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Fractional Stochastic Integro-Differential Equations with Nonintantaneous Impulses: Existence, Approximate Controllability and Stochastic Iterative Learning Control

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Abstract: In this paper, existence/uniqueness of solutions and approximate controllability concept for Caputo type stochastic fractional integro-differential equations (SFIDE) in a Hilbert space with a noninstantaneous impulsive effect are studied. In addition, we study different types of stochastic iterative learning control for SFIDEs with noninstantaneous impulses in Hilbert spaces. Finally, examples are given to support the obtained results.

Keywords: fractional calculus; stochastic equation; iterative learning control; integro-differential systems

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1. Introduction

Iterative learning control (ILC), an important type of intelligent control methodology, was introduced by Uchiyama [1] and Arimoto [2,3]. This type of technique has been widely used in solving tracking problems for different types of control systems such as networked systems, multiagent systems, various distributed parameter systems, and different types of fractional-order systems [2–8]. The simplest visualization of ILC can be found in the area of robotic assembly and mechanical test procedures where a robotic device is used to complete a specified task such as "pick and place" [9].

The differential equation with impulses has extensive applications in various fields of science, such as engineering, medicine, economics, and so on. There are two popular types of pulses in the literature:

- Instantaneous impulses—the duration of these changes is relatively short compared to the total duration of the entire process. For the differential equations with instantaneous impulses, we refer the reader to the monograph [4].
- Noninstantaneous impulses—an impulsive action that begins abruptly at a fixed point and continues on for a finite amount of time. This kind of pulse is observed in lasers, and when drugs are injected into the bloodstream intravenously, see [5]. Recently, Hernandez and O'Regan [10] analyzed a kind of differential equation with a new impulsive effect, a so-called noninstantaneous impulse.

A noninstantaneous action of impulses begins at a certain point in time and remains active for a finite time interval. It is known that drug intake has a memory impact; thus, a new class of impulses does not explain completely this type of phenomenon. In this case, fractional analysis provides a powerful tool to describe this type of phenomenon because the main feature of fractional differential equations is to describe the memory characteristics of different events. For more information on the theory of existence and controllability theory of FDEs with noninstantaneous impulses, we refer the reader to [11–34].

Recently, Huang et al. [14] studied a P-type steady-state ILC scheme for the boundary control described linear parabolic differential equations in the sense of infinity-norm. Guo et al. [15] consider ILC for a class of non-affine-in-input processes with the general plant operators in a Hilbert space. However, the results of ILC for systems with distributed parameters are rather limited due to the inherent complexity in processing multidimensional systems. Liu et al. [16] studied P-type ILC law for impulsive differential equations by using open-closed loop iterative learning schemes in L^2-norm to track the desired discontinuous output trajectory. Yu et al. [17] study P-type, PI^{α}-type, and D-type ILC for impulsive FDEs in Banach spaces in the sense of the λ-norm. Liu et al. [18] apply ILC updating law and find a desired control function that sends the error between the output and the reference trajectories to zero in the so-called λ-norm. It should be stressed out that the P-type ILC, which is employed in this contribution, is a very popular form of ILC because of its simplicity. However, a disadvantage of the P-type ILC approach is its bad learning transients for many practical applications, cf. [27,28]. To avoid this problem, here, a zero-phase filtered ILC with phase-lead compensation as presented in [29].

Theorists and control engineers have now provided detailed explanations of ILC for deterministic control systems. Many significant results have been reported and applied to real systems. However, the interference and noise are unavoidable during the practical operations. Therefore, interference rejection is an important issue for ILC studies. Hence, when considering stochastic ILC, more attention should be paid to working with random processes. However, this is only the first step towards stochastic ILC, and much more work can be conducted for this ongoing topic.

To the best of the author's knowledge, no work has been reported to study the existence, uniqueness, approximate controllability and ILC results for Caputo type SFIDEs in a Hilbert space with noninstantaneous impulses. Here are contributions of the paper:

- Sufficient conditions which guarantee the existence/uniqueness of solutions of a fractional stochastic integro-differential system with noninstantaneous impulses in a Hilbert space is presented;
- Sufficient conditions for the approximate controllability of the fractional stochastic integro-differential system with noninstantaneous impulses in a Hilbert space are derived by assuming that the associated deterministic linear system is approximately controllable;
- P-type, D-type and PI-type stochastic iterative learning control for fractional stochastic integro-differential equations with noninstantaneous impulses in Hilbert spaces are investigated. P-type, D-type and PI-type stochastic iterative learning convergence conditions are presented. These results are novel for a fractional stochastic integro-differential system with noninstantaneous impulses, even for a finite-dimensional fractional stochastic integro-differential systems.

2. Preliminaries

Here are some notations and definitions.

- $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, P)$ is a probability space.
- K, H, Z and U are real separable Hilbert spaces.
- $w(t)$ is a Q-Wiener process on $(\Omega, \mathfrak{F}, P)$

$$\langle w(t), e\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k, e\rangle \beta_k(t),\ e \in K,\ t \in [0, \tau],$$

with a linear bounded covariance operator $Q : K \to K$ such that tr$Q < \infty$. It is assumed that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in Hilbert space K, a bounded sequence of $\{\lambda_k \in R^+\}$ such that $Qe_k = \lambda_k e_k,\ \ k = 1, 2, ...,$ and a sequence $\{\beta_k\}_{k\geq 1}$ of independent real valued Brownian motions such that and $\mathfrak{F}_t = \mathfrak{F}_t^w$, where $\mathfrak{F}_t^w$ is the sigma algebra generated by $\{w(s) : 0 \leq s \leq t\}$, which is $\mathfrak{F}_t^w = \sigma\{w(s) : 0 \leq s \leq t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of P-null sets of $\mathfrak{F}$.
- L_2^0 is the space of all Hilbert–Schmidt operators $\psi : Q^{1/2}K \to H$ with the inner product $\langle \psi, \phi\rangle_{L_2^0}$ =tr$[\psi Q\phi]$.

- $L^p_{\mathfrak{F}}(0,\tau;H), p \geq 2$ is the Banach space of all pth power integrable and $\mathfrak{F}_t$-adapted processes with values in H.
- $C(0,\tau;L^p(\mathfrak{F},H))$ be the Banach space of continuous maps $\varphi : [0,\tau] \to L^p(\mathfrak{F},H)$ with the norm $\sup\left\{\mathbb{E}\|\varphi(t)\|^p_H : t \in [0,\tau]\right\} < \infty$.
  $C_{\mathfrak{F}}(0,\tau;L^p(\mathfrak{F},H))$ is the closed subspace of $C(0,\tau;L^p(\mathfrak{F},H))$ of measurable and $\mathfrak{F}_t$-adapted H-valued processes $\varphi \in C(0,\tau;L^p(\mathfrak{F},H))$ with the norm $\|\varphi\|_{C_{\mathfrak{F}}} = \left(\sup_{0\leq t\leq\tau} \mathbb{E}\|\varphi(t)\|^p_H\right)^{\frac{1}{p}}$.
- $PC_{\mathfrak{F}} := PC_{\mathfrak{F}}\left(0,\tau;L^2(\mathfrak{F},H)\right)$ is the space of all $\mathfrak{F}_t$-adapted H-valued stochastic processes φ such that φ is continuous at $t \neq t_k$, $\varphi(t_k) = \varphi(t_k^-)$ and $\varphi(t_k^+)$ exists for all $k = 1,...,N$ endowed with the norm $\|\varphi\|_{PC_{\mathfrak{F}}} = \left(\sup_{0\leq t\leq\tau} \mathbb{E}\|\varphi(t)\|^2_H\right)^{\frac{1}{2}}$.
- $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $S : H \to H$ with $M := \sup_{0\leq t\leq\tau}\|S(t)\|_{L(H)}$ and $B \in L(U,H)$.
  Define

$$
\begin{aligned}
\mathfrak{S}_\alpha(t) &= \int_0^\infty \xi_\alpha(\theta)S(t^\alpha\theta)d\theta, \quad \mathfrak{T}_\alpha(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)S(t^\alpha\theta)d\theta, \quad t \geq 0,\\
\xi_\alpha(\theta) &= \frac{1}{\alpha}\theta^{-1-1/\alpha}\varpi_\alpha\left(\theta^{-1/\alpha}\right) \geq 0,\\
\varpi_\alpha(\theta) &= \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{m-1}\theta^{-m\alpha-1}\frac{\Gamma(m\alpha+1)}{m!}\sin(m\pi\alpha), \quad \theta \in (0,\infty),
\end{aligned}
$$

where $\xi_\alpha(\theta)$ is a probability density defined on $(0,\infty)$, which is

$$
\xi_\alpha(\theta) \geq 0, \quad \theta \in (0,\infty), \quad \int_0^\infty \xi_\alpha(\theta)d\theta = 1.
$$

We use the following properties of $\mathfrak{S}_\alpha(t)$ and $\mathfrak{T}_\alpha(t)$:

- $\forall$ fixed $t \geq 0$ and $\forall\, x \in X$, $\|\mathfrak{S}_\alpha(t)x\| \leq M\|x\|$ and $\|\mathfrak{T}_\alpha(t)x\| \leq \dfrac{M}{\Gamma(\alpha)}\|x\|$.
- $\{\mathfrak{S}_\alpha(t) : t \geq 0\}$ and $\{\mathfrak{T}_\alpha(t) : t \geq 0\}$ are strongly continuous.
- $\{\mathfrak{S}_\alpha(t) : t > 0\}$ and $\{\mathfrak{T}_\alpha(t) : t > 0\}$ are compact provided that the generating semigroup $S(t), t > 0$, is compact.

In this work, we are concerned with the question of existence, approximate controllability and stochastic ILC method for a class of Caputo stochastic fractional integro-differential equations (SFIDEs) in a Hilbert space with noninstantaneous impulses of the form:

$$
\begin{cases}
D^\alpha_{s_k,t}y(t) = Ay(t) + Bu(t) + f(t,y(t)) + \int_{s_k}^t g(r,y(r))dw(r), \ \ s_k \leq t \leq t_{k+1},\\
y(t) = h_k\left(t,y\left(t_k^-\right)\right), \ \ t_k < t < s_k, \ \ k = 1,...,N,\\
y\left(s_k^+\right) = y\left(s_k^-\right), \quad k = 1,2,...,N,\\
y(0) = y_0,
\end{cases}
\tag{1}
$$

where $D^\alpha_{s_k,t}$ denotes the Caputo fractional derivative of order $\alpha \in \left(\frac{1}{2},1\right)$ for y with the lower limit s_k, $y(\tau^\pm) = \lim_{\varepsilon\to 0^+} y(\tau \pm \varepsilon)$ and t_k and s_k. The stochastic integral is understood in Ito sense; see [26].

