fractal and fractional

Article

Existence of Mild Solution of the Hilfer Fractional Differential Equations with Infinite Delay on an Infinite Interval

Chandrabose Sindhu Varun Bose [1], Ramalingam Udhayakumar [1,*], Milica Savatović [2,*], Arumugam Deiveegan [3], Vesna Todorčević [4] and Stojan Radenović [5]

[1] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632014, Tamil Nadu, India; varunbose.cs2020@vitstudent.ac.in
[2] School of Electrical Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia
[3] Department of Mathematics, Sona College of Technology, Salem 636005, Tamilnadu, India; deiveegan.a@sonatech.ac.in
[4] Mathematical Institute of the Serbian Academy of Sciences and Arts, Department of Mathematics, Faculty of Organizational Sciences, University of Belgrade, Jove Ilica 154, 11000 Belgrade, Serbia; vesna.todorcevic@fon.bg.ac.rs
[5] Faculty of Mechanical Engineering, University of Belgrade, 11000 Belgrade, Serbia; sradenovic@mas.bg.ac.rs
* Correspondence: udhayakumar.r@vit.ac.in (R.U.); milica.makragic@etf.rs (M.S.)

Abstract: In this study, we present a mild solution to the Hilfer fractional differential equations with infinite delay. Firstly, we establish the results on an infinite interval; to achieve this, we use the generalized Ascoli–Arzelà theorem and Mönch's fixed point theorem via a measure of noncompactness. Secondly, we consider the existence of a mild solution when the semigroup is compact, and the Schauder fixed-point theorem is used. The outcome is demonstrated using an infinitesimal operator, fractional calculus, semigroup theory, and abstract space. Finally, we present an example to support the results.

Keywords: Hilfer fractional derivative; mild solution; fixed-point theorem; infinite interval

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1. Introduction

In many physical processes, fractional calculus with many fractional derivatives (FD_{ve}) is highly concentrated. The fractional differential (FD_{tial}) system has recently attracted a great deal of attention due to its range of wondrous scientific and technological applications. Fractional systems may be used to solve a wide range of issues in various fields, including viscoelasticity, electrical systems, electrochemistry, fluid flow, etc. Differential inclusions, which are an extension of differential equations and inequalities and may be regarded of as a branch of control theory, have several potential applications. When one is adept at employing differential inclusions, dynamical systems with velocities that are not solely determined by the system's state are easier to analyze. Numerous studies have been undertaken to investigate boundary value problems. Additionally, several investigations have been conducted to determine if there are solutions that are applicable to FD_{tial} systems as well as FD_{tial}. In [1], the author established the concepts of semigroup theory, infinitesimal generator, and the abstract Cauchy problem. Meanwhile, researchers presented basic ideas and results related to fractional calculus and their applications [2–5]. In [6], the authors established their results related to various fractional differential systems. Meanwhile, several research papers [7–9] validate the discussion of this theory and its applications in fractional calculus.

A newly developed fractional derivative known as the Hilfer fractional derivative, which includes both the Caputo fractional and the Riemann–Liouville fractional derivatives, was proposed by Hilfer [10]. The article [11] began by determining a mild solution to the Hilfer fractional differential equations via the Laplace transform and fixed-point method.

In several recent articles [12–14], the existence and the controllability of the Hilfer fractional differential systems via fixed-point approach have been analyzed. In the article in [15], the Hilfer fractional differential system with almost sectorial operators is explained.

Recent research on fractional differential systems has predominantly focused on the existence of solutions in the limited interval [0,b]. Various fixed-point theorems and the Ascoli–Arzelà theorem are frequently used in this research. The traditional Ascoli–Arzelà theorem is a well-known technique that provides necessary and sufficient conditions to determine how abstract continuous functions relate to one another; however, it is only applicable to finite closed intervals. But, in [16], the author studied the existence of a mild solution to the Hilfer fractional differential system on a semi-infinite interval via the generalized Ascoli–Arzelà theorem. Additionally, in [17], the researchers established the existence of mild solutions for the system of Hilfer fractional derivatives (HFD_{ve}) on an infinite interval. The generalized Ascoli–Arzelà theorem and the fixed-point theorem were used to prove the existence of the mild solution.

Our article's significant contributions are as follows:

(i) For the Hilfer fractional differential system, we show the necessary and sufficient conditions for the mild solution's existence.
(ii) In this work, we study when a fractional differential system (1) has a mild solution on the infinite interval $(0, +\infty)$.
(iii) Our system (1) is defined by an infinite delay.
(iv) We show that our result is consistent with the concept of the generalized Ascoli–Arzelà theorem (8).
(v) We begin by proving the existence of the system via the measure of noncompactness by using the *Mönch's* fixed-point theorem (7).
(vi) Next, we prove the existence of a mild solution to the system for a compact semigroup. Schauder's fixed-point theorem is used in this condition.
(vii) Finally, an example is presented to illustrate the results.

In this study, by applying the generalized Ascoli–Arzelà theorem and some novel approaches, we establish the existence of mild solutions in an infinite interval via a measure of noncompactness (MNC). Consider the following system:

$$\begin{cases} {}^{H}D_{0^+}^{\mathfrak{h},\mathfrak{q}}\mathtt{x}(\mathfrak{a}) = \mathsf{A}\mathtt{x}(\mathfrak{a}) + \mathsf{F}(\mathfrak{a}, \mathtt{x}_{\mathfrak{a}}), \quad \mathfrak{a} \in (0, +\infty), \\ I_{0^+}^{(1-\mathfrak{h})(1-\mathfrak{q})}\mathtt{x}(\mathfrak{a}) = \phi(\mathfrak{a}) \in \mathsf{S}_{\lambda}, \ \mathfrak{a} \in (-\infty, 0], \end{cases} \tag{1}$$

where ${}^{H}D_{0^+}^{\mathfrak{h},\mathfrak{q}}$ is the HFD_{ve} of order $0 < \mathfrak{h} < 1$ and type $0 \leq \mathfrak{q} \leq 1$, A is the infinitesimal generator in Banach space Y, and $\mathsf{F} : [0, \infty) \times \mathsf{S}_{\lambda} \to Y$ is a function.

This paper is organized as follows: The principles of fractional calculus, abstract spaces, and semigroup are described in Section 2. In Section 3, we begin by proving the existence of the mild solution using MNC. We analyze a scenario in which the semigroup is compact and demonstrate the existence of the mild solution in Section 3.2. In Section 4, we provide an example to highlight our key principles. The final section contains the conclusions.

2. Preliminaries

We begin by defining the key concepts, theorems, and lemmas that are used throughout the whole article.

Consider Y as a Banach space, with the norm $|\cdot|$. Let $\mathtt{I} = [0, +\infty)$ and $C(\mathtt{I}, \mathsf{Y})$ be the collection of all continuous functions from $\mathtt{I}$ into Y. Now, we express

$$C_{e^{\mathfrak{a}}}(\mathtt{I}, \mathsf{Y}) = \left\{ \mathtt{x} \in C(\mathtt{I}, \mathsf{Y}) : \lim_{\mathfrak{a} \to +\infty} e^{-\mathfrak{a}} |\mathtt{x}(\mathfrak{a})| = 0 \right\}, \tag{2}$$

where $\|\mathtt{x}\|_{e^{\mathfrak{a}}} = \sup_{\mathfrak{a} \in \mathtt{I}} e^{-\mathfrak{a}} |\mathtt{x}(\mathfrak{a})| < +\infty$, which implies that $C_{e^{\mathfrak{a}}}(\mathtt{I}, \mathsf{Y})$ is a Banach space.

