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Article

Operator Kernel Functions in Operational Calculus and Applications in Fractals with Fractional Operators

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Abstract: In this study, we delve into the general theory of operator kernel functions (OKFs) in operational calculus (OC). We established the rigorous mapping relation between the kernel function and the corresponding operator through the primary translation operator e^{-pt}, which bears a striking resemblance to the Laplace transform. Our research demonstrates the uniqueness of the kernel function, determined by the rules of operational calculus and its integral representation. This discovery provides a novel perspective on how the operational calculus can be understood and applied, particularly through convolution with kernel functions. We substantiate the accuracy of the proposed method by demonstrating the consistency between the operator solution and the classical solution for the heat conduction problem. Subsequently, on the fractal tree, fractal loop, and fractal ladder structures, we illustrate the application of operational calculus in viscoelastic constitutive and hemodynamics confirming that the method proposed unifies the OKFs in the existing OC theory and can be extended to the operator field. These results underscore the practical significance of our results and open up new possibilities for future research.

Keywords: operational calculus; operator kernel function; physical fractal; integral transform; fractional operator

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1. Introduction

Operational calculus (OC), a critical pillar in mathematical physics [1–6], has seen remarkable advancements since Heaviside first introduced the concept of differential operator notation $\boldsymbol{p}$ [7]. Yet, it has never ceased to evolve and provoke intellectual curiosity [8–12], especially with its application in fractional calculus including various types [13–15] and even fractional order derivatives to functions [16]. These new concepts have renewed interest and deepened the connotation of OC. With the advent of physical fractal spaces and the fractal operators concept introduced recently, OC has attracted attention again [17–21].

Historically, Heaviside's operator algebraization has been controversial and considered by many to be fundamentally flawed [22]. While the method is overwhelmingly straightforward, it makes people figure it out mathematically [3,23–29]. The current consensus leans towards an intriguing relation between OC and integral transform [22,30], yet the nature of this relation, particularly whether a one-to-one mapping exists between the operator and the function, remains under-explored and ambiguous.

The application of OC in solving scientific and technical differential equations has been challenging, prompting a shift towards integral transformation techniques [31]. It was not until the mid-20th century that Jan Mikusiński [32] revitalized OC by treating functions and operators akin to algebraic expressions, divorcing it from integral transformations.

Despite its novelty, Mikusiński's approach had limitations, particularly its confinement to certain operators and the need for separate analysis for different operator types. The typical operator and their corresponding kernel functions are listed in Table 1. The requirement to scrutinize each operator individually precludes his theory from evolving into a comprehensive system. This modus operandi mandates embarking on fresh research for each newly encountered operator to determine its kernel function. Considering that the operator field is uncountable, exhaustively identifying all operator kernel functions is an impractical endeavor. Occasionally, the kernel function of an operator may not be discernible within the theoretical framework, necessitating reliance on external factors. For instance, in addressing the operator $e^{-ax\sqrt{p}}$ implicated in heat conduction problems [32], the author 'guessed' by juxtaposing this operator with known classical solutions, rather than deriving it computationally from his theory. This diverges considerably from the foundational premise expected of a robust theoretical system.

Table 1. Typical operator kernel functions in Mikusiński's theory.

Operator	Kernel Function
$\frac{1}{\boldsymbol{p}^n}$ [1]	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{(\boldsymbol{p}-\alpha)^n}$	$\frac{t^{n-1}}{(n-1)!}e^{\alpha t}$
$\frac{1}{\boldsymbol{p}^2+\beta^2}$	$\frac{1}{\beta}\sin\beta t$
$\frac{\boldsymbol{p}}{\boldsymbol{p}^2+\beta^2}$	$\cos\beta t$
$\left[(\boldsymbol{p}-\alpha)^2+\beta^2\right]^{-n}$	$\frac{e^{at}}{(2\beta^2)^{n-1}}\left[A_n(\beta^2t^2)\frac{1}{\beta}\sin\beta t - B_n(\beta^2t^2)t\cos\beta t\right]$ [2]
$\frac{(\sqrt{\boldsymbol{p}^2+\alpha^2}-\boldsymbol{p})^n}{\sqrt{\boldsymbol{p}^2+\alpha^2}}$	$\alpha^n J_n(\alpha t)$ [3]
$\cos\frac{1}{\boldsymbol{p}^2}$	$\sum_{i=0}^{\infty}\frac{(-1)^i t^{(2^{i+1}-1)}}{(2i)!(2^{i+1}-1)!}$
$\frac{1}{\boldsymbol{p}}e^{-\frac{\lambda}{\boldsymbol{p}}}$	$J_0(2\sqrt{\lambda t})$
$\frac{1}{\sqrt{\boldsymbol{p}}}e^{-\frac{\lambda}{\boldsymbol{p}}}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{\lambda t}$
$\frac{1}{\boldsymbol{p}^2}e^{-\frac{\lambda}{\boldsymbol{p}}}$	$\sqrt{\frac{t}{\lambda}}J_2(2\sqrt{\lambda t})$
$e^{\lambda(\boldsymbol{p}-\sqrt{\boldsymbol{p}^2+\alpha^2})}$	$1-\frac{\lambda}{\sqrt{t^2+2\lambda t}}\alpha J_1(\alpha\sqrt{t^2+2\lambda t})$
$e^{-\lambda\sqrt{\boldsymbol{p}^2+\alpha^2}}/\sqrt{\boldsymbol{p}^2+\alpha^2}$	$J_0(\alpha\sqrt{t^2-\lambda^2})\quad(0\le\lambda<t)$

[1] $n>0$ and $n\in Z$. [2] A_n and B_n are parameters. [3] J_n represents the Bessel function of order n.

The origins of these two limitations can be traced back to the existing operational calculus theory, which is exclusively rooted in algebra and exhibits relative autonomy from integral transformations. Our research addresses this issue, merging it with integral transformations to yield a unified operator form expressed via kernel functions. We aim to further bridge this gap by setting definitive transform mapping relations between operators and kernel functions, thereby generalizing Mikusiński's method. Moreover, we apply the OC to the classic heat conduction problem and three distinct fractal structures, demonstrating the method's simplicity and efficiency. The proposed OKF method for OC elucidates the expression of the operator, offering a novel paradigm for future research in this domain.

The structure of this paper is as follows: Section 2 introduces the necessary preliminaries and symbolic notation. Section 3 dedicates to proving the uniqueness of the operator satisfying given conditions, discussing the equivalence between exponential and translation operators, and substantiating the equivalence between the integral transform and OC. Finally, Section 4 demonstrates the practical application of the OKF method to the classic heat conduction problem and three distinct fractal structures, i.e., fractal tree, fractal loop and fractal ladder.

2. Preliminaries

In this section, we revisit some fundamental concepts of algebraic theory and then clarify the notation to be used in this paper. Although the content in this section is not novel, it is instrumental in assisting the reader to establish a conceptual foundation for the algebra of operational calculus.

