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Lyapunov Functions and Stability Properties of Fractional Cohen–Grossberg Neural Networks Models with Delays

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Abstract: Some inequalities for generalized proportional Riemann–Liouville fractional derivatives (RLGFDs) of convex functions are proven. As a special case, inequalities for the RLGFDs of the most-applicable Lyapunov functions such as the ones defined as a quadratic function or the ones defined by absolute values were obtained. These Lyapunov functions were combined with a modification of the Razumikhin method to study the stability properties of the Cohen–Grossberg model of neural networks with both time-variable and continuously distributed delays, time-varying coefficients, and RLGFDs. The initial-value problem was set and studied. Upper bounds by exponential functions of the solutions were obtained on intervals excluding the initial time. The asymptotic behavior of the solutions of the model was studied. Some of the obtained theoretical results were applied to a particular example.

Keywords: Cohen–Grossberg neural networks; delays; generalized proportional Riemann–Liouville fractional derivative; Lyapunov functions; Razumikhin method

MSC: 34A34; 34A08; 34D20

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1. Introduction

There are various types of fractional derivatives defined, studied, and applied in the literature. One is the Riemann–Liouville-type fractional derivative (RLTFD). The fractional differential equations with the RLTFD have been studied by many authors, such as the stability for linear systems ([1]), for nonlinear systems ([2,3]), by Lyapunov functions and comparison results ([4]), and existence and Ulam stability ([5]).

Note that the initial conditions for fractional differential equations with the RLTFD differ from the ones for differential equations with integer-order derivatives or Caputo-type fractional derivatives. This makes the independent study of the fractional differential equations with the RLTFD very important. A general RTFD of arbitrary order was defined and applied by Luchko in [6]. Recently, a generalization of the classical fractional derivatives was defined in [7,8] and named generalized proportional fractional derivatives.

We will use the generalized proportional Riemann–Liouville fractional derivative (RLGFD) to study the Cohen–Grossberg fractional model of neural networks (CGFM). A Cohen–Grossberg neural network model was investigated for ordinary derivatives and both time-varying delays and continuously distributed delays in [9], for Caputo fractional derivatives and impulses in [10], for generalized proportional Caputo fractional derivatives and impulses in [11], and for Caputo fractional derivatives and delays in [12], and a bibliographic analysis on fractional neural networks was given in [13].

We will consider a CGNN with dynamics modeled by RLGFDs. We studied the general model with both time-variable delays and continuously distributed delays. Based on the Razumikhin method and Lyapunov functions, we obtained the upper bounds of the solutions on intervals excluding the initial time. The asymptotic behavior was studied. Some of theoretical results were applied to a particular example.

The main contributions in the paper are summarized as follows:

- An inequality for the RLGFD of Lyapunov-type convex functions is proven.
- Inequalities for the RLGFD of Lyapunov functions defined by absolute values and quadratic Lyapunov functions are obtained.
- The initial-value problem for the CGNN with time-variable delays and continuously distributed delays and modeled by the RLGFD is set up.
- Two types of exponential bounds of the solutions of the model are obtained by the application of the Razumikhin method and Lyapunov functions (by absolute values and quadratic Lyapunov functions).
- Sufficient conditions for the convergence to zero of the solutions of the model are obtained.

The basic notations, definitions, and additional results are provided in Section 2. The main inequalities for Lyapunov functions with the RLGFD are proven in Section 3. As a special case, some inequalities for the Lyapunov function with the classical Riemann–Liouville fractional derivatives are provided in Section 4. In the next section, Section 5, some sufficient conditions for stability results for delay differential equations with the RLGFD are proven. These results are applied to the CGNN with the RLGFD to study the stability properties of the solutions. In the last section, some theoretical results are applied to an example.

2. Basic Definitions and Preliminary Results

Definition 1 ([7,8])**.** *The generalized proportional fractional integral (GPFI) of a function* $\mathbf{v}$*:* $[t_0, b] \to \mathbb{R}^n,\ \ 0 \le t_0 < b \le \infty$*, with* $\rho \in (0,1]$*,* $q \ge 0$*, is defined by*

$$ {}_{t_0}\mathcal{I}_t^{q,\rho}\mathbf{v}(t) = \frac{1}{\rho^q\Gamma(q)}\int_{t_0}^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{q-1}\mathbf{v}(s)\,ds, \quad t \in (t_0, b]. $$

Definition 2 ([7,8])**.** *The generalized proportional Riemann–Liouville fractional derivative (RLGFD) of a function* $\mathbf{v} : [t_0, b] \to \mathbb{R}^n,\ \ 0 \le t_0 < b \le \infty$*, with* $\rho \in (0,1]$*,* $q \in (0,1)$ *is defined by*

$$ {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho}\mathbf{v}(t) = \frac{1}{\rho^{1-q}\Gamma(1-q)}\Big((1-\rho)\int_{t_0}^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-q}\mathbf{v}(s)\,ds + \rho\frac{d}{dt}\int_{t_0}^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-q}\mathbf{v}(s)\,ds\Big), \quad t \in (t_0, b]. $$

Remark 1. *In Definitions 1 and 2, there are two parameters: q is the order of integration and differentiation; ρ is connected to the power of the exponential function. In the particular case $\rho = 1$, the defined fractional integral and derivative are reduced to the classical Riemann–Liouville fractional integral (RLFI):*

$$ {}_aI_t^q\mathbf{v}(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\mathbf{v}(s)\,ds, \ \ t \in (a, b], \tag{1} $$

and the Riemann–Liouville fractional derivative (RLFD):

$$ {}_a^{RL}D_t^q\mathbf{v}(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_a^t (t-s)^{-q}\mathbf{v}(s)\,ds, \quad t \in (a, b]. \tag{2} $$

We will provide some results known in the literature, which will be necessary for the further proofs.

Lemma 1 (Lemma 5 [14]). *Let $v \in C([t_0, b], \mathbb{R})$, $0 < t_0 < b < \infty$ be Lipschitz, and there exists a point $T \in (t_0, b]$ such that $v(T) = 0$, and $v(t) < 0$, for $t_0 \le t < T$. Then, if the RLGFD of v exists for $t = T$ with $q \in (0,1)$, $\rho \in (0,1]$, the inequality $({}^{RL}_{t_0}\mathcal{D}^{q,\rho} v)(t)|_{t=T} \ge 0$ holds.*

Lemma 2 (Lemma 2 [15]). *Let $\rho \in (0,1]$, $q \in (0,1)$ and $y \in C([t_0, b], \mathbb{R})$:*

(i) Let there exist the limit $lim_{t \to t_0+}\Big(e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}y(t)\Big) = c < \infty$. Then, ${}_{t_0}\mathcal{I}_t^{1-q,\rho} y(t)|_{t=t_0+} = c\frac{\Gamma(q)}{\rho^{1-q}}$;

(ii) Let ${}_{t_0}\mathcal{I}_t^{1-q,\rho} y(t)|_{t=t_0+} = b < \infty$. If there exists the limit $lim_{t \to 0+}\Big(e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}y(t)\Big)$, then $lim_{t \to t_0+}\Big(e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}y(t)\Big) = \frac{b\rho^{1-q}}{\Gamma(q)}$.

We define the sets:

$$C_{q,\rho}([t_0, b], \mathbb{R}^n) = \{\mathbf{v} : [t_0, b] \to \mathbb{R}^n : {}_{t_0}\mathcal{I}_t^{1-q,\rho}\mathbf{v}(t)|_{t=t_0+} < \infty,\ \forall t \in (t_0, b]\ \exists\ {}^{RL}_{t_0}\mathcal{D}_t^{q,\rho}\mathbf{v}(t) < \infty\},$$

and

$$C_q([t_0, b], \mathbb{R}^n) = \{\mathbf{v} : [t_0, b] \to \mathbb{R}^n : {}_{t_0}I_t^{1-q}\mathbf{v}(t)|_{t=t_0+} < \infty,\ \forall t \in (t_0, b]\ \exists\ {}^{RL}_{t_0}D_t^{q}\mathbf{v}(t) < \infty\}.$$

Proposition 1 ([7]). *For $\rho \in (0,1]$, $q \in (0,1)$, we have*

$$\begin{aligned} &{}^{RL}_{t_0}\mathcal{D}_t^{q,\rho}\big(e^{\frac{\rho-1}{\rho}(t-t_0)}(t-t_0)^{q-1}\big) = 0, \quad t > t_0, \\ &{}^{RL}_{t_0}\mathcal{D}_t^{q,\rho}\big(e^{\frac{\rho-1}{\rho}(t-t_0)}\big) = \frac{1}{\rho^q\Gamma(1-q)}e^{\frac{\rho-1}{\rho}(t-t_0)}(t-t_0)^{-q}, \quad t > t_0. \end{aligned} \tag{3}$$

3. Inequalities for RLGFDs

Define the set of functions:

$$\begin{aligned} \Omega = \{ &W \in C^2(\mathbb{R}^n, \mathbb{R}) : W(\mathbf{0}) = 0,\ W(\mu\mathbf{x} + (1-\mu)\mathbf{y}) \le \mu W(\mathbf{x}) + (1-\mu)W(\mathbf{y}) \\ &\text{for } \mu \in [0,1],\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\}. \end{aligned}$$

Remark 2. *Note $W \in \Omega$ iff $W \in C^2(\mathbb{R}^n, \mathbb{R})$ and $W(\mathbf{y}) \ge W(\mathbf{x}) + \sum_{i=1}^n \frac{\partial W(\mathbf{x})}{\partial x_i}(y_i - x_i)$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n)$.*

We will prove the first inequality for functions of the set Ω and their RLGFD.