Referring to the article in [18], next, we introduce an abstract phase space $\mathtt{S}_\lambda$. Let $\lambda : (-\infty, 0] \to (0, +\infty)$ be continuous along $\Omega = \int_{-\infty}^{0} \lambda(\mathfrak{a})d\mathfrak{a} < +\infty$. Now, for every $\mathtt{k} > 0$, we have

$$\mathtt{S} = \Big\{\delta : [-\mathtt{k}, 0] \to \mathsf{Y} \ : \ \delta(\mathfrak{a}) \text{ is bounded and measurable}\Big\},$$

and take the space $\mathtt{S}$ with the norm

$$\|\delta\|_{[-\mathtt{k},0]} = \sup_{\mathfrak{a}\in[-\mathtt{k},0]} \|\delta(\mathfrak{a})\|, \text{ for every } \delta \in \mathtt{S}.$$

Next, we set

$$\mathtt{S}_\lambda = \Big\{\delta : (-\infty, 0] \to \mathsf{Y} \,\big|\, \text{for all } \mathtt{k} > 0,\ \delta|_{[-\mathtt{k},0]} \in \mathtt{S} \textit{ and } \int_{-\infty}^{0} \lambda(s)\|\delta\|_{[s,0]}ds < +\infty\Big\}.$$

If $\mathtt{S}_\lambda$ is endowed with

$$\|\delta\|_\Omega = \int_{-\infty}^{0} \lambda(\mathfrak{a})\|\delta\|_{[\mathfrak{a},0]}d\mathfrak{a}, \text{for every } \delta \in \mathtt{S}_\lambda;\text{ thus, } \big(\mathtt{S}_\lambda, \|\cdot\|_\Omega\big) \text{ is a Banach space.}$$

Next, we define the set

$$\mathtt{S}'_\lambda = \Big\{\mathtt{x} \in \mathsf{C}\big(\mathbb{R}, \mathsf{Y}\big) : \lim_{\mathfrak{a}\to+\infty} e^{-\mathfrak{a}}|x(\mathfrak{a})| = 0\Big\}.$$

Let $\|\cdot\|'_\Omega$ in $\mathtt{S}'_\lambda$ be the seminorm defined as

$$\|\mathtt{x}\|'_\Omega = \|\phi\|_\Omega + \sup\big\{\|\mathtt{x}(\mathfrak{a})\| : \mathfrak{a} \in (0, +\infty)\big\},\ \mathtt{x} \in \mathtt{S}'_\lambda.$$

Lemma 1 ([18]). *If $\mathtt{x} \in \mathtt{S}'_\lambda$, then for $\mathfrak{a} \in \mathtt{I}$ $\mathtt{x}_\mathfrak{a} \in \mathtt{S}_\lambda$. Furthermore,*

$$\Omega|\mathtt{x}(\mathfrak{a})| \le \|\mathtt{x}_\mathfrak{a}\|_\Omega \le \|\phi\|_\Omega + \Omega \sup_{r\in[0,\mathfrak{a}]} |\mathtt{x}(r)|, \quad \Omega = \int_{-\infty}^{0} \lambda(\mathfrak{a})d\mathfrak{a} < +\infty.$$

Lemma 2 ([1] Hille–Yosida Theorem). *The linear operator A is the infinitesimal generator of a C_0 semigroup $\{\mathsf{T}(\mathfrak{a}),\ \mathfrak{a} \ge 0\}$ in Banach space Y if and only if*

(i) $\mathtt{A}$ *is closed and* $\overline{D(\mathtt{A})} = Y$,

(ii) $\rho(\mathtt{A})$ *is the resolvent set of* $\mathtt{A}$ *contains* $\mathbb{R}^+$ *and, for every* $\lambda > 0$*, it holds that*

$$\big\|R(\lambda, \mathtt{A})\big\| \le \frac{1}{\lambda},$$

where $R(\lambda, \mathtt{A}) = (\lambda I - \mathtt{A})^{-1}$ *and* $R(\lambda)z = \int_0^\infty e^{-\lambda z}\mathtt{T}(z)zdz$.

Lemma 3 ([11]). *The HFD_{tial} system (1) is identical to the integral equation*

$$\mathtt{x}(\mathfrak{a}) = \frac{\phi(\mathfrak{a})}{\Gamma(\mathfrak{q}(1-\mathfrak{h}) + \mathfrak{h})}\mathfrak{a}^{(\mathfrak{q}-1)(1-\mathfrak{h})} + \frac{1}{\Gamma(\mathfrak{h})}\int_0^{\mathfrak{a}} (\mathfrak{a} - \mathfrak{s})^{\mathfrak{h}-1}\big[\mathsf{A}\mathtt{x}(\mathfrak{s}) + \mathsf{F}\big(\mathfrak{s}, \mathtt{x}_\mathfrak{s}\big)\big]d\mathfrak{s}, \quad \mathfrak{a} \in \mathtt{I}.$$

Definition 1. *A function $\mathtt{x} \in \mathsf{C}(\mathbb{R}, \mathsf{Y})$ is a mild solution to the system (1), which satisfies*

$$\mathtt{x}(\mathfrak{a}) = \mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a})\phi_0 + \int_0^{\mathfrak{a}} \mathsf{P}_\mathfrak{h}(\mathfrak{a} - \mathfrak{s})\mathsf{F}\big(\mathfrak{s}, \mathtt{x}_\mathfrak{s}\big)d\mathfrak{s},\ \mathfrak{a} \in (0, +\infty), \tag{3}$$

where $\mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a}) = I_{0^+}^{\mathfrak{h}(1-\mathfrak{q})}\mathsf{P}_\mathfrak{h}(\mathfrak{a}), \quad \mathsf{P}_\mathfrak{h} = \mathfrak{a}^{\mathfrak{h}-1}\mathsf{Q}_\mathfrak{h}$*, and* $\mathsf{Q}_\mathfrak{h}(\mathfrak{a}) = \int_0^\infty \mathfrak{h}\theta\mathsf{W}_\mathfrak{h}(\theta)\mathsf{T}(\mathfrak{a}^\mathfrak{h}\theta)d\theta$.

Lemma 4 ([11]). *If $\{\mathsf{T}(\mathfrak{a}),\ \mathfrak{a} > 0\}$ is a compact operator, then $\mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a})$ and $\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a})$ are also compact operators.*

Lemma 5 ([11]). *For any fixed $\mathfrak{a} > 0$, $\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a})$, $\mathsf{P}_{\mathfrak{h}}(\mathfrak{a})$ and $\mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a})$ are linear operators, i.e., for every $\mathtt{x} \in \mathsf{Y}$,*

$$\left|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a})\mathtt{x}\right| \leq \frac{\mathsf{L}'}{\Gamma(\mathfrak{h})}|\mathtt{x}|, \quad \left|\mathsf{P}_{\mathfrak{h}}(\mathfrak{a})\mathtt{x}\right| \leq \frac{\mathsf{L}'}{\Gamma(\mathfrak{h})}\mathfrak{a}^{\mathfrak{h}-1}|\mathtt{x}| \text{ and } \left|\mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a})\mathtt{x}\right| \leq \frac{\mathsf{L}'}{\Gamma(\mathfrak{q}(1-\mathfrak{h})+\mathfrak{q})}\mathfrak{a}^{(1-\mathfrak{h})(\mathfrak{q}-1)}|\mathtt{x}|.$$