2.1. Function Space

In establishing the theoretical framework, it is crucial to select a function set that strikes a balance between practicality and inclusiveness. Ideally, one might consider introducing a space as comprehensive as possible, such as an L-integral function set [33]. Or, considering the involvement of derivative operations, one would introduce C^n–continuous functions. However, introducing the Lebesgue integral would introduce complexities and potential challenges that might hinder the overall analysis, and the choice of C^n–continuous functions would impose limitations on its range of application.

Hence, it becomes necessary to identify a suitable function subset that is both practical and sufficiently expansive. We have selected the Mikusiński function set $\mathscr{K}$, which possesses specific properties and has demonstrated its relevance and effectiveness in similar contexts [32].

Definition 1. *A function $f(t)(t \geq a)$ belongs to the space $\mathscr{K}$, if the function (real or complex-valued) satisfies:*

(i) *The function $f(t)$ has, at most, a finite number of discontinuity points in any finite interval.*
(ii) *For arbitrary $t > a$, the integration of $f(t)$ is bounded, i.e., $\int_a^t f(\tau)\mathrm{d}\tau < \infty$.*

The space $\mathscr{K}$ contains the functions that possess finite discontinuous points in any finite interval, e.g., the δ-function, which is essential in the OKF theory.

2.2. Ring and Field

We begin by introducing several basic concepts in modern algebra [34,35].

Definition 2. *A Commutative ring $\mathscr{L}$ without zero divisors is termed a integral domain, i.e., if $\boldsymbol{ab} = \mathbf{0}$, there must be either $\boldsymbol{a} = \mathbf{0}$ or $\boldsymbol{b} = \mathbf{0}$.*

Definition 3. *A set $\mathscr{Q}$ is called the field generated by the integral domains if the inverse operation of multiplication is defined on $\mathscr{L}$ by: whatever $\boldsymbol{a}, \boldsymbol{b} \in \mathscr{L}$ and $\boldsymbol{a} \neq \mathbf{0}$, there is and only is one solution $\boldsymbol{x} \in \mathscr{Q}$ that satisfies $\boldsymbol{ax} = \boldsymbol{b}$, denoted as $\boldsymbol{x} = \frac{\boldsymbol{b}}{\boldsymbol{a}}$.*

The modern algebra has shown that each commutative ring can be included by its generating field, i.e., $\mathscr{L} \subset \mathscr{Q}$. Take the set of integers $\mathcal{Z}$ as an example. By taking the usual division as the inverse of multiplication, one would generate the rational numbers $\mathcal{Q}$, and $\mathcal{Z} \subset \mathcal{Q}$. In each field, there is an identical element $\boldsymbol{I}$ satisfying $\boldsymbol{aI} = \boldsymbol{a}$ or $\boldsymbol{I} = \frac{\boldsymbol{a}}{\boldsymbol{a}}$.

2.3. Notation of Operators

In this section, we introduce conventions for the terminology and notations used throughout the paper. Functions are denoted by lowercase letters and their corresponding variables, e.g., $f(t)$, $\delta(t)$. If a specific expression is given, the variables will be omitted, e.g., e^{-t}. Operators are represented by bolded letters, e.g., $\boldsymbol{f}$, $\boldsymbol{p}$, $\boldsymbol{l}$, $\boldsymbol{T}$, with parameters explicitly included when necessary, e.g., $\boldsymbol{f}(\lambda, \mu)$. The operator field generated by Mikusiński space $\mathscr{K}$ was denoted as $\mathscr{O}$. The correspondence between the elements in the function ring $\mathscr{K}$ and their respective operator is represented as

$$\boldsymbol{f} \simeq f(t), \quad \boldsymbol{f} \in \mathscr{O}, \quad f(t) \in \mathscr{K}. \tag{1}$$

Here the symbol $\simeq$ only indicates that the function element $\mathscr{K}$ is correlated with the operator $\mathscr{O}$. The specific mapping relations will be established later in Section 3.

Definition 4. *The addition of operators is the summation of the corresponding functions, i.e.,*

$$\boldsymbol{f} + \boldsymbol{g} \simeq f(t) + g(t), \quad \boldsymbol{f}, \boldsymbol{g} \in \mathscr{O}, \quad f(t), g(t) \in \mathscr{K}. \tag{2}$$

Definition 5. *The multiplication of operators is the convolution of the corresponding functions, i.e.,*

$$\boldsymbol{f} \cdot \boldsymbol{g} = f(t) \circ g(t) = \int_0^t f(t-\tau) g(\tau) \mathrm{d}\tau, \quad \boldsymbol{f}, \boldsymbol{g} \in \mathscr{O}, \quad f(t), g(t) \in \mathscr{K}. \tag{3}$$

It can be demonstrated that the triple $(\boldsymbol{f}, +, \cdot)$ fulfills all the necessary conditions for a field, where the Definition 2 being supported by Titchmarsh's theory [32,34]. Without confusion, the multiplicative notation $(\cdot)$ can be omitted, i.e., $\boldsymbol{f} \cdot \boldsymbol{g} = \boldsymbol{f}\boldsymbol{g}$.

The corresponding function of the operator $\boldsymbol{l}$ is given by the constant value function $1(t) \equiv 1$ (note the distinction between constant value functions and numbers; a number cannot convolve with a function and can only perform multiplication operation in the algebraic sense). The action of operator $\boldsymbol{l}$ on the function $f(t) \in \mathscr{K}$ results in

$$\boldsymbol{l}\boldsymbol{f} = \int_0^t f(\tau) \mathrm{d}\tau, \quad \boldsymbol{l}, \boldsymbol{f} \in \mathscr{O}, \quad f(t) \in \mathscr{K}. \tag{4}$$

According to Equation (4), the operator $\boldsymbol{l}$ is referred to as the integral operator, and $\boldsymbol{l} \triangleq 1(t)$.

Remark 1. *According to Definition 3, there exists an element $\boldsymbol{p} \in \mathscr{O}$ such that $\boldsymbol{p}\boldsymbol{l} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identical element, and the operator $\boldsymbol{p}$ is the dual operator to $\boldsymbol{l}$. From Equation (4), we obtain $\boldsymbol{p}\boldsymbol{l}\boldsymbol{f} = \boldsymbol{I}\boldsymbol{f} = \boldsymbol{f}$; therefore, the operator $\boldsymbol{p}$ is commonly referred to as the derivative operator. Note that the operator $\boldsymbol{p}$ is not precisely equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}$, but rather, $\boldsymbol{p}\boldsymbol{f} = \frac{\mathrm{d}}{\mathrm{d}t} f(t) + \boldsymbol{p} f(0)$. There is no C^n-continuous function related to the operator $\boldsymbol{p}$ [15,28].*

Remark 2. *From Equation (3), the convolution of two constant value functions does not result in another constant value function, e.g., $C_1(t) \circ C_2(t) = C_1 C_2 t$. To draw an analogy between operations on constants in the algebraic sense, the concept of a numerical operator was introduced [32], defined as $\boldsymbol{C} = \boldsymbol{p}C(t)$. The relations between a constant C, a constant valve function $C(t)$ and a numerical operator $\boldsymbol{C}$ are: $\boldsymbol{C} = \boldsymbol{p}C(t)$, $C(t) \equiv C$.*

