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Common and Coincidence Fixed-Point Theorems for $\Im$-Contractions with Existence Results for Nonlinear Fractional Differential Equations

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Abstract: In this paper, we derive the coincidence fixed-point and common fixed-point results for $\Im$-type mappings satisfying certain contractive conditions and containing fewer conditions imposed on function $\Im$ with regard to generalized metric spaces (in terms of *Jleli Samet*). Finally, a fractional boundary value problem is reduced to an equivalent Volterra integral equation, and the existence results of common solutions are obtained with the use of proved fixed-point results.

Keywords: common fixed point; generalized metric space; coincidence fixed point; $\Im$-contraction; fractional differential equation; integral boundary conditions

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1. Introduction

One of the most significant basic fixed-point results is the well-known Banach's fixed-point theorem (abbreviated BFPT) [1]. Due to the numerous uses of this principle in other disciplines of mathematics, numerous writers have expanded, generalized, and enhanced it in numerous ways by taking into account alternative mappings or space types. Wardowski [2] provided a striking and significant generalization of this nature. He provided this to introduce the idea of $\Im$-contraction as

Definition 1. *Let (V, d) be a metric space. A mapping $\Omega : V \to V$ is said to be an $\Im$-contraction, if there exist $\Im \in \Delta(\Im)$ and $\lambda > 0$ such that for all $\mu, \varpi \in V$*

$$\lambda + \Im(d(\Omega\mu, \Omega\varpi)) \leq \Im(d(\mu, \varpi)), \tag{1}$$

where $\Delta(\Im)$ is the family of all mappings $\Im : (0, +\infty) \to (-\infty, \infty)$ meeting the criteria listed below.

($\Im_1$) $\Im(\mu) < \Im(\varpi)$ for all $\mu < \varpi$;
($\Im_2$) For all sequences $\{\varsigma_p\} \subseteq (0, +\infty)$, $\lim_{p\to+\infty} \varsigma_p = 0$, if and only if $\lim_{p\to+\infty} \Im(\varsigma_p) = -\infty$;
($\Im_3$) There is $0 < \wp < 1$ such that $\lim_{\varsigma\to 0^+} \varsigma^{\wp}\Im(\varsigma) = 0$.

Wardowski's result is given as follows:

Theorem 1 ([2]). *Let (V, d) be a complete metric space and $\Omega : V \to V$ be an $\Im$-contraction. Then, $\mu^* \in V$ is a unique fixed point of Ω and for every $\mu_0 \in V$, a sequence $\{\Omega^p\mu_0\}_{p\in\mathbb{N}}$ is convergent to μ^*.*

Secelean demonstrated in [3] that condition ($\Im_2$) can be substituted with a similar but simpler one (noted ($\Im_2'$): $\inf \Im = -\infty$). Then, instead of utilizing ($\Im_2$) and ($\Im_3$), Piri and Kumam [4] proved Wardowski's theorem using ($\Im_2$) and the continuity. Later, Wardowski [5], using λ as a function, demonstrated a fixed-point theorem for $\Im$-contractions. Recently, some authors demonstrated the Wardowski original conclusions without the criteria ($\Im_2$) and ($\Im_3$) in various ways (see, [6,7]). For more in this direction, see [8–15]. Very recently, Derouiche and Ramoul [16] introduced the notions of extended $\Im$-contractions of the Suzuki–Hardy–Rogers type, extended $\Im$-contractions of the Hardy–Rogers type, and generalized $\Im$-weak contractions of the Hardy–Rogers type as well as establishing some new fixed-point results for such kinds of mappings in the setting of complete b-metric spaces. They also dropped condition ($\Im_3$) and used a relaxed version of ($\Im_2$).

However, the concept of standard metric space is generalized in a number of ways (see [17–24]). *Jleli* and *Samet* provided one of the most common generalizations of metric spaces in [25], which recapitulates a broad class of topological spaces, including b-metric spaces, standard metric spaces, dislocated metric spaces, and modular spaces. They expanded BFPT, Cirić's fixed-point theorem and a fixed-point result attributed to Ran and Reurings, among other fixed-point theorems. Additionally, Altun et al. obtained a fixed-point theorem of the Feng–Liu type with regard to generalized metric spaces in [26], while Karapinar et al. gained fixed-point theorems within fairly broad contractive conditions in generalized metric spaces in [27]. In the framework of generalized metric spaces, Saleem et al. [28] recently demonstrated a few novel fixed-point theorems, coincidence point theorems, and a common fixed-point theorem for multivalued $\Im$-contraction involving a binary relation that is not always a partial order.

Henceforth, let V be a non-empty set and $Ł : V \times V \to [0, +\infty]$ be a given mapping. Following *Jleli* and *Samet* [25], for every $\mu \in V$, define the set

$$C(Ł, V, \mu) = \left\{ \{\mu_p\} \subset V : \lim_{p \to +\infty} Ł(\mu_p, \mu) = 0 \right\}. \tag{2}$$

Definition 2 ([25])**.** *Let V be a non-empty set and $Ł : V \times V \to [0, +\infty]$ be a function which fulfils the following criteria for all $\mu, \varpi \in V$:*

($Ł_1$) $Ł(\mu, \varpi) = 0$ implies $\mu = \varpi$;
($Ł_2$) $Ł(\mu, \varpi) = Ł(\varpi, \mu)$;
($Ł_3$) There is $\kappa > 0$ such that $(\mu, \varpi) \in V \times V$, $\{\mu_p\} \in C(Ł, V, \mu)$ implies

$$Ł(\mu, \varpi) \le \kappa \lim_{p \to +\infty} \sup Ł(\mu_p, \varpi). \tag{3}$$

Then Ł is called a generalized metric and the pair $(V, Ł)$ is called a generalized metric space. We renamed it as κ-generalized metric space (abbreviated, a κ-GMS).

Remark 1 ([25])**.** *If the set $C(Ł, V, \mu)$ is empty for every $\mu \in V$, then $(V, Ł)$ is a κ-GMS if and only if ($Ł_1$) and ($Ł_2$) are satisfied.*

Refs. [25,27,28] all contain numerous examples of GMS(JS).

Example 1 ([25])**.**

(1) A metric space is a 1-GMS.
(2) A modular metric space (V, ρ) is a ρ-GMS.
(3) A 2-metric space is a 2-GMS.

In the sequel, $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}$ indicate the set of all positive integers, the set of all non-negative integers and the set of all real numbers, respectively, and $\mathbb{R}$ indicates the set of all real numbers. Let ζ be self-mapping on a non-empty set V, $P(V)$ be the collection of all non-empty subsets of V, $C(V)$ be the collection of all non-empty closed subsets of V, and $\Omega : V \to P(V)$ be a set-valued mapping. We denoted by $\mathcal{C}oi(V, \zeta, \Omega)$ the set of all coincidence points of ζ & Ω in V and by $\mathcal{C}om(V, \zeta, \Omega)$ the set of all common fixed points of ζ & Ω in V. A non-empty subset $\sim$ of the Cartesian product $V \times V$ is a binary relation on V. For simplicity, we denote $\mu \sim \varpi$ if $(\mu, \varpi) \in \sim$. Ref. [29] contains the concepts of preorder, partial order, transitivity, reflexivity, and antisymmetry.

Definition 3 ([27])**.** *Let a binary relation on the κ-GMS $(V, Ł)$ be defined as $\sim$. If a sequence $\mu_p \subseteq V$ $\mu_p \sim \mu_{p+1}$ for all $p \in \mathbb{N}$, then the sequence is $\sim$-non-decreasing.*

Definition 4 ([27])**.** *If each $\sim$-non-decreasing and Ł-Cauchy sequence is Ł-convergent in V, then a κ-GMS $(V, Ł)$ is $\sim$-non-decreasing complete.*

Remark 2. *Keep in mind that every κ-GMS that is complete also happens to be $\sim$-non-decreasing complete, while the opposite is false, as evidenced by the case below.*

Example 2 ([28])**.** *Let $V = (0, 1]$ be furnished with the metric $d(\mu, \varpi) = |\mu - \varpi|$ for all $\mu, \varpi \in V$. Define a binary relation $\sim$ on V by*

$$\mu \sim \varpi \;\; \textit{if} \;\; 0 < \mu \leq \varpi \leq 1.$$

As a result, (V, d) is a $\sim$-non-decreasing complete κ-GMS, but it is not complete.

Definition 5 ([28])**.** *Let $\Omega : V \to P(V)$ be a multivalued mapping and $(V, Ł)$ be a κ-GMS with a preorder $\sim$. A mapping Ω is known as $\sim$-non-decreasing if for all $\mu, \varpi \in V$*

$$\mu \sim \varpi \quad \textit{implies} \quad s \sim t \quad \textit{for all} \quad s \in \Omega\mu, \;\; t \in \Omega\varpi.$$

Definition 6 ([28])**.** *Let $(V, Ł)$ be a κ-GMS furnished with a preorder $\sim$, $\zeta : V \to V$ and $\Omega : V \to P(V)$ be a multivalued mapping. A Mapping Ω is called $(\zeta, \sim)$-non-decreasing if for all $\mu, \varpi \in V$*

$$\zeta\mu \sim \zeta\varpi \quad \textit{implies} \quad s \sim t \quad \textit{for all} \quad s \in \Omega\mu, \;\; t \in \Omega\varpi.$$

By obtaining inspiration from the work of Derouiche and Ramoul [16] and by following the direction of Saleem et al. [28], in this paper, we prove the coincidence point theorem and common fixed-point theorem in generalized metric spaces for mappings satisfying certain contractive conditions and containing fewer conditions imposed on function $\Im$.

The paper is organized as follows: We renamed the generalized metric space (in the sense of *Jleli* and *Samet*) as κ-generalized metric space and consider the κ-generalized metric space for $\kappa \in (0, 1]$. Then, we derive the common fixed-point and coincidence fixed-point results in the setting of this space. Lastly, by using these results, we proved the existence results of common solutions of fractional boundary value problems.

