and a domain $\Omega \subset \mathbb{R}^N$. There have been many papers on existence and nonexistence of global solutions for these problems (see, e.g., [10,13,22,33–35]). For the time fractional diffusion equation

$${}_0^C D_t^{\alpha} u - \triangle u = {}_0 I_t^{1-\gamma}(|u|^{p-1}u), \tag{3}$$

on both $\mathbb{R}^N$ and a bounded domain $\Omega \subset \mathbb{R}^N$, where $\alpha \in (0,1]$, $\gamma \in [0,1)$ and $p > 1$, Cazenave et al. [35] obtained the critical exponents of this problem with $\alpha = 1$. For the case $0 < \alpha < 1$, Zhang and Li [22,33,34] generalized the results of [35] and obtained the Fujita critical exponents for the case $\alpha < \gamma$ and $\alpha \ge \gamma$, respectively. The results indicate that the properties of solutions for problem (3) on both $\mathbb{R}^N$ and a bounded domain $\Omega \subset \mathbb{R}^N$ can be different for these two cases.

In [13], the authors studied the blow-up of solution for the following semilinear fractional diffusion equation in a bounded domain:

$$\begin{cases} u_t = \partial_t(g_\alpha * \triangle u)(t,x) + |u|^{\rho-1}u & x \in \Omega, \ t > 0, \\ u(t,x) = 0, & x \in \partial\Omega, \ t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{4}$$

where $0 < \alpha < 1$, $\rho > 1$, $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $g_\alpha * \triangle u = \int_0^t g_\alpha(s) \triangle u(t-s,x) ds$. They obtained that, if $\alpha\rho < 1$, $u_0 \ge 0$ and $u_0 \not\equiv 0$, then any solution of (4) blows up in L^∞ norm.

Fixed $R > 0$, Asogwa et al. [10] considered

$$\begin{cases} {}_0^C D_t^{\beta} V = -(-\triangle)^{\alpha/2} V + {}_0 I_t^{1-\beta}(V^{1+\eta}), & x \in B(0,R), \ t > 0, \\ V(t,x) = 0, & x \in B(0,R)^C, \ t > 0, \\ V(0,x) = V_0(x), & x \in B(0,R), \end{cases} \quad (5)$$

where $\beta \in (0,1)$, $\alpha \in (0,2)$. They obtained that, if $0 < \eta < \frac{1}{\beta} - 1$ and $\int_{B(0,R)} V_0(x)\phi_1(x)dx > 0$, where ϕ_1 is the first eigenfunction of the above Dirichlet fractional Laplace operator, then all nonzero solutions of (5) can not exist globally in time.

For the diffusion systems with nonlinear memory terms, to our knowledge, there were only a few papers investigating the blow-up and global existence of solutions. In the limiting case $\alpha = \beta = 1$, Loayza and Quinteiro [36] proved that, if $\max\{1 - p\gamma_2 + p(1 - q\gamma_1), 1 - q\gamma_1 + q(1 - p\gamma_2)\} \geq 0$ and $u_0, v_0 \geq 0$, $u_0 + v_0 > 0$, then the correspondent solution of (1) blows up in finite time, while if $\max\{1 - p\gamma_2 + p(1 - q\gamma_1), 1 - q\gamma_1 + q(1 - p\gamma_2)\} < 0$ and the L^∞ norms of u_0 and v_0 are sufficiently small, then (1) admits a global solution.

Motivated by the aforementioned results, in this paper, we study global existence and blow-up of solutions of (1) in six different situations (see Section 3), and extend the results in [34,36].

Comparing with the results of [34,36], our conclusions of (1) show that the time fractional diffusion system (1) is more delicate. When we consider problem (1), some new cases appear and need to be studied. For example, we have to consider the case that one of the initial values identically equals zero. Indeed, in some cases, our conclusions show that the solutions of (1) can globally exist in the case $u_0, v_0 \not\equiv 0$, but all nontrivial solutions must blow up in finite time for the case $u_0 \not\equiv 0, v_0 \equiv 0$ (see Section 3). The main reason for making such difference is due to the nonlocality of time fractional derivatives. Thus, initial values have a great influence on the properties of the solution for problem (1). On the other hand, since the orders of time fractional derivatives for problem (1) can be different and time fractional derivatives are nonlocal, some methods and arguments used in [36] can not be directly applied to the study of problem (1).

This paper is organized as follows: In Section 2, we first present some definitions and properties of Riemann–Liouville fractional integrals, Caputo fractional derivatives, Mittag–Leffler function, and Wright type function. Secondly, we recall some properties of solution operators $P_\alpha(t)$ and $S_\alpha(t)$. Finally, some properties of the solution for a fractional differential inequality system are provided. The main results are given and proved in Section 3. Section 4 is devoted to a brief summary of this paper.

For simplicity of presentation, in the following sections, we use C to denote a positive constant, whose value may be not the same in different parts.

2. Preliminaries

In this section, we are ready to give some preliminaries that will be used in the following sections.

For $T > 0$, $\alpha \in (0,1]$, the Riemann–Liouville fractional integrals are defined by [5,37]

$${}_0 I_t^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \ \ {}_t I_T^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(s)}{(s-t)^{1-\alpha}} ds,$$

and the Caputo fractional derivatives are defined by [5]

$${}_0^C D_t^{\alpha} f = \frac{d}{dt} {}_0 I_t^{1-\alpha}[f(t) - f(0)], \ {}_t^C D_T^{\alpha} f = -\frac{d}{dt} {}_t I_T^{1-\alpha}[f(t) - f(T)].$$

When $\alpha = 1$, we define ${}_0^C D_t^{\alpha} f = -{}_t^C D_T^{\alpha} f = f'(t)$. Moreover, if $f \in AC([0,T])$, then ${}_0^C D_t^{\alpha} f$ and ${}_t^C D_T^{\alpha} f$ exist almost everywhere on $[0,T]$ and ${}_0^C D_t^{\alpha} f = {}_0 I_t^{1-\alpha} f'(t)$, ${}_t^C D_T^{\alpha} f = -{}_t I_T^{1-\alpha} f'(t)$ (see [5]).

Let $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assuming that $f \in L^p(0,T)$, $g \in L^q(0,T)$, we have [5]

$$\int_0^T ({}_0I_t^{\alpha} f) g(t) dt = \int_0^T ({}_tI_T^{\alpha} g) f(t) dt. \tag{6}$$

Furthermore, the following formula of integration by parts is valid [33]

$$\int_0^T g(t) ({}_0^C D_t^{\alpha} f) dt = \int_0^T (f(t) - f(0)) {}_t^C D_T^{\alpha} g dt, \tag{7}$$

provided that $f \in C([0,T])$, ${}_0^C D_t^{\alpha} f$ exists almost everywhere on $[0,T]$, ${}_0^C D_t^{\alpha} f \in L^1(0,T)$ and $g \in AC([0,T])$ with $g(T) = 0$.

Next, we recall some properties of the Mittag–Leffler function. The Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha, \beta \in \mathbb{C}, \ \mathrm{Re}(\alpha) > 0, \ \ E_{\alpha}(z) = E_{\alpha,1}(z). \tag{8}$$

$E_{\alpha,\beta}(z)$ is an entire function and has the asymptotic behavior at infinity for $0 < \alpha < 1$,

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{N} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O\left(\frac{1}{z^{N+1}}\right) \tag{9}$$

with $|z| \to +\infty$ and $\mu \leq |\arg(z)| \leq \pi$, where $\mu \in (\frac{\pi\alpha}{2}, \pi\alpha)$ is a constant (see, e.g., [5,37]). The Wright type function

$$\phi_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}, \ 0 < \alpha < 1, \ z \in \mathbb{C} \tag{10}$$

is an entire function and a probability density function, i.e., $\phi_{\alpha}(\theta) \geq 0$ for $\theta \geq 0$, $\int_0^{\infty} \phi_{\alpha}(\theta) d\theta = 1$. Moreover,

$$\int_0^{\infty} \phi_{\alpha}(\theta) e^{-z\theta} d\theta = E_{\alpha}(-z) \text{ and } \alpha \int_0^{\infty} \theta \phi_{\alpha}(\theta) e^{-z\theta} d\theta = E_{\alpha,\alpha}(-z) \tag{11}$$

for $z \in \mathbb{C}$ (see, e.g., [6,11,29]).