Lemma 6 ([11]). *Suppose $\{\mathsf{T}(\mathfrak{a}),\ \mathfrak{a} > 0\}$ is equicontinuous, then the operators $\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a})$, $\mathsf{P}_{\mathfrak{h}}(\mathfrak{a})$ and $\mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a})$ are strongly continuous, i.e., for every $\mathtt{x} \in \mathsf{Y}$ and $\mathfrak{a}_2 > \mathfrak{a}_1$, it holds*

$$\begin{aligned}
&\left|\mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a}_2)\mathtt{x} - \mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a}_1)\mathtt{x}\right| \to 0,\\
&\left|\mathsf{P}_{\mathfrak{h}}(\mathfrak{a}_2)\mathtt{x} - \mathsf{P}_{\mathfrak{h}}(\mathfrak{a}_1)\mathtt{x}\right| \to 0, \left|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2)\mathtt{x} - \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1)\mathtt{x}\right| \to 0, \text{ as } \mathfrak{a}_2 \to \mathfrak{a}_1.
\end{aligned}$$

Definition 2. *The Hausdorff measure of noncompactness $\mu(\cdot)$ is defined as $\mu(\mathsf{O}) = \inf\{\theta > 0 :$ O can be covered by a finite number of balls with radii $\theta\}$, where $\mathsf{O} \subset \mathsf{Y}$.*

Theorem 1 ([19]). *If $\{\mathtt{x}_k\}_{k=1}^{+\infty}$ is a set of Bochner integrable functions from $\mathtt{I}$ to Y with the estimate property, $\|\mathtt{x}_k(\mathfrak{a})\| \leq \mu_1(\mathfrak{a})$ for almost all $\mathfrak{a} \in \mathtt{I}$ and every $k \geq 1$, where $\mu_1 \in L^1(\mathtt{I},\mathbb{R})$, then the function $\varphi(\mathfrak{a}) = \mu(\{\mathtt{x}_k(\mathfrak{a}) : k \geq 1\})$ is in $L^1(\mathtt{I},\mathbb{R})$ and satisfies*

$$\mu\left(\left\{\int_0^{\mathfrak{a}} \mathtt{x}_k(\mathfrak{s})d\mathfrak{s} :\ k \geq 1\right\}\right) \leq 2\int_0^{\mathfrak{a}} \varphi(\mathfrak{s})d\mathfrak{s}.$$

Lemma 7 ([20]). *Suppose O is a closed convex subset of Y and $0 \in \mathsf{O}$. Suppose $\mathsf{f} : \mathsf{O} \to \mathsf{Y}$ is a continuous map which fulfills Mönch's condition; that is, if $\mathsf{O}_1 \subset \mathsf{O}$ is countable and $\mathsf{O}_1 \subset \overline{conv}(\{0\} \cup \mathsf{F}(\mathsf{O}_1))$, then $\overline{\mathsf{O}}_1$ is compact. Then, F has a fixed point in O.*

Let us consider the following hypotheses:

(H_1) $\{\mathsf{T}(\mathfrak{a}),\ \mathfrak{a} > 0\}$ is equicontinuous; that is, $\mathsf{T}(\mathfrak{a})$ is continuous in the uniform operator topology for $\mathfrak{a} > 0$ and there exists a constant $\mathtt{L} > 0$ such that $\|\mathsf{T}(\mathfrak{a})\| \leq \mathtt{L}$.

(H_2) Next, the function F fulfills following:

(a) $\mathsf{F}(\mathfrak{a},\cdot) : \mathtt{I} \times \mathtt{S}_\lambda \to \mathsf{Y}$ is Lebesgue measurable with respect to $\mathfrak{a}$ on $\mathtt{I}$, $\mathsf{F}(\cdot,\phi)$ is continuous with respect to each ϕ on $\mathtt{S}_\lambda$.

(b) There exist $\lambda_1 \in (0,\mathfrak{q}), 0 < \mathfrak{q} < 1$, the function $\mathsf{M}_{\mathsf{F}} \in L^{\frac{1}{\lambda_1}}(\mathtt{I},\mathbb{R}^+)$, and a positive integrable function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$\left|\mathsf{F}(\mathfrak{a},\mathtt{x})\right| \leq \mathsf{M}_{\mathsf{F}}(\mathfrak{a})\psi(\|\phi\|_\Omega), \text{ for all } \phi \in \mathtt{S}_\lambda, \mathfrak{a} \in (0,\infty)$$

and ψ satisfies $\liminf\limits_{n\to\infty} \frac{\psi(n)}{n} = 0$.

(c) There exist $\lambda_2 \in (0,\mathfrak{q})$ and $\mathsf{M}^*_{\mathsf{F}} \in L^{\frac{1}{\lambda_2}}(\mathtt{I},\mathbb{R}^+)$, such that $\mathtt{S}_{\lambda_2} \subset \mathtt{S}_\lambda$ is bounded:

$$\mu\left(\mathsf{F}(\mathfrak{a},\mathtt{S}_{\lambda_2})\right) \leq \mathsf{M}^*_{\mathsf{F}}(\mathfrak{a})\left[\sup_{-\infty<\phi\leq 0} \mu(\mathtt{S}_{\lambda_2}(\phi))\right]$$

for almost all $\mathfrak{a} \in \mathtt{I}$, where μ is the Hausdorff measure of noncompactness.

3. Extant

3.1. Semigroup is Noncompact

Here, we present the following generalized form of the Ascoli–Arzelà theorem.

Lemma 8 ([16]). *The set $\mathsf{G} \subset \mathsf{C}_{e^{\mathfrak{a}}}(\mathtt{I},\mathsf{Y})$ is relatively compact if and only if the succeeding conditions are satisfied:*

1. *the set* $W = \{\mathtt{k} : \mathtt{k}(\mathfrak{a}) = e^{-\mathfrak{a}}\mathtt{x}(\mathfrak{a}),\ \mathtt{x} \in \mathsf{G}\}$ *is equicontinuous on* $[0, b]$ *for any* $b > 0$*;*
2. *for any* $\mathfrak{a} \in \mathtt{I}$, $W(\mathfrak{a}) = e^{-\mathfrak{a}}\mathsf{G}(\mathfrak{a})$ *is relatively compact in* Y*;*
3. $\lim_{\mathfrak{a}\to\infty} e^{-\mathfrak{a}}|\mathtt{x}(\mathfrak{a})| = 0$ *uniformly for* $\mathtt{x} \in \mathsf{G}$.

Let us consider the operator $\Phi : \mathtt{S}'_\lambda \to \mathtt{S}'_\lambda$ defined as

$$\Phi(\mathtt{x}(\mathfrak{a})) = \begin{cases} \Phi_1(\mathfrak{a}), \quad (-\infty, 0], \\ \mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a})\phi_0 + \int_0^{\mathfrak{a}} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}-\mathfrak{s})\mathsf{F}(\mathfrak{s}, \mathtt{x}_{\mathfrak{s}})d\mathfrak{s},\ \mathfrak{a} \in (0, \infty). \end{cases} \tag{4}$$