2. Fundamental Results

We start this section by stating the following:

Lemma 1 ([16])**.** *Let $\vartheta \geq 1$ be a given real number. Let $\{\wp_p\}$ be a sequence and let $\alpha, \beta : (0, \infty) \to \mathbb{R}$ be two functions meeting the aforementioned requirements:*

- *(i)* $\alpha(\vartheta\wp_p) \leq \beta(\wp_{p-1})$*, for all* $p \in \mathbb{N}$*;*
- *(ii)* α *is non-decreasing;*
- *(iii)* $\beta(\wp) < \alpha(\wp)$ *for all* $\wp > 0$*;*
- *(iv)* $\limsup_{p \to \rho^+} \beta(\wp) < \alpha(\rho^+)$ *for all* $\rho > 0$*.*

Then, $\lim_{p \to \infty} \wp_p = 0$*.*

Consistent with [16], we set

$$\hbar_c = \{\Im : (0, \infty) \to \mathbb{R} | \Im \;\; \text{as a continuous non-decreasing function}\}.$$

Let $\varrho \geq 1$ be a given real number. We designate as Λ_ϱ the family of all functions $\lambda : (0, \infty) \to (0, \infty)$ that meet the criterion:

$$\liminf_{\wp \to s} \lambda(\wp) > 0, \quad \text{where } s \in [\gamma^+, \gamma^+ \varrho], \text{ for all } \gamma > 0. \tag{4}$$

Obviously, if $\varrho = 1$, condition (4) becomes as follows:

$$\lim_{\wp \to \gamma^+} \inf \lambda(\wp) > 0, \quad \text{for all } \gamma > 0. \tag{5}$$

Henceforth, we denote by Λ_1 the set Λ_ϱ when $\varrho = 1$. Clearly, we have $\Lambda_\varrho \subseteq \Lambda_1$. Also, observe that in the case of standard metric space, it suffices to use the condition of that $\lambda \in \Lambda_1$ instead of the condition $\lambda \in \Lambda_\varrho$.

For every $\mu \in V$, define

$$\delta(Ł, \zeta, \mu) = \sup\{Ł(\zeta^i \mu, \zeta^j \mu) : i, j \in \mathbb{N}\}.$$

Lemma 2. *Let $(V, Ł)$ be a κ-GMS and let χ be a given real number such that $\chi \geq 1$. Let $\zeta : V \to V, \Omega : V \to C(V)$ and $\{\zeta\mu_p : \zeta\mu_p \in \Omega\mu_{p-1}\}$ be the sequence based on arbitrary point $\mu_0 \in V$ such that $\sup\{Ł(\zeta\mu_l, \zeta\mu_j) : \zeta\mu_l \in \Omega\mu_{l-1}, \zeta\mu_j \in \Omega\mu_{j-1}\} < +\infty$. Assume that there exist a non-decreasing function $\Im$ and $\lambda \in \Lambda_1$ such that for all $\hat{x}, \hat{y} \in V$ with $\zeta\hat{x} \neq \zeta\hat{y}$,*

$$0 < Ł(\hat{a}, \hat{v}) < +\infty \text{ implies } \lambda(Ł(\zeta\hat{x}, \zeta\hat{y})) + \Im(\chi Ł(\hat{a}, \hat{v})) \leq \Im(Ł(\zeta\hat{x}, \zeta\hat{y})), \tag{6}$$

where $\hat{a} \in \Omega\hat{x}$ and $\hat{v} \in \Omega\hat{y}$. Then, $\lim_{p\to\infty} Ł(\zeta\mu_p, \zeta\mu_{p+1}) = 0$.

Proof. Put $Ł_p = Ł(\zeta\mu_p, \zeta\mu_{p+1})$. If $\zeta\mu_p = \zeta\mu_{p+1}$ for some $p \in \mathbb{N}_0$, then the proof is complete. So, assume that $\zeta\mu_p \neq \zeta\mu_{p+1}$ for all $p \in \mathbb{N}_0$. Since $\sup\{Ł(\zeta\mu_l, \zeta\mu_j) : \zeta\mu_l \in \Omega\mu_{l-1}, \zeta\mu_j \in \Omega\mu_{j-1}\} < +\infty$, so we have

$$Ł(\zeta\mu_p, \zeta\mu_{p+1}) < +\infty. \tag{7}$$

We also assume that $Ł(\zeta\mu_p, \zeta\mu_{p+1}) > 0$, otherwise $\zeta\mu_p = \zeta\mu_{p+1}$. Applying the inequality (6) with $\hat{x} = \mu_{p-1}$ and $\hat{y} = \mu_p$, we have for all $p \in \mathbb{N}$

$$\lambda(Ł_{p-1}) + \Im(\chi Ł_p) \leq \Im(Ł_{p-1}), \tag{8}$$

which further implies that

$$\Im(\chi Ł_p) \leq \Im(Ł_{p-1}) - \lambda(Ł_{p-1}), \quad \text{for all } p \in \mathbb{N}. \tag{9}$$

Taking $\alpha(t) = \Im(t)$ and $\beta(t) = \Im(t) - \lambda(t)$ for all $t \in (0, \infty)$, inequality (9) can be written as

$$\alpha(\chi Ł_p) \leq \beta(Ł_{p-1}), \quad \text{for all } p \in \mathbb{N}. \tag{10}$$

As $\Im$ is non-decreasing, then in view of the inequality (10) and using the fact that $\lambda \in \Lambda_1$, it is clear that all the conditions of Lemma 1 with $(\vartheta = \chi \geq 1)$ are satisfied. Thus, $\lim_{p\to\infty} Ł_p = 0$. $\square$

3. Coincidence Point Theorems

In this section, we prove the coincidence point theorems.

Theorem 2. *Let $(V, Ł)$ be a κ-GMS for $\kappa \in (0, 1]$ furnished with a preorder $\sim$, $\zeta : V \to V$ and $\Omega : V \to C(V)$. Assume that there exist $\mu_0, \mu_1 \in V$ such that $\zeta\mu_1 \in \Omega\mu_0$, $\zeta\mu_0 \sim \zeta\mu_1$,*

Ω is an $(\zeta, \sim)$-non-decreasing set-valued mapping and $\sup\{Ł(\zeta\mu_l, \zeta\mu_j) : \zeta\mu_l \in \Omega\mu_{l-1},\ \zeta\mu_j \in \Omega\mu_{j-1}\} < +\infty$*. If there exist $\Im \in \hbar_c$ and $\lambda \in \Lambda_\varrho$ satisfying*

$$0 < Ł(\mu, \varpi) < +\infty \ \text{ implies } \ \lambda(Ł(\zeta\bar{e}, \zeta\bar{f})) + \Im(Ł(\mu, \varpi)) \leq \Im(Ł(\zeta\bar{e}, \zeta\bar{f})) \tag{11}$$

for all $\bar{e}, \bar{f} \in V$ with $\bar{e} \sim \bar{f}$ and $\mu \in \Omega\bar{e}$, $\varpi \in \Omega\bar{f}$. Then there exists a sequence $\{\zeta\mu_p : \zeta\mu_p \in \Omega\mu_{p-1}\}_{p\in\mathbb{N}}$ such that

$$\lim_{p\to+\infty} Ł(\zeta\mu_p, \zeta\mu_{p+1}) = 0.$$

Moreover, if for each $p \in \mathbb{N}$, $\{\mu_p\} \subseteq \zeta(V)$, we have $\{\mu_p\} \to \mu$ implies $\mu_p \sim \mu$ and $\zeta(V)$ is $\sim$-non-decreasing-complete, then there exists $\tau \in V$ such that $\zeta\tau \in \Omega\tau$.

Proof. Let us put $\varrho_i = \zeta\mu_i$. By hypothesis, there exists $\mu_0, \mu_1 \in V$ such that $\zeta\mu_1 \in \Omega\mu_0$ and $\zeta\mu_0 \sim \mu_1$. Construct a sequence $\{\zeta\mu_p : \zeta\mu_p \in \Omega\mu_{p-1}\}$. Since $\sup\{Ł(\zeta\mu_l, \zeta\mu_j) : \zeta\mu_l \in \Omega\mu_{l-1},\ \zeta\mu_j \in \Omega\mu_{j-1}\} < +\infty$, we have

$$Ł(\zeta\mu_l, \zeta\mu_j) < +\infty, \tag{12}$$

for all $\zeta\mu_l, \zeta\mu_j \subseteq \{\zeta\mu_p\}$. There are two cases here:
Case 1:
If $\sup\{Ł(\zeta\mu_l, \zeta\mu_j) : \zeta\mu_l \in \Omega\mu_{l-1},\ \zeta\mu_j \in \Omega\mu_{j-1}\} = 0$, then for all $\zeta\mu_l, \zeta\mu_j \in \{\zeta\mu_p\}$, we obtain

$$0 \leq Ł(\zeta\mu_l, \zeta\mu_j) \leq \sup\{Ł(\zeta\mu_l, \zeta\mu_j) : \zeta\mu_l \in \Omega\mu_{l-1},\ \zeta\mu_j \in \Omega\mu_{j-1}\} = 0,$$

which further gives

$$Ł(\zeta\mu_l, \zeta\mu_j) = 0.$$

In particular,

$$0 \leq Ł(\zeta\mu_0, \Omega\mu_0) \leq Ł(\zeta\mu_0, \zeta\mu_1) = 0.$$

This implies that $Ł(\zeta\mu_0, \Omega\mu_0) = 0$. Since $\Omega\mu_0$ is closed, therefore we obtain $\zeta\mu_0 \in \Omega\mu_0$, that is, $\mu_0 \in \mathcal{C}oi(V, \zeta, \Omega)$.
Case 2:
Let $\sup\{Ł(\zeta\mu_l, \zeta\mu_j) : \zeta\mu_l \in \Omega\mu_{l-1},\ \zeta\mu_j \in \Omega\mu_{j-1}\} > 0$. Assume that $Ł(\zeta\mu_1, \zeta\mu_2) > 0$, where $\zeta\mu_1, \zeta\mu_2 \in \{\zeta\mu_p\}$ otherwise if $Ł(\zeta\mu_1, \zeta\mu_2) = 0$, then

$$0 \leq Ł(\zeta\mu_1, \Omega\mu_1) \leq Ł(\zeta\mu_1, \zeta\mu_2) = 0.$$

This gives $Ł(\zeta\mu_1, \Omega\mu_1) = 0$, since $\Omega\mu_1$ is closed, so, $\zeta\mu_1 \in \Omega\mu_1$. Since Ω is $(\zeta, \sim)$-non-decreasing set-valued mapping, therefore $\zeta\mu_1 \sim \zeta\mu_2$. Hence, from (11), we obtain

$$\lambda(Ł(\zeta\mu_0, \zeta\mu_1)) + \Im(Ł(\zeta\mu_1, \zeta\mu_2)) \leq \Im(Ł(\zeta\mu_0, \zeta\mu_1)).$$