Denote $A = \triangle$. Let $T(t)$ be the heat semigroup generated by A on $C_0(\Omega)$. Similar to [11,29,34], we define the operators $P_{\alpha}(t)$ and $S_{\alpha}(t)$ as

$$P_{\alpha}(t) u_0 = \int_0^{\infty} \phi_{\alpha}(\theta) T(t^{\alpha}\theta) u_0 d\theta, \ t \geq 0, \tag{12}$$

$$S_{\alpha}(t) u_0 = \alpha \int_0^{\infty} \theta \phi_{\alpha}(\theta) T(t^{\alpha}\theta) u_0 d\theta, \ t \geq 0. \tag{13}$$

Next, we collect some properties of the operators $P_{\alpha}(t)$ and $S_{\alpha}(t)$.

Lemma 1 ([31])**.** *The operators $P_{\alpha}(t)$ and $S_{\alpha}(t)$ have the following properties:*

(i) For $u_0 \in C_0(\Omega)$, we have $P_{\alpha}(t) u_0 \in C([0, T+\infty), C_0(\Omega))$, $P_{\alpha}(t) u_0 \in D(A)$ for all $t > 0$ and

$${}_0^C D_t^{\alpha} P_{\alpha}(t) u_0 = A P_{\alpha}(t) u_0, \ \ \frac{d}{dt} P_{\alpha}(t) u_0 = t^{\alpha-1} A S_{\alpha}(t) u_0, \ \ t > 0,$$

$$\|A P_{\alpha}(t) u_0\|_{L^{\infty}(\Omega)} + \|A S_{\alpha}(t) u_0\|_{L^{\infty}(\Omega)} \leq \frac{C}{t^{\alpha}} \|u_0\|_{L^{\infty}(\Omega)}, \ \ t > 0$$

for some constant $C > 0$.

(ii) Let $h \in L^q((0,T), C_0(\Omega))$, $q > 1$, $w = \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) h(s) ds$. Then, $w(0) = 0$ and $w \in C^{\alpha - \frac{1}{q}}([0,T], C_0(\Omega))$ if $q\alpha > 1$. Furthermore, if $h \in C^\beta([0,T], C_0(\Omega))$ for some $\beta \in (0,1)$, then

$$ {}_0^C D_t^\alpha w = Aw + h(t), \;\; t \in [0,T]. $$

In order to prove our main results, we shall borrow the idea in [35] to study properties of the solution of a fractional differential inequality system. We need to extend the Proposition 2.2 in [35].

Lemma 2. *Let $T > 0$, $0 < \alpha, \beta \le 1$, $0 \le \gamma_1, \gamma_2 < 1$, $\beta_1 = 1 - \gamma_1$, $\beta_2 = 1 - \gamma_2$, $p, q \ge 1$, $pq > 1$ and $a, b, c, d > 0$. Suppose that (u, v) satisfies $u, v \in C([0,T])$, $u, v > 0$ for $t \in (0,T]$, $u(0), v(0) \ge 0$, ${}_0I_t^{1-\alpha}(u - u(0)) \in AC([0,T])$, ${}_0I_t^{1-\beta}(v - v(0)) \in AC([0,T])$ and*

$$ \begin{cases} {}_0^C D_t^\alpha u + au \ge b({}_0I_t^{1-\gamma_1} v^p), \\ {}_0^C D_t^\beta v + cv \ge d({}_0I_t^{1-\gamma_2} u^q) \end{cases} \tag{14} $$

for almost every $t \in [0,T]$. Then,

(i) *There exists a positive constant M independent of T such that*

$$ u(0) \le M[T^{\alpha+\beta_1 - \frac{p\beta_2}{pq-1} - \frac{pq\beta_1}{pq-1}} + T^{\alpha+\beta_1 - \frac{p(\beta+\beta_2)}{pq-1} - \frac{pq\beta_1}{pq-1}} + T^{-\frac{p\beta_2}{pq-1} - \frac{(\alpha+\beta_1)}{pq-1}} + T^{-\frac{p(\beta+\beta_2)}{pq-1} - \frac{(\alpha+\beta_1)}{pq-1}}], $$

$$ v(0) \le M[T^{\beta+\beta_2 - \frac{q\beta_1}{pq-1} - \frac{pq\beta_2}{pq-1}} + T^{\beta+\beta_2 - \frac{q(\alpha+\beta_1)}{pq-1} - \frac{pq\beta_2}{pq-1}} + T^{-\frac{q\beta_1}{pq-1} - \frac{(\beta+\beta_2)}{pq-1}} + T^{-\frac{q(\alpha+\beta_1)}{pq-1} - \frac{(\beta+\beta_2)}{pq-1}}]. $$

(ii) *If $T = +\infty$, then $\liminf_{t\to+\infty} u(t) = 0$ and $\liminf_{t\to+\infty} v(t) = 0$.*

(iii) *If $T = +\infty$, then $\liminf_{t\to+\infty} t^{\gamma_1} u(t) > 0$ and $\liminf_{t\to+\infty} t^{\gamma_2} v(t) > 0$.*

(iv) *If $T = +\infty$, then $\liminf_{t\to+\infty} t^{\frac{p\beta_2+\beta_1}{pq-1}} u(t) < +\infty$ and $\liminf_{t\to+\infty} t^{\frac{q\beta_1+\beta_2}{pq-1}} v(t) < +\infty$.*

(v) *If $\max\{1 - p\gamma_2 + p(1 - q\gamma_1), 1 - q\gamma_1 + q(1 - p\gamma_2)\} \ge 0$, then $T < +\infty$.*

Proof. (i) From (14), (6) and (7), we deduce that

$$ \int_0^T [u({}_t^C D_T^\alpha \varphi) + au\varphi] dt \ge b \int_0^T v^p ({}_tI_T^{1-\gamma_1} \varphi) dt + u(0) \int_0^T {}_t^C D_T^\alpha \varphi dt, \tag{15} $$

$$ \int_0^T [v({}_t^C D_T^\beta \varphi) + cv\varphi] dt \ge d \int_0^T u^q ({}_tI_T^{1-\gamma_2} \varphi) dt + v(0) \int_0^T {}_t^C D_T^\beta \varphi dt, \tag{16} $$

where $\varphi \in AC([0,T])$ is nonnegative and $\varphi(T) = 0$. By the results in [5], we know that, for $\alpha, \beta > 0$ and $l \ge \alpha + \beta$,

$$ {}_t^C D_T^\beta (1 - \frac{t}{T})^l = \frac{\Gamma(l+1)}{\Gamma(l+1-\beta)} T^{-\beta} (1 - \frac{t}{T})^{l-\beta}, $$

$$ {}_t^C D_T^\alpha [{}_t^C D_T^\beta (1 - \frac{t}{T})^l] = \frac{\Gamma(l+1)}{\Gamma(l+1-\alpha-\beta)} T^{-\alpha-\beta} (1 - \frac{t}{T})^{l-\alpha-\beta}. $$