Remark 3. *The number multiplication operation can also be defined in the operator field, where the multiplier is not a number in the algebraic sense but a numerical operator in the operator sense, i.e., $\boldsymbol{C}\boldsymbol{f} = \boldsymbol{p}C(t) \circ f(t) = Cf(t)$. Consequently, the numerical operator can freely move between the operator field and the function ring. The symbol $\boldsymbol{C}$ should be regarded as a numerical operator on the left-hand side (LHS), while C should be considered as a constant instead of a function on the right-hand side (RHS). Because of this feature, where it is clear, we can also write $\boldsymbol{C}\boldsymbol{f} = C\boldsymbol{f} = Cf(t)$. Thus, we have $\boldsymbol{I} = \mathbf{1} = 1$ and $\boldsymbol{p} = \frac{\boldsymbol{I}}{\boldsymbol{l}} = \frac{1}{\boldsymbol{l}}$.*

3. Operator Kernel Function Method

The operator field is generated by the function ring, where the elements can be interpreted as generalized functions. A fundamental question arises: Is there a definitive mapping relation between operators and functions that specifies how operators act on functions? The method employed by Mikusiński is not applicable when dealing with the set of infinitely many elements in the operator field. Furthermore, for operators that cannot be represented by basic operators, e.g., the operator $e^{-\lambda\sqrt{\boldsymbol{p}}}$ and $\frac{1}{\sqrt{\boldsymbol{p}}} e^{-ax\sqrt{\boldsymbol{p}}}$, a universal method for finding the kernel function remains elusive. In such cases, the kernel function can only be determined by comparing it with specific practical problems.

In the upcoming section, we will present the transform relation between kernel functions and operators in a general sense, building upon several lemmas to facilitate our discussion.

3.1. Uniqueness of Solutions of Operator Differential Equations

In the theory of differential equations, the uniqueness of solutions for linear ordinary differential equations satisfying definite conditions is well-established. Similarly, this principle extends to the realm of operators, albeit with the added complexity of distinct operation rules. In the operator field, where the multiplication of operators is considered as the convolution of corresponding functions, ensuring uniqueness is not a trivial proposition. The following lemma provides crucial insights into the operator differential equations under consideration.

Lemma 1. *The operator $\boldsymbol{x}(\lambda)$ satisfy $\boldsymbol{x}'(\lambda) = \omega \boldsymbol{x}(\lambda)$ is unique given that $\boldsymbol{x}(\lambda_0) = \boldsymbol{k}$.*

The proof of Lemma 1 aligns with the proof presented by Ref. [32], which is provided in the Appendix. In the theory of differential equations, it is well-known that the general solution of the differential equation $y'(t) = \omega y(t)$ takes on an exponential form $y = e^{\omega t}$, providing further insights into the behavior of the operator differential equation.

3.2. Translation Operator

Dirac delta functions play an essential role in mathematical physics, such as Green's function method, the electric imaging method, etc. [22]. The delta function is defined by

$$\delta_\lambda(t) = \delta(t-\lambda) = \delta(\lambda - t) = \begin{cases} \infty, & t = \lambda \\ 0, & t \neq \lambda \end{cases}. \tag{5}$$

The δ-function has the integral property

$$\int_{-\infty}^{\infty} \delta_\lambda(t)\mathrm{d}t = \int_{\lambda-\epsilon}^{\lambda+\epsilon} \delta_\lambda(t)\mathrm{d}t = 1. \tag{6}$$

For a function $f(t)$ defined on the real semi-axis, we have the following convolution integral

$$\delta_\lambda(t) \circ f(t) = \int_0^t \delta(t-\tau-\lambda) f(\tau)\mathrm{d}\tau = \int_0^t \delta(\tau - t + \lambda) f(\tau)\mathrm{d}\tau = f(t-\lambda). \tag{7}$$

Equation (7) shows that the physical meaning of the convolution of the delta function $\delta_\lambda(t)$ as a kernel with $f(t)$ is to translate the function to the right by the length λ, which is why the operator $\boldsymbol{\delta}_\lambda$ is termed as the translation operator. According to the physical meaning, we have

$$\boldsymbol{\delta}_{\lambda+\mu} \boldsymbol{f} = \delta_{\lambda+\mu}(t) \circ f(t) = \delta_\lambda(t) \circ \delta_\mu(t) \circ f(t) = f(t-\lambda-\mu). \tag{8}$$

The corresponding operator $\boldsymbol{\delta}_\lambda$ has the important property

$$\boldsymbol{\delta}_{\lambda+\mu} = \boldsymbol{\delta}_\lambda \boldsymbol{\delta}_\mu. \tag{9}$$

Equation (9) has the same properties as the exponential function e^x in algebra

$$e^{\lambda+\mu} = e^\lambda e^\mu. \tag{10}$$

Remark 4. *Distinction arises between the multiplication operations defined in Equations (9) and (10). In Equation (9) the operator multiplication is characterized by the convolution of the corresponding function, while in Equation (10) the multiplication follows the conventional algebraic sense.*

Despite this difference, the striking similarity between these two operations prompts an intriguing question: Could an equivalence relation exist between them, bridging the function ring and the operator field? To explore this question further, we require an additional lemma that will shed light on their connection and potential equivalence.

Lemma 2. *The delta operator $\boldsymbol{\delta}_\lambda$ satisfy $\boldsymbol{\delta}'_\lambda = -\boldsymbol{p}\boldsymbol{\delta}_\lambda$.*

Proof. The Heaviside step function $H_\lambda(t)$ and the Delta function $\delta_\lambda(t)$ have the following property

$$H_\lambda(t) = \boldsymbol{l}\delta_\lambda(t), \quad or \quad \delta_\lambda(t) = \boldsymbol{p}H_\lambda(t). \tag{11}$$

Integration on the step function gives

$$\boldsymbol{l}H_\lambda(t) = \begin{cases} 0, & 0 \leq t < \lambda \\ t-\lambda, & 0 \leq \lambda < t \end{cases}, \tag{12}$$

$$\boldsymbol{l}^2 H_\lambda(t) = \begin{cases} 0, & 0 \leq t < \lambda \\ \frac{1}{2}(t-\lambda)^2, & 0 \leq \lambda < t \end{cases}. \tag{13}$$

Using the derivative of the Dirac operator, we obtain

$$(\boldsymbol{\delta}_\lambda)' = \left(\boldsymbol{p}^3\boldsymbol{l}^2(H_\lambda(t))\right)' = \boldsymbol{p}^3\left(\frac{\partial}{\partial\lambda}\boldsymbol{l}^2 H_\lambda(t)\right) = \boldsymbol{p}^3\begin{cases} 0, & 0 \leq t < \lambda \\ -(t-\lambda), & 0 \leq \lambda < t \end{cases}. \tag{14}$$

Further, by substituting Equations (11) and (12) into Equation (14)

$$(\boldsymbol{\delta}_\lambda)' = \boldsymbol{p}^3(-\boldsymbol{l}H_\lambda(t)) = -\boldsymbol{p}\boldsymbol{\delta}_\lambda. \tag{15}$$

The symbol $()'$ in Equations (14) and (15) denotes the derivative with respect to the parameter λ. □

The previous section has proved the uniqueness of the operator. Comparing Equations (9), (10) and (15), we can formally define

$$e^{-\boldsymbol{p}\lambda} = \boldsymbol{\delta}_\lambda \simeq \delta_\lambda(t), \quad e^{-\boldsymbol{p}\lambda}, \boldsymbol{\delta}_\lambda \in \mathscr{O}, \quad \delta_\lambda(t) \in \mathscr{K}. \tag{16}$$

Equation (16) characterizes the time-differentiable operator $\boldsymbol{p}$, which manifests as an exponential operator. The LHS of Equation (16) represents the displacement operator, while the RHS defines the corresponding kernel function.