By induction, we have $\{\zeta\mu_p\}_p \in \mathbb{N}$ satisfying $\zeta\mu_p \in \Omega\mu_{p-1}$, $\zeta\mu_p \sim \zeta\mu_{p+1}$, $Ł(\zeta\mu_p, \zeta\mu_{p+1}) > 0$ and

$$\lambda(Ł(\zeta\mu_{p-1}, \zeta\mu_p)) + \Im(Ł(\zeta\mu_p, \zeta\mu_{p+1})) \leq \Im(Ł(\zeta\mu_{p-1}, \zeta\mu_p)), \tag{13}$$

for all $p \in \mathbb{N} \setminus \{0\}$. Putting $\mu_{p-1} = \hat{x}$ and $\mu_p = \hat{y}$ in (13) and using the fact that

$$\hat{a} = \zeta\mu_p \neq \zeta\mu_{p+1} = \hat{v},$$

the inequality (13) turns into (6). Therefore, by virtue of $\Lambda_\varrho \subseteq \Lambda_1$ and Lemma 2 with $\chi = 1$, we have

$$\lim_{p\to\infty} Ł(\zeta\mu_p, \zeta\mu_{p+1}) = 0. \tag{14}$$

Next, we prove that

$$\lim_{p,q\to\infty} Ł(\zeta\mu_p, \zeta\mu_q) = 0. \tag{15}$$

If (15) is not true, then there exists $\eta > 0$ such that for all $r \geq 0$, there exist $q_\ell > p_\ell > r$

$$Ł(\zeta\mu_p, \zeta\mu_q) \geq \eta. \tag{16}$$

Also, there exists $r_0 \in \mathbb{N}$ such that

$$\eta_{r_0} = Ł(\zeta\mu_{p-1}, \zeta\mu_p) < \eta \quad \text{for all } \ p \geq r_0. \tag{17}$$

Consider two subsequences $\{\zeta\mu_{p_\ell}\}$ and $\{\zeta\mu_{q_\ell}\}$ of $\{\zeta\mu_p\}$ satisfying

$$r_0 \leq p_\ell \leq q_\ell + 1 \quad \text{and} \quad Ł(\zeta\mu_{q_\ell}, \zeta\mu_{p_\ell}) \geq \eta \quad \text{for all } \ \ell > 0. \tag{18}$$

Observe that

$$Ł(\zeta\mu_{q_\ell - 1}, \zeta\mu_{p_\ell}) < \eta \quad \text{for all } \ \ell, \tag{19}$$

where q_ℓ is chosen as minimal index for which (19) is satisfied. Also, note that because of (18) and (19), the case $p_\ell + 1 \leq p_\ell$ is impossible. Thus, $p_\ell + 1 \leq q_\ell$ for all ℓ. It implies

$$p_\ell + 1 < q_\ell < q_\ell + 1. \tag{20}$$

From (14), we have

$$\lim_{\ell\to\infty} Ł(\zeta\mu_{q_\ell - 1}, \zeta\mu_{q_\ell}) = 0. \tag{21}$$

By using (18)–(21), ($Ł_3$) and using the fact that $\kappa \in (0, 1]$, we have

$$\begin{aligned} \eta \leq \lim_{\ell\to\infty} Ł(\zeta\mu_{q_\ell}, \zeta\mu_{p_\ell}) &\leq \kappa \limsup_{\ell\to\infty} Ł(\zeta\mu_{q_\ell - 1}, \zeta\mu_{p_\ell}) \\ &\leq \kappa\eta \leq \eta. \end{aligned}$$

The above inequality leads to

$$\lim_{\ell\to\infty} Ł(\zeta\mu_{q_\ell}, \zeta\mu_{p_\ell}) = \eta. \tag{22}$$

Next, by using (14) and (22), we have

$$Ł(\zeta\mu_{q_\ell}, \zeta\mu_{p_\ell + 1}) \leq \kappa \limsup_{\ell\to\infty} Ł(\zeta\mu_{q_\ell + 1}, \zeta\mu_{p_\ell + 1}) \tag{23}$$

and

$$Ł(\zeta\mu_{q_\ell}, \zeta\mu_{p_\ell}) \leq \kappa \limsup_{\ell\to\infty} Ł(\zeta\mu_{q_\ell}, \zeta\mu_{p_\ell + 1}). \tag{24}$$

Combining (23) and (24) with (18), we obtain

$$\begin{aligned}
\eta \leq Ł(\zeta\mu_{q_\ell}, \zeta\mu_{p_\ell}) &\leq \kappa^2 \limsup_{\ell\to\infty} Ł(\zeta\mu_{q_\ell+1}, \zeta\mu_{p_\ell+1}) \\
&\leq \limsup_{\ell\to\infty} Ł(\zeta\mu_{q_\ell+1}, \zeta\mu_{p_\ell+1}) \\
&\leq \kappa^2 \limsup_{\ell\to\infty} Ł(\zeta\mu_{q_\ell}, \zeta\mu_{p_\ell}) \\
&\leq \kappa^2 \eta \\
&\leq \eta,
\end{aligned}$$

which further implies that

$$\lim_{\ell\to\infty} Ł(\zeta\mu_{q_\ell+1}, \zeta\mu_{p_\ell+1}) = \eta, \tag{25}$$

For convenience, we set

$$a_\ell = Ł(\zeta\mu_{q_\ell}, \zeta\mu_{p_\ell}), \quad \text{and} \quad b_\ell = Ł(\zeta\mu_{q_\ell+1}, \zeta\mu_{p_\ell+1}).$$

We claim that, $b_\ell > 0$. If not, then $\mu_{q_\ell+1} = \mu_{p_\ell+1}$. This gives $\zeta\mu_{q_\ell+1} \in \Omega\mu_{p_\ell}$, which is contradiction to the fact that $\zeta\mu_i \notin \Omega\mu_j$ for each $i > j$. Further, since $\sim$ is pre-order, by transitivity, we have $\zeta\mu_{q_\ell+1} \sim \zeta\mu_{p_\ell+1}$ for each $q, p \in \mathbb{N}$, $q \geq p$. Then, by using (11) and the monotonicity of $\Im$, we obtain

$$\begin{aligned}
\lim_{t\to\eta} \lambda(t) + \Im(\eta) &\leq \lim_{\ell\to\infty} \lambda(a_\ell) + \Im(\eta) \\
&\leq \liminf_{\ell\to\infty} \lambda(a_\ell) + \Im(\liminf_{\ell\to\infty} b_\ell) \\
&= \liminf_{\ell\to\infty} \lambda(a_\ell) + \liminf_{\ell\to\infty} \Im(b_\ell) \\
&= \liminf_{\ell\to\infty} (\lambda(a_\ell) + \Im(b_\ell)) \\
&\leq \lim_{\ell\to\infty} [\lambda(a_\ell) + \Im(b_\ell)] \\
&\leq \lim_{\ell\to\infty} \Im(a_\ell) \\
&= \Im\left(\lim_{\ell\to\infty} a_\ell\right) \\
&= \Im(\eta).
\end{aligned}$$

The preceding inequality implies that

$$\liminf_{t\to\eta} \lambda(t) \leq 0, \quad \text{where } t \in [\varepsilon^+, \varepsilon^+\varrho], \text{ for all } \varepsilon > 0, \tag{26}$$

which is a contradiction with (4). Hence, our assumption that (15) is not true is wrong. Thus, $\{\zeta\mu_p\}$ is Ł-Cauchy sequence. Since $\zeta(V)$ is $\sim$-non-decreasing complete, there is a point $\zeta\tau \in \zeta(V)$ such that $\{\zeta\mu_p\} \xrightarrow{Ł} \zeta\tau$. Also, by hypothesis, $\zeta\mu_p \sim \zeta\tau$, then there exists $\zeta\tau_p \in \Omega\tau$ such that $Ł(\zeta\mu_p, \zeta\tau_p) > 0$, otherwise $\zeta\mu_p = \zeta\tau_p$ and $\{\zeta\tau_p\} \xrightarrow{Ł} \zeta\tau$. Consequently, $\zeta\tau \in \Omega\tau$. Therefore, from (11), we have $\zeta\tau_p \in \Omega\tau$ satisfying

$$\lambda(Ł(\zeta\mu_{p-1}, \zeta\tau)) + \Im(Ł(\zeta\mu_p, \zeta\tau_p)) \leq \Im(Ł(\zeta\mu_{p-1}, \zeta\tau)). \tag{27}$$

By using (27) and monotonicity of $\Im$, we obtain

$$0 < Ł(\zeta\mu_p, \zeta\tau_p) \leq Ł(\zeta\mu_{p-1}, \zeta\tau). \tag{28}$$

Since $\{\zeta\mu_p\} \xrightarrow{Ł} \zeta\tau$, by letting $p \to +\infty$ in (27), we obtain

$$\lim_{p\to+\infty} Ł(\zeta\mu_p, \zeta\tau_p) = 0. \tag{29}$$

By using $(Ł_3)$, we obtain

$$0 \le Ł(\zeta\tau_p, \zeta\tau) \le \kappa \lim_{p\to+\infty} \sup Ł(\zeta\mu_p, \zeta\tau) = 0, \tag{30}$$

which implies

$$\lim_{p\to+\infty} Ł(\zeta\tau_p, \zeta\tau) = 0.$$

From the closeness of $\Omega\tau$, we have $\zeta\tau \in \Omega\tau$. Hence, $\tau \in \mathcal{C}oi(V, \zeta, \Omega)$. □

Example 3. *Let $V = \{0,1\}$ be endowed with $Ł : V \times V \to [0,\infty]$ given by*

$$Ł(0,0) = 0, \quad Ł(1,1) = 1 \quad \textit{and} \quad Ł(1,0) = Ł(0,1) = \infty.$$

Then V is a κ-GMS for $\kappa = 1$. Indeed, properties $(Ł_1)$ and $(Ł_2)$ are apparent. To prove $(Ł_3)$, let $\bar{e}, \bar{f} \in V$ and $\{\bar{e}_p\} \in C(Ł, V, \bar{e})$. Since

$$\lim_{p\to\infty} Ł(\bar{e}_p, \bar{e}) = 0,$$

there exists $p_0 \in \mathbb{N}$ such that $\bar{e}_p = \bar{e}$ for all $p \ge p_0$. If $\bar{e} = \bar{f}$, then $\bar{e}_p = \bar{e} = \bar{f}$ for all $p \ge p_0$, so $(Ł_3)$ holds for $\kappa = 1$. Similarly, if $\bar{e} \ne \bar{f}$, then $\bar{e}_p \ne \bar{f}$ for all $p \ge p_0$, so

$$Ł(\bar{e}, \bar{f}) = \infty = Ł(\bar{e}_p, \bar{f}) \quad \textit{for all} \quad p \ge p_0.$$

In any case, $(Ł_3)$ holds with $\kappa = 1$.