Lemma 3. *Let $V \in \Omega$, $\mathbf{x} \in C_{q,\rho}([t_0, b], \mathbb{R}^n)$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$, and the composite function $V(\mathbf{x}(.)) \in C_{q,\rho}([t_0, b], [0, \infty))$. Then, the inequality:*

$${}^{RL}_{t_0}\mathcal{D}_t^{q,\rho} V(\mathbf{x}(t)) \le \sum_{k=1}^n \Big({}^{RL}_{t_0}\mathcal{D}_t^{q,\rho} x_k(t)\Big) \frac{\partial V(\mathbf{x}(t))}{\partial x_k}, \quad t \in (t_0, b], \tag{4}$$

holds.

Proof. Let $T \in (t_0, b]$ be a fixed arbitrary point. The inequality (4) is equivalent to

$${}^{RL}_{t_0}\mathcal{D}_t^{q,\rho} V(\mathbf{y}(t))|_{t=T} - \sum_{k=1}^n \Big({}^{RL}_{t_0}\mathcal{D}_t^{q,\rho} y_k(t)\Big)|_{t=T} \frac{\partial V(\mathbf{y}(T))}{\partial y_k} \le 0. \tag{5}$$

Further, we will use the following equalities:

$$x_k(s) = x_k(t_0) + \int_{t_0}^s \frac{d}{d\sigma}x_k(\sigma)d\sigma, \quad k = 1, 2, \dots, n, \quad s \in [t_0, b], \tag{6}$$

and

$$V(\mathbf{x}(s)) = V(\mathbf{x}(t_0)) + \sum_{i=1}^{n} \int_{t_0}^{s} \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_i} x_i'(\sigma) d\sigma, \quad s \in [t_0, b]. \tag{7}$$

From Definition 2 and Equalities (6) and (7), it follows that

$$\begin{aligned}
&\rho^{1-q}\Gamma(1-q)\Big({}^{RL}_{t_0}\mathcal{D}^{q,\rho}_{t} V(\mathbf{x}(t))|_{t=T} - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} {}^{RL}_{t_0}\mathcal{D}^{q,\rho}_{t} x_k(t)|_{t=T}\Big) \\
&= \rho^{1-q}\Gamma(1-q)\Big\{ \frac{1}{\rho^{1-q}\Gamma(1-q)} \Big((1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} V(\mathbf{x}(s))\, ds \\
&\quad + \rho \frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} V(\mathbf{x}(s))\, ds \Big) \\
&- \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} \Big[\frac{1}{\rho^{1-q}\Gamma(1-q)} \Big((1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} x_k(s)\, ds \\
&\quad + \rho \frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} x_k(s)\, ds \Big) \Big] \Big\} \\
&= (1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big(V(\mathbf{x}(s)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k}\, x_k(s) \Big) ds \\
&\quad + \rho \frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} V(\mathbf{x}(s))\, ds \\
&- \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} \rho \frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} x_k(s)\, ds \\
&= (1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big\{ \Big(V(\mathbf{x}(t_0)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T)}{\partial x_k} x_k(t_0) \Big) \\
&\qquad + \int_{t_0}^{s} \Big(\sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k} x_k'(\sigma) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k'(\sigma) \Big) d\sigma \Big\} ds \\
&\quad + \rho \frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big((V(\mathbf{x}(t_0)) + \int_{t_0}^{s} \Big(\sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k} x_k'(\sigma) d\sigma \Big) ds \\
&\quad - \rho \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} \frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big(x_k(t_0) + \int_{t_0}^{s} x_k'(\sigma) d\sigma \Big) ds \\
&= (1-\rho) \Big((V(\mathbf{x}(t_0)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(t_0)) \Big) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} ds \\
&\quad + \rho \Big((V(\mathbf{x}(t_0)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(t_0)) \Big) \frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} ds \\
&+ (1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \sum_{k=1}^{n} \Big(\int_{t_0}^{s} \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k} x_k'(\sigma) d\sigma - \frac{\partial V(\mathbf{x}(T))}{\partial x_k} \int_{t_0}^{s} x_k'(\sigma) d\sigma \Big)\, ds \\
&\quad + \rho \frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \sum_{k=1}^{n} \int_{t_0}^{s} \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k} x_k'(\sigma) d\sigma\, ds \\
&\quad - \rho \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} \frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \int_{t_0}^{s} x_k'(\sigma) d\sigma\, ds.
\end{aligned} \tag{8}$$

Apply (6), (7), and the equalities:

$$\frac{d}{dT} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} ds = e^{\frac{\rho-1}{\rho}(T-t_0)} (T-t_0)^{-q}$$

and

$$\int_{t_0}^{T} \int_{t_0}^{s} f(s,\sigma) d\sigma ds = \int_{t_0}^{T} \int_{\sigma}^{T} f(s,\sigma) ds d\sigma$$

for the functions $f(s,\sigma) = e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}\frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k}x_k'(\sigma)$ or $f(s,\sigma) = e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}x_k'(\sigma)$ in Equation (8), and we obtain the equality:

$$\begin{aligned}
&\rho^{1-q}\Gamma(1-q)\Big({}^{RL}_{t_0}\mathcal{D}^{q,\rho}_t V(\mathbf{x}(t))|_{t=T} - \sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}\,{}^{RL}_{t_0}\mathcal{D}^{q,\rho}_t x_k(t)|_{t=T}\Big) \\
&= \rho\Big((V(\mathbf{x}(t_0)) - \sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}x_k(t_0))\Big)e^{\frac{\rho-1}{\rho}(T-t_0)}(T-t_0)^{-q} \\
&+ (1-\rho)\int_{t_0}^T \Big((V(\mathbf{x}(t_0)) + \sum_{k=1}^n \int_{t_0}^s \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k}x_k'(\sigma)d\sigma\Big)e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}ds \\
&- (1-\rho)\sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}\int_{t_0}^T \Big(x_k(t_0) + \int_{t_0}^s x_k'(\sigma)d\sigma\Big)e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}ds \\
&\quad + \rho\frac{d}{dT}\int_{t_0}^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}\sum_{k=1}^n \int_{t_0}^s \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k}x_k'(\sigma)d\sigma\, ds \\
&\quad - \rho\sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}\frac{d}{dT}\int_{t_0}^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}\int_{t_0}^s x_k'(\sigma)d\sigma\, ds \\
&= \rho\Big((V(\mathbf{x}(t_0)) - \sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}x_k(t_0))\Big)e^{\frac{\rho-1}{\rho}(T-t_0)}(T-t_0)^{-q} \\
&+ (1-\rho)\int_{t_0}^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}\Big(V(\mathbf{x}(s)) - \sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}x_k(s)\Big)ds \\
&\quad + \rho\sum_{k=1}^n \frac{d}{dT}\int_{t_0}^T \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k}x_k'(\sigma)\int_\sigma^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}ds\, d\sigma \\
&\quad - \rho\sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}\frac{d}{dT}\int_{t_0}^T x_k'(\sigma)\int_\sigma^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}ds\, d\sigma.
\end{aligned} \tag{9}$$

Note that we have

$$\begin{aligned}
&\frac{d}{dT}\int_{t_0}^T \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k}x_k'(\sigma)\int_\sigma^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}ds\, d\sigma \\
&= \int_{t_0}^T \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k}x_k'(\sigma)\frac{d}{dT}\int_\sigma^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}ds\, d\sigma \\
&= \int_{t_0}^T \frac{\partial V(\mathbf{x}(\sigma))}{\partial x_k}x_k'(\sigma)e^{\frac{\rho-1}{\rho}(T-\sigma)}(T-\sigma)^{-q}\, d\sigma
\end{aligned} \tag{10}$$

and

$$\frac{d}{dT}\int_{t_0}^T x_k'(\sigma)\int_\sigma^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}ds\, d\sigma = \int_{t_0}^T x_k'(\sigma)e^{\frac{\rho-1}{\rho}(T-\sigma)}(T-\sigma)^{-q}\, d\sigma. \tag{11}$$

Substitute Equalities (10) and (11) in (9), and we obtain

$$\begin{aligned}
&\rho^{1-q}\Gamma(1-q)\Big({}^{RL}_{t_0}\mathcal{D}^{q,\rho}_t V(\mathbf{x}(t))|_{t=T} - \sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}\,{}^{RL}_{t_0}\mathcal{D}^{q,\rho}_t x_k(t)|_{t=T}\Big) \\
&= \rho\Big(V(\mathbf{x}(t_0)) - \sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}x_k(t_0)\Big)e^{\frac{\rho-1}{\rho}(T-t_0)}(T-t_0)^{-q} \\
&+ (1-\rho)\int_{t_0}^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}\Big(V(\mathbf{x}(s)) - \sum_{k=1}^n \frac{\partial V(\mathbf{x}(T))}{\partial x_k}x_k(s)\Big)ds \\
&\quad + \rho\sum_{k=1}^n \int_{t_0}^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}\Big(\frac{\partial V(\mathbf{x}(s))}{\partial x_k} - \frac{\partial V(\mathbf{x}(T))}{\partial x_k}\Big)x_k'(s)\, ds.
\end{aligned} \tag{12}$$

We define the function $P : [t_0, T] \to \mathbb{R}$ by the equality $P(s) = V(\mathbf{x}(s)) - V(\mathbf{x}(T)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k})[x_k(s) - x_k(T)]$ for $s \in [t_0, T]$. From $V \in \Omega$, it follows that $P(s) \geq 0$ for all $s \in [t_0, T]$, $P(T) = 0$, and $\frac{dP(s)}{ds} = \sum_{k=1}^{n} \Big(\frac{\partial V(\mathbf{x}(s))}{\partial x_k} - \frac{\partial V(\mathbf{x}(T))}{\partial x_k} \Big) x_k'(s)$.