3.3. The Equivalence of Operational Calculus and Integral Transformations

The OC could be founded on a solid mathematical basis using the equivalence relation Equation (16) between the fundamental operator $e^{-\boldsymbol{p}\lambda}$ and the function $\delta_\lambda(t)$. The following theorem shows that the Laplace transform establishes a mapping relation across the operator field and the function ring.

Theorem 1. *The operators are given by the Laplace transform of the kernel function to the differential operator* $\boldsymbol{p}$.

Proof. The identical transformation is obtained by taking $\lambda = 0$ in Equation (7)

$$\boldsymbol{\delta}_0\boldsymbol{f} = \delta_0(t) \circ f(t) = \int_0^t \delta(t-\tau)f(\tau)\mathrm{d}\tau = \int_0^t \delta_\tau(t)f(\tau)\mathrm{d}\tau = f(t). \tag{17}$$

By substituting Equation (16) into Equation (17)

$$\boldsymbol{f} = \int_0^\infty e^{-\boldsymbol{p}\tau}f(\tau)\mathrm{d}\tau. \tag{18}$$

Equation (18) connects the operator field and the function ring: the LHS of the equation is the operator, and the RHS of the equation is the integral of the product of the function and the operator in the algebraic sense. This formula answers the fundamental question of constructing the corresponding operator in the operator field given a function $f(t)$ in the function ring. □

Notice that Equation (18) is formally identical to the Laplace transform [22], which is given by

$$F(p) = L[f(t)] \overset{\Delta}{=} \int_0^\infty e^{-p\tau} f(\tau)\mathrm{d}\tau. \tag{19}$$

Although Equations (18) and (19) are similar in form, their starting point and physical meaning are entirely different. In Equation (18) the formulas on the LHS are operators in $\mathscr{O}$, while those on the RHS are functions in $\mathscr{K}$. In Equation (19), the formulas on the LHS and RHS are all functions. The rules for operators in the operator field and the Laplace transform have the following formal similarities:

$$\boldsymbol{f} \simeq f(t) \quad \Leftrightarrow \quad F(p) = L[f(t)](p), \tag{20}$$

$$\boldsymbol{f} + \boldsymbol{g} \simeq f(t) + g(t) \quad \Leftrightarrow \quad L[f(t) + g(t)] = L[f(t)] + L[g(t)], \tag{21}$$

$$\boldsymbol{Cf} \simeq Cf(t) \quad \Leftrightarrow \quad L[Cf(t)] = CL[f(t)], \tag{22}$$

$$\boldsymbol{f} \cdot \boldsymbol{g} = f(t) \circ g(t) \quad \Leftrightarrow \quad L[f(t)]L[g(t)] = L[f(t) \circ g(t)], \tag{23}$$

$$\boldsymbol{pf} = f'(t) + \boldsymbol{f}(0) \quad \Leftrightarrow \quad pL[f(t)] = L[f'(t)] + f(0). \tag{24}$$

The translation operator relates the OC to the integral transform and assigns them the same rules. This also proves that the OC is a formal symbolic operation based on the Laplace transform. This assertion provides a theoretical basis for the OC. The formal simplicity of the symbolic operations of the OC compared to the integral transform makes us focus more on the algebraic computations in the operator field rather than on the partial derivative procedures coupled in time and space.

3.4. The Uniqueness of Operational Calculus and Integral Transformations

The previous section established the equivalence between OC and integral transform, which arises from the form of the chosen transform and the defined operation rules. This equivalence implies that OC, based on the fundamental transform, is also equivalent to the integral transform. A complete theory should be existence and uniqueness, and the equivalence gives the existence of the theoretical basis of the operator calculus. The following theorem guarantees the uniqueness of the integral transformations [33].

Theorem 2. *The integral transformation kernel function required to make the operator have the property* (20)–(24) *is unique, the kernel function is* $K(p,t) = e^{-pt}$, *and the integral transform constituted is the Laplace transform.*

Proof. Consider the integral transform given by the kernel function $K(p,t)$:

$$\ell[f](p) = \int_0^\infty K(p,t)f(t)\mathrm{d}t. \tag{25}$$

The transform given by Equation (25) needs to satisfy all the rules for an operator, i.e., (20)–(24). Considering the differential property Equation (24), integral $f'(t)$ by part

$$\ell[f'](p) = \int_0^\infty K(p,t)f'(t)\mathrm{d}t = (K(p,t)f(t))|_0^\infty - \int_0^\infty \frac{\partial K(p,t)}{\partial t} f(t)\mathrm{d}t. \tag{26}$$

Equation (26) should have the same form as Equation (24), so we have

$$\ell[f'] = p\ell[f] - f(0). \tag{27}$$

Comparing Equations (26) and (27), the kernel function $K(p,t)$ should satisfy

$$\frac{\partial K(p,t)}{\partial t} = -pK(p,t), \tag{28}$$

and

$$\lim_{t\to 0+} K(p,t) = 1, \quad \lim_{t\to\infty} K(p,t) = 0. \tag{29}$$

The particular unique solution of this differential equation is

$$K(p,t) = e^{-pt}. \tag{30}$$

Lastly, bringing Equation (30) back into Equation (25), we obtain the transform

$$\ell[f](p) = L[f](p) = \int_0^\infty e^{-pt} f(t)\mathrm{d}t. \tag{31}$$

So far, we have proved that the kernel function is unique based on integral transform. □

Comparing Equations (18) and (19), we immediately deduce from the uniqueness of the Laplace transform that

Theorem 3. *If there exists an inverse Laplace transform of the operator $\boldsymbol{f}$ for the differential operator $\boldsymbol{p}$ on the operator field, then the kernel function in the function ring exists and is given by the inverse Laplace transform of $\boldsymbol{f}$ for the differential operator $\boldsymbol{p}$, i.e.,*

$$f(t) = \frac{1}{2\pi i}\int_c e^{\boldsymbol{p}t}\boldsymbol{f}\mathrm{d}\boldsymbol{p}. \tag{32}$$

Readers are encouraged to independently confirm that all operators listed in Table 1 comply with the relation illustrated in Equation (32). This eliminates the necessity for cumbersome individual analyses for each new operator encountered, as the transformation relationship readily provides the expression for the OKF. The theorem further suggests that the presence of an operator kernel function necessitates the existence of its inverse Laplace transformation. A necessary condition for the inverse transformation is the equation $\lim_{\boldsymbol{p}\to\infty} \boldsymbol{f}(\boldsymbol{p}) = 0$.