For Equation (1), we consider the output equation of the form

$$
z_j(t) = Cy_j(t) + Du_j(t), \tag{2}
$$

or

$$
z_j(t) = Cy_j(t) + D\int_0^t u_j(s)ds. \tag{3}
$$

Moreover, for Equation (1), we take into consideration an open-loop P-type stochastic ILC updating law with initial state learning

$$\Delta y_j(0) = L_1 e_j(0),\ \Delta u_j(t) = \gamma_1 e_j(t) = \gamma_1 \big(z_d(t) - z_j(t)\big) \tag{4}$$

and open-loop PI^α-type stochastic ILC updating law with initial state learning

$$\Delta y_j(0) = L_2 e_j(0),\ \Delta u_j(t) = \gamma_p e_j(t) + \frac{\gamma_I}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e_j(s) ds, \tag{5}$$

where $L_1, L_2 \in L(Z, H)$ and $\gamma_1, \gamma_p, \gamma_I \in L(Z, U)$ are unknown operators to be determined.

For Equation (1), we take into consideration the following open-loop D-type stochastic ILC updating law with initial state learning

$$\Delta y_j(0) = L_3 e_j(0),\ \Delta u_j(t) = \gamma_d\, e_j'(t), \tag{6}$$

where $L_3 \in L(Z, H)$ and $\gamma_d \in L(Z, U)$ are unknown operators to be determined.

Definition 1. *Let $u \in L^2_{\mathfrak{F}}(0, \tau; U)$. We say that a function $y \in PC_{\mathfrak{F}}$ is a mild solution of (1) if y satisfies the following stochastic integral equations:*

$$y = \begin{cases} \mathfrak{S}_\alpha(t) y_0 + \int_0^t (t-s)^{\alpha-1} \mathfrak{T}_\alpha(t-s)[Bu(s) + f(s, y(s))] ds \\ \qquad + \int_0^t (t-s)^{\alpha-1} \mathfrak{T}_\alpha(t-s) \int_0^s g(r, y(r)) dw(r),\ \ t \in [0, t_1], \\ h_k\big(t, y(t_k^-)\big),\ \ t \in (t_k, s_k),\ \ k = 1, 2, \dots, N, \\ \mathfrak{S}_\alpha(t - s_k) h_k\big(s_k, y(t_k^-)\big) + \int_{s_k}^t (t-s)^{\alpha-1} \mathfrak{T}_\alpha(t-s)[Bu(s) + f(s, y(s))] ds \\ \qquad + \int_{s_k}^t (t-s)^{\alpha-1} \mathfrak{T}_\alpha(t-s) \int_{s_k}^s g(r, y(r)) dw(r) ds, \\ t \in [s_k, t_{k+1}],\ \ k = 1, 2, \dots, N. \end{cases} \tag{7}$$

3. Existence of Solutions

In order to establish the existence and uniqueness result, we will need to impose some of the following conditions.

(A1) The function $f : [0, \tau] \times H \to H$ satisfies the conditions:

- (a) $f(\cdot, y) : [0, \tau] \to H$ is measurable for all $y \in H$ and $f(t, \cdot) : H \to H$ is continuous for a.e. $t \in [0, \tau]$.
- (b) $\exists\, M_f > 0$ such that $\|f(t, y)\| \leq M_f(1 + \|y\|)$ for a.e. $t \in [0, \tau]$, for every $y \in H$.
- (c) $\exists\, L_f > 0$ such that $\|f(t, y_1) - f(t, y_2)\| \leq L_f \|y_1 - y_2\|$ for a.e. $t \in [0, \tau]$, for every $y_1, y_2 \in H$.

(A2) The function $g : [0, \tau] \times H \to L_2^0$ satisfies the conditions:

- (a) $g(\cdot, y) : [0, \tau] \to L_2^0$ is measurable for all $y \in H$ and $g(t, \cdot) : H \to L_2^0$ is continuous for a.e. $t \in [0, \tau]$.
- (b) $\exists\, M_g > 0$ such that $\|g(t, y)\|_{L_2^0} \leq M_f(1 + \|y\|)$ for a.e. $t \in [0, \tau]$, for every $y \in H$.
- (c) $\exists\, L_g > 0$ such that $\|g(t, y_1) - g(t, y_2)\|_{L_2^0} \leq L_g \|y_1 - y_2\|$ for a.e. $t \in [0, \tau]$, for every $y_1, y_2 \in H$.

(A3) $h_k : [0, \tau] \times H \to H$ are continuous and satisfy the conditions:

- (a) $\exists$ constants $M_{h_k} > 0$ such that $\|h_k(t, y)\| \leq M_{h_k}(1 + \|y\|)$ for a.e. $t \in [0, \tau]$, for every $y \in H$.
- (b) $\exists$ constants $L_{h_k} > 0$ such that $\|h_k(t, y_1) - h_k(t, y_2)\| \leq L_{h_k} \|y_1 - y_2\|$ for a.e. $t \in [0, \tau]$, for every $y_1, y_2 \in H$.

For our main consideration of problem (1), a Banach fixed point is used to investigate the existence and uniqueness of solutions for SFIDEs with noninstantaneous impulses.

Theorem 1. *Assume that conditions (A1)–(A3) are satisfied and*

$$L_c = \max\left(\left(\frac{2M_S^2 L_f^2}{\Gamma^2(\alpha)} + \frac{2M_S^2 L_g^2 t}{\Gamma^2(\alpha)} \right) \frac{t^{2\alpha}}{2\alpha - 1}, L_{h_k}^2, \left(3M_S^2 L_{h_k}^2 + \frac{3M_S^2 L_f^2}{\Gamma^2(\alpha)} + \frac{3M_S^2 L_g^2 \tau}{\Gamma^2(\alpha)} \right) \frac{\tau^{2\alpha-1} - s_k^{2\alpha-1}}{2\alpha - 1} \right) < 1. \tag{8}$$

Then, the mild solution of SFIDE (1) exists and is unique.

Proof. Consider a nonlinear operator $F : PC_{\mathfrak{F}} \to PC_{\mathfrak{F}}$ as follows:

$$(Fy)(t) = \begin{cases} \mathfrak{S}_\alpha(t)y_0 + \int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s) f(s, y(s))ds \\ \qquad + \int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s) \int_0^s g(r, y(r))dw(r)ds, \ \ t \in [0, t_1], \\ h_k\big(t, y\big(t_k^-\big)\big), \ \ t \in (t_k, s_k), \ \ k = 1, 2, ..., N, \\ \mathfrak{S}_\alpha(t - s_k)h_k\big(s_k, y\big(s_k^-\big)\big) + \int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s) f(s, y(s))ds \\ \qquad \int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s) \int_{s_k}^s g(r, y(r))dw(r)ds, \ t \in [s_k, t_{k+1}], \ \ k = 1, 2, ..., N. \end{cases}$$

From the assumption, it is easy to see that F is well defined. Now, we only need to show that F is contractive.

Case 1: For $y, z \in PC_{\mathfrak{F}}$ and $0 \le t \le t_1$, we have

$$\begin{aligned} & E\|(Fy)(t) - (Fz)(t)\|^2 \\ & \le 2E\left\| \int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)(f(s, y(s)) - f(s, z(s)))ds \right\|^2 \\ & + 2E\left\| \int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s) \int_0^s (g(r, y(r)) - g(r, z(r)))dw(r)ds \right\|^2 \\ & \le \frac{2M_S^2 L_f^2}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2\alpha-2}ds \int_0^t E\|y(s) - z(s)\|^2 ds \\ & + \frac{2M_S^2 L_g^2 t}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2\alpha-2}ds \int_0^t E\|y(s) - z(s)\|^2 ds \\ & \le \left(\frac{2M_S^2 L_f^2}{\Gamma^2(\alpha)} + \frac{2M_S^2 L_g^2 t}{\Gamma^2(\alpha)} \right) \frac{t^{2\alpha}}{2\alpha - 1} \sup_{0 \le s \le t_1} E\|y(s) - z(s)\|^2. \end{aligned}$$

We take the supremum on $[0, t_1]$ to obtain

$$\sup_{0 \le t \le t_1} E\|(Fy)(t) - (Fz)(t)\|^2 \le \left(\frac{2M_S^2 L_f^2}{\Gamma^2(\alpha)} + \frac{2M_S^2 L_g^2 \tau}{\Gamma^2(\alpha)} \right) \frac{\tau^{2\alpha}}{2\alpha - 1} \|y - z\|_{PC_{\mathfrak{F}}}^2. \tag{9}$$

Case 2: For $y, z \in PC_{\mathfrak{F}}$ and $t_k < t \le s_k$, $k = 1, ..., N$, we have

$$\begin{aligned} E\|(Fy)(t) - (Fz)(t)\|^2 & \le E\big\|h_k\big(t, y\big(t_k^-\big)\big) - h_k\big(t, z\big(t_k^-\big)\big)\big\|^2 \\ & \le L_{h_k}^2 E\big\|y\big(t_k^-\big) - z\big(t_k^-\big)\big\|^2 \le L_{h_k}^2 \|y - z\|_{PC_{\mathfrak{F}}}^2. \end{aligned} \tag{10}$$