Let $\psi_T = (1 - \frac{t}{T})^m$ ($m \ge \max\{\frac{pq(\alpha+\beta_1)}{pq-1}, \frac{pq(\beta+\beta_2)}{pq-1}\}$). It should be illustrated that choosing the test function of the type ψ_T to prove the nonexistence of global solutions to fractional differential equations firstly appeared in [18]. Here, taking $\varphi(t) = {}_t^C D_T^{\beta_1} \psi_T$ in (15) and $\varphi(t) = {}_t^C D_T^{\beta_2} \psi_T$ in (16), we deduce from Hölder's inequality that

$$b\int_0^T v^p\psi_T dt + u(0)\int_0^T {}_t^C D_T^{\alpha}({}_t^C D_T^{\beta_1}\psi_T)dt \le C(T^{\frac{q-1}{q}-\beta_1} + T^{\frac{q-1}{q}-(\alpha+\beta_1)})\Big(\int_0^T u^q\psi_T dt\Big)^{\frac{1}{q}},$$

$$d\int_0^T u^q\psi_T dt + v(0)\int_0^T {}_t^C D_T^{\beta}({}_t^C D_T^{\beta_2}\psi_T) \le C(T^{\frac{p-1}{p}-\beta_2} + T^{\frac{p-1}{p}-(\beta+\beta_2)})\Big(\int_0^T v^p\psi_T dt\Big)^{\frac{1}{p}}$$

for some constant $C > 0$, where we have used the fact that ${}_tI_T^{\beta_1}({}_t^C D_T^{\beta_1}\psi_T) = \psi_T$ and ${}_tI_T^{\beta_2}({}_t^C D_T^{\beta_2}\psi_T) = \psi_T$. This and Young's inequality with ε yield

$$\begin{aligned}\frac{b}{2}\int_0^T v^p\psi_T dt + u(0)\int_0^T {}_t^C D_T^{\alpha}({}_t^C D_T^{\beta_1}\psi_T)dt \le & C\Big(T^{1-\frac{p\beta_2}{pq-1}-\frac{pq\beta_1}{pq-1}} + T^{1-\frac{p(\beta+\beta_2)}{pq-1}-\frac{pq\beta_1}{pq-1}} \\ & + T^{1-\frac{p\beta_2}{pq-1}-\frac{pq(\alpha+\beta_1)}{pq-1}} + T^{1-\frac{p(\beta+\beta_2)}{pq-1}-\frac{pq(\alpha+\beta_1)}{pq-1}}\Big),\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{d}{2}\int_0^T u^q\psi_T dt + v(0)\int_0^T {}_t^C D_T^{\beta}({}_t^C D_T^{\beta_2}\psi_T) \le & C\Big(T^{1-\frac{q\beta_1}{pq-1}-\frac{pq\beta_2}{pq-1}} + T^{1-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{pq\beta_2}{pq-1}} \\ & + T^{1-\frac{q\beta_1}{pq-1}-\frac{pq(\beta+\beta_2)}{pq-1}} + T^{1-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{pq(\beta+\beta_2)}{pq-1}}\Big).\end{aligned} \tag{18}$$

Then, the estimates (17) and (18) imply that there exists a constant $M > 0$ such that

$$u(0) \le M[T^{\alpha+\beta_1-\frac{p\beta_2}{pq-1}-\frac{pq\beta_1}{pq-1}} + T^{\alpha+\beta_1-\frac{p(\beta+\beta_2)}{pq-1}-\frac{pq\beta_1}{pq-1}} + T^{-\frac{p\beta_2}{pq-1}-\frac{(\alpha+\beta_1)}{pq-1}} + T^{-\frac{p(\beta+\beta_2)}{pq-1}-\frac{(\alpha+\beta_1)}{pq-1}}],$$

$$v(0) \le M[T^{\beta+\beta_2-\frac{q\beta_1}{pq-1}-\frac{pq\beta_2}{pq-1}} + T^{\beta+\beta_2-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{pq\beta_2}{pq-1}} + T^{-\frac{q\beta_1}{pq-1}-\frac{(\beta+\beta_2)}{pq-1}} + T^{-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{(\beta+\beta_2)}{pq-1}}].$$

(ii) Suppose that there exist $\eta_1, \eta_2 > 0$ such that $u(t) \ge \eta_1$ or $v(t) \ge \eta_2$ for all $t \ge 1$. Then, using (17) and (18), we derive that, for $T \ge 1$

$$\begin{aligned}\frac{bT(1-\frac{1}{T})^{m+1}}{2(m+1)}\eta_2^p \le \frac{b}{2}\int_1^T v^p\psi_T dt \le & C(T^{1-\frac{p\beta_2}{pq-1}-\frac{pq\beta_1}{pq-1}} + T^{1-\frac{p(\beta+\beta_2)}{pq-1}-\frac{pq\beta_1}{pq-1}} \\ & + T^{1-\frac{p\beta_2}{pq-1}-\frac{pq(\alpha+\beta_1)}{pq-1}} + T^{1-\frac{p(\beta+\beta_2)}{pq-1}-\frac{pq(\alpha+\beta_1)}{pq-1}}),\end{aligned} \tag{19}$$

or

$$\begin{aligned}\frac{cT(1-\frac{1}{T})^{m+1}}{2(m+1)}\eta_1^q \le \frac{c}{2}\int_1^T u^q\psi_T dt \le & C(T^{1-\frac{q\beta_1}{pq-1}-\frac{pq\beta_2}{pq-1}} + T^{1-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{pq\beta_2}{pq-1}} \\ & + T^{1-\frac{q\beta_1}{pq-1}-\frac{pq(\beta+\beta_2)}{pq-1}} + T^{1-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{pq(\beta+\beta_2)}{pq-1}}).\end{aligned} \tag{20}$$

Consequently, we know $\eta_1 = 0$ or $\eta_2 = 0$ by letting $T \to +\infty$, which contradicts $\eta_1 > 0$ and $\eta_2 > 0$. Therefore, $\liminf_{t\to+\infty} u(t) = \liminf_{t\to+\infty} v(t) = 0$.

(iii) Since $E_\alpha(-\lambda t^\alpha) \ge 0$ and $E_{\alpha,\alpha}(-\lambda t^\alpha) \ge 0$ for $0 < \alpha < 1$, $\lambda > 0$ and $t \ge 0$ (see, e.g., [38]), we deduce from (14) that

$$u(t) \ge E_\alpha(-at^\alpha)u(0) + \frac{b}{\Gamma(\beta_1)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\int_0^s (s-\tau)^{-\gamma_1}v^p(\tau)d\tau ds,$$

$$v(t) \ge E_\beta(-ct^\beta)v(0) + \frac{d}{\Gamma(\beta_2)}\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-c(t-s)^\beta)\int_0^s (s-\tau)^{-\gamma_2}u^q(\tau)d\tau ds.$$

Hence, for $t \geq 3$,

$$
\begin{aligned}
u(t) &\geq \frac{b}{\Gamma(\beta_1)} \int_{t-1}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) \int_0^s (s-\tau)^{-\gamma_1} v^p(\tau) d\tau ds \\
&\geq \frac{b}{\Gamma(\beta_1)} \int_{t-1}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) \int_1^2 (s-\tau)^{-\gamma_1} v^p(\tau) d\tau ds \\
&\geq \frac{b}{\Gamma(\beta_1)} \int_{t-1}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha})(s-1)^{-\gamma_1} ds \inf_{1\leq s\leq 2} v^p(s) \\
&\geq \frac{b(t-1)^{-\gamma_1}}{\Gamma(\beta_1)} \inf_{1\leq s\leq 2} v^p(s) \int_{t-1}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) ds \\
&= \frac{b(t-1)^{-\gamma_1}}{\Gamma(\beta_1)} \inf_{1\leq s\leq 2} v^p(s) \int_0^1 \tau^{\alpha-1} E_{\alpha,\alpha}(-a\tau^{\alpha}) d\tau,
\end{aligned}
$$

which proves $\liminf_{t\to+\infty} t^{\gamma_1} u(t) > 0$ by the fact that $\int_0^1 \tau^{\alpha-1} E_{\alpha,\alpha}(-a\tau^{\alpha}) d\tau > 0$. Similarly, we can prove $\liminf_{t\to+\infty} t^{\gamma_2} v(t) > 0$.

