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Stochastic Modeling of Three-Species Prey–Predator Model Driven by Lévy Jump with Mixed Holling-II and Beddington–DeAngelis Functional Responses

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Abstract: This study examines the dynamics of a stochastic prey–predator model using a functional response function driven by Lévy noise and a mixed Holling-II and Beddington–DeAngelis functional response. The proposed model presents a computational analysis between two prey and one predator population dynamics. First, we show that the suggested model admits a unique positive solution. Second, we prove the extinction of all the studied populations, the extinction of only the predator, and the persistence of all the considered populations under several sufficient conditions. Finally, a special Runge–Kutta method for the stochastic model is illustrated and implemented in order to show the behavior of the two prey and one predator subpopulations.

Keywords: prey–predator; stochastic model; Lévy jump; Holling-II functional; Beddington–DeAngelis functional; Runge–Kutta method



Citation: Danane, J.; Yavuz, M.; Yıldız, M. Stochastic Modeling of Three-Species Prey–Predator Model Driven by Lévy Jump with Mixed Holling-II and Beddington–DeAngelis Functional Responses. *Fractal Fract.* **2023**, *7*, 751. <https://doi.org/10.3390/fractalfract7100751>

Academic Editor: Carlo Cattani

Received: 16 July 2023

Revised: 9 October 2023

Accepted: 9 October 2023

Published: 12 October 2023



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1. Introduction

Prey–predator dynamics remain a highly fruitful and attractive ecological subject [1–4]. The basic prey–predator model was proposed by Lotka and Volterra in 1956 [5]; this basic ecological problem was one of the earliest to model competition between prey and predator. One study devoted to the dynamics of fishing illustrated the mathematical and ecological properties of marine fish [6]. Another application of these mathematical tools sought to understand the interaction between wildebeest, zebra, and lion [7] in Kruger National Park in South-Africa. The natural or more realistic growth rate is a logistic suggested by Verhulst [8] for new recruitment of either prey, predators, or both. This logistic growth rate charmed several scientists in various disciplines [9–14]. Other studies have been devoted to the type of competition between prey and predators; for example, the basic prey–predator model uses a mass action computational rate [5]. In another study [15], Yavuz and Sene performed a stability analysis for the fractional predator–prey model with harvesting rate. In [16], the authors provided an analysis of the prey–predator model using the Crowley–Martin type functional response. Their main object was to study the stability behavior of the considered population. The Holling-type functional impact on the dynamics of the prey–predator relationship was studied in [17,18]. The starting predator–prey model is based on the assumption that the reaction is between one predator and one prey species. Recently, the authors of [19,20] considered two prey and one predator species under a mixed Holling-II and Beddington–DeAngelis functional response, as follows:

$$\begin{cases} \frac{d\mathcal{X}_1}{dt}(t) = \mathcal{X}_1(t) \left(v_1 - \varrho_1 \mathcal{X}_1(t) - \frac{y(t)}{l_1 + \mathcal{X}_1(t)} \right), \\ \frac{d\mathcal{X}_2}{dt}(t) = \mathcal{X}_2(t) \left(v_2 - \varrho_2 \mathcal{X}_2(t) - \frac{y(t)}{l_2 + \mathcal{X}_2(t) + \beta y(t)} \right), \\ \frac{dy}{dt}(t) = y(t) \left(-d - \varrho_3 y(t) + \frac{\vartheta_1 \mathcal{X}_1(t)}{l_1 + \mathcal{X}_1(t)} + \frac{\vartheta_2 \mathcal{X}_2(t)}{l_2 + \mathcal{X}_2(t) + \beta y(t)} \right), \end{cases} \tag{1}$$

where the two prey and one predator populations are presented by \mathcal{X}_1 , \mathcal{X}_2 , and y , respectively. The parameters of (1) and their meanings are provided in Table 1.

Table 1. The parameters of (1) and their meanings.

Parameters	Description
v_1	The intrinsic growth rate of prey \mathcal{X}_1
v_2	The intrinsic growth rate for \mathcal{X}_2
ϱ_1	The intra-specific competition for prey \mathcal{X}_1
ϱ_2	The intra-specific competition for prey \mathcal{X}_2
l_1	The half-saturation due to \mathcal{X}_1
l_2	The half-saturation due to \mathcal{X}_2
β	The impact of the predator interference
δ	The death rates of predator
ϱ_3	The rate of intra-species competition for predator
ϑ_1	The average of the transformation of predator from prey \mathcal{X}_1
ϑ_2	The average of the transformation of predator from prey \mathcal{X}_2
d	The natural mortality of the predator y

Deterministic modeling lacks one of the most important properties in prey and predator, that of randomness; for this reason, the stochastic model is preferred in modeling such problems. This is because the stochastic model takes into account the variance aspect, not just the mean, as in the deterministic case. In addition, the stochastic model may offer different predicted results for the same initial data. Certain stochastic prey–predator systems can be established by taking into consideration only the white noise for the studied population [21,22]. Recently, a Brownian random walk perturbation on model (1) was proposed in [20] by studying the persistence and extinction of the prey and predator population. A more basic modification to the modeling technique is to immediately take into account more realistic distributions in order to maintain the model’s straightforward analytic form. This naturally leads to Lévy jumps, a broader class of driving processes [23–35]. In fact, the dynamical model may encounter abrupt and significant disturbances in predator–prey development due to the intrinsic stochastic features of the related processes [36]. Here, we add the jumps to the model from [20] in response to the results of earlier research. Thus, the model becomes

$$\begin{cases} d\mathcal{X}_1(t) = \mathcal{X}_1(t) \left(v_1 - \varrho_1 \mathcal{X}_1(t) - \frac{y(t)}{l_1 + \mathcal{X}_1(t)} \right) dt + \sigma_1 \mathcal{X}_1(t) dW_1(t) \\ \quad + \int_U \Xi_1(u) \mathcal{X}_1(t-) \tilde{N}(dt, du), \\ d\mathcal{X}_2(t) = \mathcal{X}_2(t) \left(v_2 - \varrho_2 \mathcal{X}_2(t) - \frac{y(t)}{l_2 + \mathcal{X}_2(t) + \beta y(t)} \right) dt + \sigma_2 \mathcal{X}_2(t) dW_2(t) \\ \quad + \int_U \Xi_2(u) \mathcal{X}_2(t-) \tilde{N}(dt, du), \\ dy(t) = y(t) \left(-d - \varrho_3 y(t) + \frac{\vartheta_1 \mathcal{X}_1(t)}{l_1 + \mathcal{X}_1(t)} + \frac{\vartheta_2 \mathcal{X}_2(t)}{l_2 + \mathcal{X}_2(t) + \beta y(t)} \right) dt + \sigma_3 y(t) dW_3(t) \\ \quad + \int_U \Xi_3(u) y(t-) \tilde{N}(dt, du). \end{cases} \tag{2}$$

where the left limits of $\mathcal{X}_1(t)$, $\mathcal{X}_2(t)$, and $y(t)$ are denoted by $x(t-)$, $y(t-)$, and $z(t-)$, respectively, the standard Brownian motion $W_i(t)$ is defined on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ as a complete probability, and $(\mathcal{F}_t)_{t \geq 0}$ is the filtration verifying the usual conditions. We define the finite stationary compensator ν on U as the measurable subset of the non-negative half-line; then, $N(dt, du)$ is the Poisson counting $\nu(du)dt$, σ_i is the intensity of $W_i(t)$, and the jumping intensities are represented by $q_i(u)$ for $i = 1, 2, 3$.

The remaining parts of this study are organized as follows. In Section 2, we provide several properties of the solution of the model (2). The stochastic extinction of both the prey and the predator is established in Section 3. In Section 4, we prove the persistence of the prey and the extinction of the predator. The stochastic persistence of both the predator and prey is illustrated in Section 5. Finally, Section 6 of this paper is devoted to numerical simulations that validate our theoretical findings.

2. The Well-Posedness of the Solution

The following theorem ensures the well-posedness of the solution of the model (2).

Theorem 1. *For all initial conditions that are positive, model (2) admits a unique positive global solution a.s.*

Proof. Using the local Lipschitz of the diffusion and drift, for all initial positive data the problem (2) admits a local unique solution $(\mathcal{X}_1(t), \mathcal{X}_2(t), y(t))$ for $t \in [0, T^e)$, with T^e representing the time of explosion.