Case 3: For $y, z \in PC_{\mathfrak{F}}$ and $s_k < t \leq t_{k+1}, k = 1, ..., N$, we have

$$\begin{aligned}
&E\|(Fy)(t) - (Fz)(t)\|^2 \\
&\leq 3E\|S_\alpha(t - s_k)\big(h_k\big(s_k, y(s_k^-)\big) - h_k\big(s_k, z(s_k^-)\big)\big)\|^2 \\
&+ 3E\left\|\int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)(f(s,y(s)) - f(s,z(s)))ds\right\|^2 \\
&+ 3E\left\|\int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_{s_k}^s (g(r,y(r)) - g(r,z(r)))dw(r)ds\right\|^2 \\
&\leq 3M_S^2 L_{h_k}^2 \|y - z\|_{PC_{\mathfrak{F}}}^2 \\
&+ \frac{3M_S^2 L_f^2}{\Gamma^2(\alpha)} \int_{s_k}^t (t-s)^{2\alpha-2} ds \int_{s_k}^t E\|y(s) - z(s)\|^2 ds \\
&+ \frac{3M_S^2 L_g^2 t}{\Gamma^2(\alpha)} \int_{s_k}^t (t-s)^{2\alpha-2} ds \int_{s_k}^t E\|y(s) - z(s)\|^2 ds \\
&\leq \left(3M_S^2 L_{h_k}^2 + \frac{3M_S^2 L_f^2}{\Gamma^2(\alpha)} + \frac{3M_S^2 L_g^2 \tau}{\Gamma^2(\alpha)}\right)\frac{\tau^{2\alpha-1} - s_k^{2\alpha-1}}{2\alpha - 1}\|y - z\|_{PC_{\mathfrak{F}}}^2.
\end{aligned} \tag{11}$$

From (9)–(11), we obtain

$$\|Fy - Fz\|_{PC_{\mathfrak{F}}}^2 \leq L_c\|y - z\|_{PC_{\mathfrak{F}}}^2.$$

This implies that F is contractive and therefore has a unique fixed point $y \in PC_{\mathfrak{F}}(0, \tau; L^p(\mathfrak{F}, H))$, which is a mild solution of SFIDE (1). □

The second existence result of this section is based on a Krasnoselskii–Schaefer type fixed point theorem under non-Lipschitz continuity of nonlinear terms. As we can easily see, we will weaken the assumption $L_c < 1$ in Theorem 1, but at the same time we need to impose some Caratheodory and Nagumo type of assumptions as well as an additional smallness hypothesis.

(A4) There is a continuous nondecreasing functions $\psi_f, \psi_g : [0, \infty) \to [0, \infty)$ and $p_f, p_g \in L^1([0, \tau], [0, \infty))$ such that

$$\|f(t,y)\|^2 \leq p_f(t)\psi_f\left(\|y\|^2\right), \qquad \|g(t,y)\|^2 \leq p_g(t)\psi_g\left(\|y\|^2\right),$$

for a.e. $t \in [0, \tau]$ with

$$K_1 \int_{s_k}^{t_{k+1}} p_f(s)ds + K_2 \int_{s_k}^{t_{k+1}} p_g(s)ds < \int_{K_0}^\infty \frac{ds}{\psi_f(s) + \psi_g(s)},$$

where K_0, K_1, K_2 are positive constants.

Theorem 2. *Assume that $h_k(t, 0) = 0$, and hypotheses (A1a), (A2a) and (A4) hold. If*

$$L_1 = \max\left(M_S^2 L_{h_k}^2 : k = 1, ..., N\right) < 1,$$

then problem (1) possesses at least one mild solution on $[0, \tau]$.

Proof. Parallel to the proof of Theorem 1, we transform fractional stochastic problem (1) into the same equivalent fixed point formulation keeping the same operator F. Now, we split our operator F into two operators in the following way:

$$(F_1 y)(t) = \begin{cases} \mathfrak{S}_\alpha(t)y_0, \quad t \in [0, t_1], \\ h_k\big(t, y(t_k^-)\big), \ t \in (t_k, s_k), \ k = 1, 2, ..., N, \\ \mathfrak{S}_\alpha(t - s_k)h_k\big(s_k, y(s_k^-)\big), \ t \in [s_k, t_{k+1}], \ k = 1, 2, ..., N, \end{cases}$$

and

$$(F_2y)(t) = \begin{cases} \int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)f(s,y(s))ds + \int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_0^s g(r,y(r))dw(r)ds, \ \ t\in[0,t_1], \\ 0, \ \ t\in(t_k,s_k), \ \ k=1,2,...,N, \\ \int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)f(s,y(s))ds \\ \qquad \int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_{s_k}^s g(r,y(r))dw(r)ds, \ t\in[s_k,t_{k+1}], \ \ k=1,2,...,N, \end{cases}$$

To use the Krasnoselskii–Schaefer theorem, we will verify that F_1 is contractive while F_2 is a completely continuous operator. For convenience, we divided the proof into several stages.

Step 1: F_1 is contractive.

Case 1: For $y, z \in PC_{\mathfrak{F}}$ and $t_k < t \le s_k$, $k = 1, ..., N$, we have

$$\begin{aligned} E\|(F_1y)(t)-(F_1z)(t)\|^2 &\le E\left\|h_k\left(t,y\left(t_k^-\right)\right)-h_k\left(t,z\left(t_k^-\right)\right)\right\|^2 \\ &\le L_{h_k}^2 E\left\|y\left(t_k^-\right)-z\left(t_k^-\right)\right\|^2 \le L_{h_k}^2\|y-z\|^2_{PC_{\mathfrak{F}}}. \end{aligned}$$

Case 2: For $y, z \in PC_{\mathfrak{F}}$ and $s_k < t \le t_{k+1}$, $k = 1, ..., N$, we have

$$\begin{aligned} &E\|(F_1y)(t)-(F_1z)(t)\|^2 \\ &\le E\left\|S_\alpha(t-s_k)\left(h_k\left(s_k,y\left(s_k^-\right)\right)-h_k\left(s_k,z\left(s_k^-\right)\right)\right)\right\|^2 \\ &\le M_S^2 L_{h_k}^2\|y-z\|^2_{PC_{\mathfrak{F}}}. \end{aligned}$$

We take the supremum on $[0, \tau]$ and obtain

$$\|Fy-Fz\|^2_{PC_{\mathfrak{F}}} \le L_1\|y-z\|^2_{PC_{\mathfrak{F}}}.$$

Thus, F_1 is a contraction.

Step 2: F_2 is completely continuous.

The proof is omitted, since it is standard.

Step 3: A priori bound.

Show boundedness of the set

$$\Xi = \left\{z\in PC_{\mathfrak{F}} : z = \lambda F_2 z + \lambda F_1\left(\frac{z}{\lambda}\right), \text{ for some } 0<\lambda<1\right\}.$$

Case 1: For each $0 \le t \le t_1$,

$$\begin{aligned} z(t) = \mathfrak{S}_\alpha(t)y_0 &+ \lambda\int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)f(s,z(s))ds \\ &+ \lambda\int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_0^s g(r,z(r))dw(r). \end{aligned}$$

This integral representation implies that

$$\begin{aligned} E\|z(t)\|^2 \le{}& 3M\|y_0\|^2 \\ &+ 3\frac{M^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}ds \int_0^t p_f(s)\psi_f\Big(E\|z(s)\|^2\Big)ds \\ &+ 3\frac{M^2 t}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}ds \int_0^t p_g(s)\psi_g\Big(E\|z(s)\|^2\Big)ds \\ &\le \underbrace{3M\|y_0\|^2}_{K_0} + \underbrace{3\frac{M^2}{\Gamma^2(\alpha)}\frac{\tau^{2\alpha}}{2\alpha-1}}_{K_1}\int_0^t p_f(s)\psi_f\Big(E\|z(s)\|^2\Big)ds \\ &+ \underbrace{3\frac{M^2 t}{\Gamma^2(\alpha)}\frac{\tau^{2\alpha}}{2\alpha-1}}_{K_2}\int_0^t p_g(s)\psi_g\Big(E\|z(s)\|^2\Big)ds. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} E\|z(t)\|^2 \le{}& K_0 + K_1\int_0^t p_f(s)\psi_f\Big(E\|z(s)\|^2\Big)ds \\ &+ K_2\int_0^t p_g(s)\psi_g\Big(E\|z(s)\|^2\Big)ds. \end{aligned} \tag{12}$$

Let us denote the RHS of the inequality (12) by $v(t)$. Then, we have

$$v(0) = K_0,\ E\|z(t)\|^2 \le v(t), \quad 0 \le t \le t_1.$$

and

$$v'(t) = K_1 p_f(t)\psi_f\Big(E\|z(t)\|^2\Big) + K_2 p_g(t)\psi_g\Big(E\|z(t)\|^2\Big),\ \ 0 \le t \le t_1.$$

Using the increasing character of ψ_f and ψ_g, we obtain

$$v'(t) = K_1 p_f(t)\psi_f(v(t)) + K_2 p_g(t)\psi_g(v(t)),\ \ 0 \le t \le t_1.$$

This equation implies, for each $0 \le t \le t_1$,

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi_f(s)+\psi_g(s)} \le K_1\int_0^{t_1} p_f(s)ds + K_2\int_0^{t_1} p_g(s)ds < \int_{K_0}^{\infty}\frac{ds}{\psi_f(s)+\psi_g(s)}.$$

By Bihari inequality,

$$v(t) \le \Omega^{-1}\bigg(K_1\int_0^{t_1} p_f(s)ds + K_2\int_0^{t_1} p_g(s)ds\bigg),\ \ 0 \le t \le t,$$

where

$$\Omega(t) := \int_{K_0}^{t}\frac{ds}{\psi_f(s)+\psi_g(s)}.$$

Thus,

$$E\|z(t)\|^2 \le v(t) \le L_{t_o}.$$

Case 2: For each $t_k < t \le s_k,\ k = 1, ..., N$,

$$z(t) = h_k\big(t, z\big(t_k^-\big)\big).$$

This implies that

$$E\|z(t)\|^2 \le L_{h_k}^2 E\big\|z\big(t_k^-\big)\big\|^2 \le L_{h_k}^2\|z\|^2_{PC_{\mathfrak{F}}}.$$

It follows that $\|z\|^2_{PC_{\mathfrak{F}}} \leq \frac{1}{1-L^2_{h_k}}$.