For $\Phi_1 \in \mathtt{S}_\lambda$, we define $\widehat{\Phi}$ by

$$\widehat{\Phi}(\mathfrak{a}) = \begin{cases} \Phi_1(\mathfrak{a}), \quad \mathfrak{a} \in (-\infty, 0], \\ \mathsf{S}_{\mathfrak{h},\mathfrak{q}}(\mathfrak{a})\phi_0, \quad \mathfrak{a} \in \mathtt{I}, \end{cases}$$

then $\widehat{\Phi} \in \mathtt{S}'_\lambda$. Let $\mathtt{x}_{\mathfrak{a}} = [\mathtt{w}_{\mathfrak{a}} + \widehat{\Phi}_{\mathfrak{a}}]$, $-\infty < \mathfrak{a} < +\infty$. It is simple to demonstrate that $\mathtt{x}$ fulfills Equation (3) if and only if $\mathtt{w}$ satisfies $\mathtt{w}_0 = 0$ and

$$\mathtt{w}(\mathfrak{a}) = \int_0^{\mathfrak{a}} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}-\mathfrak{s})\mathsf{F}(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}})d\mathfrak{s},\ \mathfrak{a} \in (0, +\infty).$$

Let $\mathtt{S}''_\lambda = \{\mathtt{w} \in \mathtt{S}'_\lambda : \mathtt{w}_0 \in \mathtt{S}_\lambda\}$. For any $\mathtt{w} \in \mathtt{S}'_\lambda$,

$$\begin{aligned} \|\mathtt{w}\|'_\Omega &= \|\mathtt{w}_0\|_\Omega + \sup\{\|\mathtt{w}(\mathfrak{s})\| : 0 \le \mathfrak{s} < +\infty\} \\ &= \sup\{\|\mathtt{w}(\mathfrak{s})\| : 0 \le \mathfrak{s} < +\infty\}. \end{aligned}$$

Thus, $(\mathtt{S}''_\lambda, \|\cdot\|'_\Omega)$ is a Banach space.

For $\mathtt{r} > 0$, choose $\mathtt{S}_\mathtt{r} = \{\mathtt{w} \in \mathtt{S}''_\lambda : \|\mathtt{w}\|'_\Omega \le \mathtt{r}\}$. Then, $\mathtt{S}_\mathtt{r} \subset \mathtt{S}''_\lambda$ is uniformly bounded, and for $\mathtt{w} \in \mathtt{S}_\mathtt{r}$, by Lemma 1, it holds that

$$\begin{aligned} \|\mathtt{w}_{\mathfrak{a}} + \widehat{\Phi}_{\mathfrak{a}}\|_\Omega &\le \|\mathtt{w}_{\mathfrak{a}}\|_\Omega + \|\widehat{\Phi}_{\mathfrak{a}}\|_\Omega \\ &\le \Omega(\mathtt{r} + \mathsf{L}''\mathfrak{a}^{(1-\mathfrak{h})(\mathfrak{q}-1)}\phi_0) + \|\Phi_1\|_\Omega \\ &= \mathtt{r}', \end{aligned} \tag{5}$$

where $\mathsf{L}'' = \frac{\mathsf{L}'}{\Gamma(\mathfrak{q}(1-\mathfrak{h})+\mathfrak{q})}$.

Let us consider the operator $\Phi : \mathtt{S}''_\lambda \to \mathtt{S}''_\lambda$ defined by

$$\Phi'\mathtt{w}(\mathfrak{a}) = \begin{cases} 0,\ \mathfrak{a} \in (-\infty, 0], \\ \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})} \int_0^{\mathfrak{a}} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}-\mathfrak{s})\mathsf{F}(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}})d\mathfrak{s},\ \mathfrak{a} \in \mathtt{I}. \end{cases}$$

We show that Φ has a fixed point. First, we prove

Lemma 9. *Suppose that* $(H_1) - (H_2)$ *are satisfied; then,* $W = \{\mathtt{k} : \mathtt{k}(\mathfrak{a}) = e^{-\mathfrak{a}}(\Phi'\mathtt{x})(\mathfrak{a}),\ \mathtt{x} \in \mathtt{S}_\mathtt{r}\}$ *is equicontinuous on* $[0, b]$*, where* $b > 0$ *and* $\lim_{\mathfrak{a}\to\infty} e^{-\mathfrak{a}}|(\Phi'\mathtt{x})(\mathfrak{a})| = 0$ *uniformly for* $\mathtt{x} \in \mathtt{S}_\mathtt{r}$.

Proof. **Step 1.** We show that $\mathfrak{s}$ is equicontinuous. For any $\mathfrak{a}_1, \mathfrak{a}_2 \in (0, \infty)$ where $\mathfrak{a}_1 < \mathfrak{a}_2$, we obtain

$$
\begin{aligned}
&\left|e^{-\mathfrak{a}_2}(\Phi'\mathtt{x})(\mathfrak{a}_2) - e^{-\mathfrak{a}_1}(\Phi'\mathtt{x})(\mathfrak{a}_1)\right| \\
&\quad \le \left|\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_0^{\mathfrak{a}_2} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right. \\
&\qquad \left. - \mathfrak{a}_1^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_1} \int_0^{\mathfrak{a}_1} (\mathfrak{a}_1-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\quad \le \left|\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\qquad + \left|\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_0^{\mathfrak{a}_1} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\qquad - \left|\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_0^{\mathfrak{a}_1} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\qquad + \left|\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_0^{\mathfrak{a}_1} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\qquad - \left|\mathfrak{a}_1^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_1} \int_0^{\mathfrak{a}_1} (\mathfrak{a}_1-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\quad \le \left|\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\qquad + \left|\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_0^{\mathfrak{a}_1} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \big[\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) - \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s})\big] \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\qquad + \left[\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_0^{\mathfrak{a}_1} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} d\mathfrak{s} - \mathfrak{a}_1^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_1} \int_0^{\mathfrak{a}_1} (\mathfrak{a}_1-\mathfrak{s})^{\mathfrak{h}-1} d\mathfrak{s}\right] \\
&\qquad \times \left|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big)\right| \\
&\quad \le I_1 + I_2 + I_3.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
I_1 &= \left|\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\le \frac{\mathsf{L}'}{\Gamma(\mathfrak{h})} \mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{M}_{\mathsf{F}} \psi(\mathtt{r}') d\mathfrak{s} \\
&\le \frac{\mathsf{L}'}{\Gamma(\mathfrak{h})\mathfrak{h}} \mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \mathsf{M}_{\mathsf{F}} \psi(\mathtt{r}') \big|(\mathfrak{a}_2-\mathfrak{a}_1)\big|,
\end{aligned}
$$

and we obtain $I_1 \to 0$ when $\mathfrak{a}_2 \to \mathfrak{a}_1$. For arbitrary small $\epsilon > 0$ it holds

$$
\begin{aligned}
I_2 &= \left|\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_0^{\mathfrak{a}_1} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \big[\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) - \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s})\big] \mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big) d\mathfrak{s}\right| \\
&\le \mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_0^{\mathfrak{a}_1-\epsilon} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \left\|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) - \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s})\right\| \left|\mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big)\right| d\mathfrak{s} \\
&\quad + \mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_{\mathfrak{a}_1-\epsilon}^{\mathfrak{a}_1} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \left\|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) - \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s})\right\| \left|\mathsf{F}\big(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}}\big)\right| d\mathfrak{s} \\
&\le \mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_0^{\mathfrak{a}_1-\epsilon} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{M}_{\mathsf{F}} \psi(\mathtt{r}') d\mathfrak{s} \sup_{\mathfrak{s}\in[0,\mathfrak{a}_1-\epsilon]} \left\|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_2-\mathfrak{s}) - \mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s})\right\| \\
&\quad + \frac{\mathsf{L}'}{\Gamma(\mathfrak{h})\mathfrak{h}} \mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})} e^{-\mathfrak{a}_2} \int_{\mathfrak{a}_1-\epsilon}^{\mathfrak{a}_1} (\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} \mathsf{M}_{\mathsf{F}} \psi(\mathtt{r}') d\mathfrak{s}.
\end{aligned}
$$