(iv) In terms of Property (ii), there exist nondecreasing sequences $\{t_n\}$ and $\{s_n\}$ such that $u(t_n) = \min_{1\leq t\leq t_n} u(t)$, $v(s_n) = \min_{1\leq t\leq s_n} v(t)$ and $t_n \to +\infty$, $s_n \to +\infty$. It follows from (19) and (20) that

$$
\begin{aligned}
\frac{bs_n(1-\frac{1}{s_n})^{m+1}}{2(m+1)} v^p(s_n) = \frac{b}{2} v^p(s_n) \int_1^{s_n} \psi_{s_n}(t) dt \leq & C\Big(s_n^{1-\frac{p\beta_2}{pq-1}-\frac{pq\beta_1}{pq-1}} + s_n^{1-\frac{p(\beta+\beta_2)}{pq-1}-\frac{pq\beta_1}{pq-1}} \\
& + s_n^{1-\frac{p\beta_2}{pq-1}-\frac{pq(\alpha+\beta_1)}{pq-1}} + s_n^{1-\frac{p(\beta+\beta_2)}{pq-1}-\frac{pq(\alpha+\beta_1)}{pq-1}}\Big),
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\frac{ct_n(1-\frac{1}{t_n})^{m+1}}{2(m+1)} u^q(t_n) = \frac{c}{2} u^q(t_n) \int_0^{t_n} \psi_{t_n}(t) dt \leq & C\Big(t_n^{1-\frac{q\beta_1}{pq-1}-\frac{pq\beta_2}{pq-1}} + t_n^{1-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{pq\beta_2}{pq-1}} \\
& + t_n^{1-\frac{q\beta_1}{pq-1}-\frac{pq(\beta+\beta_2)}{pq-1}} + t_n^{1-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{pq(\beta+\beta_2)}{pq-1}}\Big).
\end{aligned} \tag{22}
$$

Thus, $\liminf_{t\to+\infty} t^{\frac{p\beta_2+\beta_1}{pq-1}} u(t) < +\infty$ and $\liminf_{t\to+\infty} t^{\frac{q\beta_1+\beta_2}{pq-1}} v(t) < +\infty$.

(v) Suppose the conclusion is not true. Then, $T = +\infty$. If $\max\{1-p\gamma_2+p(1-q\gamma_1), 1-q\gamma_1+q(1-p\gamma_2)\} > 0$, without loss of generality, we may assume $1-p\gamma_2+p(1-q\gamma_1) > 0$. Property (iii) implies that there exists a constant $C > 0$ such that $u(t) \geq Ct^{-\gamma_1}$ for $t \geq 2$. Then,

$$
{}^{\beta_1+p\beta_2} u^{pq-1}(t) \geq Ct^{1-p\gamma_2+p(1-q\gamma_1)} \to +\infty,\ t \to +\infty,
$$

which contradicts Property (iv). If $\max\{1-p\gamma_2+p(1-q\gamma_1), 1-q\gamma_1+q(1-p\gamma_2)\} = 0$, without loss of generality, we may assume $1-p\gamma_2+p(1-q\gamma_1) = 0$. According to Property (iii), there exists a constant $C > 0$ such that, for $t \geq 3$,

$$
\begin{aligned}
v(t) &\geq C \int_{t-1}^{t} (t-s)^{\beta-1} E_{\beta,\beta}(-c(t-s)^{\beta}) \int_1^s (s-\tau)^{-\gamma_2} \tau^{-q\gamma_1} d\tau ds \\
&\geq Ct^{-q\gamma_1} \int_{t-1}^{t} (t-s)^{\beta-1} E_{\beta,\beta}(-c(t-s)^{\beta}) \int_1^s (s-\tau)^{-\gamma_2} d\tau ds \\
&\geq Ct^{-q\gamma_1}(t-2)^{1-\gamma_2} \int_{t-1}^{t} (t-s)^{\beta-1} E_{\beta,\beta}(-c(t-s)^{\beta}) ds \\
&= Ct^{-q\gamma_1}(t-2)^{1-\gamma_2} \int_0^1 \tau^{\beta-1} E_{\beta,\beta}(-c\tau^{\beta}) d\tau.
\end{aligned}
$$

This implies that for $t \geq 4$

$$\begin{aligned} u(t) &\geq Ct^{-\gamma_1} \int_{t-1}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) \int_3^s \tau^{-pq\gamma_1} (\tau-2)^{p(1-\gamma_2)} d\tau ds \\ &\geq Ct^{-\gamma_1} \int_{t-1}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) \int_3^s \tau^{-pq\gamma_1+p(1-\gamma_2)} d\tau ds \\ &= Ct^{-\gamma_1} \int_{t-1}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) \int_3^s \tau^{-1} d\tau ds \\ &\geq Ct^{-\gamma_1} [\ln(t-1) - \ln 3] \int_{t-1}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) ds \\ &= Ct^{-\gamma_1} [\ln(t-1) - \ln 3] \int_0^1 \tau^{\alpha-1} E_{\alpha,\alpha}(-a\tau^{\alpha}) d\tau. \end{aligned}$$

Hence

$$\begin{aligned} t^{\beta_1+p\beta_2} u^{pq-1}(t) &\geq C^{pq-1} t^{\beta_1+p\beta_2-(pq-1)\gamma_1} [\ln(t-1) - \ln 3]^{pq-1} \\ &= C^{pq-1} [\ln(t-1) - \ln 3]^{pq-1} \to +\infty, \ t \to \infty. \end{aligned}$$

This contradicts Property (iv). Therefore, $T < +\infty$. □

Remark 1. *(i) In [33], for the fractional differential inequality, the authors generalized Proposition 2.2 in [35]. On the other hand, in [36], the authors extended Proposition 2.2 in [35] to the differential system. Lemma 2 further extends Lemma 5 in [33] and Proposition 6 in [36].*

(ii) When $\alpha = \beta = 1$, the key point of proving Proposition 6 in [36] is to shift the time. However, this method could not be used for our problem owing to the nonlocality of time fractional derivatives. Comparing with Lemma 5 in [33], the results of Lemma 2 are more delicate since we are dealing with a system. Moreover, some arguments used in [33] can not be directly applied.

Finally, we introduce definitions of the mild solution and weak solution of (1) and clarify their relation.

Definition 1. *Let $T > 0$, $u_0, v_0 \in C_0(\Omega)$ and $u, v \in C([0,T], C_0(\Omega))$. (u,v) is called a mild solution of problem (1) if*

$$u(t) = P_{\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_{\alpha}(t-s)|v|^{p-1}v(s)ds, \ t \in [0,T],$$

$$v(t) = P_{\beta}(t)v_0 + \int_0^t (t-s)^{\alpha-1} S_{\beta}(t-s)|u|^{q-1}u(s)ds, \ t \in [0,T].$$

Definition 2. *Let $T > 0$, $p, q \geq 1$. Assume that $u_0, v_0 \in L^1(\Omega)$. We say that (u,v) is a weak solution of (1) if $u \in L^q((0,T) \times \Omega)$ and $v \in L^p((0,T) \times \Omega)$ and*

$$\int_{\Omega} \int_0^T [{}_0I_t^{1-\gamma_1}(|v|^{p-1}u)\varphi + u_0({}_t^C D_T^{\alpha}\varphi)] dtdx = \int_{\Omega} \int_0^T u(-\triangle\varphi) dtdx + \int_{\Omega} \int_0^T u({}_t^C D_T^{\alpha}\varphi) dtdx,$$

$$\int_{\Omega} \int_0^T [{}_0I_t^{1-\gamma_2}(|u|^{q-1}u)\psi + v_0({}_t^C D_T^{\beta}\psi)] dtdx = \int_{\Omega} \int_0^T v(-\triangle\psi) dtdx + \int_{\Omega} \int_0^T v({}_t^C D_T^{\beta}\psi) dtdx$$

for every $\varphi, \psi \in C^{1,2}([0,T] \times \bar{\Omega})$ with $\varphi = \psi = 0$ on $\partial\Omega$ and $\varphi(T,x) = \psi(T,x) = 0$ for $x \in \bar{\Omega}$.

The following Lemma asserts that, for problem (1), a mild solution is a weak solution. We omit the proof of this result because it is similar to that in [22,33].

Lemma 3. *Let $T > 0$ and $p, q \geq 1$. Assume that $u_0, v_0 \in C_0(\Omega)$ and $u, v \in C([0,T], C_0(\Omega))$. If (u,v) is a mild solution of problem (1), then (u,v) is also a weak solution of problem (1).*

3. Blow-Up and Global Existence

Firstly, we can establish the following local solvability result for problem (1) by an analogous argument to that in [32,33].