3.5. Relationship with Carson-Laplace Transform

In this paper, the convolution of constant value functions is not again a constant function. There are two ways to resolve this contradiction, either by defining the concept of numerical operator here or by introducing a different definition of operator multiplication [24,33]

$$\boldsymbol{f} * \boldsymbol{g} \triangleq \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau. \tag{33}$$

In the operator multiplication defined by Equation (33), the behavior of a constant operator is identical to that of a number in the algebraic sense. According to the relation between the differential operator and the derivative, there is

$$\boldsymbol{f} * \boldsymbol{g} = \boldsymbol{p}\boldsymbol{f}\cdot\boldsymbol{g}. \tag{34}$$

Compared with the definition given by Equation (3), the differentiation in Equation (33) make one factor more in the RHS of Equation (34). At this point, the operator corresponds to the kernel function, also with an additional factor $\boldsymbol{p}$, which is given by the Carson-Laplace transform [22]

$$\tilde{\boldsymbol{f}} = \boldsymbol{p}\int_0^\infty e^{-\boldsymbol{p}\tau} f(\tau)\mathrm{d}\tau. \tag{35}$$

And the corresponding inverse transform is given by

$$f(t) = \frac{1}{2\pi i} \int_c e^{pt} \frac{\tilde{\boldsymbol{f}}}{\boldsymbol{p}} \mathrm{d}\boldsymbol{p}. \tag{36}$$

Equation (36) is consistent with the results obtained by Refs. [22,23]. The multiplication definition used in this paper gives a more direct and clear correspondence between the operator and the kernel function, although the introduction of the number operator is somewhat counterintuitive. Based on such correspondence, one can realize the operator's actions on the function directly employing the kernel function method without probing into its intrinsic details.

4. Applications of Operator Kernel Function Method

4.1. The Heat Conduction Problem

As the first example of the OKF method, let us consider the heat conduction boundary value problem. This problem was selected as our research subject based on a dual rationale: primarily, by juxtaposing the results to those yielded by classic problems, the accuracy of our proposed method can be ascertained; secondly, the OKFs implicated in this problem elude Mikusiński's theoretical constructs, whereas our approach elucidates a lucid and succinct logical basis for these kernel functions. The governing equations and boundary conditions for this problem are [22]:

$$u_{xx} = a^2 u_t; \quad u(0,t) = u_0(t), \quad \lim_{x \to \infty} u(x,t) = 0. \tag{37}$$

In the operator field, the controlling equation is rewritten as

$$\boldsymbol{u}_{xx} = a^2 \boldsymbol{p}\boldsymbol{u}. \tag{38}$$

Equation (38) is an operator differential equation; the result is given by

$$\boldsymbol{u} = C_1 e^{-ax\sqrt{\boldsymbol{p}}} + C_2 e^{ax\sqrt{\boldsymbol{p}}}. \tag{39}$$

Since the second and the third equation in Equation (37), we have

$$\boldsymbol{u} = e^{-ax\sqrt{\boldsymbol{p}}} u_0(t). \tag{40}$$

In Mikusiński's theoretical framework, the kernel function of $e^{-ax\sqrt{\boldsymbol{p}}}$ remains uninterpretable. As a workaround, Ref. [32] equated this operator with a pre-existing concrete solution, thereby inferring the operator's kernel function. However, this approach is inherently specialized and lacks broad applicability. Our goal is consistently to employ theoretical understanding to unravel unknown issues, rather than filling theoretical gaps with solutions already at hand.

According to Theorem 3, the kernel function of the operator is given by the inverse Laplace transform, that is,

$$e^{-ax\sqrt{\boldsymbol{p}}} = L^{-1}[e^{-ax\sqrt{p}}] = \frac{ax}{2\sqrt{\pi}t^{3/2}} e^{-\frac{a^2x^2}{4t}}. \tag{41}$$

Therefore, the solution is obtained by

$$u(x,t) = \left(\frac{ax}{2\sqrt{\pi}t^{3/2}} e^{-\frac{a^2x^2}{4t}} \right) \circ u_0(t) = \frac{ax}{2\sqrt{\pi}} \int_0^t u_0(t-\tau) \frac{e^{-\frac{a^2x^2}{4\tau}}}{\tau^{3/2}} \mathrm{d}\tau. \tag{42}$$

Equation (42) signifies the analytical solution for the classical boundary value problem concerning heat conduction in an infinitely long rod. Here, operational calculus is utilized to convert the governing partial differential equation into an operator algebraic or operator ordinary differential equation within the operator field. More precisely, through the method proposed herein, the kernel function related to each operator is discerned. Following this, the solution to the problem is ascertained via convolution in the function domain, aligning with the classical solution as outlined in Ref. [36].

Our approach proficiently tackles the issues associated with the construction of operator kernel functions, an area in which Mikusiński's theory falls short. Consider another typical operator, $\frac{1}{\sqrt{p}}e^{-ax\sqrt{p}}$, whose kernel function is predicated on the operator $e^{-ax\sqrt{p}}$. In this case as well, our method is uniquely capable of deriving the kernel function for this operator. Following Theorem 3, we have:

$$\frac{1}{\sqrt{p}}e^{-ax\sqrt{p}} = L^{-1}[\frac{1}{\sqrt{p}}e^{-ax\sqrt{p}}] = \frac{ax}{\sqrt{\pi t}}e^{-\frac{a^2x^2}{4t}}. \tag{43}$$

At this juncture, it should be apparent to readers that the series of OKFs, formulated on the basis of the operator $e^{-ax\sqrt{p}}$, markedly differ from those presented in Table 1. The notable distinction lies in the fact that these kernel functions are intuited or 'guessed', rather than being systematically constructed. This deviates from the typical aspirations of scientific theories for universality and objectivity. The OKF method proposed in this paper not only applies to every operator featured in Table 1 but also extends to operators encountered in heat conduction problems. As such, we contend that our OKF methodology possesses wider applicability, serving as an extension, to some extent, of Mikusiński's operational calculus.

4.2. Fractional-Order Mechanics on a Fractal Tree

Since Heymans et al. introduced the fractal tree structure [37], fractal models have continued to evolve. Researchers have developed various hierarchical models to elucidate the physical properties of complex systems, such as the fractional rheological constitutive equations of polymeric materials [38,39], spring-dashpot fractal tree networks for viscoelastic materials [17], fractional spiking properties of spiny dendritic structures [19], arterial self-similar functional circuit models for blood flow simulation [20]. These works share a common thread of incorporating physical fractal structures and utilizing operator algebraic equations based on the smallest functional cell element.