Based on Lemma 6, $I_2 \to 0$ when $\mathfrak{a}_2 \to \mathfrak{a}_1$.

$$
\begin{aligned}
I_3 = & \left[\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})}e^{-\mathfrak{a}_2}\int_0^{\mathfrak{a}_1}(\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1} - \mathfrak{a}_1^{(1-\mathfrak{h})(1-\mathfrak{q})}e^{-\mathfrak{a}_1}\int_0^{\mathfrak{a}_1}(\mathfrak{a}_1-\mathfrak{s})^{\mathfrak{h}-1}\right] \\
& \times \left|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}_1-\mathfrak{s})\mathsf{F}\big(\mathfrak{s},\mathtt{w}_{\mathfrak{s}}+\widehat{\Phi}_{\mathfrak{s}}\big)d\mathfrak{s}\right| \\
\leq & \frac{\mathsf{L}'}{\Gamma(\mathfrak{h})}\mathsf{M}_{\mathsf{F}}\psi(\mathtt{r}')\left[\mathfrak{a}_2^{(1-\mathfrak{h})(1-\mathfrak{q})}e^{-\mathfrak{a}_2}\int_0^{\mathfrak{a}_1}(\mathfrak{a}_2-\mathfrak{s})^{\mathfrak{h}-1}d\mathfrak{s} - \mathfrak{a}_1^{(1-\mathfrak{h})(1-\mathfrak{q})}e^{-\mathfrak{a}_1}\int_0^{\mathfrak{a}_1}(\mathfrak{a}_1-\mathfrak{s})^{\mathfrak{h}-1}d\mathfrak{s}\right]
\end{aligned}
$$

Clearly, $I_3 \to 0$ when $\mathfrak{a}_2 \to \mathfrak{a}_1$.

Therefore, $W = \{\mathtt{k} : \mathtt{k}(\mathfrak{a}) = e^{-\mathfrak{a}}(\Phi'\mathtt{x})(\mathfrak{a}),\ \mathtt{x} \in \mathtt{S}_{\mathtt{r}}\}$ is equicontinuous.

Step 2. Now, we prove that $\lim\limits_{\mathfrak{a}\to\infty} e^{-\mathfrak{a}}|(\Phi'\mathtt{x})(\mathfrak{a})| = 0$ uniformly for $\mathtt{x} \in \mathtt{S}_{\mathtt{r}}$. For any $\mathtt{x} \in \mathtt{S}_{\mathtt{r}}$, from Lemma 5 and (H_2), we obtain

$$
\begin{aligned}
\big|(\Phi'\mathtt{x})(\mathfrak{a})\big| & \leq \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}\int_0^{\mathfrak{a}}(\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}d\mathfrak{s}\big|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}-\mathfrak{s})\big|\big|\mathsf{F}\big(\mathfrak{s},\mathtt{w}_{\mathfrak{s}}+\widehat{\Phi}_{\mathfrak{s}}\big)\big| \\
& \leq \frac{\mathsf{L}'}{\Gamma(\mathfrak{h})}\mathsf{M}_{\mathsf{F}}\psi(\mathtt{r}')\mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}\int_0^{\mathfrak{a}}(\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}d\mathfrak{s} \\
& \leq \frac{\mathsf{L}'}{\mathfrak{h}\Gamma(\mathfrak{h})}\mathfrak{a}^{1-\mathfrak{q}(1-\mathfrak{h})}\mathsf{M}_{\mathsf{F}}\psi(\mathtt{r}'),
\end{aligned}
$$

thus

$$
\lim_{\mathfrak{a}\to\infty} e^{-\mathfrak{a}}\big|(\Phi'\mathtt{x})(\mathfrak{a})\big| \leq \frac{\mathsf{L}'}{\mathfrak{h}\Gamma(\mathfrak{h})}\lim_{\mathfrak{a}\to\infty} e^{-\mathfrak{a}}\mathfrak{a}^{1-\mathfrak{q}(1-\mathfrak{h})}\mathsf{M}_{\mathsf{F}}\psi(\mathtt{r}') = 0.
$$

Therefore, $\lim\limits_{\mathfrak{a}\to\infty} e^{-\mathfrak{a}}\big|(\Phi'\mathtt{x}(\mathfrak{a}))\big| = 0$ uniformly for $\mathtt{x} \in \mathtt{S}_{\mathtt{r}}$. □

Lemma 10. *Assume that the hypotheses (H_1) – (H_2) hold, then $\Phi'(\mathtt{S}_{\mathtt{r}}) \subset \mathtt{S}_{\mathtt{r}}$ and Φ' is continuous.*

Proof. First, we prove that Φ' maps $\mathtt{S}_{\mathtt{r}}$ into itself. For each $\mathtt{r} > 0$, assume that this is not true, i.e., there exists $\mathtt{r}^* \in \mathtt{S}_{\mathtt{r}}$ such that $\Phi'(\mathtt{r}^*) \notin \mathtt{S}_{\mathtt{r}}$. Thus,

$$
\begin{aligned}
\mathtt{r} < e^{-\mathfrak{a}}\big|(\Phi'\mathtt{x}(\mathfrak{a}))\big| & \leq \frac{\mathsf{L}'}{\Gamma(\mathfrak{h})}\mathsf{M}_{\mathsf{F}}\psi(\mathtt{r}')\mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}\int_0^{\mathfrak{a}}(\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}d\mathfrak{s} \\
& \leq \frac{\mathsf{L}'}{\mathfrak{h}\Gamma(\mathfrak{h})}\mathfrak{a}^{1-\mathfrak{q}(1-\mathfrak{h})}\mathsf{M}_{\mathsf{F}}\psi(\mathtt{r}').
\end{aligned}
$$

Dividing both sides by $\mathtt{r}$ and letting $\mathtt{r} \to \infty$, we obtain $1 < 0$, which contradicts our assumptions. Therefore, $\Phi'(\mathtt{S}_{\mathtt{r}}) \subset \mathtt{S}_{\mathtt{r}}$.

Next, we prove that Φ' is continuous. Let $\{\mathtt{x}_m\}_{m=0}^{+\infty}$ be the sequence in $\mathtt{S}_{\mathtt{r}}$, which is convergent to $\mathtt{x} \in \mathtt{S}_{\mathtt{r}}$. Then, it holds that

$$
\lim_{m\to\infty}\mathtt{x}_m(\mathfrak{a}) = \mathtt{x}(\mathfrak{a}) \Longrightarrow \lim_{m\to\infty}\mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}\mathtt{x}_m(\mathfrak{a}) = \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}\mathtt{x}(\mathfrak{a}) \text{ for } \mathfrak{a} \in (0,+\infty).
$$

Similarly,

$$
\lim_{m\to\infty}\mathsf{F}^m\big(\mathfrak{a},\mathtt{w}_{\mathfrak{a}}^m+\widehat{\Phi}_{\mathfrak{a}}\big) = \mathsf{F}\big(\mathfrak{a},\mathtt{w}_{\mathfrak{a}}+\widehat{\Phi}_{\mathfrak{a}}\big) \text{ for } \mathfrak{a} \in (0,+\infty).
$$

From (H_2), we obtain

$$
(\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\big|\mathsf{F}^m\big(\mathfrak{a},\mathtt{w}_{\mathfrak{a}}^m+\widehat{\Phi}_{\mathfrak{a}}\big)-\mathsf{F}\big(\mathfrak{a},\mathtt{w}_{\mathfrak{a}}+\widehat{\Phi}_{\mathfrak{a}}\big)\big| \leq 2(\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{M}_{\mathsf{F}}\psi(\mathtt{r}'),\ \text{for all}\ \ \mathfrak{a} \in (0,\infty).
$$