Theorem 1. *Let $0 < \alpha, \beta \leq 1$, $0 \leq \gamma_1, \gamma_2 < 1$ and $p, q \geq 1$, $pq > 1$. For given $u_0, v_0 \in C_0(\Omega)$, there exists $T = T(u_0, v_0) > 0$ such that (1) has a unique mild solution $(u, v) \in C([0, T], C_0(\Omega)) \times C([0, T], C_0(\Omega))$. The solution (u, v) can be uniquely continued up to a maximal existence interval $[0, T_*)$, where either $T_* = +\infty$ or*

$$\limsup_{t \to T_*^-} [\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}] = +\infty.$$

In addition, if $u_0, v_0 \geq 0$, $u_0 + v_0 \not\equiv 0$, then $u(t, x), v(t, x) > 0$ for $(t, x) \in (0, T_*) \times \Omega$.

We say that (u, v) blows up in a time T_* if

$$\limsup_{t \to T_*^-} \|u(t)\|_{L^\infty(\Omega)} = \limsup_{t \to T_*^-} \|v(t)\|_{L^\infty(\Omega)} = +\infty.$$

In the case $\gamma_1 \leq \alpha$, we can prove the following results.

Theorem 2. *Let $p, q \geq 1$, $pq > 1$, $\beta_1 = 1 - \gamma_1$, $\beta_2 = 1 - \gamma_2$, $\gamma_1 \leq \alpha$ and $\gamma_2 \leq \beta$. Assume that $u_0, v_0 \in C_0(\Omega)$.*

(i) *If $\frac{\beta_1 + p\beta_2}{pq - 1} \geq \gamma_1$ or $\frac{\beta_2 + q\beta_1}{pq - 1} \geq \gamma_2$, and $u_0, v_0 \geq 0$, $u_0 + v_0 \not\equiv 0$, then the mild solution of (1) blows up in a finite time.*

(ii) *If $\frac{\beta_1 + p\beta_2}{pq - 1} < \gamma_1$, $\frac{\beta_2 + q\beta_1}{pq - 1} < \gamma_2$ and $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$ are sufficiently small, then problem (1) has a global solution (u, v). Moreover, there exists a constant $C > 0$ such that $\|u(t)\|_{L^\infty(\Omega)} \leq C(1 + t)^{-\frac{\beta_1 + p\beta_2}{pq - 1}}$, $\|v(t)\|_{L^\infty(\Omega)} \leq C(1 + t)^{-\frac{\beta_2 + q\beta_1}{pq - 1}}$ for $t > 0$.*

Proof. (i) We denote by λ_1 the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$ and by φ_1 the corresponding eigenfunction. We choose $\varphi_1 > 0$ and $\int_\Omega \varphi_1(x) dx = 1$. It is easy to see that $\varphi_1 \in C^2(\bar{\Omega})$ and $\varphi_1(x) = 0, x \in \partial\Omega$. Suppose that (u, v) is the mild solution of (1) obtained by Theorem 1 and the maximal existence time $T_* = +\infty$. Then, it follows from Theorem 1 and Lemma 3 that $u, v > 0$ for $(t, x) \in (0, +\infty) \times \Omega$, and (u, v) is also a weak solution of (1) for every $T > 0$. Next, we choose $\varphi(t, x) = \psi(t, x) = \varphi_1(x)\psi_T(t)$ in Definition 2, where $\psi_T \in C^1([0, T])$ satisfies $\psi_T \geq 0$ and $\psi_T(T) = 0$, and then

$$\int_\Omega \int_0^T [{}_0I_t^{1-\gamma_1}(v^p)\varphi_1\psi_T + u_0\varphi_1({}_t^CD_T^\alpha\psi_T)]dtdx = \int_\Omega \int_0^T [\lambda_1 u\varphi_1\psi_T + u\varphi_1({}_t^CD_T^\alpha\psi_T)]dtdx. \tag{23}$$

$$\int_\Omega \int_0^T [{}_0I_t^{1-\gamma_2}(u^q)\varphi_1\psi_T + v_0\varphi_1({}_t^CD_T^\beta\psi_T)]dtdx = \int_\Omega \int_0^T [\lambda_1 v\varphi_1\psi_T + v\varphi_1({}_t^CD_T^\beta\psi_T)]dtdx. \tag{24}$$

Denote $f(t) = \int_\Omega u\varphi_1 dx$ and $g(t) = \int_\Omega v\varphi_1 dx$. It is easy to confirm that $f, g \in C([0, T])$, $f(0), g(0) \geq 0$ and $f(t), g(t) > 0$ for $t \in (0, T]$. Using (23), (24), (6) and Jensen's inequality, we can obtain that

$$\int_0^T g^p({}_tI_T^{\beta_1}\psi_T)dt + f(0)\int_0^T {}_t^CD_T^\alpha\psi_T dt \leq \lambda_1 \int_0^T f\psi_T dt + \int_0^T f({}_t^CD_T^\alpha\psi_T)dt, \tag{25}$$

$$\int_0^T f^q({}_tI_T^{\beta_2}\psi_T)dt + g(0)\int_0^T {}_t^CD_T^\beta\psi_T dt \leq \lambda_1 \int_0^T g\psi_T dt + \int_0^T g({}_t^CD_T^\beta\psi_T)dt. \tag{26}$$

In addition, Lemma 1 yields ${}_0I_t^{1-\alpha}(f - f(0)) \in AC([0, T])$ and ${}_0I_t^{1-\beta}(g - g(0)) \in AC([0, T])$. Thus, we deduce from (25), (26), (6) and (7) that

$$\int_0^T ({}_0I_t^{\beta_1} g^p)\psi_T dt \le \lambda_1 \int_0^T f\psi_T dt + \int_0^T [f(t)-f(0)]_t^C D_T^\alpha \psi_T dt = \lambda_1 \int_0^T f\psi_T dt + \int_0^T {}_0^C D_t^\alpha f \psi_T dt,$$

$$\int_0^T ({}_0I_t^{\beta_2} f^q)\psi_T dt \le \lambda_1 \int_0^T g\psi_T dt + \int_0^T [g(t)-g(0)]_t^C D_T^\beta \psi_T dt = \lambda_1 \int_0^T g\psi_T dt + \int_0^T {}_0^C D_t^\beta g \psi_T dt.$$

Due to the arbitrariness of ψ_T, we obtain

$$ {}_0^C D_t^\alpha f + \lambda_1 f \ge {}_0I_t^{\beta_1} g^p,\ \ {}_0^C D_t^\beta g + \lambda_1 g \ge {}_0I_t^{\beta_2} f^q,\ \ t \in [0,T]. \tag{27}$$

Note that $\max\{1-p\gamma_2+p(1-q\gamma_1), 1-q\gamma_1+q(1-p\gamma_2)\} \ge 0$ if and only if $\max\{\frac{p\beta_2+\beta_1}{pq-1}-\gamma_1, \frac{\beta_2+q\beta_1}{pq-1}-\gamma_2\} \ge 0$. We can obtain a contradiction by (27) and Lemma 2(v). Hence, $T_* < +\infty$ and by Theorem 1, we know

$$\limsup_{t\to T_*^-}[\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}] = +\infty.$$

Furthermore, it is easy to show that $\limsup_{t\to T_*^-}\|u(t)\|_{L^\infty(\Omega)} = \limsup_{t\to T_*^-}\|v(t)\|_{L^\infty(\Omega)} = +\infty$. In fact, if u is bounded on $[0,T_*)\times\Omega$, then the second equation would lead to a uniform bound on v, which yields a contradiction. The proof is completed.