Case 3: For each $s_k \leq t \leq t_{k+1}$, $k = 1, ..., N$

$$\begin{aligned}
&E\|(Fz)(t)\|^2 \\
&\leq 3E\left\|\mathfrak{S}_\alpha(t-s_k)\left(h_k\left(s_k, y\left(s_k^-\right)\right) - h_k\left(s_k, z\left(s_k^-\right)\right)\right)\right\|^2 \\
&+ 3E\left\|\int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)(f(s,y(s)) - f(s,z(s)))ds\right\|^2 \\
&+ 3E\left\|\int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_{s_k}^s (g(r,y(r)) - g(r,z(r)))dw(r)ds\right\|^2 \\
&\leq 3M_S^2 L_{h_k}^2 \|z\|^2_{PC_{\mathfrak{F}}} \\
&+ \frac{3M_S^2 L_f^2}{\Gamma^2(\alpha)}\int_{s_k}^t (t-s)^{2\alpha-2}ds\int_0^t p_f(s)\psi_f\left(E\|z(s)\|^2\right)ds \\
&+ \frac{3M_S^2 L_g^2 t}{\Gamma^2(\alpha)}\int_{s_k}^t (t-s)^{2\alpha-2}ds\int_0^t p_g(s)\psi_g\left(E\|z(s)\|^2\right)ds \\
&= K_0 + K_1\int_0^t p_f(s)\psi_f\left(E\|z(s)\|^2\right)ds + K_2\int_0^t p_g(s)\psi_g\left(E\|z(s)\|^2\right)ds.
\end{aligned}$$

Similar to Case 1, there exists $L_{t_{k+1}} > 0$ such that $\|z\|^2_{PC_{\mathfrak{F}}} \leq L_{t_{k+1}}$. This implies that the set Ξ is bounded.

To complete the proof, we apply the Krasnoselskii–Schaefer type fixed point theorem. Thus, F has a fixed point, which is a mild solution of the SFIDE (1). □

4. Approximate Controllability

In this section, we establish the approximate controllability of mild solutions to stochastic integro-differential equations in a Hilbert space with noninstantaneous impulses driven by Q-Wiener motions:

$$(Fy)(t) = \begin{cases} \mathfrak{S}_\alpha(t)y_0 + \int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)[Bu(s) + f(s,y(s))]ds \\ \quad + \int_0^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_0^s g(r,y(r))dw(r)ds, \ \ t \in [0,t_1], \\ h_k\left(t, y\left(t_k^-\right)\right), \ \ t \in (t_k, s_k), \ \ k = 1,2,...,N, \\ \mathfrak{S}_\alpha(t-s_k)h_k\left(s_k, y\left(t_k^-\right)\right) + \int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)[Bu(s) + f(s,y(s))]ds \\ \quad \int_{s_k}^t (t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_{s_k}^s g(r,y(r))dw(r)ds, \ t \in [s_k, t_{k+1}], \ \ k = 1,2,...,N, \end{cases}$$

We define an operator

$$\Pi_{s_N}^\tau := \int_{s_N}^\tau (\tau-s)^{\alpha-1}\mathfrak{T}_\alpha(\tau-s)BB^*\mathfrak{T}_\alpha(\tau-s)ds : H \to H.$$

It is not difficult to see that the operator $\Pi_{s_N}^\tau$ is a bounded linear operator. Indeed,

$$\begin{aligned}
\left\|\Pi_{s_N}^\tau x\right\| &\leq \int_{s_N}^\tau (\tau-s)^{\alpha-1}\|\mathfrak{T}_\alpha(\tau-s)BB^*\mathfrak{T}_\alpha(\tau-s)x\|ds \\
&\leq \frac{M^2}{\Gamma^2(\alpha)}\|B\|^2\|x\|\int_{s_N}^\tau (\tau-s)^{\alpha-1}ds \\
&= \frac{\tau^\alpha M^2}{\alpha\Gamma^2(\alpha)}\|B\|^2\|x\|.
\end{aligned}$$

It is known that the approximate controllability on $[s_N, \tau]$ of a linear system associated with (1) is equivalent to convergence of $\varepsilon\left(\varepsilon I + \Pi_{s_N}^\tau\right)^{-1} \to\to 0$ as $\varepsilon \to 0^+$ in the strong operator topology; see [25].

Definition 2. *The SFIDE (1) is said to be approximately controllable on* $[0,\tau]$ *if* $\overline{R(\tau,y_0)} = H$, *where*

$$R(\tau,y_0) = \Big\{y(\tau,y_0,u) : y \text{ is a solution of (1)}, u \in L^2([0,\tau]\times\Omega;U)\Big\}.$$

Choose any stochastic control $u_1 \in L^2_{\mathfrak{F}}(0,s_N;U)$ on the interval $[0,s_N]$ and define a stochastic control $u_{s_N}(t;y)$ on $[s_N,\tau]$ as follows:

$$\begin{aligned} u_{s_N}(t;y) &= B^*\mathfrak{T}_\alpha(\tau-t)\big(\varepsilon I + \Pi^\tau_{s_N}\big)^{-1} p_N(y), \quad t\in[s_N,\tau], \\ p_N(y) &= h - S_\alpha(\tau-s_N)h_N\big(s_N, y(t_N^-)\big) - \int_{s_N}^{\tau}(\tau-s)^{\alpha-1}\mathfrak{T}_\alpha(\tau-s)[Bu(s)+f(s,y(s))]ds \\ &\quad - \int_{s_N}^{\tau}(t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_{s_N}^{s} g(r,y(r))dw(r)ds. \end{aligned}$$

Finally, let us define

$$u(t;y) := \sum_{k=0}^{N-1} u_1(t)\chi_{[s_k,t_{k+1}]}(t) + u_{s_N}(t;y)\chi_{[s_N,\tau]}(t), \quad t\in[0,\tau],$$

where χ_A is the characteristic function of the set A.

Theorem 3. *Assume that* $h_k(t,0)=0$, *and hypotheses (A1a), (A2a) and (A4) hold. Suppose that* f, g *are uniformly bounded functions. Then, the SFIDE (1) is approximately controllable on* $[0,\tau]$ *provided that* $\varepsilon\big(\varepsilon I + \Pi^\tau_{s_N}\big)^{-1} \to 0$ *as* $\varepsilon \to 0^+$ *is strong.*

Proof. Let y^ε be a fixed point on F. By the stochastic analogue of the Fubini theorem, it is easily seen that

$$\begin{aligned} y^\varepsilon(\tau) &= S_\alpha(\tau-s_N)h_N\big(s_N, y^\varepsilon(t_N^-)\big) + \int_{s_N}^{\tau}(\tau-s)^{\alpha-1}\mathfrak{T}_\alpha(\tau-s)[Bu_{s_N}(s;y^\varepsilon)+f(s,y^\varepsilon(s))]ds \\ &+ \int_{s_N}^{\tau}(t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_{s_N}^{s} g(r,y^\varepsilon(r))dw(r)ds \\ &= S_\alpha(\tau-s_N)h_N\big(s_N, y^\varepsilon(t_N^-)\big) + \Pi^\tau_{s_N}\big(\varepsilon I + \Pi^\tau_{s_N}\big)^{-1} p_N(y^\varepsilon) \\ &+ \int_{s_N}^{\tau}(\tau-s)^{\alpha-1}\mathfrak{T}_\alpha(\tau-s)f(s,y^\varepsilon(s))ds + \int_{s_N}^{\tau}(t-s)^{\alpha-1}\mathfrak{T}_\alpha(t-s)\int_{s_N}^{s} g(r,y^\varepsilon(r))dw(r)ds \\ &= h - \varepsilon\big(\varepsilon I + \Pi^\tau_{s_N}\big)^{-1} p_N(y^\varepsilon) \end{aligned}$$

It follows from the assumptions on f and g that there exists a D such that $\|f(s,y^\varepsilon(s))\|^2 + \|g(s,y^\varepsilon(s))\|^2 \le D$. Then, there is a subsequence denoted by $\{f(s,y^\varepsilon(s)), g(s,y^\varepsilon(s))\}$ weakly converging to say $\{f(s),g(s)\}$. Now, the compactness of $S(t)$ implies that

$$\mathfrak{T}_\alpha(\tau-s)f(s,y^\varepsilon(s)) \to \mathfrak{T}_\alpha(\tau-s)f(s), \mathfrak{T}_\alpha(\tau-s)\int_{s_N}^{s} g(r,y^\varepsilon(r))dw(r) \to \mathfrak{T}_\alpha(\tau-s)\int_{s_N}^{s} g(r)dw(r).$$

From the above equation, we have

$$
\begin{aligned}
E\|y^{\varepsilon}(\tau)-h\|^2 \leq\ & 5\|\varepsilon\big(\varepsilon I+\Pi_{s_N}^{\tau}\big)^{-1}h\|^2 \\
&+5E\bigg(\int_{s_N}^{\tau}\|\varepsilon\big(\varepsilon I+\Pi_{s_N}^{\tau}\big)^{-1}\mathfrak{T}_{\alpha}(\tau-s)[f(s,y^{\varepsilon}(s))-f(s)]\|ds\bigg)^2 \\
&+5E\bigg(\int_{s_N}^{\tau}\|\varepsilon\big(\varepsilon I+\Pi_{s_N}^{\tau}\big)^{-1}\mathfrak{T}_{\alpha}(\tau-s)f(s)\|ds\bigg)^2 \\
&+5\tau E\Bigg(\int_{s_N}^{\tau}\bigg\|\varepsilon\big(\varepsilon I+\Pi_{s_N}^{\tau}\big)^{-1}\mathfrak{T}_{\alpha}(\tau-s)\int_{s_N}^{s}[g(r,y^{\varepsilon}(r))-g(r)]dr\bigg\|_{L_2^0}^2 ds\Bigg) \\
&+5\tau E\bigg(\int_{s_N}^{\tau}\Big\|\varepsilon\big(\varepsilon I+\Pi_{s_N}^{\tau}\big)^{-1}\Big\|\int_{s_N}^{s}\|\mathfrak{T}_{\alpha}(\tau-s)g(r)\|_{L_2^0}^2 drds\bigg).
\end{aligned}
$$

On the other hand, the operator $\varepsilon\big(\varepsilon I+\Pi_{s_N}^{\tau}\big)^{-1}\to 0$ behaves strongly as $\varepsilon\to 0^+$ and moreover $\|\varepsilon\big(\varepsilon I+\Pi_{s_N}^{\tau}\big)^{-1}\|\leq 1$. Thus, by the Lebesgue dominated convergence theorem, we obtain $E\|y^{\varepsilon}(\tau)-h\|^2\to 0$ as $\varepsilon\to 0^+$. This gives the approximate controllability of the control system (1). □

5. Time Invariant Stochastic ILC

In the present section, we discuss P, PI^{α}, D types of open-loop stochastic ILC methods in the sense of λ-norm. To achieve our third goal, we introduce the repeatedly running stochastic equations:

$$
\begin{cases}
D_{s_k,t}^{\varepsilon}y_j(t)=Ay_j(t)+Bu_j(t)+f\big(t,y_j(t)\big)+\int_{s_k}^{t}g(r,y(r))dw(r), & s_k\leq t\leq t_{k+1},\\
y_j(t)=h_k\big(t,y_j(t_k^-)\big), & t_k<t<s_k,\ k=1,\dots,N,\\
y_j(s_k^+)=y_j(s_k^-), & k=1,2,\dots,N,\\
z_j(t)=Cy_j(t)+Du_j(t), & t\in[0,\tau],
\end{cases}
\tag{13}
$$

Concerning (13), we consider the following open-loop P-type stochastic ILC updating law with initial state learning defined by:

$$
\Delta u_j(t)=\gamma_p e_j(t),\ \ \Delta y_j(0)=L_1e_j(0), \tag{14}
$$

where L_1,γ_p are unknown operators to be determined and $\gamma_p\in L(Z,U)$, $L_1\in L(Z,H)$.