Proving the existence globality of the solution amounts to showing that $T^e = \infty$ a.s. For a given a large number $n'_0 > 0$, for any integer $n' \geq n'_0$ the stopping time is defined as

$$T^{n'} = \inf\{t \in [0, T^e) / \mathcal{X}_1(t) \notin (\frac{1}{n'}, n') \text{ or } \mathcal{X}_2(t) \notin (\frac{1}{n'}, n') \text{ or } y(t) \notin (\frac{1}{n'}, n')\},$$

and when $n' \uparrow \infty$, the number $T^{n'}$ increases. Using $T^\infty = \lim_{n' \rightarrow \infty} T^{n'}$, with $T^\infty \leq T^e$ a.s., it is sufficient to prove that $T^\infty = \infty$, which is signified $T^e = \infty$, and $(\mathcal{X}_1(t), \mathcal{X}_2(t), y(t)) \in \mathbb{R}_+^3$ a.s. Suppose the contrary, i.e., $T^\infty < \infty$ a.s. This means that there exists some $T > 0$ and $0 < \epsilon < 1$ such that $\mathbb{P}(T^\infty \leq T) \geq \epsilon$.

Considering the functional

$$V(\mathcal{X}_1(t), \mathcal{X}_2(t), y(t)) = (\mathcal{X}_1 - 1 - \log(\mathcal{X}_1)) + (\mathcal{X}_2 - 1 - \log(\mathcal{X}_2)) + (y - 1 - \log(y)),$$

from Itô's formula [37], we obtain

$$\begin{aligned} dV(\mathcal{X}_1, \mathcal{X}_2, y) = & LV dt + \sigma_1(\mathcal{X}_1 - 1) dW_1 + \sigma_2(\mathcal{X}_2 - 1) dW_2 + \sigma_3(y - 1) dW_3 \\ & + \int_U [\Xi_1(u)\mathcal{X}_1 - \log(1 + \Xi_1(u)\mathcal{X}_1)] \\ & + \int_U [\Xi_2(u)\mathcal{X}_2 - \log(1 + \Xi_2(u)\mathcal{X}_2)] \\ & + \int_U [\Xi_3(u)y - \log(1 + \Xi_3(u)y)], \end{aligned} \tag{3}$$

with

$$\begin{aligned} LV = & \left(1 - \frac{1}{\mathcal{X}_1}\right) \left(\mathcal{X}_1(t) \left(v_1 - \varrho_1 \mathcal{X}_1 - \frac{y}{\iota_1 + \mathcal{X}_1}\right)\right) + \frac{\sigma_1^2}{2} \\ & + \left(1 - \frac{1}{\mathcal{X}_2}\right) \left(\mathcal{X}_2(t) \left(v_2 - \varrho_2 \mathcal{X}_2 - \frac{y}{\iota_2 + \mathcal{X}_2 + \beta y}\right)\right) + \frac{\sigma_2^2}{2} \\ & + \left(1 - \frac{1}{y}\right) \left(y \left(-d - \varrho_3 y + \frac{\vartheta_1 \mathcal{X}_1}{\iota_1 + \mathcal{X}_1} + \frac{\vartheta_2 \mathcal{X}_2}{\iota_2 + \mathcal{X}_2 + \beta y}\right)\right) + \frac{\rho \sigma_3^2}{2} \\ & + \int_U [\Xi_1(u) - \log(1 + \Xi_1(u))] \nu(du) \\ & + \int_U [\Xi_2(u) - \log(1 + \Xi_2(u))] \nu(du) \\ & + \int_U [\Xi_3(u) - \log(1 + \Xi_3(u))] \nu(du), \end{aligned}$$

then,

$$\begin{aligned}
 LV &\leq v_1\mathcal{X}_1 - \varrho_1\mathcal{X}_1^2 + \varrho_1\mathcal{X}_1 + \frac{\sigma_1^2}{2} \\
 &\quad + v_2\mathcal{X}_2 - \varrho_2\mathcal{X}_2^2 + \varrho_2\mathcal{X}_2 + \frac{\sigma_2^2}{2} \\
 &\quad + d + (\vartheta_1 + \vartheta_2)y - \varrho_3y^2 + \frac{\sigma_3^2}{2} + M, \\
 LV &\leq (v_1 + \varrho_1)\mathcal{X}_1 - \varrho_1\mathcal{X}_1^2 + \frac{\sigma_1^2}{2} + (v_2 + \varrho_2)\mathcal{X}_2 - \varrho_2\mathcal{X}_2^2 + \frac{\sigma_2^2}{2} \\
 &\quad + d + (\vartheta_1 + \vartheta_2)y - \varrho_3y^2 + \frac{\sigma_3^2}{2} + 3C', \\
 LV &\leq \frac{(v_1 + \varrho_1)^2}{4\varrho_1} + \frac{(v_2 + \varrho_2)^2}{4\varrho_2} + d + \frac{(\vartheta_1 + \vartheta_2)^2}{4\varrho_3} + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + M, \\
 LV &\leq M',
 \end{aligned}$$

with

$$M = 3 \max \left\{ \int_U [\Xi_1(u) - \log(1 + \Xi_1(u))] \nu(du), \int_U [\Xi_2(u) - \log(1 + \Xi_2(u))] \nu(du), \int_U [\Xi_3(u) - \log(1 + \Xi_3(u))] \nu(du) \right\}.$$

$$M = \frac{(v_1 + \varrho_1)^2}{4\varrho_1} + \frac{(v_2 + \varrho_2)^2}{4\varrho_2} + d + \frac{(\vartheta_1 + \vartheta_2)^2}{4\varrho_3} + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + M'.$$

Integrating Equation (3) between 0 and $T^{n'} \wedge T$, we obtain

$$\begin{aligned}
 0 &\leq \mathbb{E} \left(V \left(\mathcal{X}_1(T^{n'} \wedge T), \mathcal{X}_2(T^{n'} \wedge T), y(T^{n'} \wedge T) \right) \right) \\
 &\leq V(\mathcal{X}_1(0), \mathcal{X}_2(0), y(0)) + M'T.
 \end{aligned}$$

Thus, if $n' \rightarrow \infty$, then

$$\infty > V(\mathcal{X}_1(0), \mathcal{X}_2(0), y(0)) + M'CT = \infty,$$

which is in contradiction with the previous supposition. Thus, $T^\infty = \infty$ and the solution $(\mathcal{X}_1(t), \mathcal{X}_2(t), y(t))$ is unique and global a.s. \square

3. Stochastic Extinction

Now, we prove the extinction of the three subpopulations a.s. First, we define

$$\begin{aligned}
 m_1 &= v_1 + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\sigma_1^2}{2}, \\
 m_2 &= v_2 + \int_U \log(1 + \Xi_2(u)) - \Xi_2(u) \nu(du) - \frac{\sigma_2^2}{2}, \\
 m_3 &= \frac{\vartheta_1}{\varrho_1} \left(v_1 + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\sigma_1^2}{2} \right) \\
 &\quad + \frac{\vartheta_2}{\varrho_2} \left(v_2 + \int_U \log(1 + \Xi_2(u)) - \Xi_2(u) \nu(du) - \frac{\sigma_2^2}{2} \right) - d \\
 &\quad + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) - \frac{\sigma_3^2}{2}.
 \end{aligned}$$

Theorem 2. *If $\max\{m_1, m_2, m_3\} < 0$, then for any initial positive condition we have*

$$\begin{aligned} \lim_{t \rightarrow +\infty} \langle \mathcal{X}_1 \rangle_t &= 0, \\ \lim_{t \rightarrow +\infty} \langle \mathcal{X}_2 \rangle_t &= 0, \\ \lim_{t \rightarrow +\infty} \langle y \rangle_t &= 0. \end{aligned}$$