(ii) Set

$$X = \{(u,v) \in L^\infty((0,\infty), L^\infty(\Omega)) \times L^\infty((0,\infty), L^\infty(\Omega)) \mid \|(u,v)\| < \infty\},$$

where

$$\|(u,v)\| = \max\Big\{\sup_{t>0}(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\|u(t)\|_{L^\infty(\Omega)}, \sup_{t>0}(1+t)^{\frac{\beta_2+q\beta_1}{pq-1}}\|v(t)\|_{L^\infty(\Omega)}\Big\}.$$

We define the operator Φ on X as $\Psi(u,v)(t) = (\Psi_1(v), \Psi_2(u))$,

$$\Psi_1(v)(t) = P_\alpha(t)u_0 + \frac{1}{\Gamma(\beta_1)}\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\int_0^s (s-\tau)^{-\gamma_1}|v|^{p-1}v(\tau)d\tau ds,$$

$$\Psi_2(u)(t) = P_\beta(t)u_0 + \frac{1}{\Gamma(\beta_2)}\int_0^t (t-s)^{\beta-1}S_\beta(t-s)\int_0^s (s-\tau)^{-\gamma_1}|u|^{q-1}u(\tau)d\tau ds.$$

Fix $K>0$ and let $\mathcal{B}_K = \{(u,v) \in X \mid \|(u,v)\| \le K\}$. Note that $\frac{p(\beta_2+q\beta_1)}{pq-1} < 1$ if and only if $\frac{p\beta_2+\beta_1}{pq-1} < \gamma_1$, and $\frac{q(\beta_1+p\beta_2)}{pq-1} < 1$ if and only if $\frac{q\beta_1+\beta_2}{pq-1} < \gamma_2$. Hence, it follows from (9) and (11) that there exists a constant $C>0$ such that, for $u_0, v_0 \in C_0(\Omega)$,

$$\begin{aligned}(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\|P_\alpha(t)u_0\|_{L^\infty(\Omega)} &\le C(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\int_0^{+\infty}\phi_\alpha(\theta)e^{-\lambda_1 t^\alpha\theta}d\theta\|u_0\|_{L^\infty(\Omega)} \\ &= C(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}E_\alpha(-\lambda_1 t^\alpha)\|u_0\|_{L^\infty(\Omega)} \\ &\le C(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}-\alpha}\|u_0\|_{L^\infty(\Omega)},\end{aligned} \tag{28}$$

$$(1+t)^{\frac{\beta_2+q\beta_1}{pq-1}}\|P_\beta(t)v_0\|_{L^\infty(\Omega)} \le C(1+t)^{\frac{\beta_2+q\beta_1}{pq-1}-\beta}\|v_0\|_{L^\infty(\Omega)}. \tag{29}$$

Moreover, for any $(u,v) \in \mathcal{B}_K$, we deduce from (11) that

$$
\begin{aligned}
&(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\|\Psi_1(v))-P_\alpha(t)u_0\|_{L^\infty(\Omega)}\\
\leq &C(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\int_0^t\int_0^s(t-s)^{\alpha-1}(s-\tau)^{-\gamma_1}\int_0^{+\infty}\theta\phi_\alpha(\theta)e^{-\lambda_1(t-s)^\alpha\theta}d\theta\|v(\tau)\|^p_{L^\infty(\Omega)}d\tau ds\\
\leq &C(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\int_0^t\int_0^s(t-s)^{\alpha-1}(s-\tau)^{-\gamma_1}E_{\alpha,\alpha}(-\lambda_1(t-s)^\alpha)\|v(\tau)\|^p_{L^\infty(\Omega)}d\tau ds\\
\leq &CK^p(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\int_0^t\int_0^s(t-s)^{\alpha-1}(s-\tau)^{-\gamma_1}E_{\alpha,\alpha}(-\lambda_1(t-s)^\alpha)(1+\tau)^{-\frac{p(\beta_2+q\beta_1)}{pq-1}}d\tau ds\\
\leq &CK^p(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_1(t-s)^\alpha)\int_0^s(s-\tau)^{-\gamma_1}\tau^{-\frac{p(\beta_2+q\beta_1)}{pq-1}}d\tau ds\\
= &CK^p(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_1(t-s)^\alpha)s^{\beta_1-\frac{p(\beta_2+q\beta_1)}{pq-1}}ds,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&(1+t)^{\frac{\beta_2+q\beta_1}{pq-1}}\|\Psi_2(u))-P_\beta(t)u_0\|_{L^\infty(\Omega)}\\
\leq &CK^q(1+t)^{\frac{\beta_2+q\beta_1}{pq-1}}\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(-\lambda_1(t-s)^\beta)s^{\beta_2-\frac{q(\beta_1+p\beta_2)}{pq-1}}ds.
\end{aligned}
\tag{31}
$$

For any $(u_1,v_1),(u_2,v_2)\in\mathcal{B}_K$, using some arguments analogous to those used above, we derive that there exists a constant $C>0$ such that

$$
\begin{aligned}
&(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\|\Psi_1(v_1))-\Psi_1(v_2))\|_{L^\infty(\Omega)}\\
\leq &CK^{p-1}(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_1(t-s)^\alpha)\int_0^s(s-\tau)^{-\gamma_1}\tau^{-\frac{p(\beta_2+q\beta_1)}{pq-1}}d\tau ds\|u-v\|\\
\leq &CK^{p-1}(1+t)^{\frac{\beta_1+p\beta_2}{pq-1}}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_1(t-s)^\alpha)s^{\beta_1-\frac{p(\beta_2+q\beta_1)}{pq-1}}ds\|u-v\|,\ t>0,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
&(1+t)^{\frac{\beta_2+q\beta_1}{pq-1}}\|\Psi_2(u_1))-\Psi_2(u_2))\|_{L^\infty(\Omega)}\\
\leq &CK^{q-1}(1+t)^{\frac{\beta_2+q\beta_1}{pq-1}}\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(-\lambda_1(t-s)^\beta)s^{\beta_2-\frac{q(\beta_1+p\beta_2)}{pq-1}}ds\|u-v\|,\ t>0.
\end{aligned}
\tag{33}
$$

Since $\max\{\frac{p(\beta_2+q\beta_1)}{pq-1},\frac{q(\beta_1+p\beta_2)}{pq-1}\}<1$ and $E_{\alpha,\alpha}(z)$, $E_{\beta,\beta}(z)$ are entire functions, we know that, for given $t>0$,

$$
(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda_1(t-s)^\alpha)s^{\beta_1-\frac{p(\beta_2+q\beta_1)}{pq-1}}\in L^1(0,t),
$$

$$
(t-s)^{\beta-1}E_{\beta,\beta}(\lambda_1(t-s)^\beta)s^{\beta_2-\frac{q(\beta_1+p\beta_2)}{pq-1}}\in L^1(0,t).
$$

Thus, the dominated convergence theorem and (8) imply that

$$
\begin{aligned}
&\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_1(t-s)^\alpha)s^{\beta_1-\frac{p(\beta_2+q\beta_1)}{pq-1}}ds\\
&=\sum_{k=0}^{\infty}\int_0^t\frac{(-\lambda_1)^k(t-s)^{\alpha k+\alpha-1}s^{\beta_1-\frac{p(\beta_2+q\beta_1)}{pq-1}}}{\Gamma(\alpha k+\alpha)}ds\\
&=\Gamma(1+\beta_1-\frac{p(\beta_2+q\beta_1)}{pq-1})t^{\alpha-\frac{\beta_1+p\beta_2}{pq-1}}E_{\alpha,\alpha+1+\beta_1-\frac{p(\beta_2+q\beta_1)}{pq-1}}(-\lambda_1t^\alpha),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(-\lambda_1(t-s)^\beta)s^{\beta_2-\frac{q(\beta_1+p\beta_2)}{pq-1}}ds\\
&=\Gamma(1+\beta_2-\frac{q(\beta_1+p\beta_2)}{pq-1})t^{\beta-\frac{\beta_2+q\beta_1}{pq-1}}E_{\beta,\beta+1+\beta_2-\frac{q(\beta_1+p\beta_2)}{pq-1}}(-\lambda_1t^\beta).
\end{aligned}
\tag{34}
$$