For simplification, we set

$$
\begin{aligned}
&\rho_1:=\frac{\tau^{2\alpha-1}}{2\alpha-1}\frac{6M^2}{\Gamma^2(\alpha)},\ \ \rho_2:=\frac{\tau^{2\alpha-1}}{2\alpha-1}\frac{6M^2L_f^2}{\Gamma^2(\alpha)},\ \ \rho_3:=\frac{\tau^{2\alpha}}{2\alpha-1}\frac{3M^2L_g^2}{\Gamma^2(\alpha)},\\
&\rho_4:=\rho_1/\Big(1-3M^2\exp(\rho_2+\rho_3\tau)e^{-\lambda\tau}\Big).
\end{aligned}
$$

Firstly, we give an estimation of $\Delta y_j(t)$ in terms of an integral of Δu_j.

Lemma 1. *Under the conditions (A1)–(A3), the following estimation holds:*

$$
E\big\|\Delta y_j(t)\big\|^2\leq
\begin{cases}
\Big(3M^2\big\|\Delta y_j(0)\big\|^2+\rho_1\dfrac{e^{\lambda t}-1}{\lambda}\big\|B\Delta u_j\big\|_{\lambda}^2\Big) & \\
\quad\times\exp((\rho_2+\rho_3)\tau), & 0\leq t\leq t_1,\\
L_{h_k}^2E\big\|\Delta y_j\big(t_k^-\big)\big\|^2, & t_k\leq s\leq s_k\\
\Big(3M^2L_{h_k}E\big\|y_{j+1}\big(t_k^-\big)-y_j\big(t_k^-\big)\big\|^2+\rho_1\dfrac{e^{\lambda t}-1}{\lambda}\big\|B\Delta u_j\big\|_{\lambda}^2\Big) & \\
\quad\exp((\rho_2+\rho_3)(\tau-s_k)) & s_k\leq s\leq t_{k+1}.
\end{cases}
\tag{15}
$$

Proof. We consider the following three cases.

Case 1: $0 \le t \le t_1$:
From the solution of the state equation for (13), for any $0 \le t \le t_1$, we have

$$E\|\Delta y_j(t)\|^2 \le 3\|\mathfrak{S}_\alpha(t)\Delta y_j(0)\|^2$$
$$+3E\left(\int_0^t (t-s)^{\alpha-1}\|\mathfrak{T}_\alpha(t-s)\left[B\Delta u_j(s)+f(s,y_{j+1}(s))-f(s,y_j(s))\right]\|ds\right)^2$$
$$+3E\left(\int_0^t (t-s)^{2(\alpha-1)}\left\|\mathfrak{T}_\alpha(t-s)\int_0^s \left(g(r,y_{j+1}(r))-g(r,y_j(r))\right)dw(r)\right\|ds\right)^2 := I_1+I_2+I_3. \tag{16}$$

Now, we estimate I_1, I_2 and I_3:

$$\begin{aligned}
I_1 &\le 3M^2\|\Delta y_j(0)\|^2,\\
I_2 &\le \frac{3M^2}{\Gamma^2(\alpha)}E\left(\int_0^t (t-s)^{\alpha-1}\left(\|B\Delta u_j(s)\|+\|f(s,y_{j+1}(s))-f(s,y_j(s))\|\right)ds\right)^2\\
&\le \frac{6M^2}{\Gamma^2(\alpha)}E\left(\int_0^t (t-s)^{\alpha-1}\|B\Delta u_j(s)\|ds\right)^2+\frac{6M^2L_f^2}{\Gamma^2(\alpha)}E\left(\int_0^t (t-s)^{\alpha-1}\|\Delta y_j(s)\|ds\right)^2\\
&\le \frac{6M^2}{\Gamma^2(\alpha)}E\left(\int_0^t (t-s)^{\alpha-1}\|B\Delta u_j(s)\|ds\right)^2+\frac{6M^2L_f^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)}ds\int_0^t E\|\Delta y_j(s)\|^2ds\\
&= \frac{6M^2}{\Gamma^2(\alpha)}E\left(\int_0^t (t-s)^{\alpha-1}\|B\Delta u_j(s)\|ds\right)^2+\frac{t^{2\alpha-1}}{2\alpha-1}\frac{6M^2L_f^2}{\Gamma^2(\alpha)}\int_0^t E\|\Delta y_j(s)\|^2ds && (17)\\
&\le \frac{t^{2\alpha-1}}{2\alpha-1}\frac{e^{\lambda t}-1}{\lambda}\int_0^t \|B\Delta u_j(s)\|^2ds, && (18)
\end{aligned}$$

$$\begin{aligned}
I_3 &\le \frac{3M^2}{\Gamma^2(\alpha)}E\left(\int_0^t (t-s)^{\alpha-1}\left\|\int_0^s \left(g(r,y_{j+1}(r))-g(r,y_j(r))\right)dw(r)\right\|ds\right)^2\\
&\le \frac{3M^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)}ds\int_0^t E\left\|\int_0^s \left(g(r,y_{j+1}(r))-g(r,y_j(r))\right)dw(r)\right\|^2ds\\
&\le \frac{t^{2\alpha-1}}{2\alpha-1}\frac{3M^2L_g^2}{\Gamma^2(\alpha)}\int_0^t\int_0^s E\|\Delta y_j(r)\|^2drds\\
&\le \frac{t^{2\alpha}}{2\alpha-1}\frac{3M^2L_g^2}{\Gamma^2(\alpha)}\int_0^t E\|\Delta y_j(r)\|^2dr. && (19)
\end{aligned}$$

Combining (16)–(19), we obtain

$$E\|\Delta y_j(t)\|^2 \le 3M^2\|\Delta y_j(0)\|^2+\frac{t^{2\alpha-1}}{2\alpha-1}\frac{e^{\lambda t}-1}{\lambda}\int_0^t \|B\Delta u_j(s)\|^2ds+\frac{t^{2\alpha-1}}{2\alpha-1}\frac{6M^2L_f^2}{\Gamma^2(\alpha)}\int_0^t E\|\Delta y_j(s)\|^2ds$$
$$+\frac{t^{2\alpha}}{2\alpha-1}\frac{3M^2L_g^2}{\Gamma^2(\alpha)}\int_0^t E\|\Delta y_j(r)\|^2dr.$$

Applying the Gronwall inequality, we obtain

$$E\|\Delta y_j(t)\|^2 \le \left(3M^2\|\Delta y_j(0)\|^2+\rho_1\frac{e^{\lambda t}-1}{\lambda}\int_0^t \|B\Delta u_j(s)\|^2ds\right)$$
$$\times \exp((\rho_2+\rho_3)t).$$

Multiplying the above inequality through by $e^{-\lambda t}$ and taking the λ-norm, we obtain the desired inequality on $[0, t_1]$.

Case 2: if $t_k \leq s \leq s_k$, $k = 1, ..., N$, from the solution of the state equation for (13), we have

$$E\|\Delta y_j(t)\|^2 \leq L_{h_k}^2 E\|y_{j+1}(t_k^-) - y_j(t_k^-)\|^2 = L_{h_k}^2 E\|\Delta y_j(t_k^-)\|^2.$$

Case 3: $s_k \leq t \leq t_{k+1}$; $k = 1, ..., N$: In a similar manner, we obtain

$$\begin{aligned} E\|\Delta y_j(t)\|^2 &\leq 3M^2 L_{h_k} E\|y_{j+1}(t_k^-) - y_j(t_k^-)\|^2 \\ &\quad + \rho_1(t)\int_{s_k}^t E\|B\Delta u_j(s)\|^2 ds + (\rho_2(t) + \rho_3(t))\int_{s_k}^t E\|\Delta y_j(s)\|^2 ds. \end{aligned}$$

It follows that

$$E\|\Delta y_j(t)\|^2 \leq \left(3M^2 L_{h_k} E\|y_{j+1}(t_k^-) - y_j(t_k^-)\|^2 + \rho_1 \int_{s_k}^t E\|B\Delta u_j(s)\|^2 ds\right)\exp((\rho_2 + \rho_3)(\tau - s_k)).$$

□

Theorem 4. *Assume that the conditions (A1)–(A3) hold. Under the conditions*

$$\begin{cases} \|(I - CL - D\gamma_p)\| < 1, \\ 2\|I - D\gamma_p\|^2 < 1, \end{cases} \tag{20}$$

we have

$$\lim_{j\to\infty} \|e_j\|_\lambda = 0.$$

Proof. For the tracking error, learning law (14), we have

$$\begin{aligned} e_{j+1}(0) &= z_d(0) - z_j(0) - z_{j+1}(0) + z_j(0) \\ &= e_j(0) - C\Delta y_j(0) - D\Delta u_j(0) \\ &= e_j(0) - C\Delta y_j(0) - D\gamma_p e_j(0) \\ &= (I - CL - D\gamma_p)e_j(0). \end{aligned}$$

It follows (20) that

$$\lim_{j\to\infty} \|e_{j+1}(0)\|_H \leq \lim_{j\to\infty} \|(I - CL - D\gamma_p)\|^j \|e_1(0)\| = 0. \tag{21}$$

Following the learning law (14) and the output equation for (13), for any $t \in [0, \tau]$, we have

$$\begin{aligned} e_{j+1}(t) &= z_d(t) - z_j(t) - z_{j+1}(t) + z_j(t) \\ &= e_j(t) - C\Delta y_j(t) - D\Delta u_j(t) \\ &= e_j(t) - C\Delta y_j(t) - D\gamma_p e_j(t) \\ &= (I - D\gamma_p)e_j(t) - C\Delta y_j(t). \end{aligned} \tag{22}$$