Using integration by parts, the equalities $lim_{s \to T-} \frac{P(s)}{(T-s)^q} = lim_{s \to T-} \frac{P''(s)}{q(q-1)} (T-s)^{2-q} = 0$ and $\frac{d}{ds} \Big(e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big) = e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} (\frac{1-\rho}{\rho} + q(T-s)^{-1})$, and we obtain

$$
\begin{aligned}
&\int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \sum_{k=1}^{n} \Big(\frac{\partial V(\mathbf{x}(s))}{\partial x_k} - \frac{\partial V(\mathbf{x}(T))}{\partial x_k} \Big) x_k'(s)\, ds \\
&= \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \, \frac{dP(s)}{ds} ds \\
&= e^{\frac{\rho-1}{\rho}(T-s)} \frac{P(s)}{(T-s)^q} \Big|_{t_0}^{T} - \int_{t_0}^{T} \Big(\frac{d}{ds} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big) P(s) ds \\
&= -e^{\frac{\rho-1}{\rho}(T-t_0)} \frac{P(t_0)}{(T-t_0)^q} - \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big(\frac{1-\rho}{\rho} + q(T-s)^{-1} \Big) P(s) ds \\
&\leq -e^{\frac{\rho-1}{\rho}(T-t_0)} \frac{P(t_0)}{(T-t_0)^q} - \frac{1-\rho}{\rho} \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} P(s) ds.
\end{aligned} \tag{13}
$$

From $V \in \Omega$ and Remark 2 with $\mathbf{y} = \mathbf{0}$, we obtain $\sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(T) \geq V(\mathbf{x}(T))$, and thus, from (12) and (13), we obtain

$$
\begin{aligned}
&\rho^{1-q} \Gamma(1-q) \Big({}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} V(\mathbf{x}(t))|_{t=T} - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} x_k(t)|_{t=T} \Big) \\
&\leq \rho \Big(V(\mathbf{x}(t_0)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(t_0) \Big) \frac{e^{\frac{\rho-1}{\rho}(T-t_0)}}{(T-t_0)^q} \\
&\quad + (1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big(V(\mathbf{x}(s)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(s) \Big) ds \\
&\quad - \rho e^{\frac{\rho-1}{\rho}(T-t_0)} \frac{P(t_0)}{(T-t_0)^q} - (1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} P(s) ds \\
&= \rho \Big(V(\mathbf{x}(t_0)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(t_0) \Big) \frac{e^{\frac{\rho-1}{\rho}(T-t_0)}}{(T-t_0)^q} \\
&\quad + (1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big(V(\mathbf{x}(s)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(s) \Big) ds \\
&\quad - \rho \frac{e^{\frac{\rho-1}{\rho}(T-t_0)}}{(T-t_0)^q} \Big(V(\mathbf{x}(t_0)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(t_0) - V(\mathbf{x}(T)) + \sum_{k-1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(T) \Big) \\
&\quad - (1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big(V(\mathbf{x}(s)) - V(\mathbf{x}(T)) - \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k})[x_k(s) - x_k(T)] \Big) ds \\
&= -\rho \frac{e^{\frac{\rho-1}{\rho}(T-t_0)}}{(T-t_0)^q} \Big(- V(\mathbf{x}(T)) + \sum_{k-1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k} x_k(T) \Big) \\
&\quad - (1-\rho) \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q} \Big(- V(\mathbf{x}(T)) + \sum_{k=1}^{n} \frac{\partial V(\mathbf{x}(T))}{\partial x_k}) x_k(T) \Big) ds \leq 0.
\end{aligned}
$$

Therefore, Inequality (5) is proven, and the claim of Lemma 3 is true. □

As special cases of the result in Lemma 3, we obtain some results about Lyapunov functions.

First, we consider the Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{x} = \sum_{k=1}^{n} x_k^2 \in \Omega$, where $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} = (x_1, x_2, \dots, x_n)$.

Lemma 4 ([14])**.** *Suppose the function* $\mathbf{v} \in C_{q,\rho}([t_0, b], \mathbb{R}^n)$ *and* $\mathbf{v}^T\mathbf{v} = \sum_{k=1}^{n} v_k^2 \in C_{q,\rho}([t_0, b], \mathbb{R})$, $\mathbf{v} = (v_1, v_2, \dots, v_n)$*. Then, the inequality*

$$\sum_{k=1}^{n} {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} v_k^2(t) \leq 2 \sum_{k=1}^{n} v_k(t) {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} v_k(t), \quad t \in (t_0, b],$$

holds.

Consider the Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x} \in \Omega$ with $\mathbf{x} \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$ a positive semidefinite, symmetric, square, and constant matrix.

Lemma 5. *Suppose the function* $\mathbf{v} \in C_{q,\rho}([t_0, b], \mathbb{R}^n)$, $\mathbf{v}^T\mathbf{v} \in C_{q,\rho}([t_0, b], \mathbb{R})$, $\mathbf{v} = (v_1, v_2, \dots, v_n)$ *and* $P \in \mathbb{R}^{n\times n}$ *is a positive semidefinite, symmetric, square, and constant matrix. Then, the inequality:*

$${}_{t_0}^{RL}\mathcal{D}_t^{q,\rho}\Big(\mathbf{v}^T(t)P\,\mathbf{v}(t)\Big) \leq 2\Big(\mathbf{v}^T(t)P\,{}_{t_0}^{RL}\mathcal{D}_t^{q,\rho}\mathbf{v}(t)\Big), \quad t \in (t_0, b],$$

holds.

Consider the Lyapunov function $V(\mathbf{x}) = \sum_{i=1}^{n} x_i^4 \in \Omega$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

Lemma 6. *Suppose the function* $\mathbf{v} \in C_{q,\rho}([t_0, b], \mathbb{R}^n)$, $\mathbf{v} = (v_1, v_2, \dots, v_n)$ *and* $v_k^4 \in C_{q,\rho}([t_0, b], \mathbb{R})$, $k = 1, 2, \dots, n$*. Then, the inequality*

$$\sum_{i=1}^{n} {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} v_i^4(t) \leq 4 \sum_{i=1}^{n} v_i^3(t) {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} v_i(t), \quad t \in (t_0, b],$$

holds.

Consider the Lyapunov function $V(\mathbf{x}) = e^{\sum_{i=1}^{n} x_i} \in \Omega$ with $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

Lemma 7. *Suppose the function* $\mathbf{v} \in C_{q,\rho}([t_0, b], \mathbb{R}^n)$, $\mathbf{v} = (v_1, v_2, \dots, v_n)$*. Then, the inequality*

$${}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} e^{\sum_{i=1}^{n} v_i(t)} \leq \sum_{i=1}^{n} e^{v_i(t)} {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} v_i(t), \quad t \in (t_0, b],$$

holds.

Consider the Lyapunov function $V(\mathbf{x}) = \sum_{i=1}^{n} |x_i|$ with $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$. This function is not differentiable at 0, so Lemma 3 could not be applied directly.

Lemma 8. *Let* $v \in C_{q,\rho}([t_0, b], \mathbb{R})$*. Then, for any* $t \in (t_0, b] : v(t) \neq 0$*, the inequality:*

$${}_{t_0}^{RL}\mathcal{D}_t^{q,\rho}|v(t)| \leq sign(v(t))\, {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} v(t) \tag{14}$$

holds.

Proof. The proof is similar to the one of Lemma 3 with the function $V(y) = |y|$ and for any fixed point $T \in [t_0, b]$ applying the function $P(s) = |v(s)| - |v(T)| - sign(v(T))[v(s) - v(T)] = |v(s)| - sign(v(T))v(s) \geq 0$ for all $s \in [t_0, T]$. □

Corollary 1. *Let* $\mathbf{y} \in C_{q,\rho}([t_0, b], \mathbb{R}^n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n)$*. Then, for any point* $t \in (t_0, b]$ *such that* $y_i(t) \neq 0$, $i = 1, 2, \dots, n$*, the inequality:*

$$\sum_{i=1}^{n} {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho}|y_i(t)| \leq \sum_{i=1}^{n} sign(y_i(t))\, {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho} y_i(t) \tag{15}$$

holds.