However, the operators obtained in these studies are often fractional or irrational, rendering them incompatible with primary operators in Mikusiński's theory. To address this issue, researchers have employed different strategies, such as treating fractional operators as fractional derivatives (e.g., Riemann-Liouville or Caputo fractional order derivatives) [17], using series expansion to approximate target operators [19], or utilizing the Babenko operator and Heaviside translation principle to derive kernel functions [20]. Despite these efforts, existing methods have their limitations. For example, directly utilizing fractional order derivatives lacks uniqueness, operator series expansion offers limited approximations or may even be nonconvergent, and the Heaviside translation principle only works with specific operators.

To demonstrate the general applicability of the OKF method, this section uses the fractal-tree structure [17,19] as an example to solve the kernel function of the fractal operator (see Figure 1). The results reveal how the OKF connects the fractal structure with fractional order calculus, illustrating its broad potential in addressing complex systems.

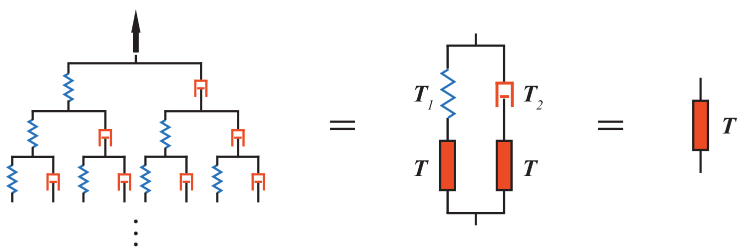

Figure 1. Schematic diagram of the fractal-tree structure and fractal cell elements.

Based on the mechanical-electrical analogy, the relation between stiffness operator $\boldsymbol{T}$ of the structure and the component elements satisfy [17,19]

$$\boldsymbol{T} = \frac{\boldsymbol{T}\boldsymbol{T}_1}{\boldsymbol{T}+\boldsymbol{T}_1} + \frac{\boldsymbol{T}\boldsymbol{T}_2}{\boldsymbol{T}+\boldsymbol{T}_2}. \tag{44}$$

Solving the operator algebraic Equation (44) gives the stiffness operator as

$$\boldsymbol{T} = \sqrt{\boldsymbol{T}_1\boldsymbol{T}_2}. \tag{45}$$

To facilitate the understanding for the readers, we provide a detailed explanation of this method in Appendix A. Because the stiffness should be positive, only the positive root remains in Equation (45). By taking $\boldsymbol{T} = E_1$, $\boldsymbol{T_2} = \eta\boldsymbol{p}$, we obtain

$$\boldsymbol{T} = \sqrt{E_1\eta\boldsymbol{p}}. \tag{46}$$

The stress of the model is given by

$$\sigma(t) = \sqrt{E_1\eta\boldsymbol{p}}\epsilon(t). \tag{47}$$

Guo et al. [19] and Peng et al. [20] utilized the Babenko operator [40]

$$\sqrt{\boldsymbol{p}+\gamma} = e^{-\gamma t}({}^{RL}\boldsymbol{D}^{\frac{1}{2}})e^{\gamma t}. \tag{48}$$

to obtain the result of Equation (47), where ${}^{RL}\boldsymbol{D}^{\frac{1}{2}}$ denotes the one-second Riemann-Liouville fractional derivative. Equation (48) was termed the Heaviside translation principle by Hilbert and Courant [22].

Here, we do not presuppose the operator $\boldsymbol{p}$ as Riemann-Liouville fractional derivative; instead, the expression of Equation (47) is obtained using the OKF method. Notice that the inverse Laplace transform of $\boldsymbol{p}$ does not exist. Thus, we rewrite Equation (47) using multiplication decomposition as

$$\sigma(t) = \sqrt{E_1\eta\boldsymbol{p}}\left(\boldsymbol{p}^{-\frac{1}{2}}\epsilon(t)\right). \tag{49}$$

In Equation (49), we have $\boldsymbol{p}f = \frac{\mathrm{d}}{\mathrm{d}t}f(t) + \boldsymbol{p}f(0)$ and $\boldsymbol{p}^{-\frac{1}{2}} = \boldsymbol{l}^{\frac{1}{2}} = \frac{1}{\sqrt{\pi t}}$. Substituting into Equation (47), we obtain the response expression in the function domain

$$\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi t}} \circ \epsilon(t)\right) + \boldsymbol{p}\left(\frac{1}{\sqrt{\pi t}} \circ \epsilon(t)\right)\Bigg|_{t=0}. \tag{50}$$

Equation (50) includes an improper integral because the integrand of the convolution might be discontinuous at $\tau = t$. The improper integral of Equation (50) converges if $\lim\limits_{t\to 0}\lim\limits_{\tau\to t} t^{\frac{1}{2}}\epsilon(\tau) = 0$, and we have $\left(\frac{1}{\sqrt{\pi t}} \circ \epsilon(t)\right)\Big|_{t=0} = 0$, then Equation (50) becomes

$$\sigma(t) = \sqrt{E_1\eta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{1}{\sqrt{\pi}\sqrt{t-\tau}}\epsilon(\tau)\mathrm{d}\tau = \sqrt{E_1\eta}\left({}^{RL}\boldsymbol{D}^{\frac{1}{2}}\epsilon(t)\right). \tag{51}$$

This result is consistent with Ref. [19]. We next consider the stress relaxation of the fractal-tree, obtained by setting $\epsilon \equiv \epsilon_0$:

$$\sigma(t) = \frac{\sqrt{E_1\eta}}{\sqrt{\pi t}}\epsilon_0. \tag{52}$$

Corresponding to the stress relaxation, let us consider its inverse effect, i.e., the strain creep response given by the flexibility operator $\frac{1}{\boldsymbol{T}}$:

$$\frac{1}{\boldsymbol{T}} = \frac{1}{\sqrt{E_1\eta}}\boldsymbol{p}^{-\frac{1}{2}}. \tag{53}$$

And the expression in the operator field is

$$\epsilon(t) = \frac{1}{\boldsymbol{T}}\sigma(t) = \frac{1}{\sqrt{E_1\eta}}\boldsymbol{p}^{-\frac{1}{2}}\sigma(t). \tag{54}$$

By using $\boldsymbol{p}^{-\frac{1}{2}} = \boldsymbol{l}^{\frac{1}{2}} = \frac{1}{\sqrt{\pi t}}$ again, we have

$$\epsilon(t) = \frac{1}{\boldsymbol{T}}\sigma(t) = \frac{1}{\sqrt{\pi E_1\eta}}\int_0^t \frac{1}{\sqrt{t-\tau}}\sigma(\tau)\mathrm{d}\tau. \tag{55}$$

Similarly, the convergence condition of the integration result of Equation (64) is $\lim_{t\to 0}\lim_{\tau\to t} t^{\frac{1}{2}}\sigma(\tau) = 0$. The strain creep is obtained by setting $\sigma \equiv \sigma_0$ in Equation (55)

$$\epsilon(t) = \frac{2\sigma_0 t^{\frac{1}{2}}}{\sqrt{\pi E_1\eta}}. \tag{56}$$

The fractal stiffness operator $\boldsymbol{T}$ represents a 1/2-order differential operator, while the fractal flexibility operator $\frac{1}{\boldsymbol{T}}$ corresponds to a 1/2-order integral operator. Both of these operators exhibit apparently fractional order characteristics, specifically a 1/2-order type. Consequently, the kernel function associated with these operators includes a power function with an order of 1/2. Notably, the stress relaxation and strain creep response observed in the fractal tree structure demonstrates fractional order behavior. The response for stress relaxation and strain creep of the fractal tree model, the Kelvin-Voigt model, and the Maxwell model are shown in Figure 2. Only when the system consists of fractal systems with infinite components would the irrational fractal-type operator arise [21], showcasing a profound inherent difference from the traditional Kelvin-Voigt model.