Let $\zeta : V \to V$ and $\Omega : V \to C(V)$ be mappings given by

$$\zeta(0) = 1, \quad \zeta(1) = 0 \quad \textit{and} \quad \Omega(0) = \Omega(1) = \{0,1\}.$$

Define a relation $\sim$ on V by

$$\bar{e} \sim \bar{f} \quad \textit{if} \quad \bar{e} = \bar{f},$$

then $\sim$ is a preorder, Ω is an $(\zeta, \sim)$-non-decreasing set-valued mapping, and $\zeta(V)$ is $\sim$-non-decreasing complete.

Observe that $0 < Ł(\mu, \varpi) < +\infty$ for $\mu \in \Omega\bar{e}$ and $\varpi \in \Omega\bar{f}$ with $\bar{e} \sim \bar{f}$ only when $\bar{e} = \bar{f} \in \{0,1\}$. So, there arise two cases:

Case: I When $\bar{e} = \bar{f} = 0$, then

$$\begin{aligned} Ł(\zeta\bar{e}, \zeta\bar{f})(Ł(\mu, \varpi) + 1) &= Ł(1,1)(Ł(1,1) + 1) \\ &= 2 \\ &= Ł(1,1) + 1 \\ &= Ł(\zeta\bar{e}, \zeta\bar{f}) + 1. \end{aligned}$$

Hence, in this case (11) holds true for $\Im(t) = \ln(t+1)$ and $\lambda(t) = \ln(t)$ for all $t \in (0,\infty)$.

Case: II When $\bar{e} = \bar{f} = 1$, then

$$\begin{aligned} Ł(\zeta\bar{e}, \zeta\bar{f})(Ł(\mu, \varpi) + 1) &= Ł(0,0)(Ł(1,1) + 1) \\ &= 0 \\ &< 1 \\ &= Ł(0,0) + 1 \\ &= Ł(\zeta\bar{e}, \zeta\bar{f}) + 1. \end{aligned}$$

So, in this case, inequality (11) holds true for $\Im(t) = \ln(t+1)$ and $\lambda(t) = \ln(t)$ for all $t \in (0,\infty)$.
Hence, all the conditions of Theorem 2 are fulfilled and $\{0,1\}$ is the set of coincidence points of ζ and Ω.

Remark 3. *Note that in Example 3, the function $\Im : (0,+\infty) \to \mathbb{R}$ defined by $\Im(t) = \ln(t+1)$ belongs to $\hbar_c$. But $\Im$ does not satisfy $(\Im_2)$. Indeed, for any sequence $\pi_p \in (0,+\infty)$ such that $\lim_{p\to\infty} \pi_p = 0$, we have*

$$\begin{aligned}\lim_{p\to\infty} \Im(\pi_p) &= \lim_{p\to\infty} \ln(1+\pi_p) \\ &= \ln(1+\lim_{p\to\infty} \pi_p) \\ &= 0 \neq -\infty.\end{aligned}$$

Next, from Theorem 2 we obtain the following by using the fact that a partial order $\ll$ is a preorder $\sim$.

Corollary 1. *Let $(V,Ł)$ be a κ-GMS for $\kappa \in (0,1]$ furnished with a partial order $\ll$, $\zeta : V \to V$ and $\Omega : V \to C(V)$. Assume that there exist $\mu_0, \mu_1 \in V$ such that $\zeta\mu_1 \in \Omega\mu_0$, $\zeta\mu_0 \ll \zeta\mu_1$, Ω is an $(\zeta,\ll)$-non-decreasing set-valued mapping and $\sup\{Ł(\zeta\mu_l,\zeta\mu_j) : \zeta\mu_l \in \Omega\mu_{l-1},\ \zeta\mu_j \in \Omega\mu_{j-1}\} < +\infty$. If there exist $\Im \in \hbar_c$ and $\lambda \in \Lambda_\varrho$ satisfying (11) for all $\bar{e},\bar{f} \in V$ with $\bar{e} \ll \bar{f}$ and $\mu \in \Omega\bar{e}$, $\varpi \in \Omega\bar{f}$, then there exists a sequence $\{\zeta\mu_p : \zeta\mu_p \in \Omega\mu_{p-1}\}_{P\in\mathbb{N}}$ such that*

$$\lim_{p\to+\infty} Ł(\zeta\mu_p,\zeta\mu_{p+1}) = 0.$$

Moreover, if for all $\{\mu_p\} \subseteq \zeta(V)$ we have $\{\mu_p\} \to \mu$ implies $\mu_p \sim \mu$ for all $p \in \mathbb{N}$ and $\zeta(V)$ is $\ll$-non-decreasing-complete, then there exists $\tau \in V$ such that $\zeta\tau \in \Omega\tau$.

In the light of Remark 2, Theorem 2 gives the following corollary:

Corollary 2. *Let $(V,Ł)$ be a κ-GMS for $\kappa \in (0,1]$, $\zeta : V \to V$ and $\Omega : V \to C(V)$. Assume that there exist $\mu_0, \mu_1 \in V$ such that $\zeta\mu_1 \in \Omega\mu_0$ and $\sup\{Ł(\zeta\mu_l,\zeta\mu_j) : \zeta\mu_l \in \Omega\mu_{l-1},\ \zeta\mu_j \in \Omega\mu_{j-1}\} < +\infty$. If there exist $\Im \in \hbar_c$ and $\lambda \in \Lambda_\varrho$ satisfying*

$$0 < Ł(\mu,\varpi) < +\infty \ \textit{ implies } \ \lambda(Ł(\zeta\bar{e},\zeta\bar{f})) + \Im(Ł(\mu,\varpi)) \le \Im(Ł(\zeta\bar{e},\zeta\bar{f})) \tag{31}$$

for all $\bar{e},\bar{f} \in V$ and $\mu \in \Omega\bar{e}$, $\varpi \in \Omega\bar{f}$. Then there exists a sequence $\{\zeta\mu_p : \zeta\mu_p \in \Omega\mu_{p-1}\}_{P\in\mathbb{N}}$ such that

$$\lim_{p\to+\infty} Ł(\zeta\mu_p,\zeta\mu_{p+1}) = 0.$$

Moreover, if $\zeta(V)$ is complete, then there exists $\tau \in V$ such that $\xi\tau \in \Omega\tau$.

By defining $\zeta = I$ (identity mapping) in Theorem 2, we obtain the following:

Corollary 3. *Let $(V,Ł)$ be a κ-GMS for $\kappa \in (0,1]$ furnished with a preorder $\sim$ and $\Omega : V \to C(V)$. Assume that there exist $\mu_0, \mu_1 \in V$ such that $\mu_1 \in \Omega\mu_0$, $\mu_0 \sim \mu_1$, Ω is an $\sim$-non-decreasing set-valued mapping and $\sup\{Ł(\mu_l,\mu_j) : \mu_l \in \Omega\mu_{l-1},\ \mu_j \in \Omega\mu_{j-1}\} < +\infty$. If there exist $\Im \in \hbar_c$ and $\lambda \in \Lambda_\varrho$ satisfying*

$$0 < Ł(\mu,\varpi) < +\infty \ \textit{ implies } \ \lambda(Ł(\bar{e},\bar{f})) + \Im(Ł(\mu,\varpi)) \le \Im(Ł(\bar{e},\bar{f})) \tag{32}$$

for all $\bar{e},\bar{f} \in V$ with $\bar{e} \sim \bar{f}$ and $\mu \in \Omega\bar{e}$, $\varpi \in \Omega\bar{f}$. Then, there exists a sequence $\{\mu_p : \mu_p \in \Omega\mu_{p-1}\}_{P\in\mathbb{N}}$ such that

$$\lim_{p\to+\infty} Ł(\mu_p, \mu_{p+1}) = 0.$$

Moreover, if for all $p \in \mathbb{N}$, $\{\mu_p\} \subseteq V$ *we have* $\{\mu_p\} \to \mu$ *implies* $\mu_p \sim \mu$ *and* V *is* $\sim$*-non-decreasing-complete, then there exists* $\tau \in V$ *such that* $\tau \in \Omega\tau$.

Since a standard metric space is a κ-GMS for $\kappa = 1$, by the virtue of Theorem 2 we obtain the following:

Corollary 4. *Let* (V, d) *be a metric space furnished with a preorder* $\sim$, $\zeta : V \to V$ *and* $\Omega : V \to C(V)$. *Assume that there exist* $\mu_0, \mu_1 \in V$ *such that* $\zeta\mu_1 \in \Omega\mu_0$, $\zeta\mu_0 \sim \zeta\mu_1$, Ω *is an* $(\zeta, \sim)$*-non-decreasing set-valued mapping. If there exist* $\Im \in \hbar_c$ *and* $\lambda \in \Lambda_1$ *satisfying*

$$d(\mu, \varpi) > 0 \ \textit{implies} \ \lambda(d(\xi\bar{e}, \zeta\bar{f})) + \Im(d(\mu, \varpi)) \leq \Im(d(\zeta\bar{e}, \zeta\bar{f})) \tag{33}$$

for all $\bar{e}, \bar{f} \in V$ *with* $\bar{e} \sim \bar{f}$ *and* $\mu \in \Omega\bar{e}$, $\varpi \in \Omega\bar{f}$. *Then, there exists a sequence* $\{\zeta\mu_p : \zeta\mu_p \in \Omega\mu_{p-1}\}_{P\in\mathbb{N}}$ *such that*

$$\lim_{p\to+\infty} d(\zeta\mu_p, \zeta\mu_{p+1}) = 0.$$

Moreover, if for each $p \in \mathbb{N}$, $\{\mu_p\} \subseteq \zeta(V)$ *we have* $\{\mu_p\} \to \mu$ *implies* $\mu_p \sim \mu$ *and* $\zeta(V)$ *is* $\sim$*-non-decreasing-complete, then there exists* $\tau \in V$ *such that* $\zeta\tau \in \Omega\tau$.