Taking the λ-norm for (22), we have

$$\|e_{j+1}\|_\lambda^2 \leq 2\|I - D\gamma_p\|^2 \|e_j\|_\lambda^2 + 2\|C\|^2 \|\Delta y_j\|_\lambda^2. \tag{23}$$

Case 1: $0 \leq t \leq t_1$:

$$\|\Delta y_j(0)\| = \|Le_j(0)\|. \tag{24}$$

From (15), it follows that

$$Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \leq \left(3M^2 e^{-\lambda t}\|\Delta y_j(0)\|^2 + \rho_1 \frac{1 - e^{-\lambda t}}{\lambda}\|B\Delta u_j\|_\lambda^2\right)\exp((\rho_2 + \rho_3)\tau)$$

Taking λ-norm on $[0, t_1]$, we obtain

$$\sup_{0\le t\le t_1} e^{-\lambda t}E\|\Delta y_j(t)\|^2 \le 3M^2\exp((\rho_2+\rho_3)\tau)\sup_{0\le t\le t_1} e^{-\lambda t}E\|\Delta y_j(t)\|^2 + \rho_1\frac{1-e^{-\lambda t}}{\lambda}\|B\Delta u_j\|_\lambda^2\exp((\rho_2+\rho_3)\tau).$$

Solving the above inequality, we have

$$\sup_{0\le t\le t_1} e^{-\lambda t}E\|\Delta y_j(t)\|^2 \le \rho_4(\lambda)\|B\gamma_p\|^2\|e_j\|_\lambda^2\exp((\rho_2+\rho_3)\tau)s.$$

Combining the expressions (21) and (23), we have

$$\begin{aligned}\sup_{0\le t\le t_1} e^{-\lambda t}E\|e_{j+1}(t)\|^2 &\le 2\|I-D\gamma_p\|^2\sup_{0\le t\le t_1} e^{-\lambda t}E\|e_j(t)\|^2 + 2\|C\|^2\sup_{0\le t\le t_1} e^{-\lambda t}E\|\Delta y_j(t)\|^2\\ &\le 2\|I-D\gamma_p\|^2\sup_{0\le t\le t_1} e^{-\lambda t}E\|e_j(t)\|^2 + 2\|C\|^2\rho_4(\lambda)\|B\gamma_p\|^2\sup_{0\le t\le t_1} e^{-\lambda t}E\|e_j(t)\|^2\exp((\rho_2+\rho_3)\tau)\\ &\le \left[2\|I-D\gamma_p\|^2 + 2\frac{1-e^{-\lambda t}}{\lambda}\|C\|^2\rho_4(\lambda)\|B\gamma_p\|^2\exp((\rho_2+\rho_3)\tau)\right]\sup_{0\le t\le t_1} e^{-\lambda t}E\|e_j(t)\|^2. \end{aligned}\tag{25}$$

For large λ and by assumption (20), the coefficient of $\sup_{0\le t\le t_1} e^{-\lambda t}E\|e_j(t)\|^2$ is less than 1. Thus, $\lim_{j\to\infty}\sup_{0\le t\le t_1} e^{-\lambda t}E\|e_{j+1}(t)\|^2 = 0$.

Case 2: if $t_k \le s \le s_k, k = 1, ..., N$,

From Lemma 1, we have

$$E\|\Delta y_j(t)\|^2 \le L_{h_k}^2 E\|y_{j+1}(t_k^-) - y_j(t_k^-)\|^2 = L_{h_k}^2 E\|\Delta y_j(t_k^-)\|^2.$$

Hence,

$$\sup_{t_k\le t\le s_k} e^{-\lambda t}E\|e_{j+1}(t)\|^2 \le 2\|I-D\gamma_p\|^2\sup_{t_k\le t\le s_k} e^{-\lambda t}E\|e_j(t)\|^2 + 2\|C\|^2\sup_{s_{k-1}\le t\le t_k} e^{-\lambda t}E\|\Delta y_j(t)\|^2. \tag{26}$$

Case 3: $s_k \le t \le t_{k+1}$; $k = 1, ..., N$: In a similar manner to (15), we obtain

$$E\|\Delta y_j(t)\|^2 \le \left(3M^2L_{h_k}E\|y_{j+1}(t_k^-) - y_j(t_k^-)\|^2 + \rho_1\frac{e^{\lambda t}-1}{\lambda}\|B\Delta u_j\|_\lambda^2\right)\exp((\rho_2+\rho_3)(\tau-s_k)).$$

It follows that

$$\sup_{s_k\le t\le t_{k+1}} e^{-\lambda t}E\|\Delta y_j(t)\|^2 \le 3M^2L_{h_k}\sup_{s_{k-1}\le t\le t_k} e^{-\lambda t}E\|\Delta y_j(t)\|^2\exp((\rho_2+\rho_3)(\tau-s_k)) + \rho_1\frac{e^{\lambda t}-1}{\lambda}\|B\Delta u_j\|_\lambda^2\exp((\rho_2+\rho_3)(\tau-s_k))$$

Taking the λ-norm on $[s_k, t_{k+1}]$, we have

$$\sup_{s_k\le t\le t_{k+1}} e^{-\lambda t}E\|e_{j+1}(t)\|^2 \le 2\|I-D\gamma_p\|^2\sup_{s_k\le t\le t_{k+1}} e^{-\lambda t}\|e_j(t)\|^2 + 2\|C\|^2\sup_{s_k\le t\le t_{k+1}} e^{-\lambda t}E\|\Delta y_j(t)\|^2. \tag{27}$$

Now, for large λ and by assumption (20), the coefficient of $\sup_{0\le t\le t_1} e^{-\lambda t}E\|e_j(t)\|^2$ in (25) is less than 1. Thus,

$$\lim_{j\to\infty}\sup_{0\le t\le t_1} e^{-\lambda t}E\|e_{j+1}(t)\|^2 = 0. \tag{28}$$

Similarly, using inequalities (26) and (27), one can see that (28) is true on every $[t_1, s_1]$, $[s_1, t_2], ..., [t_N, s_N]$, $[s_N, t_{N+1}]$. The theorem is proved. □

Secondly, concerning (13), we consider the following open-loop PI^{α}-type learning law to meet the require control function and initial state learning law:

$$\begin{cases} u_{j+1}(t) - u_j(t) = \gamma_p e_j(t) + \gamma_1 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e_j(s) ds, \ \ t \in [0, \tau], \\ y_{j+1}(0) - y_j(0) = L_2 e_j(0). \end{cases} \tag{29}$$

Next, we have the theorem related to stochastic ILC problem (13) with (29)

Theorem 5. *Assume that the conditions (A1)–(A3) hold. Under the conditions*

$$\begin{cases} \|(I - CL_2 - D\gamma_p)\| < 1, \\ 2\|I - D\gamma_p\|^2 < 1, \end{cases}$$

we have

$$\lim_{j \to \infty} \|e_j\|_{\lambda} = 0.$$

Proof. It is obvious that

$$\begin{aligned} e_{j+1}(t) &= z_d(t) - z_{j+1}(t) \\ &= z_d(t) - z_j(t) + z_j(t) - z_{j+1}(t) \\ &= e_j(t) + z_j(t) - z_{j+1}(t) \\ &= e_j(t) + [Cy_j(t) + Du_j(t)] - [Cy_{j+1}(t) + Du_{j+1}(t)] \\ &= e_j(t) - C[y_{j+1}(t) - y_j(t)] - D[u_{j+1}(t) - u_j(t)] \\ &= e_j(t) - C\Delta y_j(t) - D\Delta u_j(t) \\ &= e_j(t) - C\Delta y_j(t) - D\left[\gamma_p e_j(t) + \gamma_1 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e_j(s) ds\right] \\ &= (I - D\gamma_p)e_j(t) - C\Delta y_j(t) - D\gamma_1 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e_j(s) ds. \end{aligned}$$

For $t = 0$, we obtain

$$\begin{aligned} e_{j+1}(0) &= (I - D\gamma_p)e_j(0) - C\Delta y_j(0) \\ &= (I - D\gamma_p)e_j(0) - CL_2 e_j(0) \\ &= (I - CL_2 - D\gamma_p)e_j(0). \end{aligned}$$

Then, we have by our assumption

$$\lim_{j \to \infty} \|e_{j+1}(0)\|_H \leq \|e_1(0)\|_H \lim_{j \to \infty} \|(I - CL_2 - D\gamma_p)\|^j = 0.$$

Now, we consider

$$e_{j+1}(t) = (I - D\gamma_p)e_j(t) - C\Delta y_j(t) - D\gamma_1 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e_j(s) ds.$$

It follows that

$$\begin{aligned} \|e_{j+1}\|_{\lambda} &\leq \|I - D\gamma_p\| \|e_j\|_{\lambda} + \|C\| \|\Delta y_j\|_{\lambda} + \frac{\|D\| \|\gamma_1\|}{\lambda^{\alpha}} \|e_j\|_{\lambda} \\ &= \left(\|I - D\gamma_p\| + \frac{\|D\| \|\gamma_1\|}{\lambda^{\alpha}}\right) \|e_j\|_{\lambda} + \|C\| \|\Delta y_j\|_{\lambda}, \end{aligned}$$

and

$$\|e_{j+1}\|_\lambda^2 \le 2\left(\|I - D\gamma_p\| + \frac{\|D\|\|\gamma_1\|}{\lambda^\alpha}\right)^2 \|e_j\|_\lambda^2 + 2\|C\|^2\|\Delta y_j\|_\lambda^2 \tag{30}$$

Case 1: $0 \le t \le t_1$:

$$E\|\Delta y_j(t)\|^2 \le \left(3M^2\|\Delta y_j(0)\|^2 + \rho_1 \frac{e^{\lambda t} - 1}{\lambda} E\|B\Delta u_j(t)\|^2\right) e^{(\rho_2+\rho_3)\tau}$$

Multiplying by $e^{-\lambda t}$ both sides of the inequality above

$$Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \le \left(3M^2 e^{-\lambda t}\|\Delta y_j(0)\|^2 + \rho_1 \frac{1 - e^{-\lambda t}}{\lambda} \|B\Delta u_j(t)\|_\lambda^2\right) e^{(\rho_2+\rho_3)\tau}$$

Taking λ-norm on $[0, t_1]$, we obtain

$$\begin{aligned}
&\sup_{0\le t\le t_1} Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \\
&\le 3M^2 e^{(\rho_2+\rho_3)\tau} \sup_{0\le t\le t_1} Ee^{-\lambda t}\|\Delta y_j(t)\|^2 + \rho_1 \frac{1 - e^{-\lambda t}}{\lambda}\|B\Delta u_j\|_\lambda^2 e^{(\rho_2+\rho_3)\tau} \\
&\Longrightarrow \left(1 - 3M^2 e^{(\rho_2+\rho_3)\tau}\right) \sup_{0\le t\le t_1} Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \le \rho_1 \frac{1 - e^{-\lambda t}}{\lambda}\|B\Delta u_j\|_\lambda^2 e^{(\rho_2+\rho_3)\tau} \\
&\Longrightarrow \sup_{0\le t\le t_1} Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \le \rho_6 \frac{1 - e^{-\lambda t}}{\lambda}\|B\Delta u_j\|_\lambda^2 e^{(\rho_2+\rho_3)\tau}
\end{aligned}$$

where $\rho_6 = \frac{\rho_1}{1 - 3M^2 e^{(\rho_2+\rho_3)\tau}}$.