Proof. Considering

$$\mathcal{F}(\mathcal{X}_1) = \log(\mathcal{X}_1(t)),$$

from Itô’s formula we have

$$d\mathcal{F} = L\mathcal{F} dt + \sigma_1 dW_1(t) + \int_U \log(1 + \Xi_1(u)) \tilde{N}(dt, du), \tag{4}$$

with

$$L\mathcal{F} = v_1 - \varrho_1 \mathcal{X}_1 - \frac{y}{t_1 + \mathcal{X}_1} - \frac{\sigma_1^2}{2} + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du),$$

then,

$$L\mathcal{F} \leq v_1 - \varrho_1 \mathcal{X}_1 - \frac{\sigma_1^2}{2} + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du),$$

and,

$$\begin{aligned} \frac{\log(\mathcal{X}_1(t)) - \log(\mathcal{X}_1(0))}{t} &\leq v_1 - \frac{\sigma_1^2}{2} + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) \\ &\quad - \varrho_1 \langle \mathcal{X}_1 \rangle_t + \int_0^t \frac{\sigma_1 dW_1(s)}{t} ds \\ &\quad + \frac{1}{t} \int_0^t \int_U \log(1 + \Xi_1(u)) \tilde{N}(ds, du) ds, \end{aligned}$$

therefore,

$$\begin{aligned} \langle \mathcal{X}_1 \rangle_t &\leq \frac{1}{\varrho_1} \left(v_1 + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\sigma_1^2}{2} \right) \\ &\quad + \frac{1}{\varrho_1} \left(\int_0^t \frac{\sigma_1 dW_1(s)}{t} ds + \frac{1}{t} \int_0^t \int_U \log(1 + \Xi_1(u)) \tilde{N}(ds, du) ds \right) \\ &\quad - \frac{1}{\varrho_1} \frac{(\log(\mathcal{X}_1(t)) - \log(\mathcal{X}_1(0)))}{t}. \end{aligned}$$

Let us now take

$$M_t = \int_0^t \sigma_1 dW_1(s),$$

then,

$$\limsup_{t \rightarrow +\infty} \frac{\langle M_t, M_t \rangle}{t} = \limsup_{t \rightarrow +\infty} \frac{\sigma_1^2}{t} < \infty.$$

Using Martingale’s theorem (the strong law of large numbers), we obtain

$$\limsup_{t \rightarrow +\infty} \frac{M_t}{t} = 0.$$

Thus,

$$\lim_{t \rightarrow +\infty} \langle \mathcal{X}_1 \rangle_t \leq \frac{1}{\varrho_1} \left(v_1 + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\sigma_1^2}{2} \right).$$

Following the same technique, we obtain

$$\liminf_{t \rightarrow +\infty} \langle \mathcal{X}_2 \rangle_t \leq \frac{1}{\varrho_2} \left(\varrho_2 + \int_U \log(1 + \Xi_2(u)) - \Xi_2(u) \nu(du) - \frac{\sigma_2^2}{2} \right).$$

We next consider

$$\mathcal{V}(t) = \log(y).$$

Itô's formula implies that

$$d\mathcal{V} = L\mathcal{V} dt + \sigma_3 dW_3(t) + \int_U \log(1 + \Xi_3(u)) \tilde{N}(dt, du), \tag{5}$$

with

$$\begin{aligned} L\mathcal{V} &= -d - \varrho_3 z + \frac{\vartheta_1 \mathcal{X}_1}{\iota_1 + \mathcal{X}_1} + \frac{\vartheta_2 \mathcal{X}_2}{\iota_2 + \mathcal{X}_2 + \beta y} - \frac{\sigma_3^2}{2} + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) \\ &\leq -d - \varrho_3 y + \vartheta_1 \mathcal{X}_1 + \vartheta_2 \mathcal{X}_2 + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) - \frac{\sigma_3^2}{2}, \end{aligned}$$

then,

$$\begin{aligned} \frac{\mathcal{V}(t) - \mathcal{V}(0)}{t} &\leq \vartheta_1 \langle \mathcal{X}_1 \rangle + \vartheta_2 \langle \mathcal{X}_2 \rangle - d - \varrho_3 \langle y \rangle_t + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) \\ &\quad - \frac{\sigma_3^2}{2} + \frac{1}{t} \int_0^t \left(\sigma_3 dW_3(t) + \int_U \log(1 + \Xi_3(u)) \tilde{N}(dt, du) \right), \end{aligned}$$

then,

$$\begin{aligned} \langle y \rangle_t \leq \frac{1}{\varrho_3} \left(\vartheta_1 \langle \mathcal{X}_1 \rangle + \vartheta_2 \langle \mathcal{X}_2 \rangle - d - \frac{\sigma_3^2}{2} + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) \right) \\ - \frac{1}{\varrho_3} \frac{\mathcal{V}(t) - \mathcal{V}(0)}{t} + \frac{1}{\alpha_3} \frac{1}{t} \int_0^t \left(\sigma_3 dW_3(t) + \int_U \log(1 + \Xi_3(u)) \tilde{N}(dt, du) \right). \end{aligned}$$

Similar to $\langle \mathcal{X}_1 \rangle_t$, we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \langle y \rangle_t \leq \frac{1}{\varrho_3} \left(\frac{\vartheta_1}{\varrho_1} \left(v_1 + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\sigma_1^2}{2} \right) \right. \\ \left. + \frac{\vartheta_2}{\varrho_2} \left(v_2 + \int_U \log(1 + \Xi_2(u)) - \Xi_2(u) \nu(du) - \frac{\sigma_2^2}{2} \right) \right. \\ \left. + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) - d - \frac{\sigma_3^2}{2} \right). \end{aligned}$$

□

4. Stochastic Extinction of Predator

In order to prove the extinction of the predator, we need to define

$$\begin{aligned}
 m'_1 &= \frac{m_1}{2q_1 - v_1}, \\
 m'_2 &= \frac{m_2}{2q_2 - v_2}, \\
 m_4 &= \vartheta_1 + \vartheta_2 - d + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u)v(du) - \frac{\sigma_3^2}{2}.
 \end{aligned}$$

Theorem 3. Suppose that $m_4 \leq 0$ and $\min\{m'_1, m'_2\} > 0$; then,

$$\lim_{t \rightarrow \infty} \langle y \rangle_t = 0,$$

and

$$\begin{aligned}
 \liminf_{t \rightarrow \infty} \langle \mathcal{X}_1 \rangle_t &\geq m'_1 \text{ a.s.} \\
 \liminf_{t \rightarrow \infty} \langle \mathcal{X}_2 \rangle_t &\geq m'_2 \text{ a.s.}
 \end{aligned}$$

Proof. The first Equation of (2) implies that

$$\begin{aligned}
 \frac{\mathcal{X}_1(t) - \mathcal{X}_1(0)}{t} &= \frac{1}{t} \int_0^t \left(\mathcal{X}_1(s)(v_1 - q_1\mathcal{X}_1(s)) - \frac{y(s)}{l_1 + \mathcal{X}_1(s)} \right) ds + \sigma_1 x(s) dW_1(s) \\
 &\quad + \frac{1}{t} \int_0^t \left(\int_U \Xi_1(u)\mathcal{X}_1(s-)\tilde{N}(ds, du) \right) ds.
 \end{aligned}$$

Let us consider the following:

$$\mathcal{F}(\mathcal{X}_1) = \log(\mathcal{X}_1(t)),$$

from It'ô's formula, we have

$$\begin{aligned}
 d\mathcal{F} &= v_1 - q_1\mathcal{X}_1 - \frac{\sigma_1^2}{2} + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u)v(du) \\
 &\quad - \frac{y}{l_1 + \mathcal{X}_1} dt + \sigma_1 dW_1(t) + \int_U \log(1 + \Xi_1(u))\tilde{N}(dt, du),
 \end{aligned}$$

then,

$$\begin{aligned}
 \frac{\mathcal{X}_1(t) - \mathcal{X}_1(0)}{t} + \frac{\mathcal{F}(t) - \mathcal{F}(0)}{t} &\geq v_1 \langle \mathcal{X}_1 \rangle_t - 2q_1 \langle \mathcal{X}_1 \rangle_t - \frac{1}{t} \int_0^t \frac{(\mathcal{X}_1 + 1)y}{l_1 + \mathcal{X}_1} ds \\
 &\quad + v_1 - \frac{\sigma_1^2}{2} + \int_0^t \int_U \log(1 + \Xi_1(u)) - \Xi_1(u)v(du) ds \\
 &\quad + \frac{1}{t} \int_0^t \sigma_1(1 + \mathcal{X}_1(s)) dW_1(s) \\
 &\quad + \frac{1}{t} \int_0^t \left(\int_U \log(1 + \Xi_1(u)) + \Xi_1(u)\mathcal{X}_1(s-)\tilde{N}(ds, du) \right) ds
 \end{aligned}$$

and we know that

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \frac{(\mathcal{X}_1 + 1)y}{l_1 + \mathcal{X}_1} ds \leq \lim_{t \rightarrow +\infty} \frac{1}{t} \left(\int_0^t \left(\frac{(\mathcal{X}_1 + 1)}{l_1 + \mathcal{X}_1} \right)^2 ds \right)^{1/2} \times \left(\int_0^t y^2 ds \right)^{1/2} = 0.$$