4. Inequalities for RLFD

According to Remark 1, as special cases of the results in the previous section, we obtain some inequalities for the RLFD. We will only set up the statements without the proof because they are similar to the ones in the previous section.

Lemma 9. *Suppose the function* $V \in \Omega$, $y \in C_q([t_0, b], \mathbb{R})$*, and* $V(y(.)) \in C_q([t_0, b], \mathbb{R})$*. Then, the inequality:*

$$\left({}_{t_0}^{RL}D^q V(y(t))\right) \leq \left({}_{t_0}^{RL}D^q y(t)\right) \frac{\partial V(y(t))}{\partial y}, \quad t \in (t_0, b], \tag{16}$$

holds.

Lemma 10. *Suppose the function* $\mathbf{v} \in C_q([t_0, b], \mathbb{R}^n)$ *and* $\nu_k^2 \in C_q([t_0, b], \mathbb{R})$, $k = 1, 2, \dots, n$,, $\mathbf{v} = (\nu_1, \nu_2, \dots, \nu_n)$*. Then, the inequality*

$$\sum_{k=1}^{n} {}_{t_0}^{RL}D_t^q \nu_k^2(t) \leq 2\sum_{k=1}^{n} \nu_k(t) {}_{t_0}^{RL}D_t^q \nu_k(t), \quad t \in (t_0, b],$$

holds.

Lemma 11. *Suppose the function* $\mathbf{v} \in C_q([t_0, b], \mathbb{R}^n)$, $\nu_k^2 \in C_q([t_0, b], \mathbb{R})$, $k = 1, 2, \dots, n$, $\mathbf{v} = (\nu_1, \nu_2, \dots, \nu_n)$*, and* $P \in \mathbb{R}^{n\times n}$ *is a positive semidefinite, symmetric, square, and constant matrix. Then, the inequality:*

$${}_{t_0}^{RL}D_t^q\left(\mathbf{v}^T(t) P\, \mathbf{v}(t)\right) \leq 2\left(\mathbf{v}^T(t) P\, {}_{t_0}^{RL}D_t^q \mathbf{v}(t)\right), \quad t \in (t_0, b], \tag{17}$$

holds.

Remark 3. *The inequality (17) was applied by several authors to the RLFD, but the authors used (inappropriately) the version for Caputo fractional derivatives, proven in [16] (see, for example, Lemma 2.2 [17], Lemma 3 [18], Lemma 2.2 [19], and Lemma 2 [20]).*

Remark 4. *Inequality (17) was proven for generalized proportional Caputo fractional derivative in Lemma 3.2 [21].*

Lemma 12. *Suppose* $\mathbf{v} \in C_q([t_0, \infty), \mathbb{R}^n)$, $\mathbf{v} = (\nu_1, \nu_2, \dots, \nu_n)$*. Then, for any* $t \in (t_0, b]$: $\mathbf{v}(t) \neq 0$*, the inequality:*

$$\sum_{i=1}^{n} {}_{t_0}^{RL}D_t^q|\nu_i(t)| \leq \sum_{i=1}^{n} sign(\nu_i(t))\, {}_{t_0}^{RL}D_t^q \nu_i(t), \quad t \in (t_0, b],$$

holds.

Lemma 13. *Suppose the function* $\mathbf{v} \in C_q([t_0, b], \mathbb{R}^n)$, $\mathbf{v} = (\nu_1, \nu_2, \dots, \nu_n)$*, and* $\nu_k^4 \in C_q([t_0, b], \mathbb{R})$, $k = 1, 2, \dots, n$*. Then, the inequality:*

$$\sum_{i=1}^{n} {}_{t_0}^{RL}D_t^q \nu_i^4(t) \leq 4\sum_{i=1}^{n} \nu_i^3(t) {}_{t_0}^{RL}D_t^q \nu_i(t), \quad t \in (t_0, b],$$

holds.

Remark 5. *All the results in this paper about the Riemann–Liouville-type fractional derivatives are for functions in the set* Ω.

5. Stability Results for Delay Differential Equations with RLGFD

Consider the following nonlinear delay differential equation with the RLGFD:

$$ {}_{t_0}^{RL}\mathcal{D}_t^{q,\rho}\mathbf{y}(t) = F(t,\mathbf{y}(t),\mathbf{y}_t) \text{ for } t > t_0, \tag{18} $$

with the initial conditions:

$$ \begin{aligned} &\mathbf{y}(t_0+t) = \phi(t), \text{ for } t \in [-\tau, 0), \\ &lim_{t\to t_0+}[e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}\mathbf{y}(t)] = \frac{\phi(0)\rho^{1-q}}{\Gamma(q)}, \end{aligned} \tag{19} $$

where $q \in (0,1)$, $\rho \in (0,1]$, $\mathbf{y}_t(\eta) = \mathbf{y}(t+\eta)$, $\eta \in [-\tau, 0]$, $\phi : [-\tau, 0] \to \mathbb{R}^n$, and $F : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $F = (F_1, F_2, \dots, F_n)$.

Remark 6. *The second line of the initial condition* (19) *could be replaced by the equivalent equality (see Lemma 3):*

$$ {}_{t_0}\mathcal{I}_t^{1-q,\rho}\mathbf{y}(t)|_{t=t_0+} = \phi(0). $$

We will assume that, for any initial function $\phi \in C([-\tau, 0], \mathbb{R}^n))$, the problem (18), (19) has a solution $\mathbf{y}(t;\phi) \in C_{q,\rho}([t_0,\infty), \mathbb{R}^n)$.

Remark 7. *For any vector* $\mathbf{x} = (x_1, x_2, \dots, x_n)$, *we will use the norm* $||\mathbf{x}||$. *It could be* $||\mathbf{x}||_1 = \sum_{i=1}^n |x_i|$ *or* $||\mathbf{x}||_2 = \sqrt{\sum_{i=1}^n x_i^2}$.

Denote $||\phi||_0 = \max_{t\in[-\tau,0]} ||\phi(t)||$, where $\phi \in C([-\tau, 0], \mathbb{R}^n)$ and $||.||$ is a norm in $\mathbb{R}^n$.

Theorem 1. *Suppose there exists a function* $V \in \Omega$ *such that:*

(i) *There exists a function* $a \in \mathcal{K}: \ a(||\mathbf{x}||) \le V(\mathbf{x})$ *for* $\mathbf{x} \in \mathbb{R}^n$;

(ii) *For any solution* $\mathbf{y} \in C_{q,\rho}([t_0,\infty), \mathbb{R}^n)$ *of* (18), (19), *the following conditions hold:*

- *(a)* *For all* $t > t_0$, *the fractional derivative* ${}_{t_0}^{RL}\mathcal{D}_t^{q,\rho}V(\mathbf{y}(t))$ *exists;*
- *(b)* *There exists an increasing function* $g \in C([0,\infty), \mathbb{R})$, $g(0) = 0$:

$$ lim_{t\to t_0+}e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}V(\mathbf{y}(t)) \le g(||\phi(0)||); $$

- *(c)* *For any* $t > 0$ *such that*

$$ \begin{aligned} e^{\frac{1-\rho}{\rho}(t+\Theta-t_0)}(t+\Theta-t_0)^{1-q}V(\mathbf{y}(t+\Theta)) &< e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}V(\mathbf{y}(t)) \\ \text{for } \ \Theta &\in (-\min\{t-t_0, \tau\}, 0), \end{aligned} \tag{20} $$

the inequality:

$$ \sum_{k=1}^n F_k(t,\mathbf{y}(t),\mathbf{y}_t)\frac{\partial V(\mathbf{y}(t))}{\partial y_k} < 0 \tag{21} $$

holds.

Then, there exists a point $T_q > 0$ such that any solution $\mathbf{y}(t)$ of (18), (19) satisfies the inequality:

$$||\mathbf{y}(t)|| < a^{-1}\Big(g(||\phi||_0)e^{\frac{\rho-1}{\rho}(t-t_0)}\Big) \text{ for } t > t_0 + T_q.$$

Proof. Let $\mathbf{y}(t) = \mathbf{y}(t;\phi)$ be a solution of (18), (19) with the initial function $\phi \in C([-\tau, 0], \mathbb{R}^n)$. From Conditions (iia) and (iib) and Lemma 2, it follows that $V(\mathbf{y}(.)) \in C_{q,\rho}([t_0, \infty), \mathbb{R}_+)$.

From Condition (iib), we obtain $lim_{t\to t_0+}e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}V(\mathbf{y}(t)) \le g(||\phi||_0)$, and therefore, there exists a number $\delta > 0$ such that

$$V(\mathbf{y}(t)) < e^{\frac{\rho-1}{\rho}(t-t_0)}(t-t_0)^{q-1}g(||\phi||_0) \text{ for } t \in (t_0, t_0+\delta). \tag{22}$$

Consider the function $H(t) = g(||\phi||_0)e^{\frac{\rho-1}{\rho}(t-t_0)}(t-t_0)^{q-1},\ t \in [t_0, \infty)$.