4. Common Fixed-Point Theorems

Theorem 3. *Let* $(V, Ł)$ *be a complete* κ*-GMS for* $\kappa \in (0, 1]$ *and* $\Omega_1, \Omega_2 : V \to C(V)$. *Assume that there exist* $\mu_0, \mu_1, \mu_2 \in V$ *such that* $\mu_1 \in \Omega_1\mu_0$, $\mu_2 \in \Omega_2\mu_1$ *and* $\sup\{Ł(\mu_{2k+1}, \mu_{2j+2}) : \mu_{2k+l} \in \Omega_1\mu_{2k},\ \mu_{2j+2} \in \Omega_2\mu_{2j+1}\} < +\infty$. *If there exist* $\Im \in \hbar_c$ *and* $\lambda \in \Lambda_\varrho$ *satisfying*

$$0 < Ł(\mu, \varpi) < +\infty \ \textit{implies} \ \lambda(Ł(\bar{e}, \bar{f})) + \Im(Ł(\mu, \varpi)) \leq \Im(Ł(\bar{e}, \bar{f})) \tag{34}$$

for all $\bar{e}, \bar{f} \in V$ *with* $\mu \in \Omega_1\bar{e}$ *and* $\varpi \in \Omega_2\bar{f}$. *Then*

1. *There exists a sequence* $\{\mu_p : \mu_{2p+1} \in \Omega_1\mu_{2p},\ \mu_{2p+2} \in \Omega_2\mu_{2p+1}\}_{p\in\mathbb{N}}$ *such that*

$$\lim_{p\to+\infty} Ł(\mu_p, \mu_{p+1}) = 0;$$

2. $\{\mu_p\}$ *is Ł-Cauchy;*
3. Ω_1 *and* Ω_2 *owns a common fixed point in* V.

Proof. By hypothesis, there exist $\mu_0, \mu_1, \mu_2 \in V$ such that $\mu_1 \in \Omega_1\mu_0$ and $\mu_2 \in \Omega_2\mu_1$. Construct a sequence $\{\mu_p : \mu_{2p+1} \in \Omega_1\mu_{2p},\ \mu_{2p+2} \in \Omega_2\mu_{2p+1}\}_{p\in\mathbb{N}}$. Firstly, note that

$$Ł(\mu_{2k+1}, \mu_{2j+2}) < +\infty, \tag{35}$$

for all $\mu_{2k+l} \in \Omega_1\mu_{2k},\ \mu_{2j+2} \in \Omega_2\mu_{2j+1}$ because $\sup\{Ł(\mu_{2k+1}, \mu_{2j+2}) : \mu_{2k+l} \in \Omega_1\mu_{2k},\ \mu_{2j+2} \in \Omega_2\mu_{2j+1}\} < +\infty$.

Now if $\mu_1 \in \Omega_1\mu_1 \cap \Omega_2\mu_1$, then μ_1 is a common fixed point of Ω_1 and Ω_2, so let $\mu_1 \notin \Omega_1\mu_1$. Consequently, we assert that $Ł(\mu_1, \mu_2) > 0$. Hence, from (34), we obtain

$$\lambda(Ł(\mu_0, \mu_1)) + \Im(Ł(\mu_1, \mu_2)) \leq \Im(Ł(\mu_0, \mu_1)).$$

Next, if $\mu_2 \in \Omega_1\mu_2 \cap \Omega_2\mu_2$, then μ_2 is a common fixed point of Ω_1 and Ω_2, so let $\mu_2 \notin \Omega_1\mu_2$. Consequently, we assert that $Ł(\mu_2, \mu_3) > 0$. Hence, from (34), we obtain

$$\lambda(Ł(\mu_1, \mu_2)) + \Im(Ł(\mu_2, \mu_3)) \leq \Im(Ł(\mu_1, \mu_2)).$$

By induction, we have $\{\mu_p\}_p \in \mathbb{N}$ such that $\mu_{2p+1} \in \Omega_1\mu_{2p}$ and $\mu_{2p+2} \in \Omega_2\mu_{2p+1}$ with $\mu_{2p+1} \notin \Omega_1\mu_{2p+1}$, $\mu_{2p} \notin \Omega_2\mu_{2p}$ and $Ł(\zeta\mu_p, \zeta\mu_{p+1}) > 0$ satisfying

$$\lambda(Ł(\mu_{2p-1}, \mu_{2p})) + \Im(Ł(\mu_{2p}, \mu_{2p+1})) \leq \Im(Ł(\mu_{2p-1}, \mu_{2p})), \tag{36}$$

for all $p \in \mathbb{N} \setminus \{0\}$. Putting $\mu_{2p-1} = \hat{x}$ and $\mu_{2p} = \hat{y}$ in (36) and using the fact that

$$\hat{a} = \zeta\mu_{2p} \neq \zeta\mu_{2p+1} = \hat{v},$$

the inequality (36) turns into (6). Therefore, by virtue of $\Lambda_\varrho \subseteq \Lambda_1$ and Lemma 2 with $\chi = 1$ and $\zeta = \mathcal{I}$(identity mapping), we have

$$\lim_{p\to\infty} Ł(\mu_2 p, \mu_{2p+1}) = 0. \tag{37}$$

Next, we prove that

$$\lim_{p,q\to\infty} Ł(\mu_{2q+1}, \mu_{2p}) = 0. \tag{38}$$

If (38) is not true, then there exists $\eta > 0$ such that for all $r \geq 0$, there exist $q_\ell > p_\ell > r$

$$Ł(\mu_{2q+1}, \mu_{2p}) > \eta. \tag{39}$$

Also, there exists $r_0 \in \mathbb{N}$ such that

$$\eta_{r_0} = Ł(\zeta\mu_{2p-1}, \zeta\mu_{2p}) < \eta \quad \text{for all } \ 2p \geq r_0. \tag{40}$$

Consider two subsequences $\{\mu_{2p_\ell}\}$ and $\{\mu_{2q_\ell}\}$ of $\{\mu_p\}$ satisfying

$$r_0 \leq 2p_\ell \leq 2q_\ell + 2 \quad \text{and} \quad Ł(\zeta\mu_{2q_\ell+1}, \zeta\mu_{2p_\ell}) > \eta \quad \text{for all } \ \ell > 0. \tag{41}$$

Observe that

$$Ł(\zeta\mu_{2q_\ell}, \zeta\mu_{2p_\ell}) \leq \eta \quad \text{for all } \ \ell, \tag{42}$$

where $2q_\ell$ is chosen as minimal index for which (42) is satisfied. Also, note that because of (41) and (42), the case $2p_\ell + 1 \leq 2p_\ell$ is impossible. Thus, $2p_\ell + 1 \leq 2q_\ell$ for all ℓ. It implies

$$2p_\ell + 1 < 2q_\ell < 2q_\ell + 1. \tag{43}$$

From (37), we have

$$\lim_{\ell\to\infty} Ł(\mu_{2q_\ell+1}, \mu_{2q_\ell}) = 0. \tag{44}$$

By using (41)–(44), we obtain

$$\begin{aligned} \eta < Ł(\mu_{2q_\ell+1}, \mu_{2p_\ell}) &\leq \kappa \limsup_{\ell\to\infty} Ł(\mu_{2q_\ell}, \mu_{2p_\ell}) \\ &\leq \kappa\eta. \end{aligned}$$

By using the fact that $\kappa \in (0, 1]$, above inequality leads to

$$\lim_{\ell\to\infty} Ł(\mu_{2q_\ell+1}, \mu_{2p_\ell}) = \eta. \tag{45}$$

Next, by using (37), we have

$$Ł(\mu_{2q_\ell+1}, \mu_{2p_\ell+1}) \leq \kappa \limsup_{\ell\to\infty} Ł(\mu_{2q_\ell+2}, \mu_{2p_\ell+1}) \tag{46}$$

and

$$Ł(\mu_{2q_\ell+1}, \mu_{2p_\ell}) \leq \kappa \limsup_{\ell\to\infty} Ł(\mu_{2q_\ell+1}, \mu_{2p_\ell+1}). \tag{47}$$

Combining (46) and (47) with (45), we obtain

$$\begin{aligned} \eta < Ł(\mu_{2q_\ell+1}, \mu_{2p_\ell}) &\leq \kappa^2 \limsup_{\ell\to\infty} Ł(\mu_{2q_\ell+2}, \mu_{2p_\ell+1}) \\ &\leq \limsup_{\ell\to\infty} Ł(\mu_{2q_\ell+2}, \mu_{2p_\ell+1}) \\ &\leq \kappa^2 \limsup_{\ell\to\infty} Ł(\mu_{2q_\ell+1}, \mu_{2p_\ell}) \\ &\leq \kappa^2 \eta \\ &\leq \eta, \end{aligned}$$

which further implies that

$$\eta < \lim_{\ell\to\infty} Ł(\mu_{2q_\ell+2}, \mu_{2p_\ell+1}) \leq \eta. \tag{48}$$

Thus, we have

$$\lim_{\ell\to\infty} Ł(\mu_{2q_\ell+2}, \mu_{2p_\ell+1}) = \eta. \tag{49}$$

For convenience, we set

$$\alpha_\ell = Ł(\mu_{2q_\ell+1}, \mu_{2p_\ell}), \quad \beta_\ell = Ł(\mu_{2q_\ell+2}, \mu_{2p_\ell+1}).$$

We claim that $\beta_\ell > 0$. If not then $\mu_{2q_\ell+2} = \mu_{2p_\ell+1}$. This gives $\mu_{2q_\ell+2} \in \mu_{2p_\ell}$, which is contradiction. Then, by using (34) and the monotonicity of $\Im$, we obtain

$$\begin{aligned} \lim_{t\to\eta} \lambda(t) + \Im(\eta) &\leq \lim_{\ell\to\infty} \lambda(\alpha_\ell) + \Im(\eta) \\ &\leq \liminf_{\ell\to\infty} \lambda(\alpha_\ell) + \Im(\liminf_{\ell\to\infty} \beta_\ell) \\ &= \liminf_{\ell\to\infty} \lambda(\alpha_\ell) + \liminf_{\ell\to\infty} \Im(\beta_\ell) \\ &= \liminf_{\ell\to\infty} (\lambda(\alpha_\ell) + \Im(\beta_\ell)) \\ &\leq \lim_{\ell\to\infty} [\lambda(\alpha_\ell) + \Im(\beta_\ell)] \\ &\leq \lim_{\ell\to\infty} \Im(\alpha_\ell) \\ &= \Im\left(\lim_{\ell\to\infty} \alpha_\ell\right) \\ &= \Im(\eta). \end{aligned}$$

The preceding inequality implies that

$$\liminf_{t\to\eta} \lambda(t) \leq 0, \quad \text{where } t \in [\varepsilon^+, \varepsilon^+ \varrho], \text{ for all } \varepsilon > 0, \tag{50}$$

which is a contradiction with (4). This contradiction shows that $\{\mu_p\}$ is Ł-Cauchy sequence.