Thus,

$$\liminf_{t \rightarrow +\infty} \langle \mathcal{X}_1 \rangle_t \geq \frac{v_1 + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\sigma_1^2}{2}}{2\varrho_1 - v_1}.$$

Using the same technique, we can show that

$$\liminf_{t \rightarrow +\infty} \langle \mathcal{X}_2 \rangle_t \geq \frac{v_2 + \int_U \log(1 + \Xi_2(u)) - \Xi_2(u) \nu(du) - \frac{\sigma_2^2}{2}}{2\varrho_2 - v_2}.$$

Let us now take

$$\mathcal{V}(t) = \log(y),$$

from Itô's formula, we have

$$d\mathcal{V} = L\mathcal{V} dt + \sigma_3 dW_3(t) + \int_U \log(1 + \Xi_3(u)) \tilde{N}(dt, du), \tag{6}$$

with

$$\begin{aligned} L\mathcal{V} &= -d - \varrho_3 y + \frac{\vartheta_1 \mathcal{X}_1}{\iota_1 + \mathcal{X}_1} + \frac{\vartheta_2 \mathcal{X}_2}{\iota_2 + \mathcal{X}_2 + \beta y} - \frac{\sigma_3^2}{2} + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) \\ &\leq -d - \varrho_3 y + v_1 + v_2 - \frac{\sigma_3^2}{2}, \end{aligned}$$

therefore,

$$\begin{aligned} \frac{\mathcal{V}(t) - \mathcal{V}(0)}{t} &\leq \vartheta_1 + \vartheta_2 - d - \varrho_3 \langle y \rangle_t + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) - \frac{\sigma_3^2}{2} \\ &\quad + \frac{1}{t} \int_0^t \left(\sigma_3 dW_3(t) + \int_U \log(1 + \Xi_3(u)) \tilde{N}(dt, du) \right), \end{aligned}$$

then,

$$\begin{aligned} \langle y \rangle_t &\leq \frac{1}{\varrho_3} \left(\vartheta_1 + \vartheta_2 - d + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) - \frac{\sigma_3^2}{2} \right) - \frac{1}{\varrho_3} \frac{\mathcal{V}(t) - \mathcal{V}(0)}{t} \\ &\quad + \frac{1}{\varrho_3} \frac{1}{t} \int_0^t \left(\sigma_3 dW_3(t) + \int_U \log(1 + \Xi_3(u)) \tilde{N}(dt, du) \right), \end{aligned}$$

and

$$\lim_{t \rightarrow +\infty} \langle y \rangle_t \leq \frac{1}{\varrho_3} \left(\vartheta_1 + \vartheta_2 - d + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) - \frac{\sigma_3^2}{2} \right).$$

□

5. Stochastic Persistence

In this section, we show that all three subpopulations persist.

Theorem 4. For $m'_1 > 0, m'_2 > 0$, and

$$\begin{aligned} & \frac{1}{\varrho_3} \left(\frac{v_1\vartheta_1 + \vartheta_1 \int_U \log(1 + \Xi_1(u)) - \Xi_1(u)v(du) - \frac{\vartheta_1\sigma_1^2}{2}}{\left(\iota_1 + \frac{v_1}{2\varrho_1}\right)(v_1 - 2\varrho_1)} \right. \\ & + \frac{v_2\vartheta_2 + \vartheta_2 \int_U \log(1 + \Xi_2(u)) - \Xi_2(u)v(du) - \frac{\vartheta_2\sigma_2^2}{2}}{\left(\iota_2 + \frac{v_2}{2\varrho_2} + \beta \frac{v_2}{2\varrho_2}\right)(v_2 - 2\varrho_2)} \\ & \left. - d + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u)v(du) - \frac{\sigma_3^2}{2} \right) > 0, \end{aligned}$$

which means that all of the subpopulations persist in the mean. In addition,

$$\begin{aligned} \liminf_{t \rightarrow +\infty} \langle \mathcal{X}_1 \rangle_t & \geq \frac{v_1 + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u)v(du) - \frac{\sigma_1^2}{2}}{2\varrho_1 - v_1} \text{ a.s.} \\ \liminf_{t \rightarrow +\infty} \langle \mathcal{X}_2 \rangle_t & \geq \frac{v_2 + \int_U \log(1 + \Xi_2(u)) - \Xi_2(u)v(du) - \frac{\sigma_2^2}{2}}{2\varrho_2 - v_2} \text{ a.s.} \\ \liminf_{t \rightarrow +\infty} \langle y \rangle_t & \geq \frac{1}{\varrho_3} \left(\frac{v_1\vartheta_1 + \vartheta_1 \int_U \log(1 + \Xi_1(u)) - \Xi_1(u)v(du) - \frac{\vartheta_1\sigma_1^2}{2}}{\left(\iota_1 + \frac{v_1}{2\varrho_1}\right)(v_1 - 2\varrho_1)} \right. \\ & + \frac{v_2\vartheta_2 + \vartheta_2 \int_U \log(1 + \Xi_2(u)) - \Xi_2(u)v(du) - \frac{\vartheta_2\sigma_2^2}{2}}{\left(\iota_2 + \frac{v_2}{2\varrho_2} + \beta \frac{v_2}{2\varrho_2}\right)(v_2 - 2\varrho_2)} \\ & \left. - d + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u)v(du) - \frac{\sigma_3^2}{2} \right) \text{ a.s.} \end{aligned}$$

Proof. The first Equation of (2) implies that

$$\begin{aligned} \frac{\mathcal{X}_1(t) - \mathcal{X}_1(0)}{t} & = \frac{1}{t} \int_0^t \left(\mathcal{X}_1(s)(v_1 - \varrho_1\mathcal{X}_1(s)) - \frac{y(s)}{\iota_1 + \mathcal{X}_1(s)} \right) ds + \sigma_1 x(s) dW_1(s) \\ & + \frac{1}{t} \int_0^t \left(\int_U \Xi_1(u)\mathcal{X}_1(s-) \tilde{N}(ds, du) \right) ds. \end{aligned}$$

Let us consider the following:

$$\mathcal{F}(\mathcal{X}_1) = \log(\mathcal{X}_1(t)),$$

from Itô's formula, we have

$$\begin{aligned} d\mathcal{F} & = v_1 - \varrho_1\mathcal{X}_1 - \frac{\sigma_1^2}{2} + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u)v(du) \\ & - \frac{y}{\iota_1 + \mathcal{X}_1} dt + \sigma_1 dW_1(t) + \int_U \log(1 + \Xi_1(u)) \tilde{N}(dt, du), \end{aligned}$$

then,

$$\begin{aligned} \frac{\mathcal{X}_1(t) - \mathcal{X}_1(0)}{t} + \frac{\mathcal{F}(t) - \mathcal{F}(0)}{t} &\geq v_1 < \mathcal{X}_1 >_t - 2\varrho_1 < \mathcal{X}_1 >_t - \frac{1}{t} \int_0^t \frac{(\mathcal{X}_1 + 1)y}{\iota_1 + \mathcal{X}_1} ds \\ &+ v_1 - \frac{\sigma_1^2}{2} + \int_0^t \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) ds \\ &+ \frac{1}{t} \int_0^t \sigma_1(1 + \mathcal{X}_1(s)) dW_1(s) \\ &+ \frac{1}{t} \int_0^t \left(\int_U \log(1 + \Xi_1(u)) + \Xi_1(u) \mathcal{X}_1(s-) \tilde{N}(ds, du) \right) ds. \end{aligned}$$

We know that

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \frac{(\mathcal{X}_1 + 1)y}{\iota_1 + \mathcal{X}_1} ds \leq \lim_{t \rightarrow +\infty} \frac{1}{t} \left(\int_0^t \left(\frac{(\mathcal{X}_1 + 1)}{\iota_1 + \mathcal{X}_1} \right)^2 ds \right)^{1/2} \times \left(\int_0^t y^2 ds \right)^{1/2} = 0.$$

Thus,

$$\liminf_{t \rightarrow +\infty} < \mathcal{X}_1 >_t \geq \frac{v_1 + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\sigma_1^2}{2}}{2\varrho_1 - v_1}.$$