Also, the function $\mathfrak{s} \to 2(\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{M}_{\mathsf{F}}\psi(\mathtt{r}')$ is integrable for $\mathfrak{s} \in [0,\mathfrak{a})$, $\mathfrak{a} \in [0,\infty)$. Using the Lebesgue-dominated convergent theorem, we obtain

$$\int_0^{\mathfrak{a}} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\big|\mathsf{F}^m\big(\mathfrak{a}, \mathtt{w}_{\mathfrak{a}}^m + \widehat{\Phi}_{\mathfrak{a}}\big) - \mathsf{F}\big(\mathfrak{a}, \mathtt{w}_{\mathfrak{a}} + \widehat{\Phi}_{\mathfrak{a}}\big)\big| d\mathfrak{s} \to 0,\ m \to \infty.$$

Therefore,

$$\begin{aligned}
&\Big| e^{-\mathfrak{a}}\big(\Phi'\mathtt{x}_m\big)(\mathfrak{a}) - e^{-\mathfrak{a}}\big(\Phi'\mathtt{x}\big)(\mathfrak{a}) \Big| \\
&\qquad \le \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}e^{-\mathfrak{a}}\int_0^{\mathfrak{a}} \big|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}-\mathfrak{s})\big[\mathsf{F}^m\big(\mathfrak{a}, \mathtt{w}_{\mathfrak{a}}^m + \widehat{\Phi}_{\mathfrak{a}}\big) - \mathsf{F}\big(\mathfrak{a}, \mathtt{w}_{\mathfrak{a}} + \widehat{\Phi}_{\mathfrak{a}}\big)\big]\big| d\mathfrak{s} \\
&\qquad \le \frac{\mathsf{L}}{\Gamma(\mathfrak{h})}\mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}e^{-\mathfrak{a}}\int_0^{\mathfrak{a}} \big|\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}-\mathfrak{s})\big[\mathsf{F}^m\big(\mathfrak{a}, \mathtt{w}_{\mathfrak{a}}^m + \widehat{\Phi}_{\mathfrak{a}}\big) - \mathsf{F}\big(\mathfrak{a}, \mathtt{w}_{\mathfrak{a}} + \widehat{\Phi}_{\mathfrak{a}}\big)\big]\big| d\mathfrak{s} \\
&\qquad \to 0,\ \textit{when } m \to \infty.
\end{aligned}$$

Hence, Φ' is continuous. Thus, the proof is completed. □

Theorem 2. *Suppose that $(H_1)-(H_2)$ hold. If Φ' satisfies Mönch′s condition, then the system (1) has at least one mild solution.*

Proof. Considering the set $W = \{\mathtt{k} : \mathtt{k}(\mathfrak{a}) = e^{-\mathfrak{a}}(\Phi'\mathtt{x})(\mathfrak{a}),\ \mathtt{x} \in \mathtt{S}_{\mathtt{r}}\}$, we show that W is relatively compact.

According to Lemmas 5 and 6, the set W is equicontinuous and $\lim\limits_{\mathfrak{a}\to+\infty} e^{-\mathfrak{a}}|\Phi'\mathtt{x}(\mathfrak{a})| = 0$ uniformly for $\mathtt{x} \in \mathtt{S}_{\mathtt{r}}$. Thus, it remains to verify that the set W is relatively compact. Suppose that $\mathtt{O}_1^* = \{\mathtt{w}_{\mathfrak{s}}^m + \widehat{\Phi}_{\mathfrak{s}}\}_{m=0}^{+\infty} \subseteq \mathtt{S}_{\mathtt{r}}$ is countable and $\mathtt{O}_1^* \subseteq conv(\{0\} \cup \Phi'(\mathtt{O}_1^*))$. We must prove that $\mu(\mathtt{O}_1^*) = 0$, where μ is the Hausdorff measure of noncompactness. Based on Theorem 1 and (H_2), we obtain

$$\begin{aligned}
\mu\big(\mathtt{O}_1^*\big) = \mu\big(\{\mathtt{w}_{\mathfrak{s}}^m + \widehat{\Phi}_{\mathfrak{s}}\}_{m=0}^{+\infty}\big) &= \mu\Big(\big(\mathtt{w}_0 + \widehat{\Phi}\big)(\mathfrak{a}) \cup \{\mathtt{w}_{\mathfrak{s}}^m + \widehat{\Phi}_{\mathfrak{s}}\}_{m=0}^{+\infty}\Big) \\
&= \mu\Big(\widehat{\Phi}(\mathfrak{a}) \cup \{\mathtt{w}_{\mathfrak{s}}^m + \widehat{\Phi}_{\mathfrak{s}}\}_{m=0}^{+\infty}\Big),
\end{aligned}$$

then,

$$\begin{aligned}
\mu\big(W(\mathfrak{a})\big) &= \mu\big(\{e^{-\mathfrak{a}}\Phi'\mathtt{x}^m(\mathfrak{a})\}_{m=0}^{+\infty}\big) \\
&\le \mu\Big(\mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}e^{-\mathfrak{a}}\int_0^{\mathfrak{a}} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}-\mathfrak{s})\mathsf{F}\big(\mathfrak{s}, \{\mathtt{w}_{\mathfrak{s}}^m + \widehat{\Phi}\}_{m=0}^{+\infty}\big) d\mathfrak{s}\Big) \\
&\le \frac{\mathsf{L}}{\Gamma(\mathfrak{h})}\mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}e^{-\mathfrak{a}}\int_0^{\mathfrak{a}} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mu\Big(\mathsf{F}\big(\mathfrak{a}, \{\mathtt{w}_{\mathfrak{s}}^m + \widehat{\Phi}_{\mathfrak{s}}\}_{m=0}^{+\infty}\big)\Big) d\mathfrak{s} \\
&\le \frac{\mathsf{L}}{\Gamma(\mathfrak{h})}\mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}e^{-\mathfrak{a}}\int_0^{\mathfrak{a}} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{M}_{\mathsf{F}}^* \times \sup_{-\infty<\mathfrak{s}\le 0} \mu\Big(\{\mathtt{w}_{\mathfrak{s}}^m + \widehat{\Phi}_{\mathfrak{s}}\}_{m=0}^{+\infty}\Big) d\mathfrak{s} \\
&\le \frac{\mathsf{L}'}{\mathfrak{h}\Gamma(\mathfrak{h})}\mathfrak{a}^{1-\mathfrak{q}(1-\mathfrak{h})}\mathsf{M}_{\mathsf{F}}^* \times \sup_{-\infty<\mathfrak{s}\le 0} \mu\Big(\{\mathtt{w}_{\mathfrak{s}}^m + \widehat{\Phi}_{\mathfrak{s}}\}_{m=0}^{+\infty}\Big) \\
\mu\big(W(\mathfrak{a})\big) &\le \frac{\mathsf{L}'}{\mathfrak{h}\Gamma(\mathfrak{h})}\mathfrak{a}^{1-\mathfrak{q}(1-\mathfrak{h})}\mathsf{M}_{\mathsf{F}}^* \times \sup_{-\infty<\mathfrak{s}\le 0} \mu\big(\mathtt{O}_1^*\big).
\end{aligned}$$

Thus, we obtain $\mu(W(\mathfrak{a})) = 0$, which implies that $W(\mathfrak{a})$ is relatively compact. Therefore, based on Lemma 8, the set W is relatively compact. Hence, using Lemma 7, we conclude that the fractional differential system (1) has at least one mild solution. □

3.2. Semigroup is Compact

In this part we assume that for $t > 0$, the semigroup $T(t)$ is compact on X. Hence, the compactness of $\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a})$ follows.