We obtain $lim_{t\to t_0+}\Big(e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}H(t)\Big) < \infty$, and from Proposition 1, we obtain

$${}^{RL}_{t_0}\mathcal{D}^{q,\rho}_t H(t) = g(||\phi||_0){}^{RL}_{t_0}\mathcal{D}^{q,\rho}(e^{\frac{\rho-1}{\rho}(t-t_0)}(t-t_0)^{q-1}) = 0.$$

Thus, $H \in C_{q,\rho}([t_0, \infty), \mathbb{R}_+)$.

There exists $T_\rho > 0$ such that $(t-t_0)^{q-1} < 1$ for $t > t_0 + T_\rho$, and thus,

$$H(t) < g(||\phi||_0)e^{\frac{\rho-1}{\rho}(t-t_0)} \text{ for } t > t_0 + T_\rho. \tag{23}$$

We now prove that

$$V(\mathbf{y}(t)) < H(t), \quad t > t_0. \tag{24}$$

The inequality (24) holds for $t \in (t_0, t_0+\delta)$ according to (22). Assume (24) does not hold for all $t > t_0$. Thus, there exists $\eta \ge t_0 + \delta > t_0$ such that

$$V(\mathbf{y}(\eta)) = H(\eta), \text{ and } V(\mathbf{y}(t)) < H(t),\ t \in (t_0, \eta). \tag{25}$$

Therefore, $V(\mathbf{y}(.)) - H(.) \in C_{q,\rho}([t_0, \eta], \mathbb{R})$, and applying Lemma 1 with $T = \eta, v(t) \equiv V(y(t)) - H(t)$, we obtain the inequality ${}^{RL}_{t_0}\mathcal{D}^{q,\rho}_t\Big(V(\mathbf{y}(t)) - H(t)\Big)|_{t=\eta} \ge 0$. Thus,

$${}^{RL}_{0}\mathcal{D}^{q,\rho}_t V(\mathbf{y}(t))|_{t=\eta} = {}^{RL}_{0}\mathcal{D}^{q,\rho}_t\Big(V(\mathbf{y}(t)) - H(t)\Big)|_{t=\eta} \ge 0. \tag{26}$$

Apply Lemma 3 to inequality (26), and obtain

$$\sum_{k=1}^{n} f_k(\eta, \mathbf{y}(\eta), \mathbf{y}_\eta)\,\frac{\partial V(\mathbf{y}(\eta))}{\partial y_k} \ge 0. \tag{27}$$

Consider the following two possible cases:

Case 1. Let $\eta > t_0 + \tau$. Then, $\min\{\eta - t_0, \tau\} = \tau$, and for $\Theta \in [-\tau, 0)$, we have $\eta + \Theta \in (t_0, \eta)$. From (25), it follows that, for $\Theta \in [-\tau, 0]$,

$$\begin{aligned}
&e^{\frac{1-\rho}{\rho}(\eta+\Theta-t_0)}(\eta+\Theta-t_0)^{1-q}V(\mathbf{y}(\eta+\Theta)) < e^{\frac{1-\rho}{\rho}(\eta+\Theta-t_0)}(\eta+\Theta-t_0)^{1-q}H(\eta+\Theta)\\
&= e^{\frac{1-\rho}{\rho}(\eta+\Theta-t_0)}(\eta+\Theta-t_0)^{1-q}g(||\phi||_0)e^{\frac{\rho-1}{\rho}(\eta+\Theta-t_0)}(\eta+\Theta-t_0)^{q-1}\\
&= g(||\phi||_0) = e^{\frac{1-\rho}{\rho}(\eta-t_0)}(\eta-t_0)^{1-q}H(\eta) = e^{\frac{1-\rho}{\rho}(\eta-t_0)}(\eta-t_0)^{1-q}V(\mathbf{y}(\eta)).
\end{aligned} \tag{28}$$

According to Condition (iic) for $t = \eta$, the inequality:

$$\sum_{k=1}^{n} F_k(\eta, \mathbf{y}(\eta), \mathbf{y}_\eta)\,\frac{\partial V(\mathbf{y}(\eta))}{\partial y_k} < 0 \tag{29}$$

holds.

The inequality (29) contradicts (27).

Case 2. Let $\eta \leq t_0 + \tau$. Then, $\min\{\eta - t_0, \tau\} = \eta$, and for $\Theta \in [-\eta, 0)$, we have $\eta + \Theta \in (t_0, \eta)$. From (25), we have the inequality (28) for $\Theta \in (-\eta, 0)$. Similar to Case 1, we obtain a contradiction.

From Inequalities (23), (24) and Condition (i), it follows that

$$a(||\mathbf{y}(t)||) \leq V(\mathbf{y}(t)) < H(t) < g(||\phi||_0)e^{\frac{\rho-1}{\rho}(t-t_0)} \text{ for } t > t_0 + T_q. \tag{30}$$

This proves the claim of Theorem 1. □

Corollary 2. *Suppose the conditions of Theorem 1 are fulfilled, except here, we replace the inequality (20) with*

$$V(\mathbf{y}(t+\Theta)) < V(\mathbf{y}(t)) \quad \text{for } \Theta \in (-\min\{t-t_0, \tau\}, 0). \tag{31}$$

Then, any solution $\mathbf{y}(t)$ of (18), (19) satisfies the inequality

$$||\mathbf{y}(t)|| < a^{-1}\Big(g(||\phi||_0)e^{\frac{\rho-1}{\rho}(t-t_0)}\Big) \text{ for } t > t_0 + T_q.$$

Proof. Let $t \geq t_0$ be an arbitrary fixed point. Define the increasing function $\Xi : (-\min\{t - t_0, \tau\}, 0) \to \mathbb{R}$ by the equality $\Xi(\Theta) = e^{\frac{1-\rho}{\rho}(t+\Theta-t_0)}(t + \Theta - t_0)^{1-q}$. From Inequality (31), we have that

$$\begin{aligned} &\Xi(\Theta)V(\mathbf{y}(t+\Theta)) = e^{\frac{1-\rho}{\rho}(t+\Theta-t_0)}(t+\Theta-t_0)^{1-q}V(\mathbf{y}(t+\Theta)) \\ &< e^{\frac{1-\rho}{\rho}(t+\Theta-t_0)}(t+\Theta-t_0)^{1-q}V(\mathbf{y}(t)) \leq e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}V(\mathbf{y}(t)), \\ &\quad \text{for } \Theta \in (-\min\{t-t_0, \tau\}, 0), \end{aligned} \tag{32}$$

i.e., Inequality (20) holds, and we could apply Theorem 1. □

Corollary 3. *Suppose for any solution $\mathbf{y} \in C_{\rho,q}([t_0, \infty), \mathbb{R}^n)$ of (18), (19) and for any $t > t_0$ such that $||\mathbf{y}(t+\Theta)||_1 < ||\mathbf{y}(t)||_1$ for $\Theta \in (-\min\{t-t_0, \tau\}, 0)$ the inequality:*

$$\sum_{k=1}^{n} F_k(t, \mathbf{y}(t), \mathbf{y}_t)\, sign(y_k(t)) < 0 \tag{33}$$

holds.

Then, there exists a point $T_q > 0$ such that

$$||\mathbf{y}(t)||_1 < \frac{\sum_{i=1}^{n} \max_{t\in[\tau,0]} |\phi_i(t)|\, \rho^{1-q}}{\Gamma(q)} e^{\frac{\rho-1}{\rho}t} \text{ for } t > t_0 + T_q.$$

The proof follows from Theorem 1 with the Lyapunov function $V(\mathbf{x}) = \sum_{i=1}^{n} |x_i|$ and applying $lim_{t\to 0+} e^{\frac{1-\rho}{\rho}t} t^{1-q}|y_i(t)| = \frac{|\phi_i(0)|\rho^{1-q}}{\Gamma(q)}$, i.e., $g(u) = \frac{u\,\rho^{1-q}}{\Gamma(q)}$, $a(u) \equiv u$.

Corollary 4. *Suppose for any solution $\mathbf{y} \in C_{q,\rho}([t_0, \infty), \mathbb{R}^n)$ of (18), (19), the following conditions are fulfilled:*

- $\mathbf{y}^T\mathbf{y} \in C_{q,\rho}([t_0, \infty), \mathbb{R})$;
- *There exists an increasing function $g \in C([0, \infty), \mathbb{R}) : g(0) = 0$ such that*

$$lim_{t\to 0+} e^{\frac{1-\rho}{\rho}t} t^{1-q}\mathbf{y}^T(t)\mathbf{y}(t) \leq g(||\phi(0)||_2);$$

- *for any $t > t_0$ such that*

$$\mathbf{y}^T(t+\Theta)\mathbf{y}(t+\Theta) < \mathbf{y}^T(t)\mathbf{y}(t), \quad \Theta \in (-\min\{t-t_0,\tau\},0), \tag{34}$$

the inequality

$$\sum_{k=1}^{n} y_k(t)\, F_k(t,\mathbf{y}(t),\mathbf{y}_t) < 0 \tag{35}$$

holds.

Then, there exists a point $T_q > 0$ such that

$$||\mathbf{y}(t)||_2 < \sqrt{g\Big(\max_{t\in[-\tau,0]} ||\phi||_2\Big)}e^{\frac{\rho-1}{2\rho}t} \text{ for } t > t_0 + T_q.$$

The proof of Corollary 4 follows from Theorem 1 with the application of the Lyapunov function $V(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$.