Since V is complete, there exists a point $\tau \in V$ such that $\{\mu_p\} \xrightarrow{Ł} \tau$. From (34), for $p \in \mathbb{N}$ and $\mu_{2p+1} \in \Omega_1 \mu_{2p}$, there exists $\tau_{2p} \in \Omega_2 \tau$ satisfying

$$\lambda(Ł(\mu_{2p}, \tau)) + \Im(Ł(\mu_{2p+1}, \tau_{2p})) \leq \Im(Ł(\mu_{2p}, \tau)). \tag{51}$$

By using (51) and monotonicity of $\Im$, we obtain

$$0 < Ł(\mu_{2p+1}, \tau_{2p}) \leq Ł(\mu_{2p}, \tau). \tag{52}$$

Since $\{\mu_p\} \xrightarrow{Ł} \tau$, so by letting $p \to +\infty$ in (55), we obtain

$$\lim_{p\to+\infty} Ł(\mu_{2p+1}, \tau_{2p}) = 0. \tag{53}$$

By using $(Ł_3)$, we obtain

$$0 \leq Ł(\tau_{2p}, \tau) \leq \kappa \lim_{p\to+\infty} \sup Ł(\mu_{2p+1}, \tau) = 0, \tag{54}$$

which implies

$$\lim_{p\to+\infty} Ł(\tau_{2p}, \tau) = 0.$$

Since $\Omega_2\tau$ is closed, we have $\tau \in \Omega_2\tau$.

Similarly, from (34), for $p \in \mathbb{N}$ and $\mu_{2p} \in \Omega_2\mu_{2p-1}$, there exists $\tau_{2p+1} \in \Omega_1\tau$ satisfying

$$\lambda(Ł(\tau, \mu_{2p-1})) + \Im(Ł(\tau_{2p+1}, \mu_{2p})) \leq \Im(Ł(\tau, \mu_{2p-1})). \tag{55}$$

By using (55) and monotonicity of $\Im$, we obtain

$$0 < Ł(\tau_{2p+1}, \mu_{2p}) \leq Ł(\tau, \mu_{2p-1}). \tag{56}$$

Since $\{\mu_p\} \xrightarrow{Ł} \tau$, so by letting $p \to +\infty$ in (56), we obtain

$$\lim_{p\to+\infty} Ł(\tau_{2p+1}, \mu_{2p}) = 0. \tag{57}$$

By using $(Ł_3)$, we obtain

$$0 \leq Ł(\tau_{2p+1}, \tau) \leq \kappa \lim_{p\to+\infty} \sup Ł(\mu_{2p}, \tau) = 0, \tag{58}$$

which implies

$$\lim_{p\to+\infty} Ł(\tau_{2p+1}, \tau) = 0.$$

Since $\Omega_1\tau$ is closed, we have $\tau \in \Omega_1\tau$. Hence, $\tau \in \Omega_1\tau \cap \Omega_2\tau$. $\square$

Example 4. *Let $V = \{0, 1, 2\}$ be endowed with $Ł : V \times V \to [0, \infty]$ given by*

$$Ł(\bar{e}, \bar{f}) = \begin{cases} \bar{e} & \text{if} \quad \bar{e} = \bar{f} \\ \infty & \text{if} \quad \bar{e} \neq \bar{f}. \end{cases}$$

Then, V is a κ-GMS for $\kappa = 1$. Indeed, properties $(Ł_1)$ and $(Ł_2)$ are apparent. To prove $(Ł_3)$, let $\bar{e}, \bar{f} \in V$ and $\{\bar{e}_p\} \in C(Ł, V, \bar{e})$. Since

$$\lim_{p\to\infty} Ł(\bar{e}_p, \bar{e}) = 0,$$

there exists $p_0 \in \mathbb{N}$ such that $\bar{e}_p = \bar{e}$ for all $p \geq p_0$. If $\bar{e} = \bar{f}$, then $\bar{e}_p = \bar{e} = \bar{f}$ for all $p \geq p_0$, so $(Ł_3)$ holds for $\kappa = 1$. Similarly, if $\bar{e} \neq \bar{f}$, then $\bar{e}_p \neq \bar{f}$ for all $p \geq p_0$, so

$$Ł(\bar{e}, \bar{f}) = \infty = Ł(\bar{e}_p, \bar{f}) \quad \text{for all} \quad p \geq p_0.$$

In any case, $(Ł_3)$ holds with $\kappa = 1$.

Let $\Omega_1, \Omega_2 : V \to C(V)$ be mappings given by

$$\Omega_1(\bar{e}) = \begin{cases} \{0,1\} & \text{if} \quad \bar{e} \in \{0,1\} \\ \{2\} & \text{if} \quad \bar{e} = 2 \end{cases}$$

and

$$\Omega_2(\bar{e}) = \begin{cases} \{0,2\} & \text{if} \quad \bar{e} \in \{0,1\} \\ \{1\} & \text{if} \quad \bar{e} = 2. \end{cases}$$

Suppose that $0 < Ł(\mu, \varpi) < +\infty$ for $\mu \in \Omega_1\bar{e}$ and $\varpi \in \Omega_2\bar{f}$. Then, we have the following cases:

Case: I *When $\bar{e} = 0, \ \bar{f} = 2$, then there exist $1 \in \Omega_1\bar{e}$ and $1 \in \Omega_2\bar{f}$ such that $0 < Ł(1,1) = 1 < +\infty$. So, for any $\theta > 0$, we have*

$$Ł(\mu, \varpi) = Ł(1,1) = 1 < \infty = Ł(0,2)e^{-\theta}.$$

Hence, in this case (34) holds true for $\Im(t) = \ln(t)$ and $\lambda(t) = \vartheta$ for all $t \in (0,\infty)$ and $\theta > 0$.

Case: II *When $\bar{e} = 1, \ \bar{f} = 2$, then there exist $1 \in \Omega_1\bar{e}$ and $1 \in \Omega_2\bar{f}$ such that $0 < Ł(1,1) = 1 < +\infty$. So, for any $\theta > 0$, we have*

$$Ł(\mu, \varpi) = Ł(1,1) = 1 < \infty = Ł(1,2)e^{-\theta}.$$

Hence, in this case (34) holds true for $\Im(t) = \ln(t)$ and $\lambda(t) = \vartheta$ for all $t \in (0,\infty)$ and $\theta > 0$.

Case: III *When $\bar{e} = 2, \ \bar{f} = 0$, then there exist $2 \in \Omega_1\bar{e}$ and $2 \in \Omega_2\bar{f}$ such that $0 < Ł(2,2) = 2 < +\infty$. So, for any $\theta > 0$, we have*

$$Ł(\mu, \varpi) = Ł(2,2) = 2 < \infty = Ł(2,0)e^{-\theta}.$$

Hence, in this case (34) holds true for $\Im(t) = \ln(t)$ and $\lambda(t) = \vartheta$ for all $t \in (0,\infty)$ and $\theta > 0$.

Case: IV *When $\bar{e} = 2, \ \bar{f} = 1$, then there exist $2 \in \Omega_1\bar{e}$ and $2 \in \Omega_2\bar{f}$ such that $0 < Ł(2,2) = 2 < +\infty$. So, for any $\theta > 0$, we have*

$$Ł(\mu, \varpi) = Ł(2,2) = 2 < \infty = Ł(2,1)e^{-\theta}.$$

Hence, in this case (34) holds true for $\Im(t) = \ln(t)$ and $\lambda(t) = \vartheta$ for all $t \in (0,\infty)$ and $\theta > 0$.

Hence, all the conditions of Theorem 3 are fulfilled and 0 is the common fixed point of Ω_1 and Ω_2.

By defining $C(V) = V$ in Theorem 3, we obtain the following:

Corollary 5. *Let $(V, Ł)$ be a complete κ-GMS for $\kappa \in (0,1]$ and $\Omega_1, \Omega_2 : V \to V$. Assume that there exist $\mu_0, \mu_1, \mu_2 \in V$ such that $\mu_1 = \Omega_1\mu_0$, $\mu_2 = \Omega_2\mu_1$ and $\sup\{Ł(\mu_{2k+1}, \mu_{2j+2}) : \mu_{2k+l} = \Omega_1\mu_{2k}, \ \mu_{2j+2} = \Omega_2\mu_{2j+1}\} < +\infty$. If there exists a function $\Im \in \hbar_c$ and $\lambda \in \Lambda_\varrho$ satisfying*

$$0 < Ł(\mu, \varpi) < +\infty \ \text{ implies } \ \lambda(Ł(\bar{e}, \bar{f})) + \Im(Ł(\Omega_1\bar{e}, \Omega_2\bar{f})) \leq \Im(Ł(\bar{e}, \bar{f})) \tag{59}$$

for all $\bar{e}, \bar{f} \in V$. Then,

1. *There exists a sequence $\{\mu_p : \mu_{2p+1} = \Omega_1\mu_{2p}, \ \mu_{2p+2} = \Omega_2\mu_{2p+1}\}_{p\in\mathbb{N}}$ such that*

$$\lim_{p\to+\infty} Ł(\mu_p, \mu_{p+1}) = 0;$$

2. *$\{\mu_p\}$ is Ł-Cauchy;*
3. *Ω_1 and Ω_2 owns a common fixed-point in V.*

By considering $\Omega_1 = \Omega_2$ in Theorem 3, we obtain the following:

Corollary 6. *Let* $(V, Ł)$ *be a complete* κ*-GMS for* $\kappa \in (0,1]$ *and* $\Omega_1 : V \to C(V)$*. Assume that there exist* $\mu_0, \mu_1, \mu_2 \in V$ *such that* $\mu_1 \in \Omega_1\mu_0$, $\mu_2 \in \Omega_1\mu_1$ *and* $\sup\{Ł(\mu_{2k+1}, \mu_{2j+2}) : \mu_{2k+l} \in \Omega_1\mu_{2k},\ \mu_{2j+2} \in \Omega_1\mu_{2j+1}\} < +\infty$*. If there exists a function* $\Im \in \hbar_c$ *and* $\lambda \in \Lambda_\varrho$ *satisfying*

$$0 < Ł(\mu, \varpi) < +\infty \ \ \textit{implies} \ \ \lambda(Ł(\bar{e}, \bar{f})) + \Im(Ł(\mu, \varpi)) \leq \Im(Ł(\bar{e}, \bar{f})) \tag{60}$$

for all $\bar{e}, \bar{f} \in V$ *with* $\mu \in \Omega_1\bar{e}$ *and* $\varpi \in \Omega_1\bar{f}$*. Then,*

1. *There exists a sequence* $\{\mu_p : \mu_{2p+1} \in \Omega_1\mu_{2p},\ \mu_{2p+2} \in \Omega_1\mu_{2p+1}\}_{p\in\mathbb{N}}$ *such that*
$$\lim_{p\to+\infty} Ł(\mu_p, \mu_{p+1}) = 0;$$
2. $\{\mu_p\}$ *is* $Ł$*-Cauchy;*
3. Ω_1 *owns a fixed-point in* V*.*