From

$$\Delta u_j(t) = \gamma_p e_j(t) + \gamma_1 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e_j(s) ds,$$

we obtain

$$\begin{aligned}
\|\Delta u_j\|_\lambda &\le \|\gamma_p\|\|e_j\|_\lambda + \frac{\|\gamma_1\|\|e_j\|_\lambda}{\lambda^\alpha} \\
&= \left(\|\gamma_p\| + \frac{\|\gamma_1\|}{\lambda^\alpha}\right)\|e_j\|_\lambda
\end{aligned}$$

Using the last inequality in the previous inequality, we have

$$\sup_{0\le t\le t_1} Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \le \rho_6 \frac{1 - e^{-\lambda t}}{\lambda}\|B\|^2\left(\|\gamma_p\| + \frac{\|\gamma_1\|}{\lambda^\alpha}\right)^2 \|e_j\|_\lambda^2 e^{(\rho_2+\rho_3)\tau}$$

Using (30), we obtain

$$\|e_{j+1}\|_\lambda^2 \le 2\left(\|I - D\gamma_p\| + \frac{\|D\|\|\gamma_1\|}{\lambda^\alpha}\right)^2 \|e_j\|_\lambda^2 + 2\|C\|^2 E\|\Delta y_j\|_\lambda^2,$$

$$\begin{aligned}
\|e_{j+1}\|_\lambda^2 &\le 2\left(\|I - D\gamma_p\| + \frac{\|D\|\|\gamma_1\|}{\lambda^\alpha}\right)^2 E\|e_j\|_\lambda^2 \\
&\quad + 2\|C\|^2 \rho_6 \frac{1 - e^{-\lambda t}}{\lambda}\|B\|^2\left(\|\gamma_p\| + \frac{\|\gamma_1\|}{\lambda^\alpha}\right)^2 e^{(\rho_2+\rho_3)\tau} E\|e_j\|_\lambda^2,
\end{aligned}$$

$$\|e_{j+1}\|_\lambda^2 \le \Bigg[2\bigg(\|I - D\gamma_p\| + \frac{\|D\|\|\gamma_1\|}{\lambda^\alpha}\bigg)^2 + 2\|C\|^2\rho_6\frac{1-e^{-\lambda t}}{\lambda}\|B\|^2\bigg(\|\gamma_p\| + \frac{\|\gamma_1\|}{\lambda^\alpha}\bigg)^2 e^{(\rho_2+\rho_3)\tau}\Bigg]E\|e_j\|_\lambda^2,$$

we obtain

$$\|e_{j+1}\|_\lambda^2 \le 2\bigg(\|I - D\gamma_p\| + \frac{\|D\|\|\gamma_1\|}{\lambda^\alpha}\bigg)^2\|e_j\|_\lambda^2,$$

and clearly we have

$$\lim_{j\to\infty}\|e_{j+1}\|_\lambda^2 = 0.$$

Case 3: $s_k \le t \le t_{k+1}$; $k = 1, ..., N$: From (1) again, we consider

$$E\|\Delta y_j(t)\|^2 \le \bigg(3M^2L_{h_k}\|\Delta y_j(t_k^-)\|^2 + \rho_1\frac{e^{\lambda t}-1}{\lambda}\|B\Delta u_j(t)\|^2\bigg)e^{(\rho_2+\rho_3)(\tau-s_k)}$$

Multiplying by $e^{-\lambda t}$ both sides of the inequality above

$$Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \le \bigg(3M^2L_{h_k}e^{-\lambda t}\|\Delta y_j(t_k^-)\|^2 + \rho_1\frac{1-e^{-\lambda t}}{\lambda}\|B\Delta u_j(t)\|_\lambda^2\bigg)e^{(\rho_2+\rho_3)(\tau-s_k)}.$$

Taking λ-norm on $[s_k, t_{k+1}]$, we obtain

$$\sup_{s_k\le t\le t_{k+1}} Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \le 3M^2L_{h_k}e^{(\rho_2+\rho_3)(\tau-s_k)}\sup_{s_k\le t\le t_{k+1}} Ee^{-\lambda t}\|\Delta y_j(t)\|^2 + \rho_1\frac{1-e^{-\lambda t}}{\lambda}\|B\Delta u_j\|_\lambda^2 e^{(\rho_2+\rho_3)(\tau-s_k)}$$

Solving the last inequality

$$\Rightarrow \Big(1 - 3M^2L_{h_k}e^{(\rho_2+\rho_3)(\tau-s_k)}\Big)\sup_{s_k\le t\le t_{k+1}} Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \le \rho_1\frac{1-e^{-\lambda t}}{\lambda}\|B\Delta u_j\|_\lambda^2 e^{(\rho_2+\rho_3)(\tau-s_k)}$$

$$\sup_{s_k\le t\le t_{k+1}} Ee^{-\lambda t}\|\Delta y_j(t)\|^2 \le \rho_7\frac{1-e^{-\lambda t}}{\lambda}\|B\Delta u_j\|_\lambda^2 e^{(\rho_2+\rho_3)(\tau-s_k)}$$

where $\rho_7 = \frac{\rho_1}{1-3M^2L_{h_k}e^{(\rho_2+\rho_3)(\tau-s_k)}}$.

Employing this in (30) shifting the intervals, we gain

$$\begin{aligned}
&\sup_{s_k\le t\le t_{k+1}} e^{-\lambda t}E\|e_{j+1}(t)\|^2 \\
&\le 2\bigg(\|I - D\gamma_p\| + \frac{\|D\|\|\gamma_1\|}{\lambda^\alpha}\bigg)^2\sup_{s_k\le t\le t_{k+1}} E\|e_j(t)\|^2 \\
&+ 2\|C\|^2\sup_{s_k\le t\le t_{k+1}} E\|\Delta y_j(t)\|^2 \\
&\le 2\bigg(\|I - D\gamma_p\| + \frac{\|D\|\|\gamma_1\|}{\lambda^\alpha}\bigg)^2\sup_{s_k\le t\le t_{k+1}} E\|e_j(t)\|^2 \\
&+ 2\|C\|^2\rho_6\frac{1-e^{-\lambda t}}{\lambda}\|B\|^2\bigg(\|\gamma_p\| + \frac{\|\gamma_1\|}{\lambda^\alpha}\bigg)^2 e^{(\rho_2+\rho_3)(\tau-s_k)}\sup_{s_k\le t\le t_{k+1}} e^{-\lambda t}E\|e_j(t)\|^2,
\end{aligned}$$

and therefore

$$\sup_{s_k \le t \le t_{k+1}} e^{-\lambda t} E\|e_{j+1}\|^2$$
$$\le \left[2\left(\|I - D\gamma_p\| + \frac{\|D\|\|\gamma_1\|}{\lambda^\alpha}\right)^2 + 2\|C\|^2 \rho_7 \frac{1-e^{-\lambda t}}{\lambda}\|B\|^2\left(\|\gamma_p\| + \frac{\|\gamma_1\|}{\lambda^\alpha}\right)^2 e^{(\rho_2+\rho_3)(\tau - s_k)}\right] \sup_{s_k \le t \le t_{k+1}} e^{-\lambda t} E\|e_j\|^2,$$

and, as a result of choosing a sufficiently large λ, we obtain

$$\lim_{j\to\infty} \sup_{s_k \le t \le t_{k+1}} e^{-\lambda t} E\|e_{j+1}\|_\lambda^2 = 0.$$

□

Lastly, concerning (13), we consider the following open-loop D-type learning law to meet the require control function and initial state learning law:

$$\Delta y_j(0) = L_3 e_j(0),\ \Delta u_j(t) = \gamma_d\, e'_j(t).$$

The next theorem is related to stochastic ILC problem (13) with (29) and output Equation (4).

Theorem 6. *Assume that the conditions (A1)–(A3) hold. Under the conditions*

$$\begin{cases} \|(I - CL_3)\| < 1, \\ \|I - D\gamma_d\| < 1, \end{cases}$$

we have

$$\lim_{j\to\infty} \|e_j\|_\lambda = 0.$$

The proof is similar to that of Theorem 5 and omitted.