6. CGNN Model with Delays and RLGFD

6.1. Model Description

The general model of the CGNN with the RLGFD and with time-variable delays and distributed delays is described by the following state equations (GGFDs):

$$\begin{aligned} {}_0^{RL}\mathcal{D}_t^{q,\rho}u_i(t) &= A_i(u_i(t))\Big(- B_i(u_i(t)) + \sum_{k=1}^{n} a_{i,k}(t)f_k(u_k(t)) \\ &+ \sum_{k=1}^{n} b_{i,k}(t)g_k(u_k(t-\eta(t))) + \sum_{k=1}^{n} c_{i,k}(t)\int_{t-\Theta(t)}^{t} h_k(u_k(s))ds\Big), \quad t>0, \quad i=1,2,\ldots,n, \end{aligned} \tag{36}$$

where $u_i(t), i = 1,2,\ldots,n$ are the state variables of the i-th neuron at time $t > 0$, $A_i(x)$ are the amplification functions, $B_i(x)$ are the behaved functions, $a_{ij}(t), b_{ij}(t), c_{ij}(t)$ represent the strengths of the neuron interconnection at time t (assuming they are time changeable), n is the number of units in the neural network, ${}_0^{RL}\mathcal{D}^{q,\rho}$ denotes the RLGFD of order $q \in (0,1)$, $\rho \in (0,1]$, $f_j(u)$, $g_j(u)$ and $h_j(u)$ denote the activation functions of the j-th neuron, $\eta(t)$ is the time-varying delay, and $\Theta(t)$ denotes the distributed time-varying delay with $0 \leq \eta(t) \leq \eta$, $0 \leq \Theta(t) \leq \Theta$ and $\tau = \max\{\eta,\Theta\}$.

The presence of delays and the applied RLGFD lead to a singularity of the solutions at the initial time 0 and the following initial conditions associated with the model: (36):

$$\begin{aligned} &{}_0\mathcal{I}_t^{1-q,\rho}u_i(t)|_{t=0} = \phi_i(0), \\ &u_i(t) = \phi_i(t) \text{ for } t \in [-\tau,0), \quad i=1,2,3,\ldots,n, \end{aligned} \tag{37}$$

where $\phi_i \in C([-\tau,0],\mathbb{R})$, $i = 1,2,\ldots,n$.

Remark 8. *The first equality in the initial condition (37) could be replaced (see Lemma 3):*

$$lim_{t\to 0+}\Big(e^{\frac{1-\rho}{\rho}t}t^{1-q}u_i(t)\Big) = \phi_i(0)\frac{\rho^{q-1}}{\Gamma(q)}, \quad i=1,2,\ldots,n. \tag{38}$$

We will introduce the following assumptions:

A1. The function $A_i \in C(\mathbb{R},[\mu_i,\lambda_i])$, where μ_i, λ_i, $i = 1,2,\ldots,n$ are positive constants.

A2. The functions $f_i, g_i, h_i \in C(\mathbb{R}, \mathbb{R})$, and there exist positive constants $\alpha_i, \beta_i, \gamma_i,\ i = 1, 2, \dots, n$ such that

$$|f_i(x) - f_i()| \leq \alpha_i |x - y|, \ \ x, y \in \mathbb{R},$$
$$|g_i(x) - g_i(y)| \leq \beta_i |x - y|, \ \ x, y \in \mathbb{R},$$
$$|h_i(x) - h_i(y)| \leq \gamma_i |x - y|, \ \ x, y \in \mathbb{R}.$$

A3. The functions $B_i \in C(\mathbb{R}, \mathbb{R})$, and there exist positive constants κ_i such that

$$\kappa_i \leq \frac{B_i(x) - B_i(y)}{x - y}, \quad x, y \in \mathbb{R},\ x \neq y,\ \ i = 1, 2, \dots, n.$$

A4. The functions $a_{ij}, b_{ij}, c_{ij} \in C([0, \infty), \mathbb{R}),\ i, j = 1, 2, \dots, n.$

Remark 9. *Let $B_i(0) = 0$. Then, Assumption A3 is satisfied iff $(sign\ x)B_i(x) \geq \kappa_i |x|,\ x \in \mathbb{R}$.*

Remark 10. *Let $B_i(0) = 0$. Then, Assumption A3 is satisfied iff $xB_i(x) \geq \kappa_i x^2,\ x \in \mathbb{R}$.*

6.2. Stability of the Model

The goal is to study the stability properties of the CGNN model (36) with the initial conditions (37). We will apply the Razumikhin method and some of the proven inequalities for the appropriate Lyapunov functions.

We will emphasize some particularities of the studied model (36). The applied RLGFD leads to a singularity of the solutions at the initial time. It requires this point to be excluded in consideration of the stability properties. Note that it is totally different than the case of the Caputo-type fractional derivative or the derivative of any integer order. In case the Riemann–Liouville type of fractional derivative is applied, there are expressions $(t - t_0)^{-q}$ and $(t - t_0)^{q-1}$ in the integral presentation of the solutions, and they are not bounded for points close enough to the initial time t_0 (for example, in [22–24], this was not taken into consideration).

6.3. Lyapunov Functions Defined by Absolute Values

Theorem 2. *Suppose the assumptions A1–A4 are fulfilled and:*

1. *The functions $B_i(0) = 0, f_i(0) = 0, g_i(0) = 0, h_i(0) = 0,\ i = 1, 2, \dots, n$.*
2. *For all $i = 1, 2, \dots, n$ and $t \geq 0$, the inequalities:*

$$-\kappa_i \mu_i + \alpha_i \sum_{k=1}^{n} \lambda_k |a_{k,i}(t)| + \beta \sum_{k=1}^{n} \lambda_k \max_{i=1,2,\dots,n} |b_{k,i}(t)| + \gamma\tau \sum_{k=1}^{n} \lambda_k \max_{i=1,2,\dots,n} |c_{k,i}(t)| < 0 \tag{39}$$

hold, where $\beta = \max_{i=1,2,\dots,n} \beta_i$ and $\gamma = \max_{i=1,2,\dots,n} \gamma_i$.

Then, there exists a point $T_q > 0$ such that any solution $\mathbf{u} \in C_{q,\rho}([0, \infty), \mathbb{R}^n)$ of (36), (37) satisfies the inequality:

$$||\mathbf{u}(t)||_1 < \frac{\sum_{i=1}^{n} \max_{t \in [-\tau, 0]} |\phi_i(t)|\, \rho^{1-q}}{\Gamma(q)} e^{\frac{\rho-1}{\rho} t} \ \textit{for } t > T_q.$$

Proof. Let $\mathbf{u}(t) \in C_{q,\rho}([0, \infty), \mathbb{R}^n)$ be a solution of (36), (37).

Denote

$$\begin{aligned} F_i(t, \mathbf{u}(t), \mathbf{u}_t) = A_i(u_i(t))\Big(- B_i(u_i(t)) + \sum_{k=1}^{n} a_{i,k}(t) f_k(u_k(t)) \\ + \sum_{k=1}^{n} b_{i,k}(t) g_k(u_k(t - \eta(t))) + \sum_{k=1}^{n} c_{i,k}(t) \int_{t-\Theta(t)}^{t} h_k(u_k(s)) ds\Big), \\ t > 0,\ \ i = 1, 2, \dots, n, \end{aligned} \tag{40}$$

Let the point $t > 0$ be such that $||\mathbf{u}(t+\Theta)||_1 < ||\mathbf{u}(t)||_1$ for $\Theta \in [-\min\{t,\tau\},0)$. According to Assumptions A1–A3, Remark 9, and the inequalities (39), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} sign(u_i(t))F_i(t,\mathbf{u}(t),\mathbf{u}_t)\\
&= \sum_{i=1}^{n}\Big\{sign(u_i(t))A_i(u_i(t))\Big(-B_i(u_i(t)) + \sum_{k=1}^{n} a_{i,k}(t)f_k(u_k(t))\\
&+ \sum_{k=1}^{n} b_{i,k}(t)g_k(u_k(t-\eta(t))) + \sum_{k=1}^{n} c_{i,k}(t)\int_{t-\Theta(t)}^{t} h_k(u_k(s))ds\Big)\Big\}\\
&\le \sum_{i=1}^{n}\Big\{-\kappa_i\mu_i|u_i(t)| + \lambda_i\sum_{k=1}^{n}|a_{i,k}(t)|\alpha_k|u_k(t)|\\
&+ \lambda_i\sum_{k=1}^{n}|b_{i,k}(t)|\beta_k|u_k(t-\eta(t))| + \lambda_i\sum_{k=1}^{n}|c_{i,k}(t)|\int_{t-\Theta(t)}^{t}\gamma_k|u_k(s)|ds\Big\}\\
&\le \sum_{i=1}^{n}\Big(-\kappa_i\mu_i|u_i(t)| + \alpha_i|u_i(t)|\sum_{k=1}^{n}\lambda_k|a_{k,i}(t)|\Big)\\
&+ \beta\sum_{k=1}^{n}\lambda_k\max_{i=1,2,\dots,n}|b_{k,i}(t)|\sum_{i=1}^{n}|u_i(t)| + \gamma\Theta(t)\sum_{i=1}^{n}|u_i(t)|\sum_{k=1}^{n}\lambda_k\max_{i=1,2,\dots,n}|c_{k,i}(t)|\\
&\le \sum_{i=1}^{n}\Big(-\kappa_i\mu_i + \alpha_i\sum_{k=1}^{n}\lambda_k|a_{k,i}(t)|\\
&+ \beta\sum_{k=1}^{n}\lambda_k\max_{i=1,2,\dots,n}|b_{k,i}(t)| + \gamma\Theta(t)\sum_{k=1}^{n}\lambda_k\max_{i=1,2,\dots,n}|c_{k,i}(t)|\Big)|u_i(t)| < 0.
\end{aligned} \tag{41}$$

From Corollary 3 with $t_0 = 0$, we have the claim of Theorem 2. □

Corollary 5. *Let the conditions of Theorem 2 be satisfied. Then, any solution* $\mathbf{u} \in C_{q,\rho}([0,\infty),\mathbb{R}^n)$ *of* (36), (37) *satisfies* $lim_{t\to\infty}\mathbf{u}(t) = 0$.