Note that $\gamma_1 \leq \alpha$, $\gamma_2 \leq \beta$. Then, $\frac{\beta_1+p\beta_2}{pq-1} < \gamma_1 \leq \alpha$ and $\frac{\beta_2+q\beta_1}{pq-1} < \gamma_2 \leq \beta$. Hence, it follows from (9) and (28)–(33) that we can choose $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$ and K small enough so that Ψ is a contraction on $\mathcal{B}_K$. As a result, Ψ possesses a unique fixed point $(u,v) \in \mathcal{B}_K$. Evidently, $u, v \in C([0,\infty), C_0(\Omega))$. We have thus proved the theorem. □

Theorem 3. *Let $p, q \geq 1$, $pq > 1$, $\gamma_1 \leq \alpha$, $\gamma_2 > \beta$ and $u_0, v_0 \in C_0(\Omega)$.*

(i) *If $\frac{\beta_1+p\beta_2}{pq-1} \geq \gamma_1$ or $\frac{\beta_2+q\beta_1}{pq-1} > \beta$, and $u_0, v_0 \geq 0$, $v_0 \not\equiv 0$, then the corresponding mild solution (u,v) of (1) blows up in a finite time.*

(ii) *If $\frac{\beta_1+p\beta_2}{pq-1} < \gamma_1$ and $\frac{\beta_2+q\beta_1}{pq-1} \leq \beta$, then problem (1) admits a global solution (u,v) when $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{L^\infty(\Omega)}$ are sufficiently small. Moreover, $\|u(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_1+p\beta_2}{pq-1}}$, $\|v(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_2+q\beta_1}{pq-1}}$ for some constant $C > 0$.*

Proof. (i) Suppose that the maximal existence interval of (u,v) is $[0, T_*)$. According to the proof of Theorem 2(i), we find that inequality (27) still holds in this case. Then, Lemma 2(i) implies $g(0) = 0$ if $\frac{\beta_2+q\beta_1}{pq-1} - \beta > 0$. In addition, $\frac{\beta_1+p\beta_2}{pq-1} \geq \gamma_1$ if and only if $1 - p\gamma_2 + p(1 - q\gamma_1) \geq 0$. Hence, it follows from Lemma 2(v) that $T_* < +\infty$.

(ii) In this case, our assumptions imply that $\frac{\beta_1+p\beta_2}{pq-1} < \gamma_1 \leq \alpha$ and $\frac{\beta_2+q\beta_1}{pq-1} \leq \beta < \gamma_2$. Then, $\frac{p(\beta_2+q\beta_1)}{pq-1} < 1$, $\frac{q(\beta_1+p\beta_2)}{pq-1} < 1$. Proceeding as in the proof of Theorem 2(ii), we can carry out the proof of this theorem. □

When $\gamma_1 \leq \alpha$ and $v_0 \equiv 0$, we can obtain that Theorem 2 remains true for every $\beta \in (0,1)$ and $\gamma_2 \in [0,1)$.

Theorem 4. *Let $p, q \geq 1$, $pq > 1$, $\gamma_1 \leq \alpha$ and $u_0 \in C_0(\Omega)$, $v_0 \equiv 0$.*

(i) *If $\frac{\beta_1+p\beta_2}{pq-1} \geq \gamma_1$ or $\frac{\beta_2+q\beta_1}{pq-1} \geq \gamma_2$, and $u_0 \geq 0$, $u_0 \not\equiv 0$, then the corresponding mild solution (u,v) of (1) blows up in a finite time.*

(ii) *If $\frac{\beta_1+p\beta_2}{pq-1} < \gamma_1$ and $\frac{\beta_2+q\beta_1}{pq-1} < \gamma_2$, then problem (1) admits a global solution (u,v) providing that $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{L^\infty(\Omega)}$ are sufficiently small. Moreover, $\|u(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_1+p\beta_2}{pq-1}}$, $\|v(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_2+q\beta_1}{pq-1}}$ for some constant $C > 0$.*

Proof. (i) The result follows from Theorem 2(i).

(ii) In this case, the estimate (29) holds for $0 < \beta \leq 1$ due to $v_0 \equiv 0$. Moreover, our assumptions imply $\frac{\beta_1+p\beta_2}{pq-1} < \gamma_1 \leq \alpha$, $\frac{p(\beta_2+q\beta_1)}{pq-1} < 1$ and $\frac{q(\beta_1+p\beta_2)}{pq-1} < 1$. Then, when $\beta \geq \frac{\beta_2+q\beta_1}{pq-1}$, we can obtain the desired conclusion by some arguments analogous to those in Theorem 2(ii). For the case $\beta < \frac{\beta_2+q\beta_1}{pq-1}$, we can estimate (31) and (33) for $t > 1$ by using (34). When $0 \leq t \leq 1$, we can easily see that the term of the right hand of (31) is less than CM^q and the term of the right hand of (34) is less than $CK^{q-1}\|u - v\|_Y$. Thus, the conclusion of this theorem also holds in the case $\beta < \frac{\beta_2+q\beta_1}{pq-1}$. □

Remark 2. *(i) We deduce from the proof of Theorem 4 that the conclusions remain true for the case $\gamma_2 \leq \beta$, $\alpha \in (0,1)$ and $u_0 \equiv 0$, $v_0 \not\equiv 0$.*

(ii) It follows from Theorem 2 that our results coincide with those in [36] when $\alpha = \beta = 1$. Hence, our results extend those in [36].

(iii) Theorems 3 and 4 imply that, for the case $\gamma_1 \leq \alpha$ and $\gamma_2 > \beta$, the properties of solutions of (1) can be different if one of the initial values is identically vanishing. This is impossible for the classical reaction diffusion system (i.e., (1) with $\alpha = \beta = 1$) because of $\gamma_2 < 1 = \beta$.

Finally, we consider the case $\gamma_1 > \alpha$, and have the following results.

Theorem 5. *Let $p, q \geq 1$, $pq > 1$, $\gamma_1 > \alpha$, $\gamma_2 > \beta$ and $u_0, v_0 \in C_0(\Omega)$.*

(i) *If $\frac{\beta_1+p\beta_2}{pq-1} > \alpha$ or $\frac{\beta_2+q\beta_1}{pq-1} > \beta$ and $u_0, v_0 \geq 0$, $u_0, v_0 \not\equiv 0$, then the corresponding mild solution (u, v) of (1) blows up in a finite time.*

(ii) *If $\frac{\beta_1+p\beta_2}{pq-1} \leq \alpha$, $\frac{\beta_2+q\beta_1}{pq-1} \leq \beta$ and $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$ are sufficiently small, then problem (1) has a global solution (u, v). Moreover, there exists a constant $C > 0$ such that $\|u(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_1+p\beta_2}{pq-1}}$, $\|v(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_2+q\beta_1}{pq-1}}$ for $t > 0$.*

Proof. (i) Suppose that the maximal existence interval of (u, v) is $[0, +\infty)$. In terms of the proof of Theorem 2(i), we see that inequality (27) remains valid for every $T > 0$ in this case. Note that Lemma 2(i) implies that $f(0) = 0$ or $g(0) = 0$ if $\max\{\frac{\beta_1+p\beta_2}{pq-1} - \alpha, \frac{\beta_2+q\beta_1}{pq-1} - \beta\} > 0$. This yields a contradiction. The proof is completed.