Using the same technique, we can show that

$$\liminf_{t \rightarrow +\infty} < \mathcal{X}_2 >_t \geq \frac{v_2 + \int_U \log(1 + \Xi_2(u)) - \Xi_2(u) \nu(du) - \frac{\sigma_2^2}{2}}{2\varrho_2 - v_2}.$$

Now, we need to take into account that

$$\mathcal{V}(t) = \log(y),$$

where Itô's formula implies that

$$d\mathcal{V} = L\mathcal{V} dt + \sigma_3 dW_3(t) + \int_U \log(1 + \Xi_3(u)) \tilde{N}(dt, du), \tag{7}$$

with

$$L\mathcal{V} = -d - \varrho_3 y + \frac{\vartheta_1 \mathcal{X}_1}{\iota_1 + \mathcal{X}_1} + \frac{\vartheta_2 \mathcal{X}_2}{\iota_2 + \mathcal{X}_2 + \beta y} - \frac{\sigma_3^2}{2} + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du),$$

and,

$$\begin{aligned} \frac{\mathcal{V}(t) - \mathcal{V}(0)}{t} &\geq \frac{\vartheta_1}{\iota_1 + \frac{v_1}{2\varrho_1}} \left(\frac{v_1 + \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\sigma_1^2}{2}}{v_1 - 2\varrho_1} \right) \\ &+ \frac{\vartheta_1}{\iota_2 + \frac{v_2}{2\varrho_2} + \beta \frac{v_2}{2\varrho_2}} \left(\frac{v_2 + \int_U \log(1 + \Xi_2(u)) - \Xi_2(u) \nu(du) - \frac{\sigma_2^2}{2}}{v_2 - 2\varrho_2} \right) \\ &- d - \varrho_3 < y >_t + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) - \frac{\sigma_3^2}{2} \\ &+ \frac{1}{t} \int_0^t \left(\sigma_3 dW_3(t) + \int_U \log(1 + \Xi_3(u)) \tilde{N}(dt, du) \right), \end{aligned}$$

therefore,

$$\begin{aligned} \langle y \rangle_t \geq & \frac{1}{\varrho_3} \left(\frac{\vartheta_1 v_1 + \vartheta_1 \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\vartheta_1 \sigma_1^2}{2}}{(\iota_1 + N)(v_1 - 2\varrho_1)} \right) \\ & + \frac{1}{\varrho_3} \left(\frac{\vartheta_2 v_2 + \vartheta_2 \int_U \log(1 + \Xi_2(u)) - \Xi_2(u) \nu(du) - \frac{\vartheta_2 \sigma_2^2}{2}}{(\iota_2 + \frac{v_2}{2\varrho_2} + \beta \frac{v_2}{2\varrho_2})(v_2 - 2\varrho_2)} \right) \\ & - \frac{1}{\varrho_3} \left(d + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) - \frac{\sigma_3^2}{2} - \frac{V(t) - V(0)}{t} \right) \\ & + \frac{1}{\varrho_3} \frac{1}{t} \int_0^t \left(\sigma_3 dW_3(t) + \int_U \log(1 + \Xi_3(u)) \tilde{N}(dt, du) \right), \end{aligned}$$

and

$$\begin{aligned} \liminf_{t \rightarrow +\infty} \langle y \rangle_t \geq & \frac{1}{\varrho_3} \left(\frac{v_1 \vartheta_1 + \vartheta_1 \int_U \log(1 + \Xi_1(u)) - \Xi_1(u) \nu(du) - \frac{\vartheta_1 \sigma_1^2}{2}}{(\iota_1 + \frac{v_1}{2\varrho_1})(v_1 - 2\varrho_1)} \right) \\ & + \frac{v_2 \vartheta_2 + \vartheta_2 \int_U \log(1 + \Xi_2(u)) - \Xi_2(u) \nu(du) - \frac{\vartheta_2 \sigma_2^2}{2}}{(\iota_2 + \frac{v_2}{2\varrho_2} + \beta \frac{v_2}{2\varrho_2})(v_2 - 2\varrho_2)} \\ & - d + \int_U \log(1 + \Xi_3(u)) - \Xi_3(u) \nu(du) - \frac{\sigma_3^2}{2}. \end{aligned}$$

□

6. Numerical Analysis

In this section, we illustrate the algorithm used in the numerical analysis of the considered model (2). First, we use the following equation:

$$dZ(t) = \mathcal{G}(t, Z(t))dt + \sigma(t, Z(t))dW_t + \int_U \Pi(t, u) \mathcal{H}(t-, Z(t-)) \tilde{N}(dt, du). \tag{8}$$

Thus,

$$\begin{aligned} Z(t) = Z(0) + & \underbrace{\int_0^t \mathcal{G}(\xi, Z(\xi))d\xi}_{\text{Part I}} + \underbrace{\int_0^t \sigma(\xi, Z(\xi))dW_\xi}_{\text{Part I}} \\ & + \underbrace{\mathcal{H}(t-, Y(t-)) \int_0^t \int_U \Pi(\xi, u) \tilde{N}(d\xi, du)}_{\text{Part II}}. \end{aligned} \tag{9}$$

Using the Runge–Kutta method, we can provide an approximation of part I in (8):

$$Z_{j+1} = Z_j + \frac{1}{6}(M_1 + 2M_2 + 2M_3 + M_4),$$

with

$$\begin{aligned} M_1 &= \mathcal{G}(t_j, Z_j)h_t + \sigma(t_j, Z_j)(W_{j+1} - W_j), \\ M_2 &= \mathcal{G}(t_j + \frac{h_t}{2}, Z_j + \frac{M_1}{2})h_t + \sigma(t_j + \frac{h_t}{2}, Z_j + \frac{M_1}{2})(W_{j+1} - W_j), \\ M_3 &= \mathcal{G}(t_j + \frac{h_t}{2}, Z_j + \frac{M_2}{2})h_t + \sigma(t_j + \frac{h_t}{2}, Z_j + \frac{M_2}{2})(W_{j+1} - W_j), \\ M_4 &= \mathcal{G}(t_j + h_t, Z_j + M_3)h_t + \sigma(t_j + h_t, Z_j + M_3)(W_{j+1} - W_j). \end{aligned}$$

We consider the infinitesimal interval $[T_j, T_{j+1}) \subset (t_j, t_{j+1})$; then, for the second part, we have the following:

With no jumping on $[T_j, T_{j+1})$, we obtain

$$\mathcal{H}(t-, Z(t-)) \int_{T_j}^{T_{j+1}} \int_U \Pi(\xi, u) \check{N}(d\xi, du) = 0.$$

With one jump at the point $t_i \in [T_i, T_{i+1})$, we obtain

$$\mathcal{H}(t-, Z(t-)) \int_{T_j}^{T_{j+1}} \int_U \Pi(u) \check{N}(d\xi, du) = \mathcal{H}(t_j-, Z(t_j-)) \Pi(t_j, \zeta(t_j)),$$

and,

$$\mathcal{H}(t-, Z(t-)) \int_0^t \int_U \Pi(\xi, u) \check{N}(d\xi, du) = \sum_{j=0}^{j=n} \mathcal{H}(t_j-, Z(t_j-)) \Pi(t_j, \zeta(t_j)).$$

Therefore,

$$Z_{j+1} = Z_j + \frac{1}{6} (M'_1 + 2M'_2 + 2M'_3 + M'_4)$$

is the Runge–Kutta version of Equation (8), with

$$\begin{aligned} M'_1 &= \mathcal{G}(t_j, Z_j)h_t + \sigma(t_j, Z_j)(W_{j+1} - W_j) \\ &\quad + \mathcal{H}(t_j-, Z(t_j))\Pi(t_j, \zeta(t_j)), \\ M'_2 &= \mathcal{G}(t_j + \frac{h_t}{2}, Z_j + \frac{M'_1}{2})h_t + \sigma(t_j + \frac{h_t}{2}, Z_j + \frac{M'_1}{2})(W_{j+1} - W_j) \\ &\quad + \mathcal{H}(t_j + \frac{h_t}{2}, Z(t_j) + \frac{M'_1}{2})\Pi(t_j + \frac{h_t}{2}, \zeta(t_j + \frac{h_t}{2})), \\ M'_3 &= \mathcal{G}(t_j + \frac{h_t}{2}, Z_j + \frac{M'_2}{2})h_t + \sigma(t_j + \frac{h_t}{2}, Y_j + \frac{M'_2}{2})(W_{j+1} - W_j) \\ &\quad + \mathcal{H}(t_j + \frac{h_t}{2}, Z(t_j) + \frac{M'_2}{2})\Pi(t_j + \frac{h_t}{2}, \zeta(t_j + \frac{h_t}{2})), \\ M'_4 &= \mathcal{G}(t_j + h_t, Z_j + M'_3)h_t + \sigma(t_j + h_t, Z_j + M'_3)(W_{j+1} - W_j) \\ &\quad + f(t_j + h_t, Z(t_j) + M'_3)\Pi(t_j + h_t, \zeta(t_j + h_t)). \end{aligned}$$

The different values used for the numerical results are provided in Table 2.