6. Example

Example 1. Consider the following fractional stochastic partial integro-differential equations with noninstantaneous impulses of the form

$$\begin{cases} {}^C D_{s_1}^{\frac{2}{3}} y(t,\theta) = y_{\theta\theta}(t,\theta) + \mu(t,\theta) + K_1(t, y(t,\theta)) + \int_0^t K_2(s, y(s,\theta))dw(s), (t,\theta) \in (s_1, t_2] \times [0,\pi], \\ y(t,\theta) = H_1(t, y(t_1^-, \theta)), \quad 0 < \theta < \pi, \quad t_1 \le t < s_1 \\ y(t,0) = y(t,\pi) = 0,\ 0 \le t \le 1, \\ y(0,\theta) = y_0(\theta),\ 0 < \theta < \pi, \end{cases} \tag{31}$$

and

$$z(t,\theta) = cy(t,\theta) + du(t,\theta), \quad c,d \in \mathbb{R}^+,\ t \in [0,1],\ \theta \in (0,\pi) \tag{32}$$

or

$$z(t,\theta) = cy(t,\theta) + d\int_0^\pi \int_0^t u(s,\theta) ds d\theta, \quad t \in [0,1],\ \theta \in (0,\pi) \tag{33}$$

where $w(t)$ denotes a standard real valued Wiener process on $(\Omega, F, \{F_t\}, P)$ and $y_0 \in L^2(0,\pi)$; $\mu : [0,1] \times (0,\pi) \to (0,1)$ is continuous in t; $K_1, K_2 : R \to R$ is continuous. Let $Z = H = U = L^2(0,1)$, $\tau = 1$, $N = 1$, $t_1 = 0$, $s_1 = \frac{1}{4}$, $t_2 = 1$ and $\alpha = \frac{2}{3}$. Define the operator $A : H \to H$ by $Ay = \frac{\partial^2}{\partial\theta^2} y = y_{\theta\theta}$ with domain $D(A) = \{y \in H, y, y_\theta$ being absolutely continuous $, y_{\theta\theta} \in H,\ y(t,0) = y(t,\pi) = 0\}$. Then, A can be expressed as

$$Ay = -\sum_{n=1}^{\infty} n^2 < y, e_n > e_n, \quad y \in H.$$

where $e_n(\theta) = \sqrt{\frac{2}{\pi}}\sin(n\theta), n = 1, 2, \ldots$ is a complete orthonormal set of eigenvectors of A. In addition, $-A$ generates an analytic C_0-semigroup $\{S(t), t \geq 0\}$ given by

$$S(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} < y, e_n > e_n, \quad y \in H, \text{ with } \|S(t)\|_{L(H,H)} \leq e^{-t} \leq 1 = M$$

It follows that $\{S(t), t > 0\}$ is uniformly bounded. Then, one can write the known operators $\mathfrak{S}_\alpha(t)$ and $\tau_\alpha(t)$ as

$$\mathfrak{S}_{\frac{2}{3}}(t) := \int_0^\infty \xi_{\frac{2}{3}}(\theta) S\left(t^{\frac{2}{3}}\theta\right) d\theta$$

$$\mathfrak{T}_{\frac{2}{3}}(t) := \frac{2}{3}\int_0^\infty \theta \xi_{\frac{2}{3}}(\theta) S\left(t^{\frac{2}{3}}\theta\right) d\theta$$

Then, we readily obtain $\left\|\mathfrak{S}_{\frac{2}{3}}(t)\right\|_{L(H,H)} \leq 1$ and $\left\|\mathfrak{T}_{\frac{2}{3}}(t)\right\|_{L(H,H)} \leq \frac{1}{\Gamma(\frac{2}{3})}$ for $t \in [0,1]$.

Let $y(t)(\theta) = y(t,\theta)$ and define the bounded linear operator $B : U \to H$ by $Bu(t)(\theta) = \mu(t,\theta), \ 0 \leq \theta \leq 1$. Furthermore, define $f(t, y(t))(\theta) = K_1(t, y(t,\theta)) = \eta_1 \sin(y(t,\theta))$, $g(t, y(t))(\theta) = K_2(t, y(t,\theta)) = \eta_2 \sin(y(t,\theta))$ and $h_1(t, y(t))(\theta) = H(t, y(t,\theta)) = \eta_3 y(t,\theta)$ where $\eta_i, i = 1,2,3 \in R^+$. Then, with these choices, system (31) can be written in the abstract form of (1). Thus, the conditions (A1)–(A3) and (8) are satisfied. Hence, by Theorem 1, the stochastic control integro-differential system (31) is approximately controllable on $[0,1]$.

Denote $z(t)(\theta) = z(t,\theta)$ and take $C = cI_\gamma$, and $D = dI_\gamma$. Then, Equations (32) and (33) can be rewritten as (2) and (3), respectively. Thus, $(1 - c - d)I_\gamma \in L(Z,Z)$ and $(1 - d)I_\gamma \in L(Z,Z)$. Set $L_1 = L_2 = L_3 = L_H \in L(Z,H)$, $\gamma_1 = \gamma_p = \gamma_l = \gamma_d \in L(Z,U)$. If $1 > c + d > 0$ and $d > \frac{1}{2}$; then, the statements of Theorem 4 and 5 hold. Thus, the mentioned theorems guarantee that z_j tends to z_d as $j \to \infty$, or, if $1 > \max\{c,d\} > 0$, then the conditions of Theorem 6 hold. This theorem gives that z_j tends to z_d as $j \to \infty$ too.

Example 2. As a second example, consider the following iterated control system of the fractional stochastic partial integro-differential equations with noninstantaneous impulses of the form

$$\begin{cases} D^\alpha y_j(t,\theta) = \frac{\partial^2}{\partial\theta^2} y_j(t,\theta) + \frac{1}{5}e^{-2\theta} + \dfrac{|y_j(t,\theta)|}{|y_j(t,\theta)| + 5}, & 0 < t \leq 0.6 \text{ or } 0.9 \leq t < 0.5 \\ y_j(t,\theta) = 0.1\int_0^1 cos(z)\ln(1 + zcos(t - 0.6)|y_j(t_i^-,\theta)|)dz, & 0.6 < t < 0.9, \ 0 < \theta < 1 \\ y_j(t,0) = y_j(t,1), \quad 0 \leq t \leq 1.5, \\ y(0,\theta) = y_0(\theta) = 0, \quad 0 \leq \theta \leq 1 \end{cases} \tag{34}$$

and

$$z_j(t,\theta) = 0.5y_j(t,\theta) + 0.8u_j(t,\theta) \ \ 0 \leq t \leq 1.5, \ \ 0 \leq \theta \leq 1.$$

and

$$u_{j+1}(t,\theta) = u_j(t,\theta) + e_j(t,\theta) \ \ 0 \leq t \leq 1.5, \ \ 0 \leq \theta \leq 1.$$

Let $Z = H = U = L^2(0,1)$. Define the operator $A : H \to H$ by $Ay = \frac{\partial^2}{\partial\theta^2} y = y_{\theta\theta}$ with $D(A) = \{y \in H, y, y_\theta$ are absolutely continuous $, y_{\theta\theta} \in H, \ y(t,0) = y(t,\pi) = 0\}$. Then, A can be expressed as

$$Ay = -\sum_{n=1}^{\infty} n^2 < y, e_n > e_n, \quad y \in H.$$

where $e_n(\theta) = \sqrt{\frac{2}{\pi}}\sin(n\theta), n = 1, 2, \ldots$ is a complete orthonormal set of eigenvectors of A. In addition, $-A$ generates an analytic C_0-semigroup $\{S(t), t \geq 0\}$ given by

$$S(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} < y, e_n > e_n, \quad y \in H, \text{ with } \|S(t)\|_{L(H,H)} \leq e^{-t} \leq 1 = M$$

It follows that $\{S(t), t > 0\}$ is uniformly bounded. Then, one can write the known operators $\mathfrak{S}_\alpha(t)$ and $\tau_\alpha(t)$ as

$$\mathfrak{S}_\alpha(t)y := \int_0^\infty \xi_\alpha(\theta)S(t^\alpha\theta)d\theta := \sum_{n=1}^{\infty} E_\alpha\left(-n^2t^\alpha\right) < y, e_n > e_n$$

$$\mathfrak{T}_\alpha(t)y := \alpha\int_0^\infty \theta\xi_\alpha(\theta)S(t^\alpha\theta)d\theta := \sum_{n=1}^{\infty} \mathbb{E}_{\alpha,\alpha}\left(-n^2t^\alpha\right) < y, e_n > e_n$$

Then, we readily obtain $\|\mathfrak{S}_\alpha(t)\|_{L(H,H)} \leq 1$ and $\|\mathfrak{T}_\alpha(t)\|_{L(H,H)} \leq \frac{1}{\Gamma(\alpha)}$ for $t \in [0, 1.5]$, then we put $M_S = 1$, $M_\tau = \frac{1}{\Gamma(\alpha)}$. Define $f : [0, 1.5] \times H \to H$ by $f(t, y(t,\theta)) = \frac{1}{5}e^{-2\theta} + \frac{|y_j(t,\theta)|}{|y_j(t,\theta)|+5}$ and $B : U \to H$ by $B = 0.25I$, and $h_k : [0, 1.5] \times H \to H$ by $h_k\big(t, y_j(t_i^-, \theta)\big) = 0.1\int_0^1 cos(z)\ln(1 + zcos(t - 0.6)\left|y_j(t_i^-, \theta)\right|)dz$ Clearly, $\|f(t, y)\|_H \leq 0.2(1 + \|y\|_H)$ and $\|f(t, y_1) - f(t, y_2)\|_H \leq 0.2\|y_1 - y_2\|_H$. Then, we set $M_f = L_f = 0.2$. $\|h_k(t, y)\|_H \leq 0.1(1 + \|y\|_H)$ and $\|h_k(t, y_1) - h_k(t, y_2)\|_H \leq 0.1\|y_1 - y_2\|_H$ and then we take $M_{h_k} = L_{h_k} = 0.1$, $C = 0.5I$, $D = 0.8I$. Then, (A1)–(A3) hold.

If L_{c_α} is L_c of Theorem 1 with respect to α, $L_{c_{0.5}} = 0.0167, L_{c_{0.3}} = 0.0144, L_{c_{0.1}} = 0.0136 < 1$, then the conditions of Theorem 1 hold, so it has a unique solution by Theorem 1. It is easy to check that $\left\|I - CL_i - D\gamma_p\right\| = |1 - 0.5 - 0.8| = 0.3 < 1$, $2\left\|I - D\gamma_p\right\|^2 = 2\left|1 - 0.8\gamma_p\right|^2 = 0.08 < 1$, $\|I - CL_3\| = |1 - 0.5| = 0.5 < 1$, $\left\|I - D\gamma_p\right\| = |1 - 0.8| = 0.2 < 1$—hence all conditions of Theorems 4–6.

7. Conclusions

Existence uniqueness of solutions and the approximate controllability concept for Caputo type SFIDEs in a Hilbert space with a noninstantaneous impulsive effect are studied. The sufficient conditions for existence uniqueness and approximate controllability are proved. Moreover, the stochastic ILC problem has been addressed in this paper for SFIDEs with a noninstantaneous impulsive effect. A different type stochastic ILC such as P-type, PI^α-type and D-type iterative learning schemes are proposed with an initial state learning mechanism. The sufficient conditions for guaranteeing the asymptotical convergence are provided and proved.

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