(ii) From our assumptions, we have $\frac{\beta_1+p\beta_2}{pq-1} \leq \alpha < \gamma_1$, $\frac{\beta_2+q\beta_1}{pq-1} \leq \beta < \gamma_2$ and $\frac{p(\beta_2+q\beta_1)}{pq-1} < 1$, $\frac{q(\beta_1+p\beta_2)}{pq-1} < 1$. Then, by proceeding as in the proof of Theorem 2(ii), the conclusion holds. □

Theorem 6. *Let $p, q \geq 1$, $pq > 1$, $\gamma_1 > \alpha$, $\gamma_2 \leq \beta$ and $u_0, v_0 \in C_0(\Omega)$. Assume that (u, v) is the corresponding mild solution of (1).*

(i) *If $\frac{\beta_1+p\beta_2}{pq-1} > \alpha$ or $\frac{\beta_2+q\beta_1}{pq-1} \geq \gamma_2$ and $u_0, v_0 \geq 0$, $u_0 \not\equiv 0$, then (u, v) blows up in a finite time.*

(ii) *If $\frac{\beta_1+p\beta_2}{pq-1} \leq \alpha$, $\frac{\beta_2+q\beta_1}{pq-1} < \gamma_2$ and $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$ are sufficiently small, then the maximal existence time $T_* = +\infty$ and there exists a constant $C > 0$ such that $\|u(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_1+p\beta_2}{pq-1}}$, $\|v(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_2+q\beta_1}{pq-1}}$ for $t > 0$.*

Proof. (i) Suppose that the maximal existence time $T_* = +\infty$. In view of the proof of Theorem 2(i), we see that inequality (27) remains true for every $T > 0$ in this case. Hence, it follows from Lemma 2(i) that $f(0) = 0$ if $\frac{\beta_1+p\beta_2}{pq-1} > \alpha$, which contradicts $u_0 \not\equiv 0$. On the other hand, if $\frac{\beta_2+q\beta_1}{pq-1} \geq \gamma_2$, we can obtain a contradiction by Lemma 2(v).

(ii) Since our assumptions imply that $\frac{\beta_1+p\beta_2}{pq-1} \leq \alpha < \gamma_1$, $\frac{\beta_2+q\beta_1}{pq-1} < \gamma_2 \leq \beta$ and $\frac{p(\beta_2+q\beta_1)}{pq-1} < 1$, $\frac{q(\beta_1+p\beta_2)}{pq-1} < 1$. Then, we obtain the desired conclusion by some arguments similar to the proof of Theorem 2(ii). □

Theorem 7. *Let $p, q \geq 1$, $pq > 1$, $\gamma_1 > \alpha$ and $u_0 \in C_0(\Omega)$, $v_0 \equiv 0$. Assume that (u, v) is the corresponding mild solution of (1).*

(i) *If $\frac{\beta_1+p\beta_2}{pq-1} > \alpha$ or $\frac{\beta_2+q\beta_1}{pq-1} \geq \gamma_2$, and $u_0 \geq 0$, $u_0 \not\equiv 0$, then (u, v) blows up in a finite time.*

(ii) *If $\frac{\beta_1+p\beta_2}{pq-1} \leq \alpha$ and $\frac{\beta_2+q\beta_1}{pq-1} < \gamma_2$, then (u, v) is a global solution of problem (1) when $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{L^\infty(\Omega)}$ are sufficiently small. Moreover, $\|u(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_1+p\beta_2}{pq-1}}$, $\|v(t)\|_{L^\infty(\Omega)} \leq C(1+t)^{-\frac{\beta_2+q\beta_1}{pq-1}}$ for some constant $C > 0$.*

Proof. (i) The result follows from the proof of Theorem 6(i).

(ii) In this case, the estimate (29) holds for every $0 < \beta \leq 1$. In addition, our assumptions imply $\frac{\beta_1+p\beta_2}{pq-1} \leq \alpha < \gamma_1$, $\frac{p(\beta_2+q\beta_1)}{pq-1} < 1$ and $\frac{q(\beta_1+p\beta_2)}{pq-1} < 1$. Hence, we obtain the desired conclusion by repeating some arguments in the proof of Theorem 2(ii) and Theorem 4(ii). □

Remark 3. *Our results coincide with those in [34] when $\alpha = \beta$, $\gamma_1 = \gamma_2$ and $p = q > 1$, and those in [36] when $\alpha = \beta = 1$. Thus, we extend the results in [34,36]. Comparing the classical diffusion system (i.e., (1) with $\alpha = \beta = 1$), some new cases appear for problem (1). Moreover, we* *obtain some results, which are different from the classical diffusion system. Hence, our results are not just direct generalizations of the case $\alpha = \beta = 1$.*

Remark 4. *For every $p, q \geq 1$, $pq > 1$ and $T > 0$, we deduce from Lemma 2 and the proof of Theorem 2(i) that if $u_0, v_0 \geq 0$ satisfy*

$$\int_\Omega u_0(x)\varphi_1(x)dx \leq M[T^{\alpha+\beta_1-\frac{p\beta_2}{pq-1}-\frac{pq\beta_1}{pq-1}} + T^{\alpha+\beta_1-\frac{p(\beta+\beta_2)}{pq-1}-\frac{pq\beta_1}{pq-1}} + T^{-\frac{p\beta_2}{pq-1}-\frac{(\alpha+\beta_1)}{pq-1}} + T^{-\frac{p(\beta+\beta_2)}{pq-1}-\frac{(\alpha+\beta_1)}{pq-1}}],$$

or

$$\int_\Omega v_0(x)\varphi_1(x)dx \leq M[T^{\beta+\beta_2-\frac{q\beta_1}{pq-1}-\frac{pq\beta_2}{pq-1}} + T^{\beta+\beta_2-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{pq\beta_2}{pq-1}} + T^{-\frac{q\beta_1}{pq-1}-\frac{(\beta+\beta_2)}{pq-1}} + T^{-\frac{q(\alpha+\beta_1)}{pq-1}-\frac{(\beta+\beta_2)}{pq-1}}],$$

where M is a positive constant given in Lemma 2, then the maximal existence time T_* of the mild solution satisfies $T_* < T$.

4. Conclusions

The main aim of this paper is to investigate the blow-up and global existence of the solution of the initial boundary value problem (1). We firstly prove Lemma 2, where some properties of the solutions for a fractional differential inequality system are studied. Our result extends Lemma 5 in [36]. The proof of these estimates is based on the test function method, the representation of solutions of the nonhomogeneous fractional differential equations with constant coefficients, and the nonnegativity of the Mittag–Leffler functions $E_{\alpha,\alpha}(t)$ and $E_\alpha(t)$ for the case $0 < \alpha < 1$. Due to the memory effect of time fractional derivative, the standard method by shifting the time could not be available for our problem. We overcome this technical difficulty by the test function method. Moreover, since the orders of time fractional derivatives for problem (1) can be different, some methods and arguments used in [36] can not be directly applied to the study of problem (1). Secondly, we assert that the mild solution is the weak solution and give the local solvability result for problem (1). Finally, the blow-up results for problem (1) in different situations are proved by the eigenfunction method combined with the estimates in Lemma 2. Furthermore, by using the estimates of the solution operators $P_\alpha(t)$ and $S_\alpha(t)$, the asymptotic behavior of the Mittag–Leffler function and a fixed point argument, we obtain the existence of global solutions and the decay estimates of the solutions in the space $L^\infty(\Omega)$ when $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{L^\infty(\Omega)}$ are sufficiently small. As a result, we determine the critical exponents of parameters α, β, γ_1 and γ_2 in six different situations.

Our results extend ones in [34,36]. Some new results different from the ones of classical diffusion systems are obtained. Comparing with the results of classical diffusion system and time fractional diffusion equation, we find that the critical exponents of problem (1) are more delicate. Our results show that, in some cases, whether one of the initial values is identically equal to zero has a great influence on blow-up and global existence of the solutions for problem (1). However, this conclusion is false for the classical diffusion system because we can shift the time for the classical diffusion system. This indicates that the nonlocality of time fractional derivatives really affect properties of the solutions for time fractional diffusion systems.

In terms of practical applications and theoretical interests, ones may be more concerned with the space-time fractional diffusion system than what we have studied in the current paper. However, from the proof of our results, we know that the conclusions of this paper are still valid when the Laplace operator is replaced by the fractional Laplace operator supplemented with the exterior Dirichlet condition on $\mathbb{R}^N \setminus \Omega$. On the other hand, ones may be concerned with the sup-diffusion case of problem (1), i.e., $1 < \alpha < 2$ or $1 < \beta < 2$. However, it will be definitely more challenging. For example, the nonnegativity of the Mittag–Leffler function $E_{\alpha,\alpha}(t)$ is invalid in the case $1 < \alpha < 2$, and thus the method used in this paper can not be applied to study the sup-diffusion case. Nevertheless, we are still considering this more generalized case, and we expect to establish parallel results.

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