Table 2. Values of the parameters used in the numerical simulations.

Parameters	Figure 1	Figure 2	Figure 3
v_1	0.7	1.7	2
ϱ_1	2	3	3
v_2	0.65	1.8	2.3
ϱ_2	1	4	4
ϑ_1	0.1	0.15	0.2
ϑ_2	0.12	0.17	0.3
ϱ_3	2	2	1.5
β	0.002	0.0015	0.001
d	0.4	0.35	0.1
σ_1	10^{-3}	10^{-3}	10^{-2}
σ_2	2×10^{-5}	2×10^{-4}	2×10^{-3}
σ_3	2×10^{-5}	2×10^{-3}	2×10^{-2}
$\Xi_1(u)$	0.006	0.005	0.005
$\Xi_2(u)$	0.03	0.03	0.03
$\Xi_3(u)$	0.07	0.06	0.05

The dynamics of the two prey subpopulations \mathcal{X}_1 and \mathcal{X}_2 and the predator subpopulation y can be seen in Figure 1, where the extinction of both prey species and the predator species can be observed.

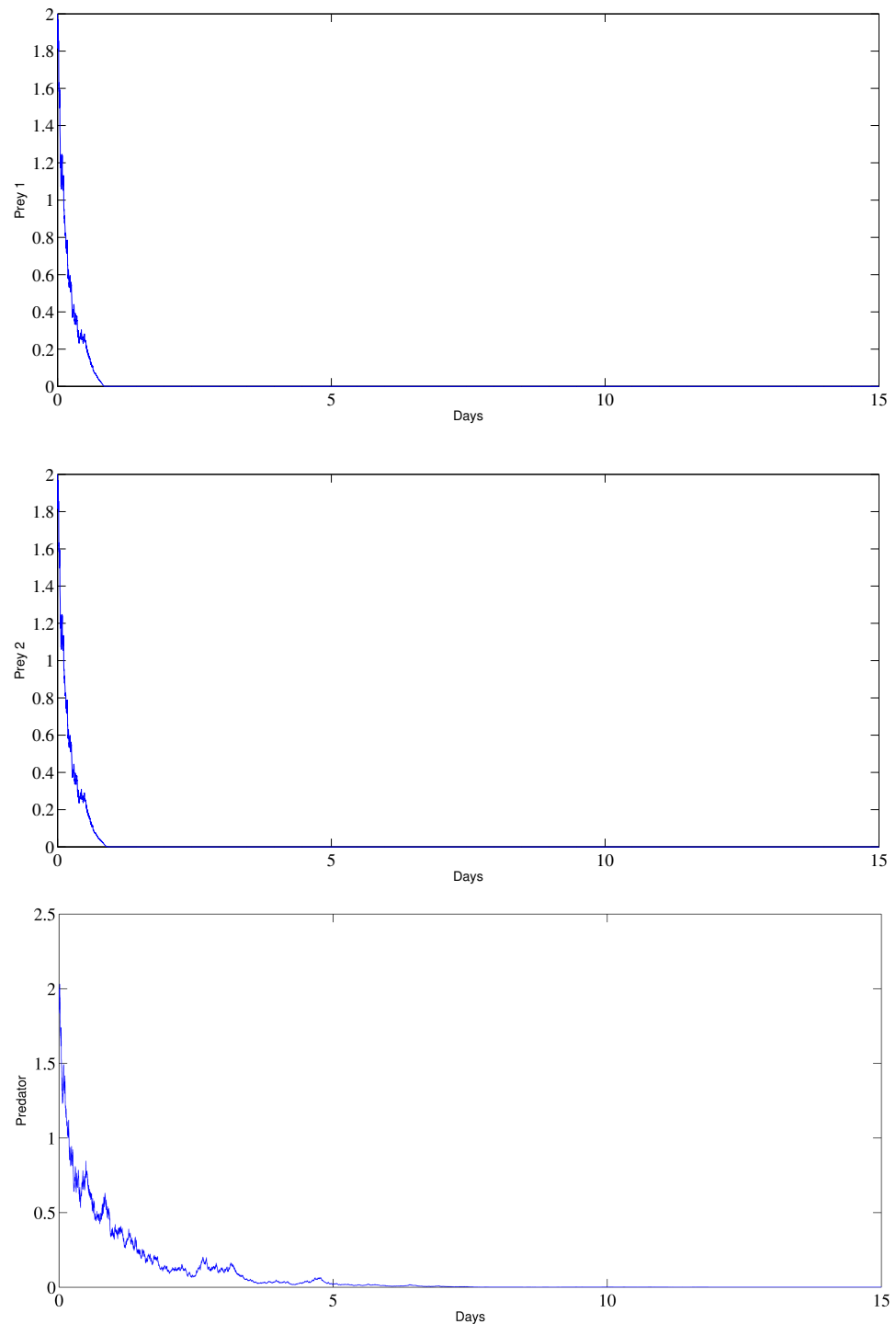


Figure 1. The behavior of the prey and predator subpopulations in the extinction situation.

Figure 2 illustrates the behavior of \mathcal{X}_1 , \mathcal{X}_2 and y ; it can be observed that the prey subpopulations remain strictly positive and the predator subpopulation vanishes, indicates the extinction of the predator and is opposite to the scenario in which the predator and prey populations persist.