Theorem 3. *If the assumptions* (H_1)–(H_2) *are true and* $\mathsf{T}(\mathfrak{a})$ *is compact, the system (1) has a mild solution.*

Proof. Obviously, it is sufficient to show that $\Phi'(\mathtt{x})$ has a fixed point in $\mathtt{S}_{\mathtt{r}}$. Here, we assume that the semigroup $\mathsf{T}(\mathfrak{a})$ is compact and fulfills (H_1). Then, based on Lemmas 5 and 6, the set W is equicontinuous and $\lim_{\mathfrak{a}\to\infty} e^{-\mathfrak{a}}|\Phi'\mathtt{x}(\mathfrak{a})| = 0$ uniformly for $\mathtt{x} \in \mathtt{S}_{\mathtt{r}}$. Thus, it remains to verify that the set W is relatively compact. To achieve this, we introduce a new operator $\Phi'_{\epsilon,\delta}$, such that $0 < \epsilon < \mathfrak{a}$ and $\delta > 0$. Take $W_{\epsilon,\delta} = \{\mathtt{k}_{\epsilon,\delta} : \mathtt{k}_{\epsilon,\delta}(\mathfrak{a}) = e^{-\mathfrak{a}}(\Phi'_{\epsilon,\delta}\mathtt{x})(\mathfrak{a}),\ \mathtt{x} \in \mathtt{S}_{\mathtt{r}}\}$. Then, we consider

$$
\begin{aligned}
(\Phi'_{\epsilon,\delta}\mathtt{x})(\mathfrak{a}) &= \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})} \int_0^{\mathfrak{a}-\epsilon} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a}-\mathfrak{s})\mathsf{F}(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}})d\mathfrak{s} \\
&= \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})} \left(\int_0^{\mathfrak{a}-\epsilon} \int_{\delta}^{\infty} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathfrak{h}\theta\mathsf{T}(\epsilon^{\alpha}\delta)\mathsf{W}_{\mathfrak{h}}((\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}}\theta)d\theta\mathsf{F}(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}})d\mathfrak{s} \right) \\
&= \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})} \left(\mathsf{T}(\epsilon^{\mathfrak{h}}\delta) \int_0^{\mathfrak{a}-\epsilon} \int_{\delta}^{\infty} \mathfrak{h}\theta(\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{W}_{\mathfrak{h}}((\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}}\theta)d\theta\mathsf{F}(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}})d\mathfrak{s} \right).
\end{aligned}
$$

Since, according to Lemma 4, $\mathsf{T}(\mathfrak{a})$ is compact, this implies that $\mathsf{Q}_{\mathfrak{h}}(\mathfrak{a})$ is compact for $\mathfrak{a} > 0$. Therefore, $\mathsf{T}(\epsilon^{\alpha}\delta)$ is compact, so that $W_{\epsilon,\delta}$ is relatively compact. Furthermore, for $\mathtt{x} \in \mathtt{S}_{\mathtt{r}}$, we obtain that:

$$
\begin{aligned}
&\left| e^{-\mathfrak{a}}(\Phi\mathtt{x})(\mathfrak{a}) - e^{-\mathfrak{a}}(\Phi'_{\epsilon,\delta}\mathtt{x})(\mathfrak{a}) \right| \\
&\quad \le \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})e^{\mathfrak{a}}} \left| \mathsf{T}(\epsilon^{\mathfrak{h}}\delta) \int_0^{\mathfrak{a}} \int_0^{\infty} \mathfrak{h}\theta(\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{W}_{\mathfrak{h}}((\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}}\theta)d\theta\mathsf{F}(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}})d\mathfrak{s} \right| \\
&\qquad + \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})} \left| \mathsf{T}(\epsilon^{\mathfrak{h}}\delta) \int_{\mathfrak{a}-\epsilon}^{\mathfrak{a}} \int_{\delta}^{\infty} \mathfrak{h}\theta(\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{W}_{\mathfrak{h}}((\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}}\theta)d\theta\mathsf{F}(\mathfrak{s}, \mathtt{w}_{\mathfrak{s}} + \widehat{\Phi}_{\mathfrak{s}})d\mathfrak{s} \right| \\
&\quad \le \mathfrak{a}^{(1-\mathfrak{h})(1-\mathfrak{q})}\mathsf{L}\mathfrak{h}e^{-\mathfrak{a}} \left[\int_0^{\mathfrak{a}} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{M}_{\mathsf{F}}(\mathfrak{a})\psi(\mathtt{r}')d\mathfrak{s} \int_0^{\delta} \theta\mathsf{W}_{\mathfrak{h}}((\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}}\theta)d\theta \right. \\
&\qquad \left. + \int_{\mathfrak{a}-\epsilon}^{\mathfrak{a}} (\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}-1}\mathsf{M}_{\mathsf{F}}(\mathfrak{a})\psi(\mathtt{r}')d\mathfrak{s} \int_0^{\delta} \theta\mathsf{W}_{\mathfrak{h}}((\mathfrak{a}-\mathfrak{s})^{\mathfrak{h}}\theta)d\theta \right], \\
\Longrightarrow &\left| e^{-\mathfrak{a}}(\Phi\mathtt{x})(\mathfrak{a}) - e^{-\mathfrak{a}}(\Phi'_{\epsilon,\delta}\mathtt{x})(\mathfrak{a}) \right| \to 0 \textit{ when } \epsilon, \delta \to 0.
\end{aligned}
$$

Therefore, the set W is relatively compact in Y. Thus, using the Schauder fixed-point theorem, we prove that Φ' has a fixed point, so the system (1) has a mild solution. This completes the proof. □

4. Application

Let us consider the following HFD_{tial} system with infinite delay on an infinite interval:

$$
\begin{cases}
{}^{H}D_{0^+}^{\mathfrak{h},\mathfrak{q}}(\mathfrak{Z}(\mathfrak{a},\tau)) = \frac{\partial^2}{\partial\tau^2}(\mathfrak{Z}(\mathfrak{a},\tau)) + \int_{-\infty}^{\mathfrak{a}} \mathsf{F}^*(\mathfrak{a},\tau,s-\mathfrak{a})\mathsf{H}(\mathfrak{Z}(s,\mathsf{d}))ds, \quad \mathsf{d} \in [0,\pi],\ \mathfrak{a} > 0, \\
\mathfrak{Z}(\mathfrak{a},0) = \mathfrak{Z}(\mathfrak{a},\pi) = 0,\ \mathfrak{a} \ge 0, \\
\mathfrak{Z}(\mathfrak{a},\tau) = \phi(\mathfrak{a},\tau),\ \mathfrak{a} \in (-\infty,0],\ \tau \in [0,\pi].
\end{cases} \tag{6}
$$

Let us take $\mathsf{Y} = L^2([0,\pi],\mathbb{R})$ to satisfy the norm $|\cdot|$ and $\mathsf{A} : \mathsf{Y} \to \mathsf{Y}$ defined by $\mathsf{Au} = \mathsf{u}''$, such that the domain

$$\mathtt{D}(\mathsf{A}) = \{\mathsf{u} \in \mathsf{Y} :\ \mathsf{u}, \mathsf{u}' \text{ are absolutely continuous}, \mathsf{u}'' \in \mathsf{Y},\ \mathsf{u}(0) = \mathsf{u}(\pi) = 0\}$$

contains the orthogonal set of eigenvectors u_k of A and

$$\mathsf{A}\mathsf{u} = \sum_{k=1}^{\infty} k^2 \langle \mathsf{u}, \mathsf{u}_k \rangle \mathsf{u}_k,\ \mathsf{u} \in D(A),$$

where $\mathsf{u}_k(s) = \sqrt{2/\pi}\sin ks,\ k = 1, 2, \cdots$. Then, A generates a compact, analytic, self-adjoint semigroup $\{\mathsf{T}(\mathfrak{a}),\ \mathfrak{a} > 0\}$; that is, $\mathsf{T}(\mathfrak{a})\mathsf{u} = \sum_{k=1}^{\infty} e^{-k^2\mathfrak{a}} \langle \mathsf{u}, \mathsf{u}_k \rangle \mathsf{u}_k,\ \mathsf{u} \in \Upsilon$. Therefore, there is a constant $\mathtt{L} > 0$, such that $\|\mathsf{T}(\mathfrak{a})\| \leq \mathtt{L}$.