Since a standard metric space is a κ-GMS for $\kappa = 1$, so form Theorem 3 we obtain the following:

Corollary 7. *Let* (V, d) *be a complete metric space and* $\Omega_1, \Omega_2 : V \to C(V)$*. Assume that there exist* $\mu_0, \mu_1, \mu_2 \in V$ *such that* $\mu_1 \in \Omega_1\mu_0$ *and* $\mu_2 \in \Omega_2\mu_1$*. If there exists a function* $\Im \in \hbar_c$ *and* $\lambda \in \Lambda_1$ *satisfying*

$$d(\mu, \varpi) > 0 \ \ \textit{implies} \ \ \lambda(d(\bar{e}, \bar{f})) + \Im(d(\mu, \varpi)) \leq \Im(d(\bar{e}, \bar{f})) \tag{61}$$

for all $\bar{e}, \bar{f} \in V$ *with* $\mu \in \Omega_1\bar{e}$ *and* $\varpi \in \Omega_2\bar{f}$*. Then,*

1. *There exists a sequence* $\{\mu_p : \mu_{2p+1} \in \Omega_1\mu_{2p},\ \mu_{2p+2} \in \Omega_2\mu_{2p+1}\}_{p\in\mathbb{N}}$ *such that*
$$\lim_{p\to+\infty} d(\mu_p, \mu_{p+1}) = 0;$$
2. $\{\mu_p\}$ *is Cauchy;*
3. Ω_1 *and* Ω_2 *owns a common fixed-point in* V*.*

5. Existence of Common Solution of Nonlinear Fractional Differential Equations with Nonlocal Boundary Conditions

In this section, we present the application of our results to prove the existence of the common solutions for the following boundary value problems involving Caputo fractional derivative.

$$\begin{cases} ({}^c\mathbb{D}^\alpha u)(\ell) = h(\ell)f(\ell, u(\ell)) \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0 \\ u(1) = \gamma \int_0^\eta u(s)ds, \end{cases} \tag{62}$$

where $\ell, \eta \in [0,1]$, $n-1 < \alpha \leq n$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$.

$$\begin{cases} ({}^c\mathbb{D}^\alpha v)(\ell) = h(\ell)g(\ell, v(\ell)) \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0 \\ v(1) = \gamma \int_0^\eta v(s)ds, \end{cases} \tag{63}$$

where $\ell, \eta \in [0,1]$, $n-1 < \alpha \leq n$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$.

Firstly, we recall the definition of Caputo fractional derivative and related concepts [30–32].

Definition 7. *For a continuous function* $u : [0, \infty) \to \mathbb{R}$*, the Caputo derivative of fractional order* α *is defined as*

$$ {}^{c}\mathbb{D}^{\alpha}u(\ell) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\ell} (\ell - s)^{n-\alpha-1} u^{(n)}(s) ds, \quad n-1 < \alpha < n, \ n = [\alpha] + 1, \tag{64} $$

where $[\alpha]$ *denotes the integer part of the real number* α*.*

Definition 8. *The Riemann–Liouville fractional integral of order* α *is defined as*

$$ I^{\alpha}u(\ell) = \frac{1}{\Gamma(\alpha)} \int_0^{\ell} \frac{u(s)}{(\ell - s)^{1-\alpha}} ds, \quad \alpha > 0, \tag{65} $$

provided the integral exists.

Lemma 3 ([31])**.** *For* $\alpha > 0$*, the general solution of the fractional differential equation* ${}^{c}\mathbb{D}^{\alpha}x(\ell) = 0$ *is given by*

$$ x(\ell) = c_0 + c_1\ell + c_2\ell^2 + \cdots + c_{n-1}\ell^{n-1}, \tag{66} $$

where $c_i \in \mathbb{R}$*,* $i = 0, 1, 2, \cdots, n-1$ $(n = [\alpha] + 1)$*.*

In view of Lemma 3, it follows that

$$ I^{\alpha c}\mathbb{D}^{\alpha}x(\ell) = x(\ell) + c_0 + c_1\ell + c_2\ell^2 + \cdots + c_{n-1}\ell^{n-1}, \tag{67} $$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n-1$ $(n = [\alpha] + 1)$.

In the following, we obtain the Volterra integral equation of the fractional differential equation boundary value problem.

Lemma 4. *Given* $y \in [0, 1]$*. The problem*

$$ \begin{cases} {}^{c}\mathbb{D}^{\alpha}\wp(\ell) = \varrho(\ell) \\ \wp(0) = \wp'(0) = \cdots = \wp^{(n-2)}(0) = 0 \\ \wp(1) = \gamma \int_0^{\eta} \wp(s) ds, \end{cases} \tag{68} $$

where $\ell, \eta \in [0, 1]$*,* $n - 1 < \alpha \le n$ *and* $\varrho : [0, 1] \times \mathbb{R} \to \mathbb{R}$*, is equivalent to the*

$$ \begin{aligned} \wp(\ell) = & \frac{n\ell^{n-1}}{(n - \gamma\eta^n)\Gamma(\alpha)} \int_0^{\eta} \int_0^{s} (s - \tau)^{\alpha-1} \varrho(\tau) d\tau ds \\ & \frac{n\ell^{n-1}}{(n - \gamma\eta^n)\Gamma(\alpha)} \int_0^{1} (1 - s)^{\alpha-1} \varrho(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^{\ell} (\ell - s)^{\alpha-1} \varrho(s) ds. \end{aligned} \tag{69} $$

Proof. From Lemma 3, the general solution for the problem (69) is

$$ \wp(\ell) = b_0 + b_1\ell + b_2\ell^2 + \cdots + b_{n-1}\ell^{n-1} + \frac{1}{\Gamma(\alpha)} \int_0^{1} (1 - s)^{\alpha-1} \varrho(s) ds, \tag{70} $$

where $b_i \in \mathbb{R}$. By using the boundary conditions $\wp(0) = \wp'(0) = \cdots = \wp^{(n-2)}(0) = 0$, we have $b_0 = b_1 = \cdots = b_{n-2} = 0$. Now to possess the coefficient b_{n-1}, we use the boundary condition $\wp(1) = \gamma \int_0^{\eta} \wp(s) ds$ to obtain

$$ b_{n-1} = \frac{1}{\Gamma(\alpha)} \int_0^{1} (1 - s)^{\alpha-1} \varrho(s) ds - \wp(1), $$

where

$$\begin{aligned}\wp(1) &= \gamma \int_0^\eta \wp(s)ds \\ &= \gamma \int_0^\eta \left(b_{n-1}s^{n-1} + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \varrho(\tau) d\tau \right) ds \\ &= \frac{\gamma\eta^n}{n} b_{n-1} + \frac{1}{\Gamma(\alpha)} \int_0^\eta \int_0^s (s-\tau)^{\alpha-1} \varrho(\tau) d\tau ds.\end{aligned}$$

Hence,

$$\begin{aligned}b_{n-1} = &\frac{n}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^\eta \int_0^s (s-\tau)^{\alpha-1} \varrho(\tau) d\tau ds \\ &\frac{n}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varrho(s) ds.\end{aligned}$$

Substituting the value of $b_0, b_1, \cdots, b_{n-1}$ in (70), we obtain

$$\begin{aligned}\wp(\ell) = &\frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^\eta \int_0^s (s-\tau)^{\alpha-1} \varrho(\tau) d\tau ds \\ &\frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varrho(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^\ell (\ell-s)^{\alpha-1} \varrho(s) ds.\end{aligned}$$

□

Let $V = C(I, \mathbb{R})$ be the space of all continuous real valued functions on I, where $I = [0,1]$. Then, V is a complete metric space with respect to metric $Ł(x,y) = \sup_{\ell \in I} |x(\ell) - y(\ell)|$. Since every metric space is κ-GMS for $\kappa = 1$; henceforth, we assume that $(V, Ł)$ is complete is κ-GMS. Define the operators $A, L : V \to V$ as follows:

$$\begin{aligned}A\wp(\ell) = &\frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^\eta \int_0^s (s-\tau)^{\alpha-1} h(\tau) f(\tau, \wp(\tau)) d\tau ds \\ &\frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) f(s, \wp(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^\ell (\ell-s)^{\alpha-1} h(s) f(s, \wp(s)) ds.\end{aligned} \tag{71}$$

and

$$\begin{aligned}A\wp(\ell) = &\frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^\eta \int_0^s (s-\tau)^{\alpha-1} h(\tau) g(\tau, \wp(\tau)) d\tau ds \\ &\frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) g(s, \wp(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^\ell (\ell-s)^{\alpha-1} h(s) g(s, \wp(s)) ds.\end{aligned} \tag{72}$$

Note that a common fixed point of operators (71) and (72) is the common solutions of (62) and (63). We consider the following set of assumptions in the following:

Hypothesis 1. *$h : [0,1] \to [0,\infty)$ is continuous with $0 < \int_0^1 h(\ell) d\ell < \infty$.*

Hypothesis 2. *$|f(\ell, u(\ell)) - g(\ell, v(\ell))| \le |u(\ell) - v(\ell)| + 1$ for all $\ell \in [0,1]$.*

Hypothesis 3. *$\|h\|_\infty \le \frac{1}{\Upsilon}$, where*

$$\begin{aligned}\Upsilon = \ &\tfrac{n}{(n-\gamma\eta^n)\Gamma(\alpha)} \sup_{\ell \in (0,1)} \Big(\int_0^\eta \int_0^s (s-\tau)^{\alpha-1} d\tau ds + \int_0^1 (1-s)^{\alpha-1} ds \\ &+ \tfrac{(n-\gamma\eta^n)}{n} \int_0^\ell (\ell-s)^{\alpha-1} ds \Big).\end{aligned}$$

Theorem 4. *Suppose that hypothesis* $(H1)$–$(H3)$ *hold. Then, the boundary value problems* (62) *and* (63) *have a common solution in* V.