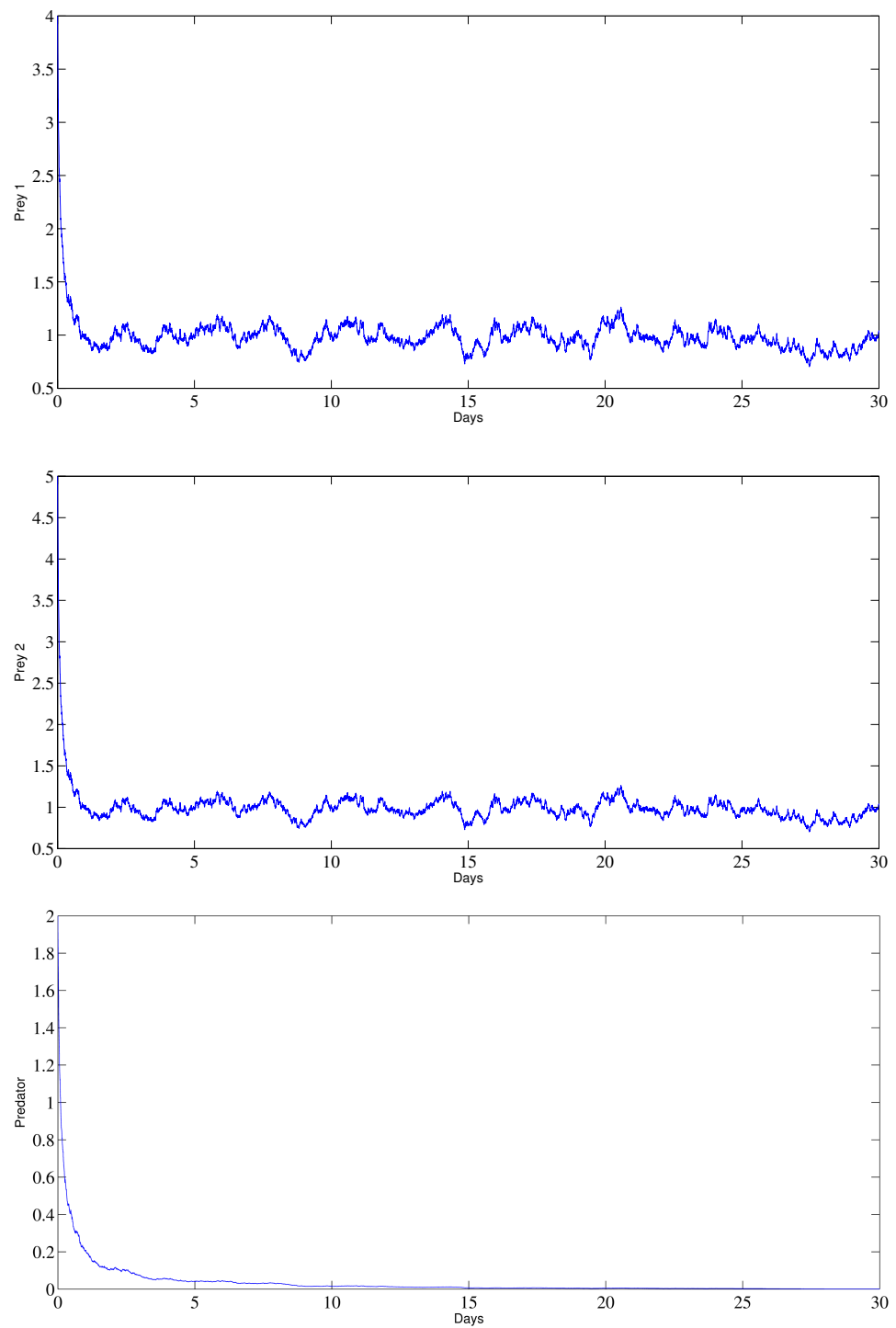


Figure 2. The behavior of the prey and predator populations in the extinction–persistence situation.

The behavior of \mathcal{X}_1 , \mathcal{X}_2 and y is illustrated in Figure 3, observing that all the considered populations persist, which coincides with our theoretical result in the case of persistence.

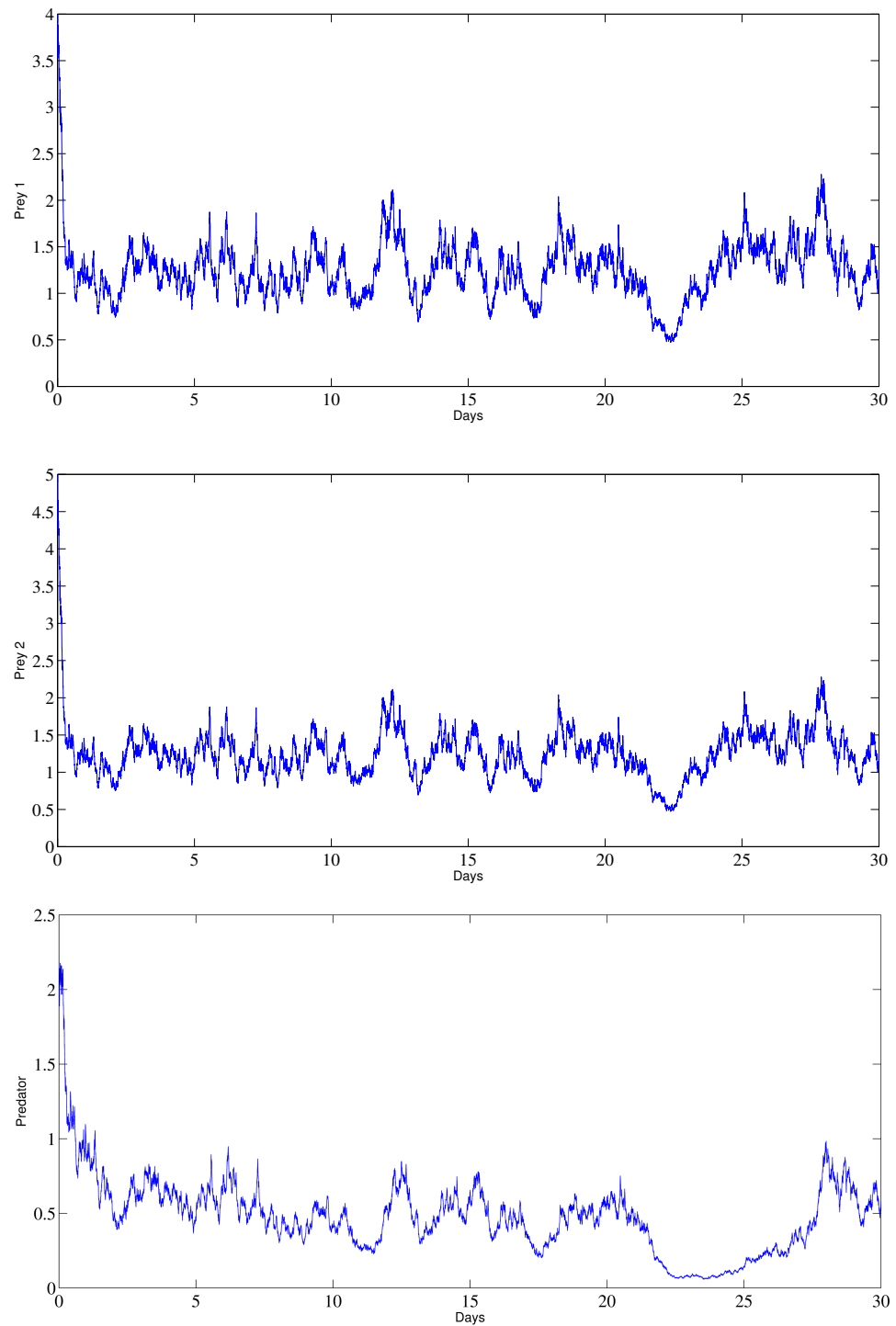


Figure 3. The behavior of the prey and predator populations in the case of persistence.

In Figure 4, we show a comparison between the stochastic model (2) and deterministic model (1) in the persistence case. It can be observed that the curve of the stochastic case has perturbations around the deterministic curve, which proves that the stochastic model can more efficiently describe the dynamics of the prey–predator model.