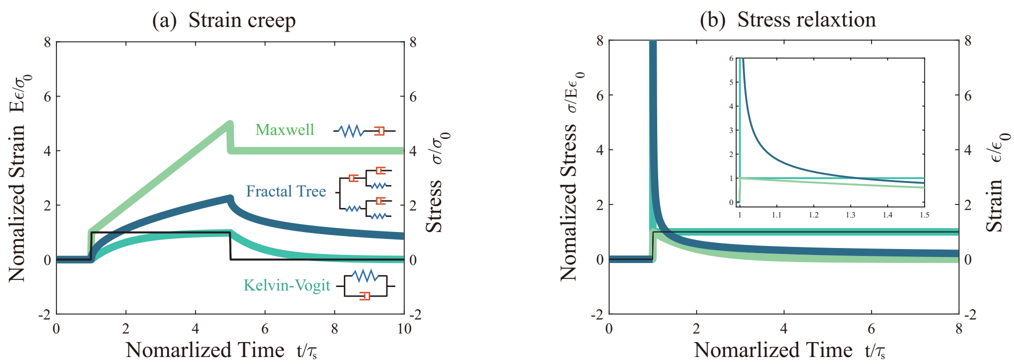

Figure 2. Response curves for stress relaxation and strain creep of the fractal tree model, the Kelvin-Voigt, and the Maxwell models. (**a**) The black line represented in the figure indicates the process of applying and releasing step stress; (**b**) The black line shown in the figure denotes the application of step strain.

4.3. Fractional-Order Mechanics on a Fractal Loop

In addition to the fractal tree-like structure, we further examined another typical cellular architecture, as described by Yin et al. [21]. A representative 2-loop fractal loop cell is illustrated in Figure 3.

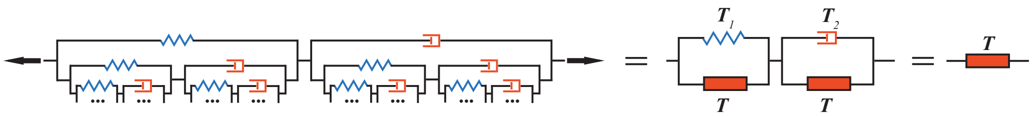

Figure 3. Schematic diagram of the fractal loop structure and fractal cell elements.

The fractal loop structure presents a distinct topology in comparison to the fractal tree. Consequently, this implies that the stress flow dynamics between these two structures would differ. By employing the analytical methodology delineated in the preceding section, we derived the stiffness operator algebraic equation as follows:

$$\frac{1}{\boldsymbol{T}+\boldsymbol{T}_1}+\frac{1}{\boldsymbol{T}+\boldsymbol{T}_2}=\frac{1}{\boldsymbol{T}}. \tag{57}$$

Solving Equation (57), we established that the integral stiffness is given by:

$$\boldsymbol{T}=\sqrt{\boldsymbol{T}_1\boldsymbol{T}_2}. \tag{58}$$

Intriguingly, the 2-loop fractal cell exhibits an identical stiffness to the fractal tree, as described by Equation (45). Ordinarily, structures with distinct topologies would respond differently to the same input. Nonetheless, both the fractal loop and fractal tree structures possess the same stiffness operators, suggesting identical responses to arbitrary input signals. This may seem counterintuitive at first glance. However, this anomaly can be attributed to the fact that both structures share the same topological invariants. The topology index for the fractal tree was defined based on its branches, while for the fractal loop, it was based on its number of loops [21]. For the self-similar 2-branch fractal tree and fractal loop, one structure can be reorganised into another, demonstrating the flexibility and interchangeability of the two structures. The notion of representing the response of a structure in terms of operators lends an intuitive understanding to this discussion. Analogous to the previous section, the response functions for stress relaxation and strain creep were precisely solved.

4.4. Hemodynamics on a Fractal Ladder

Hemodynamics can be studied in two main ways. The first is the classical continuous medium hemodynamic model, which uses the principles of mass, momentum, and energy balance to create a series of partial differential equations based on the Navier-Stokes equation [41]. The second is the physical circuit simulation method, a systemic approach to simulating total systemic arterial blood flow. This method has been widely used to understand various physiological and pathological behaviors of the arterial circulatory system [42].

One of the physical circuit simulation methods is the Elastic-cavity functional circuit model, also known as the lumped parameter electrical model or Windkessel model. This model has evolved over time with different researchers contributing to its development [43–45]. Most recently, Peng et al. [20] expanded the model by increasing the number of elastic cavities to infinity, creating what is known as a fractal ladder model.

The derivative operator of this model is not an integer-order operator and does not fit into Mikusiński's operational calculus theory. To overcome this, we use the OKF method—an approach that allows us to create an expression for the hemodynamic conductivity operator and identify the kernel function of the operator. The analogue circuit diagram and fractal operator schematic are shown in Figure 4.

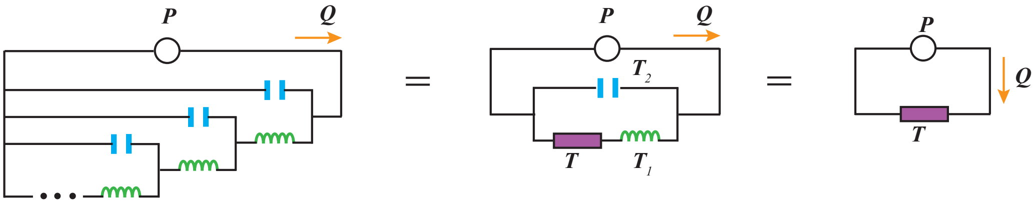

Figure 4. Schematic diagram of the fractal ladder structure and fractal cell elements.