6.4. Quadratic Lyapunov Functions

When the quadratic Lyapunov function is applied and given its RLGFD, we need to be sure that the RLGFD of the squared function also exits. This assumption has to be also added.

Theorem 3. *Suppose the assumptions A1–A4 are fulfilled and:*

1. *The functions* $B_i(0) = 0, f_i(0) = 0, g_i(0) = 0, h_i(0) = 0,\ i = 1,2,\dots,n$.
2. *Any solution* $u \in C_{q,\rho}([0,\infty),\mathbb{R}^n)$ *of* (36), (37) *is such that* $u^T u \in C_{q,\rho}([0,\infty),\mathbb{R}^n)$.
3. *For all* $i = 1,2,\dots,n$, *the inequalities:*

$$\begin{aligned}
&\lambda\sum_{k=1}^{n}\Big(\alpha_k\max_{i=1,2,\dots,n}|a_{i,k}(t)| + \beta_k\max_{i=1,2,\dots,n}|b_{i,k}(t)| + \gamma\tau\max_{i=1,2,\dots,n}|c_{i,k}(t)|\Big)\\
&+ \sum_{k=1}^{n}\Big(\alpha\lambda_k\max_{i=1,2,\dots,n}|a_{k,i}(t)| + \beta\lambda_k\max_{i=1,2,\dots,n}|b_{k,i}(t)| + \gamma\tau\lambda_k\max_{i=1,2,\dots,n}|c_{k,i}(t)|\Big)\\
&< 2\mu\kappa
\end{aligned} \tag{42}$$

hold, where $\mu = \min_{i=1,2,\dots,n}\mu_i$, $\kappa = \min_{i=1,2,\dots,n}\kappa_i$, $\alpha = \max_{i=1,2,\dots,n}\alpha_i$, $\beta = \max_{i=1,2,\dots,n}\beta_i$, $\lambda = \max_{i=1,2,\dots,n}\lambda_i$, *and* $\gamma = \max_{i=1,2,\dots,n}\gamma_i$.

Then, there exists a point $T_q > 0$ such that any solution $\mathbf{u} \in C_{q,\rho}([0,\infty),\mathbb{R}^n)$ of (36), (37) satisfies the inequality:

$$||\mathbf{u}(t)||_2 < \frac{\sum_{i=1}^n \max_{t\in[\tau,0]} |\phi_i(t)|\, \rho^{1-q}}{\Gamma(q)} e^{\frac{\rho-1}{\rho}t} \text{ for } t > T_q.$$

Proof. Let $\mathbf{u}(t) \in C_{q,\rho}([0,\infty),\mathbb{R}^n)$ be a solution of (36), (37).

Let the point $t > 0$ be such that $||\mathbf{u}(t+\Theta)||_2 < ||\mathbf{u}(t)||_2$ for $\Theta \in [-\min\{t,\tau\},0)$. Apply Assumptions A1–A3, Remark 10, and the inequalities (42), and we obtain for the function $F(t,\mathbf{u}(t),\mathbf{u}_t)$ defined by (40) that

$$\begin{aligned}
&\sum_{i=1}^n u_i(t)F_i(t,\mathbf{u}(t),\mathbf{u}_t) \\
&\leq -\sum_{i=1}^n \mu_i\kappa_i u_i^2(t) + 0.5\sum_{i=1}^n \lambda_i u_i^2(t)\sum_{k=1}^n |a_{i,k}(t)|\alpha_k + 0.5\sum_{i=1}^n \lambda_i \sum_{k=1}^n |a_{i,k}(t)|\alpha_k u_k^2(t) \\
&+ 0.5\sum_{i=1}^n \lambda_i u_i^2(t)\sum_{k=1}^n |b_{i,k}(t)|\beta_k + 0.5\sum_{i=1}^n \lambda_i \sum_{k=1}^n |b_{i,k}(t)|\beta_k u_k^2(t-\eta(t)) \\
&+ 0.5\sum_{i=1}^n \lambda_i \sum_{k=1}^n |c_{i,k}(t)| u_i^2(t)\Theta(t)\gamma_k + 0.5\sum_{i=1}^n \lambda_i \sum_{k=1}^n |c_{i,k}(t)| \int_{t-\Theta(t)}^t \gamma_k u_k^2(s)ds \\
&\leq 0.5\sum_{i=1}^n \Big\{ -2\mu\kappa + \lambda\sum_{k=1}^n \max_{i=1,2,\ldots,n} |a_{i,k}(t)|\alpha_k + \alpha\sum_{k=1}^n \lambda_k \max_{i=1,2,\ldots,n} |a_{k,i}(t)| \\
&+ \lambda\sum_{k=1}^n \max_{i=1,2,\ldots,n} |b_{i,k}(t)|\beta_k + \beta\sum_{k=1}^n \lambda_k \max_{i=1,2,\ldots,n} |b_{k,i}(t)| \\
&+ \lambda\tau\gamma\sum_{k=1}^n \max_{i=1,2,\ldots,n} |c_{i,k}(t)| + \gamma\tau\sum_{k=1}^n \lambda_k \max_{i=1,2,\ldots,n} |c_{k,i}(t)| \Big\} u_i^2(t) < 0.
\end{aligned}$$

Apply Corollary 4, and obtain the claim of Theorem 3. □

Corollary 6. *Let the conditions of Theorem 3 be fulfilled. Then, any solution $\mathbf{u} \in C_{q,\rho}([0,\infty),\mathbb{R}^n)$ of (36), (37) satisfies $lim_{t\to\infty}\mathbf{u}(t) = 0$.*

Remark 11. *Note that all sufficient conditions given in Theorems 2 and 3 do not depend on the fractional order and the parameter of the fractional derivative.*

7. Application

Example 1. *Consider the following CGNN model with three neurons and delays and modeling the states' dynamics by the RLGFD:*

$$\begin{aligned}
{}^{RL}_{\ 0}\mathcal{D}_t^{0.3,0.5} u_i(t) &= A_i(t)u_i(t)\Big(B_i(u_i(t)) + \sum_{k=1}^3 a_{i,k}(t)f_k(u_k(t)) \\
&+ \sum_{k=1}^3 b_{i,k}(t)g_k(u_k(t-1)) + \sum_{k=1}^3 c_{i,k}(t)\int_{t-e^{-t}}^t h_k(u_k(s))ds\Big), \quad t>0, \quad i=1,2,3,
\end{aligned} \tag{43}$$

with $\rho = 0.5$, $q = 0.3$, $\eta(t) = 1$, $\Theta(t) = e^{-t}$, $\tau = 1$, $A_1(t) = 2$, $A_2(t) = 0.5(1 + e^{-t})$, $A_3(t) = 1 + 0.05e^{-t}$, and $B_i(u) = ue^{|u|}$, $u \in \mathbb{R}$, $i = 1,2,3$. Then, $\lambda_1 = \mu_1 = 2$, $\mu_2 = 0.5$, $\lambda_2 = 1$, $\mu_3 = 1$, $\lambda_3 = 1.05$, $sign(u)B_i(u) \geq |u|$, $u \in \mathbb{R}, \kappa_i = 1$, $i = 1,2,3$.