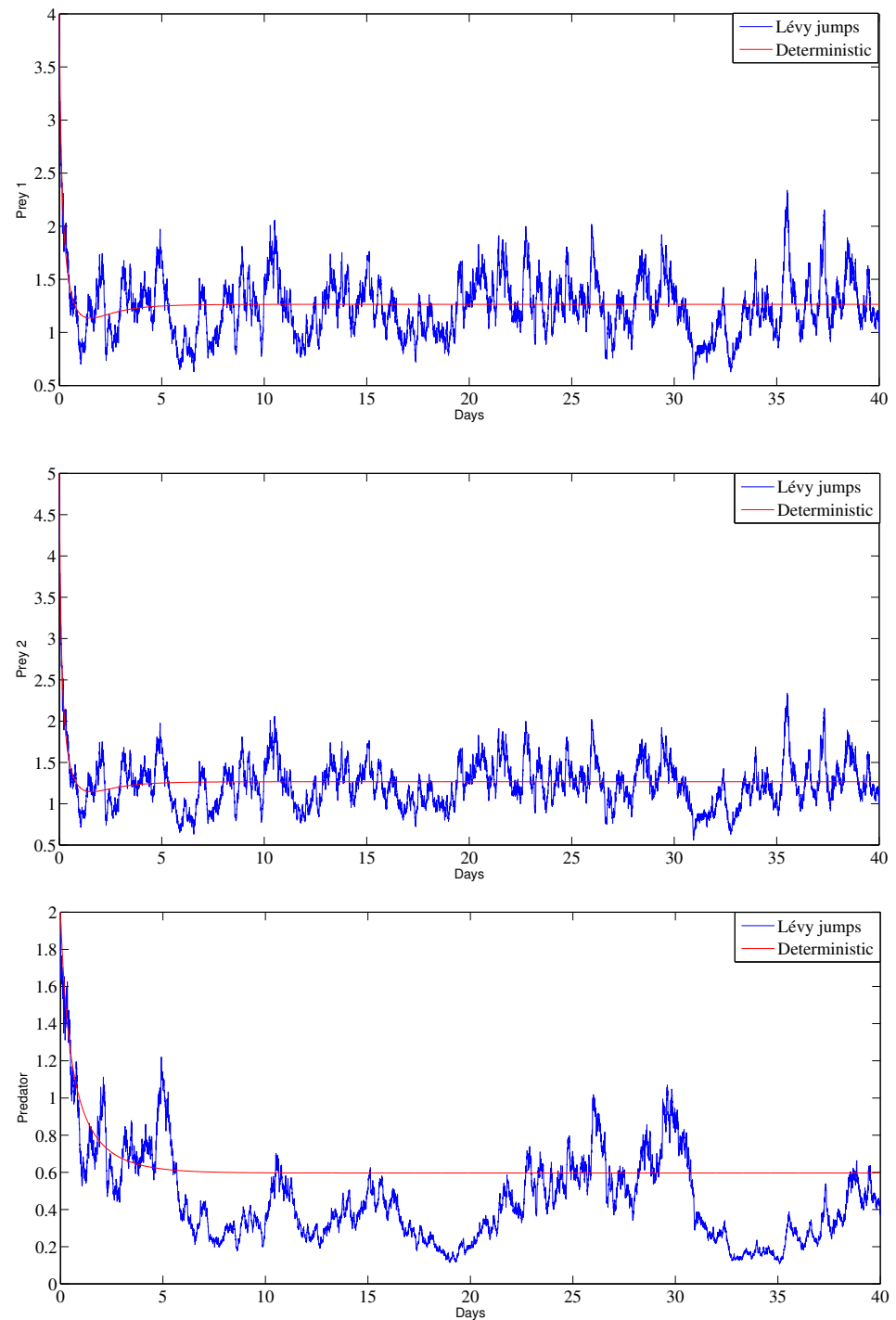


Figure 4. Comparison of the behavior of the prey and predator populations the deterministic model and stochastic model in the case of persistence between.

Figure 5 shows a comparison of the stochastic model (2) in the persistence case with and without Lévy jumps. It is notable that the curve of the stochastic model with jumping has a number of jumped-up perturbations, while the curve without Lévy jumps has normal perturbations. This observation demonstrates that certain cases can lead to Lévy jumps in the behavior of the prey–predator model.

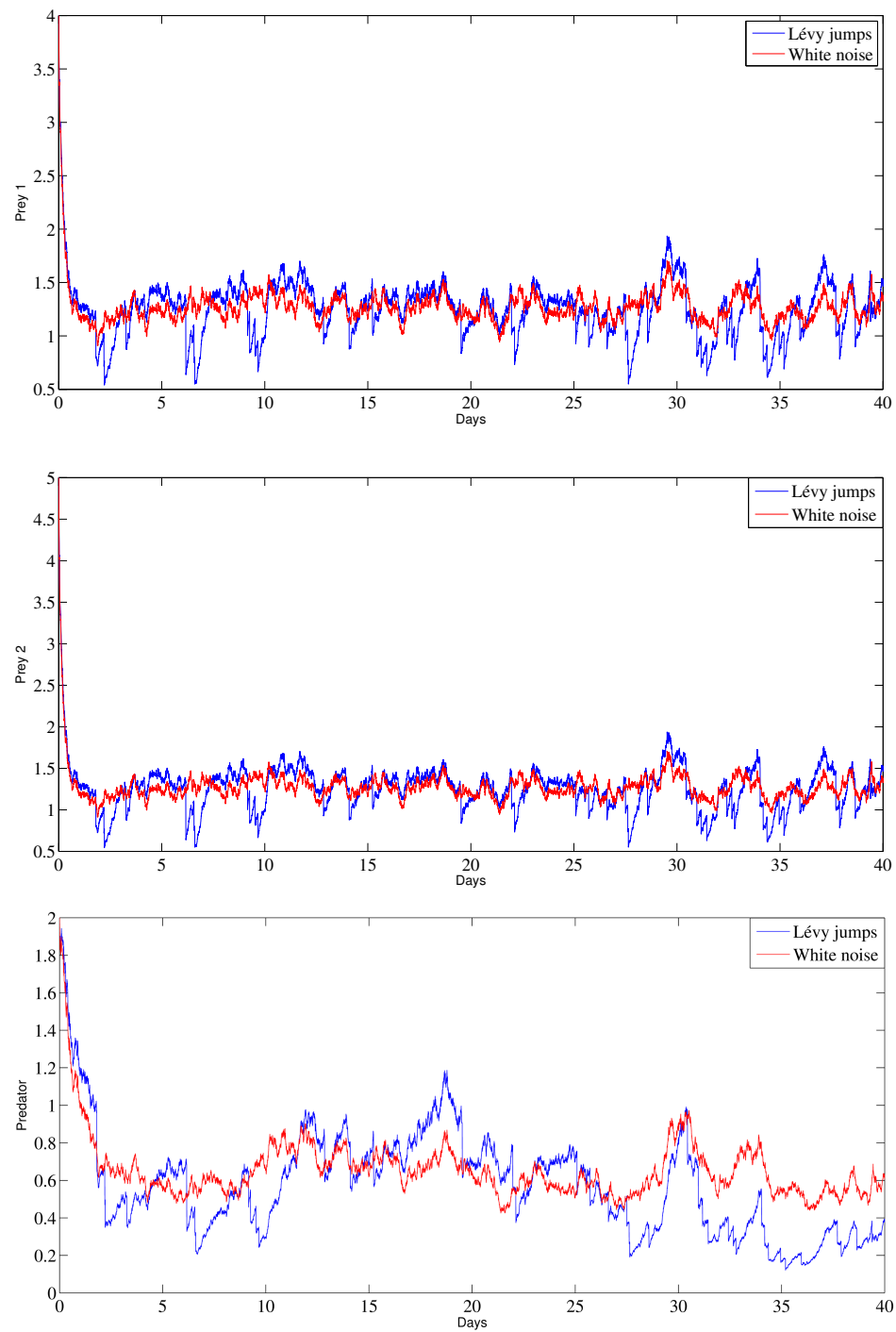


Figure 5. Comparison of the behavior of the prey and predator populations in the case of persistence with and without Lévy jumps.

7. Conclusions and Discussion

This paper has investigated a stochastic prey–predator model driven by Lévy noise in the presence of two prey subpopulations and one predator subpopulation. The model under consideration uses two forms of functional responses, namely, Holling-II $\frac{y\mathcal{X}_2}{i_2 + \mathcal{X}_2 + \beta y}$ and Beddington–DeAngelis $\frac{y\mathcal{X}_1}{i_1 + \mathcal{X}_1}$, due to the saturation behavior. These two mixed functionals can better describe the reality of the computation between the prey and predator subpopulations. First, we have established the well-posedness of the solution of the

model (2). An analysis of persistence and extinction led us to three possible cases: the first is devoted to the extinction of all the considered population when $\max\{m_1, m_2, m_3\} \leq 0$; the second case is dedicated to the persistence of the two prey species and the extinction of the predator species when $\min\{m'_1, m'_2\} \geq 0$ and $m_4 \leq 0$; and in the last case, we show the persistence of all the studied populations under several conditions. In our numerical analysis, we illustrate a special Runge–Kutta method for the stochastic model under Brownian and Lévy jump perturbations as implemented in Matlab R2019b software.

In this way, we considered the three cases with respect to extinction (persistence of the two prey species, extinction of the predator, and persistence of all three species) in simulations in order to validate our theoretical findings. Finally, we present several numerical comparisons. The first is between the deterministic and stochastic cases, which shows the usefulness of considering the stochastic model. The second comparison is between the cases with and without Lévy jumps, which proves that certain cases naturally lead to Lévy jumping in the behavior of the prey–predator model. Future efforts should address remaining issues; one of these is how the presented predator–prey model spreads via Levy jumps. In addition, we intend to study the chaotic behavior of the stochastic predator–prey system. Moreover, it is possible to model the prey–predator relationship using the space–time spectral order Sinc-collocation method (see [36] for example) and to illustrate a new modified stochastic predictor–corrector compact difference method, such as the one suggested in [38].

Author Contributions: Conceptualization, M.Y. (Mehmet Yavuz); Methodology, J.D. and M.Y. (Mustafa Yıldız); Software, J.D.; Validation, M.Y. (Mustafa Yıldız); Formal analysis, M.Y. (Mehmet Yavuz); Investigation, M.Y. (Mustafa Yıldız); Resources, M.Y. (Mehmet Yavuz); Data curation, J.D. and M.Y. (Mehmet Yavuz); Writing—original draft, J.D.; Writing—review & editing, M.Y. (Mehmet Yavuz) and M.Y. (Mustafa Yıldız). All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Our manuscript has no associated data.

Conflicts of Interest: The authors declare no conflict of interest.

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