As depicted in the figure, when the number of stages approaches infinity, a fractal cell element remains self-similar, even when it is arranged in series with an inductive element and subsequently in parallel with a capacitive element. This condition gives rise to the following equation for the fractal operator:

$$\frac{\boldsymbol{T}\boldsymbol{T}_1}{\boldsymbol{T}+\boldsymbol{T}_1}+\boldsymbol{T}_2=\boldsymbol{T}. \tag{59}$$

Upon solving the algebraic equation for the operator, we derive the expression for the fractal step operator as follows:

$$\boldsymbol{T}=\frac{\boldsymbol{T}_2-\sqrt{\boldsymbol{T}_2^2+4\boldsymbol{T}_1\boldsymbol{T}_2}}{2}. \tag{60}$$

Within Equation (60), a single solution is retained, which aligns with the physical interpretation of the structure. The admittance operator $\boldsymbol{T}_1$, which corresponds to the inductance L element, is defined as:

$$\boldsymbol{T}_1=\frac{1}{L\boldsymbol{p}}. \tag{61}$$

The admittance operator $\boldsymbol{T}_2$ corresponding to the capacitance C element is:

$$\boldsymbol{T}_2=C\boldsymbol{p}. \tag{62}$$

By substituting Equations (61) and (62) back into Equation (60), we can derive the expression for the aortic fractal operator:

$$\boldsymbol{T}=\frac{C\boldsymbol{p}-\sqrt{C^2\boldsymbol{p}^2+4C/L}}{2}. \tag{63}$$

The total blood pressure-flow response of the arterial is governed by

$$Q(t)=\boldsymbol{T}(\boldsymbol{p})P(t). \tag{64}$$

Consequently, given the known arterial input pressure signal $P(t)$, the output flow signal $Q(t)$ can be readily determined through the modulation of the operator $\boldsymbol{T}$. Employing the OKF method, as used in this study, we express the kernel function of the fractal ladder operator as follows:

$$\boldsymbol{T}=\frac{C\boldsymbol{p}-\sqrt{C^2\boldsymbol{p}^2+4C/L}}{2}=-\sqrt{\frac{C}{L}}\frac{1}{t}J_1\left(\frac{2t}{\sqrt{LC}}\right). \tag{65}$$

In Equation (65), $J_1(x)$ represents the Bessel function of the first type. This results in the following pressure-flow relation:

$$Q(t)=-\sqrt{\frac{C}{L}}\int_0^t\frac{1}{\tau}J_1\left(\frac{2\tau}{\sqrt{LC}}\right)P(t-\tau)\mathrm{d}\tau. \tag{66}$$

In this section, we have primarily focused on the application of the operator kernel function method. We have demonstrated its utility using examples and discussed its implications for dealing with structures or behaviors exhibiting self-similarity. For a more comprehensive understanding of the process of establishing the operator algebra equations and the hemodynamic results, we recommend referring to the seminal work by Peng et al. [20].

The OKF method in this study provides an effective and straightforward approach to analyzing systems. Unlike the integral transform method, which requires modifications to the system's inputs and outputs, the OKF method treats the system as a distinct object of study. This approach focuses on the behavior of the system as represented through the operator, and as a result, the functional response of the system is obtained according to the OKF method. This advancement in the methodological approach serves to enrich our understanding and analysis of complex systems.

5. Conclusions

This paper presents a symbolic algebraic operation method based on the OC operator kernel function, combined with the integral transform but has a wider generality than the integral transform since it does not need any transform of the input and output. This approach streamlines the expression form and ensures that the solution process aligns closely with its physical implications.

Our research establishes a novel correspondence between the exponential operator and the delta function. Using the fundamental translation operator and the identical transform, we have established the form of the operator generated by the kernel function in the function ring. By employing the inverse Laplace transform of the operator, we were able to obtain the kernel function of the operator. This process verifies the unique equivalence between the operator and the kernel function under the rules of OC and the transformed form.

Without exception, every operator implicated in Table 1 can be uniformly tackled using our approach. The operators discussed in Section 4.1 compellingly illustrate that the acquisition of OKFs is not contingent upon external factors. These functions can be computed directly within the theoretical framework, a capability that surpasses the limitations of the current systems. It provides a unified approach to obtaining kernel functions, thereby eliminating the need to study specific operators individually. Significantly, our work broadens the scope of Mikusiński's finite operator expression, extending it to the entire operator field. This expansion marks a significant contribution to the field, providing a novel approach to solving differential and integral equations in mechanics. Finally, through three self-similar fractal structures, we exemplify how OC morphs differential equation issues into operator algebraic problems, showcasing the clear and concise logic of OC. Following this, the unidentified operators are resolved using the OKF method.

In our follow-up work, utilizing the OKF method delineated in this paper, we have successfully represented a series of classical fractional calculus as operators in OC. Furthermore, we have extended this representation to encompass the generalized fractional calculus theory, as denoted by the Sonine kernel. Looking ahead, we intend to further apply the OKF method in conducting asymptotic analysis research on operators that are not amenable to inverse Laplace transformations.

Looking forward, our study paves the way for future research, presenting new questions about the potential applications of this method in other mathematical or physical contexts. We anticipate that the operational calculus OKF method can open new avenues for problem-solving in mechanics and beyond.

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Abbreviations

The following abbreviations are used in this manuscript:

OKF	Operator kernal funtion
OC	Operational Calculus
LHS	Left hand side
RHS	Right hand side

Appendix A

Proof of Lemma 1. Suppose that there exist two different operators $\boldsymbol{x}_1(\lambda)$ and $\boldsymbol{x}_2(\lambda)$ satisfy

$$\boldsymbol{x}'(\lambda) = \omega \boldsymbol{x}(\lambda), \quad \boldsymbol{x}(\lambda_0) = k. \tag{A1}$$

Thus, the operator

$$\bar{\boldsymbol{x}}_\lambda = \boldsymbol{x}_1(\lambda) - \boldsymbol{x}_2(\lambda), \tag{A2}$$

also satisfy the equation. The definite condition for $\bar{\boldsymbol{x}}_\lambda$ becomes

$$\bar{\boldsymbol{x}}(\lambda_0) = \boldsymbol{x}_1(\lambda_0) - \boldsymbol{x}_2(\lambda_0) = 0. \tag{A3}$$

We need to prove that $\bar{\boldsymbol{x}}_\lambda \equiv 0$. Construct an auxiliary operator [32]

$$\boldsymbol{y}(\lambda) = \bar{\boldsymbol{x}}(\lambda)\bar{\boldsymbol{x}}(2\mu - \lambda). \tag{A4}$$

where the parameter μ is an arbitrary real number. The derivative of $\boldsymbol{y}(\lambda)$ with respect to λ gives

$$\begin{aligned} \boldsymbol{y}'(\lambda) &= \bar{\boldsymbol{x}}'(\lambda)\bar{\boldsymbol{x}}(2\mu - \lambda) - \bar{\boldsymbol{x}}(\lambda)\bar{\boldsymbol{x}}'(2\mu - \lambda) \\ &= \omega\bar{\boldsymbol{x}}(\lambda)\bar{\boldsymbol{x}}(2\mu - \lambda) - \bar{\boldsymbol{x}}(\lambda)\omega\bar{\boldsymbol{x}}(2\mu - \lambda) \\ &= 0. \end{aligned} \tag{A5}$$

The symbol $()'$ in Equations (A1) and (A5) denotes the derivative with respect to λ. The auxiliary operator satisfies

$$\boldsymbol{y}(\lambda) = \bar{\boldsymbol{x}}(\lambda)\bar{\boldsymbol{x}}(2\mu - \lambda) \equiv 0. \tag{A6}$$

Using Titchmarsh's theorem, there is no zero factor on the operator field, so there must be $\bar{\boldsymbol{x}} \equiv 0$ for arbitrary μ. This proves that the operator satisfying the conditions of the lemma is unique. □

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