The activation functions $f_1(x) = g_1(x) = h_1(x) = \frac{x}{1+e^{-x}}$ are the Swish functions with constants $\alpha_1 = \beta_1 = \gamma_1 = 1.1$, $f_2(x) = g_2(x) = h_2(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, are the tanh functions with *constants* $\alpha_2 = \beta_2 = \gamma_2 = 1$*, and* $f_3(x) = g_3(x) = h_3(x) = 0.5(|x+1| - |x-1|)$ *with* $\alpha_3 = \beta_3 = \gamma_3 = 1$*, and the matrices of the strengths of the interconnections are given by*

$$\{a_{i,k}(t)\} = \begin{bmatrix} 0.01 & 0 & 0.02 \\ 0.01e^{-t} & 0.01 & 0 \\ 0 & \frac{0.01t}{1+t} & 0 \end{bmatrix}, \quad \{b_{i,k}(t)\} = \begin{bmatrix} -0.01 & 0.1 & 0.1 \\ -0.1e^{-t} & 0 & 0.1e^{-2t} \\ 0 & 0.001\sin(t) & 0 \end{bmatrix},$$

$$\{c_{i,k}(t)\} = \begin{bmatrix} 0 & 0.01 & -0.01e^{-t} \\ -0.01 & \frac{0.0022t}{1+t} & 0 \\ -0.001 & 0 & 0.005e^{-t} \end{bmatrix}$$

with

$$\sum_{k=1}^{3} \lambda_k \max_{i=1,2,3} |b_{k,i}(t)| \leq 0.30105, \quad \sum_{k=1}^{3} \lambda_k \max_{i=1,2,3} |c_{k,i}(t)| \leq 0.03525,$$

$$\alpha_1 \sum_{k=1}^{n} \lambda_k |a_{k,1}(t)| \leq 0.033, \quad \alpha_2 \sum_{k=1}^{n} \lambda_k |a_{k,2}(t)| \leq 0.0205, \quad \alpha_3 \sum_{k=1}^{n} \lambda_k |a_{k,3}(t)| = 0.04.$$

Then, for all $i = 1,2,3$*, the inequalities* (39) *hold, i.e.,*

$$\alpha_i \sum_{k=1}^{n} \lambda_k |a_{k,i}(t)| + 1.1 \sum_{k=1}^{n} \lambda_k \max_{i=1,2,\ldots,n} |b_{k,i}(t)| + 1.1 * 2 \sum_{k=1}^{n} \lambda_k \max_{i=1,2,\ldots,n} |c_{k,i}(t)| < \mu_i \tag{44}$$

with $\beta = \gamma = 1.1$.

We apply the Lyapunov function $V(\mathbf{u}) = \sum_{\mathbf{i=1}}^{\mathbf{3}} |\mathbf{u_i}|$.

According to Theorem 2*, there exists a point* $T_{0.3} = 1 > 0 : \ t^{-0.3} > 1$ *such that any solution* $u \in C_{0.3,0.5}([0,\infty), \mathbb{R}^3)$ *of* (43), (37) *satisfies the inequality:*

$$||\mathbf{u}(t)||_1 = |u_1(t)| + |u_2(t)| + |u_2(t)| < \frac{\sum_{i=1}^{3} \max_{t\in[-1,0]} |\phi_i(t)| \, (0.5)^{0.7}}{\Gamma(0.3)} e^{-t} \text{ for } t > 1,$$

and $lim_{t\to\infty} u(t) = 0$.

Remark 12. *Note that Inequality* (44) *is very important for the stability properties.*

8. Conclusions

The main aim of this paper was to prove some inequalities for the RLGFD of Lyapunov-type convex functions. As a special case, we obtained some inequalities for the widely applied Lyapunov functions defined by the absolute values and the quadratic Lyapunov functions. These inequalities were used to study the behavior of the solutions of Cohen–Grossberg neural network models with variable delays, distributed delays, and RLGFDs. To be more general, we considered the model with coefficients that were variable in time. The applied derivative gave us the opportunity to model more adequately the behavior with anomalies at the initial time. Some upper bounds with exponential function of the solutions were obtained on intervals excluding the initial time. The base of the investigation was the Razumikhin method and Lyapunov functions. Some of theoretical results were illustrated with an example.

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References

1. Qian, D.; Li, C.; Agarwal, R.P.; Wong, P.J.Y. Stability analysis of fractional differential system with Riemann–Liouville derivative. *Math. Comput. Model.* **2010**, *52*, 862–874. [CrossRef]
2. Agarwal, R.; Hristova, S.; O'Regan, D. Stability Concepts of Riemann–Liouville Fractional-Order Delay Nonlinear Systems. *Mathematics* **2021**, *9*, 435. [CrossRef]
3. Hristova, S.; Tersian, S.; Terzieva, R. Lipschitz Stability in Time for Riemann–Liouville Fractional Differential Equations. *Fractal Fract.* **2021**, *5*, 37. [CrossRef]
4. Agarwal, R.; Hristova, S.; O'Regan, D. Practical stability for Riemann–Liouville delay fractional differential equations. *Arab. J. Math.* **2021**, *10*, 271–283. [CrossRef]
5. Benchohra, M.; Bouriah, S.; Nieto, J.J. Existence and Ulam stability for nonlinear implicit differential equations with Riemann–Liouville fractional derivative. *Demonstr. Math.* **2019**, *52*, 437–450. [CrossRef]
6. Luchko, Y. Fractional Differential Equations with the General Fractional Derivatives of Arbitrary Order in the Riemann–Liouville Sense.*Mathematics* **2022**, *10*, 849. [CrossRef]
7. Jarad, F.; Abdeljawad, T.; Alzabut, J. Generalized fractional derivatives generated by a class of local proportional derivatives. *Eur. Phys. J. Spec. Top.* **2017**, *226*, 3457–3471. [CrossRef]
8. Jarad, F.; Abdeljawad, T. Generalized fractional derivatives and Laplace transform. *Discret. Contin. Dyn. Syst. Ser. S* **2020**, *13*, 709–722. [CrossRef]
9. Song, Q.; Cao, J. Stability analysis of Cohen–Grossberg neural network with both time-varying and continuously distributed delays. *J. Comput. Appl. Math.* **2006**, *197*, 188–203. [CrossRef]
10. Stamova, I.; Sotirov, S.; Sotirova, E.; Stamov, G. Impulsive Fractional Cohen–Grossberg Neural Networks: Almost Periodicity Analysis. *Fractal Fract.* **2021**, *5*, 78. [CrossRef]
11. Agarwal, R.; Hristova, S. Impulsive Memristive Cohen– Grossberg Neural Networks Modeled by Short Term Generalized Proportional Caputo Fractional Derivative and Synchronization Analysis. *Mathematics* **2022**, *10*, 2355. [CrossRef]
12. Yang, Z.; Zhang, J.; Hu, J.; Mei, J. Finite-Time Stability Criteria for a Class of High-Order Fractional Cohen–Grossberg Neural Networks with Delay. *Complexity* **2020**, *2020*, 3604738. [CrossRef]
13. Viera-Martin, E.; Gómez-Aguilar, J.F.; Solís-Pérez, J.E.; Hernández-Pérez, J.A.; Escobar-Jiménez, R.F. Artificial neural networks: A practical review of applications involving fractional calculus. *Eur. Phys. J. Spec. Top.* **2022**, *231*, 2059–2095. [CrossRef] [PubMed]
14. Agarwal, R.P.; Hristova, S.; O'Regan, D. Mittag-Leffler-Type Stability of BAM Neural Networks Modeled by the Generalized Proportional Riemann–Liouville Fractional Derivative. *Axioms* **2023**, *12*, 588. [CrossRef]
15. Hristova, S.; Abbas, M.I. Explicit Solutions of Initial Value Problems for Fractional Generalized Proportional Differential Equations with and without Impulses. *Symmetry* **2021**, **13**, 996. [CrossRef]
16. Duarte-Mermoud, M.A.; Aguila-Camacho, N.; Gallegos, J.A.; Castro-Linares, R. Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *22*, 650–659. [CrossRef]
17. Liu, S.; Wu, X.; Zhang, Y.-J.; Yang, R. Asymptotical stability of Riemann–Liouville fractional neutral systems. *Appl. Math. Lett.* **2017**, *69*, 168–173. [CrossRef]
18. Korkmaz, E.; Ozdemir, A.; Yildirim, K. Asymptotical Stability of Riemann–Liouville Nonlinear Fractional Neutral Neural Networks with Time-Varying Delays. *J. Math.* **2022**, *2022*, 6832472. [CrossRef]
19. Altun, Y. Further results on the asymptotic stability of Riemann–Liouville fractional neutral systems with variable delays. *Adv. Differ. Equ.* **2019**, *2019*, 437. [CrossRef]
20. Chartbupapan, W.; Bagdasar, O.; Mukdasai, K. A Novel Delay-Dependent Asymptotic Stability Conditions for Differential and Riemann–Liouville Fractional Differential Neutral Systems with Constant Delays and Nonlinear Perturbation. *Mathematics* **2020**, *8*, 82. [CrossRef]
21. Bohner, M.; Hristova, S. Stability for generalized Caputo proportional fractional delay integro-differential equations. *Bound Value Probl.* **2022**, *2022*, 14. [CrossRef]
22. Alidousti, J.; Khoshsiar Ghaziani, R.; Eshkaftaki, A.B. Stability analysis of nonlinear fractional differential order systems with Caputo and Riemann–Liouville derivatives. *Turk. J. Math.* **2017**, *41*, 1260–1278. [CrossRef]
23. Qin, Z.; Wu, R.; Lu, Y. Stability analysis of fractional order systems with the Riemann–Liouville derivative. *Syst. Sci. Control Eng. Open Access J.* **2014**, *2*, 727–731. [CrossRef]
24. Zhang, R.; Yang, S.; Feng, S. Stability analysis of a class of nonlinear fractional differential systems with Riemann–Liouville derivative. *EE/CAA J. Autom. Sin.* 2016, *early access*. [CrossRef]