Let $\lambda(s) = e^{2s},\ s < 0$, then $\Lambda = \int_{-\infty}^{0} \lambda(s) ds = \frac{1}{2}$ and define

$$\|\phi\|_{\Omega} = \int_{-\infty}^{0} \lambda(s) \sup_{s \leq \theta \leq 0} |\phi(\theta)|_{L^2} ds.$$

Let us take

$$\begin{aligned}
&\mathfrak{Z}(\mathfrak{a})(\tau) = \mathfrak{Z}(\mathfrak{a}, \tau),\ \mathfrak{a} \in [0, \infty),\\
&\phi(\theta)(\tau) = \phi(\theta, \tau),\ (\theta, \tau) \in (-\infty, 0] \times [0, \pi],\\
&\mathsf{F}(\mathfrak{a}, \mathtt{x}_{\mathfrak{a}}) = \int_{-\infty}^{\mathfrak{a}} \mathsf{F}^*(\mathfrak{a}, \tau, s - \mathfrak{a}) \mathtt{H}(\mathfrak{Z}(s)(\tau)) ds.
\end{aligned}$$

Thus, the Equation (6) is represented in the abstract form of the Equation (1). Furthermore, the system satisfies the following:

1. $\mathsf{F}^*(\mathfrak{a}, \tau, \theta)$ is continuous in $[0, \infty) \times [0, \pi] \times (-\infty, 0]$ and $\mathsf{F}^* \geq 0$, $\int_{-\infty}^{0} \mathsf{F}^*(\mathfrak{a}, \tau, \theta) d\theta = \mathsf{M}_1(\mathfrak{a}, \mathsf{d}) < +\infty$.
2. $\mathtt{H}(\cdot)$ is continuous and for $(\theta, \tau) \in (-\infty, 0] \times [0, \pi]$ it holds that $0 \leq \mathtt{H}(\mathfrak{Z}(\theta)(\mathsf{d})) \leq \Xi\big(\int_{-\infty}^{0} e^{2s} \|\mathfrak{Z}(s, \cdot)\|_{L^2} ds\big)$, where $\Xi : [0, +\infty) \to (0, +\infty)$ is a continuous increasing function.

We verify the following:

$$\begin{aligned}
|\mathsf{F}(\mathfrak{a}, \phi)|_{L^2} &= \left[\int_0^{\pi} \left(\int_{-\infty}^{0} \mathsf{F}^*(\mathfrak{a}, \tau, s - \mathfrak{a}) \mathtt{H}(\mathfrak{Z}(s)(\tau)) d\theta\right)^2 d\tau\right]^{\frac{1}{2}}\\
&\leq \left[\int_0^{\pi} \left(\int_{-\infty}^{0} \mathsf{F}^*(\mathfrak{a}, \tau, s - \mathfrak{a}) \Xi\left(\int_{-\infty}^{0} e^{2s} \|\phi(s, \cdot)\|_{L^2} ds\right) d\theta\right)^2 d\tau\right]^{\frac{1}{2}}\\
&\leq \left[\int_0^{\pi} \left(\int_{-\infty}^{0} \mathsf{F}^*(\mathfrak{a}, \tau, s - \mathfrak{a}) \Xi\left(\int_{-\infty}^{0} e^{2s} \sup_{\theta \leq s \leq 0} \|\phi(s, \cdot)\|_{L^2} ds\right) d\theta\right)^2 d\tau\right]^{\frac{1}{2}}\\
&\leq \left[\int_0^{\pi} \left(\int_{-\infty}^{0} \mathsf{F}^*(\mathfrak{a}, \tau, s - \mathfrak{a}) d\theta\right)^2 d\mathsf{d}\right]^{\frac{1}{2}} \Xi(\|\phi\|_{\Omega})\\
&\leq \left[\int_0^{\pi} (\mathsf{M}_1(\mathfrak{a}, \tau))^2 d\tau\right]^{\frac{1}{2}} \Xi(\|\phi\|_{\Omega})\\
&\equiv \mathsf{M}^*(\mathfrak{a}) \Xi(\|\phi\|_{\Omega}),
\end{aligned}$$

such that $\lim\limits_{n\to\infty} \frac{\Xi(n)}{n} = 0$. Furthermore, we can write

$$\begin{aligned}
\mu\big(F(\mathfrak{a}, \mathtt{x}_{\mathfrak{a}})\big) &= \mu\bigg(\int_{-\infty}^{\mathfrak{a}} \mathsf{F}^*(\mathfrak{a}, \mathsf{d}, s-\mathfrak{a})\mathtt{H}(\mathfrak{Z}(s)(\tau))ds\bigg) \\
&\leq \mu\bigg(\int_{-\infty}^{0} \mathsf{F}^*(\mathfrak{a}, \tau, s-\mathfrak{a})\Xi\bigg(\int_{-\infty}^{0} e^{2s}\|\phi(s,\cdot)\|_{L^2}ds\bigg)d\tau\bigg) \\
&\leq \mu\bigg(\int_{-\infty}^{0} \mathsf{F}^*(\mathfrak{a}, \tau, s-\mathfrak{a})\Xi\bigg(\int_{-\infty}^{0} e^{2s}\sup_{\theta\leq s\leq 0}\|\phi(s,\cdot)\|_{L^2}ds\bigg)d\tau\bigg) \\
&\leq \mathsf{M}^{**}(\mathfrak{a}) \sup_{-\infty<\phi\leq 0} \mu\big(\mathtt{O}_1^{**}(\phi)\big), \text{ where } \mathtt{O}_1^{**} \text{ is the subset of } \mathtt{S}_{\mathtt{r}}.
\end{aligned}$$

Therefore, (H_2) is satisfied, proving that the system (1) has a mild solution on the infinite interval $(0, +\infty)$.

5. Conclusions

In this work, we studied the existence of a mild solution to the Hilfer fractional differential system on an infinite interval via the generalized Ascoli–Arzelà theorem and fixed-point method. First, we proved the existence of a mild solution to an infinite delay system using the measure of noncompactness; after that, we established the compactness of the semigroup via the Schauder fixed-point technique; and finally, an example was provided. In the future, we will study the controllability of a Hilfer fractional differential system on an infinite interval via the generalized Ascoli–Arzelà theorem and fixed-point approach.

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Abbreviations

The following abbreviations are used in this manuscript:

HFD_{ve}	Hilfer Fractional Derivative
HFD_{tial}	Hilfer Fractional Differential
MNC	Measure of Noncompactness.

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