Proof. Observe that for all $u, v \in V$ and $t \in [0,1]$, we have

$$\begin{aligned}|Au(\ell) - Lv(\ell)| =& \bigg| \frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^\eta \int_0^s (s-\tau)^{\alpha-1} h(\tau)[f(\tau,u(\tau)) - g(\tau,v(\tau))] d\tau ds \\ & \frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)[f(s,u(s)) - g(s,v(s))] ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^\ell (\ell-s)^{\alpha-1} h(s)[f(s,u(s)) - g(s,v(s))] ds \bigg| \\ \leq & \frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^\eta \int_0^s (s-\tau)^{\alpha-1} |h(\tau)||f(\tau,u(\tau)) - g(\tau,v(\tau))| d\tau ds \\ & \frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |h(s)||f(s,u(s)) - g(s,v(s))| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^\ell (\ell-s)^{\alpha-1} |h(s)||f(s,u(s)) - g(s,v(s))| ds \\ \leq & \frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^\eta \int_0^s (s-\tau)^{\alpha-1} \|h)\|_\infty (|u(\tau) - v(\tau)| + 1) d\tau ds \\ & \frac{n\ell^{n-1}}{(n-\gamma\eta^n)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \|h\|_\infty (|u(s) - v(s)| + 1) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^\ell (\ell-s)^{\alpha-1} \|h\|_\infty (|u(s) - v(s)| + 1) ds.\end{aligned}$$

Which implies that

$$|Au(\ell) - Lv(\ell)| \leq \Upsilon \|h\|_\infty (\|u - v\|_\infty + 1), \tag{73}$$

From (H3) and (73), we have

$$\begin{aligned}|Au(\ell) - Lv(\ell)| \leq & (\|u - v\|_\infty + 1) \\ \leq & \frac{\|u - v\|_\infty + 1}{1 + \|u - v\|_\infty (1 + \|u - v\|_\infty)}.\end{aligned}$$

Hence, (59) is satisfied for $\Im(x) = -\frac{1}{x+1}$ and $\lambda(\ell) = t$. Thus, all hypotheses of Corollary 5 are satisfied, and therefore boundary value problems (62) and (63) have a common solution in I. □

Remark 4. *Note that in Theorem 4,* $\Im \in \hbar_c$ *but* $\Im$ *does not satisfy* $(\Im_2)$ *(see Example 3.2 in [16]).*

Example 5. *Consider the following fractional differential equations:*

$$\begin{cases} ({}^c\mathbb{D}^{\frac{7}{2}} u)(\ell) = \frac{\ell^{-\frac{1}{2}}(1-\ell)^{-\frac{1}{2}}}{6}(3u(\ell) + 4), \quad 0 \leq \ell \leq 1 \\ u(0) = u'(0) = u''(0) = 0 \\ u(1) = \frac{3}{4} \int_0^{\frac{2}{3}} u(s) ds, \end{cases} \tag{74}$$

and

$$\begin{cases} ({}^c\mathbb{D}^{\frac{7}{2}} v)(\ell) = \frac{\ell^{-\frac{1}{2}}(1-\ell)^{-\frac{1}{2}}}{2}(v(\ell) + 1), \quad 0 \leq \ell \leq 1 \\ v(0) = v'(0) = v''(0) = 0 \\ v(1) = \frac{3}{4} \int_0^{\frac{2}{3}} v(s) ds. \end{cases} \tag{75}$$

Observe that $\alpha = \frac{7}{2}$, $\eta = \frac{2}{3}$, $\gamma = \frac{2}{3}$, $n = 4$, $h(\ell) = \frac{\ell^{-\frac{1}{2}}(1-\ell)^{-\frac{1}{2}}}{2}$, $f(\ell, u(\ell)) = \frac{1}{3}(3u(\ell) + 4)$ *and* $g(\ell, v(\ell)) = (v(\ell) + 1)$.

So $(H1)$ *holds; indeed, h is continuous with* $0 < \int_0^1 h(\ell)d\ell < \infty$. *Also,*

$$\begin{aligned}
\|h\|_\infty = |h(s)| &= \left|\frac{s^{-\frac{1}{2}}(1-s)^{-\frac{1}{2}}}{2}\right| \\
&\leq \frac{3\Gamma(5.5)}{4 + (4.5)7} \\
&= \frac{4}{3.87\Gamma\left(\frac{7}{2}\right)} \sup_{\ell \in (0,1)} \left(\int_0^{\frac{2}{3}} \int_0^s (s-\tau)^{\frac{5}{2}} d\tau ds + \int_0^1 (1-s)^{\frac{5}{2}} ds \right. \\
&\left. + \frac{3.87\Gamma\left(\frac{7}{2}\right)}{4} \int_0^\ell (\ell - s)^{\frac{5}{2}} ds \right),
\end{aligned}$$

and so $(H3)$ *holds. Lastly,*

$$\begin{aligned}
|f(\ell, u(\ell)) - g(\ell, v(\ell))| &= |\frac{1}{3}(3u(\ell) + 4) - (v(\ell) + 1)| \\
&= \frac{1}{3}|3u(\ell) + 4 - 3v(\ell) - 3| \\
&= \frac{1}{3}|3u(\ell) - 3v(\ell) + 1| \\
&\leq |u(\ell) - v(\ell) + \frac{1}{3} \\
&< |u(\ell) - v(\ell) + 1,
\end{aligned}$$

and hence, $(H3)$ *holds. Consequently, it follows from Theorem 4 that boundary value problems (74) and (75) have common solutions.*

6. Common Solution to Integral Inclusions

In this section, we present the existence of common solutions to the integral inclusions. For this, let $V = C(J, \mathbb{R})$ be the space of all continuous real valued functions on J, where $J = [a, b]$. Then, V is a complete metric space with respect to metric $Ł(x, y) = \sup_{t \in J} |x(t) - y(t)|$. Since every metric space is GMS(JS), throughout this section we assume that $(V, Ł)$ is complete and is GMS(JS). Consider the following integral inclusions:

$$\pi(t) \in q(t) + \int_{\alpha(t)}^{\beta(t)} k(t,s)L(s, \pi(s))ds \tag{76}$$

and

$$\xi(t) \in q(t) \int_{\alpha(t)}^{\beta(t)} k(t,s)M(s, \xi(s))ds \tag{77}$$

for $t \in J$, where $\alpha, \beta : J \to J$, $q : J \to V$, $k : I \times J \to \mathbb{R}$ are continuous and L, $M : J \times V \to P(\mathbb{R})$, $P(\mathbb{R})$ denotes the collection of all nonempty, compact, and convex subsets of $\mathbb{R}$. For each $x \in V$, the operators $L(., x)$ and $M(., y)$ are lower semi-continuous.

Define the multivalued operators $\Omega, \Omega_1 : V \to C(V)$ as follows:

$$\Omega\pi(t) = \left\{ u \in V \ : \ u \in q(t) + \int_{\alpha(t)}^{\beta(t)} k(t,s)L(s, \pi(s))ds, t \in J \right\} \tag{78}$$

and

$$\Omega_1 \xi(t) = \left\{ v \in V \ : \ v \in q(t) + \int_{\alpha}^{\beta(t)} k(t,s) M(s, \xi(s)) ds, t \in J \right\} \tag{79}$$

Note that a common fixed point of multivalued operators (78) and (79) is the common solution of integral inclusions (76) and (77). We consider the following set of assumptions in the following.

Hypothesis 4. *The function $k(t,s)$ is continuous and nonnegative on $J \times J$ with* $\|k\|_\infty = \sup\{k(t,s) : t,s \in J\}$.

Hypothesis 5. $|l_x - m_y| \leq |x(s) - y(s)|$ *for all* $l_x(s) \in L(s, x(s))$ *and* $m_y(s) \in M(s, y(s))$.

Hypothesis 6. $\|k\|_\infty \leq e^{-\theta}$ *for some* $\theta > 0$.

Theorem 5. *Assume that hypothesis* $(H4)$–$(H6)$ *hold. Then, integral inclusions* (76) *and* (77) *have a common solution in* V.

Proof. Let $x, y \in V$. Denote $L_x = L_x(s, x(s))$ and $M_y = M_y(s, y(s))$. Now for $L_x : J \to P(\mathbb{R})$ and $M_y : J \to P(\mathbb{R})$, by Micheal's selection theorem, there exists continuous operators $l_x, m_y : J \times J \to \mathbb{R}$ with $l_x(s) \in L_x(s)$ and $m_y(s) \in M_y(s)$ for $s \in J$. So, we have $u = \int_{\alpha(t)}^{\beta(t)} k(t,s) l_x(s) ds + q(t) \in \Omega \bar{e}(t)$ and $v = \int_{\alpha(t)}^{\beta(t)} k(t,s) m_y(s) ds + q(t) \in \Omega_1 \bar{f}(t)$. Thus, the operators $\Omega \bar{e}$ and $\Omega_1 \bar{f}$ is nonempty and closed (see [33]). By hypothesis (H4)–(H6) and by using Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}
Ł(u,v) &= \sup_{t \in J} |u(t) - v(t)| \\
&= \sup_{t \in J} \left| \int_{\alpha(t)}^{\beta(t)} k(t,s) \big(l_x(s) - m_y(s)\big) ds \right| \\
&\leq \sup_{t \in J} \int_{\alpha(t)}^{\beta(t)} k(t,s) |l_x(s) - m_y(s)| ds \\
&\leq \sup_{t \in J} \int_{\alpha(t)}^{\beta(t)} k(t,s) |x(s) - y(s)| ds \\
&\leq \sup_{t \in J} \left(\int_{\alpha(t)}^{\beta(t)} k^2(t,s) ds \right)^{\frac{1}{2}} \left(\int_{\alpha(t)}^{\beta(t)} |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}} \\
&\leq \|k\|_\infty \sup_{t \in J} |x(t) - y(t)| \\
&\leq e^{-\theta} \sup_{t \in J} |x(t) - y(t)| \\
&= e^{-\theta} Ł(x,y).
\end{aligned}$$

Hence, (34) is satisfied for $\Im(\wp) = \ln(\wp)$ and $\lambda(\wp) = \theta > 0$ for all $\wp \in (0, \infty)$. Thus, all hypotheses of Theorem 3 are satisfied, and therefore Ω and Ω_1 have a common fixed point. It further implies that integral inclusions (76) and (77) have a common solution in I. □

Lastly, we present an open problem for future work as follows:

Open Problem

Let $(V, Ł)$ be a κ-GMS for any $\kappa > 0$l then, can Theorems 2 and 3 still be proved?

7. Conclusions

We have proved the coincidence fixed-point and common fixed-point theorems in the setting of generalized metric spaces (in the terms of *Jleli* and *Samet*) for $\Im$-type mappings satisfying certain contractive conditions. To prove these results, we have used fewer conditions imposed on function $\Im$. We have also provided the supportive examples of obtained results to illustrate the usability. Moreover, the existence results of common solutions for fractional boundary value problems and integral inclusions are obtained by the use of proved common fixed-point results. Finally, we have also presented two open problems for future work in this direction.

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