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Difference Equations and Julia Sets of Several Functions for Degenerate q-Sigmoid Polynomials

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Abstract: In this article, we construct a new type of degenerate q-sigmoid (DQS) polynomial for sigmoid functions containing quantum numbers and find several difference equations related to it. We check how each point moves by iteratively synthesizing a quartic degenerate q-sigmoid (DQS) polynomial that appears differently depending on q in the space of a complex structure. We also construct Julia sets associated with quartic DQS polynomials and find their features. Based on this, we make some conjectures.

Keywords: (q,h)-derivative; (q,h)-difference equations; DQS polynomials; Julia set

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1. Introduction

In quantum physics and chemistry, a quantum number represents a conserved quantity in the dynamics of a quantum system. Quantum numbers often specifically describe the energy levels of electrons in an atom, but other possibilities include the spin, angular momentum, and more. Since quantum numbers are defined in various ways depending on the subject, the quantum numbers with which we deal here are the following.

The quantum number (q-number) introduced by Jackson [1,2] around 1900 is

$$[\kappa]_q = \frac{1-q^\kappa}{1-q} = q^{\kappa-1} + \cdots + 1,$$

for any positive integer κ with $q \neq 1$. Here, we note that $\lim_{q\to 1}[\kappa]_q = \kappa$. In particular, for $\rho \in \mathbb{Z}$, $[\rho]_q$ is called a q-integer.

The quantum Gaussian binomial coefficient is defined by

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[m-r]_q![r]_q!},$$

where m and r are non-negative integers; see [2–4]. Note that $[\kappa]_q! = [\kappa]_q[\kappa-1]_q \cdots [2]_q[1]_q$ and $[0]_q! = 1$.

Many contributors have developed new theories related to quantum numbers in differential equations, integration, discrete distributions, series, etc.; see [5–9]. Mathematicians working in this area have studied polynomial families such as those of Bernoulli, Euler, and Genocchi by using quantum numbers; see [10,11]. The authors of [4] introduced q-sigmoid (QS) polynomials while building on [12].

Definition 1 ([4])**.** *Let $0 < q < 1$. QS numbers and polynomials are defined as*

$$\sum_{\kappa=0}^{\infty} \mathcal{S}_{\kappa,q}\frac{\mu^\kappa}{[\kappa]_q!} = \frac{2}{e_q(-\mu)+1}, \quad \sum_{\kappa=0}^{\infty} \mathcal{S}_{\kappa,q}(\varphi)\frac{\mu^\kappa}{[\kappa]_q!} = \frac{2}{e_q(-\mu)+1}e_q(\mu\varphi),$$

respectively.

We note that $\lim_{q\to 1} e_q(-\mu) = e^{-\mu}$. In addition, it can be seen that QS numbers lead to a q-sigmoid function; see [13,14]. The sigmoid function plays an important role as an activation function in deep learning, and it is currently the subject of active study; see [15,16]. For instance, Narayan [17] used generalized nonlinear and differential sigmoid activation functions in a multilayer perceptron (MLP) network to demonstrate improved functionality, and they analyzed a well-known classification problem. Mulindwa and Du [18] proposed the n-sigmoid. Based on this, they introduced the results of further improving a network by introducing a new SE block, and they improved the learning and generalization abilities of two-dimensional and three-dimensional neural networks by reducing the gradient loss problem of the proposed function. There is an ongoing research effort to combine the sigmoid function and quantum numbers, and many new approaches are being explored; see [19,20].

Benaoum [21] defined and studied the generalized (q,h)-Newton binomial formula of Schützenberger's formula. In [22], Čermák and Nechvátal discussed (q,h)-integrals and derivatives involving the h parameter in quantum numbers. They also defined Nabla (q,h)-fractional integrals and Delta (q,h)-fractional integrals based on (q,h)-calculus and presented their basic properties by introducing related derivatives. A two-parameter time scale $\mathbf{T}_{q,h}$ was introduced as follows:

$$\mathbf{T}_{q,h} := \{q^n x + [n]_q h \mid x \in \mathbb{R},\ n \in \mathbb{Z},\ h, q \in \mathbb{R}^+,\ q \neq 1\} \cup \{\frac{h}{1-q}\}.$$

Definition 2 ([22,23])**.** *Let $f : \mathbf{T}_{q,h} \to \mathbb{R}$ be any function. Thus, the delta (q,h)-derivative of f $D_{q,h}(f)$ is defined by*

$$D_{q,h} f(\varphi) := \frac{f(q\varphi + h) - f(\varphi)}{(q-1)\varphi + h}.$$

Definition 3 ([21,23])**.** *The generalized quantum binomial $(x - x_0)^{\kappa}_{q,h}$ is defined by*

$$(\boldsymbol{x} - \boldsymbol{x_0})^{\kappa}_{q,h} := \begin{cases} 1, & \text{if } \kappa = 0, \\ \prod_{i=1}^{\kappa} (x - (q^{i-1} x_0 + [i-1]_q h)), & \text{if } \kappa > 0, \end{cases}$$

where $x_0 \in \mathbb{R}$.

Definition 4 ([10,11,23])**.** *The (q,h)-exponential function $e_{q,h}(\varphi)$ is defined by*

$$e_{q,h}(\varphi : \mu) = \sum_{\kappa=0}^{\infty} (\boldsymbol{\varphi})^{\kappa}_{q,h} \frac{\mu^{\kappa}}{[\kappa]_q!}.$$

Definition 5 ([24,25])**.** *Let $c \in \mathbb{C}$ be a fixed point. The Julia sequence is then*

$$\mathfrak{J}_0(z,c) = z,\ \ \mathfrak{J}_{n+1}(z,c) = \mathfrak{J}_n^2(z,c) + c \quad (n = 0, 1, 2, 3, \cdots).$$

We also know the Julia set for a complex plane, i.e., $\mathfrak{K}_c := \mathbb{C} - \mathfrak{B}_{\infty}$, where $\mathfrak{B}_{\infty} := \{c \in\ : |\mathfrak{J}_n(c)| \to \infty (n \to \infty)\}$.

Based on the above concepts, the main goal of this study is to construct the most general DQS polynomial that can express both degenerate polynomials and polynomials combined with quantum numbers. Then, we obtain various forms of higher-order difference equations whose solutions are general DQS polynomials and show the properties of these equations. Furthermore, we explore their dynamic behavior by identifying the specific form of these polynomials. We use Newtonian methods to visualize Newtonian fractals and construct Julia sets to determine whether dynamic function systems appear in DQS polynomials.

This study's structure is as follows. In Section 2, we find different types of difference equations that have DQS polynomials as solutions. We also look for the symmetric properties of these difference equations. Section 3 examines the results of changing the value of q by selecting quartic polynomials from among the broader class of DQS polynomials. We use the Newton method to check for self-similarity and find several figures in the Julia set.

2. Several Difference Equations That are Related to DQS Polynomials

In this section, we define DQS polynomials and examine DQS numbers by using these polynomials. By using the (q,h)-derivative, we also derive several difference equations that are related to DQS polynomials. We discuss some relations among DQS polynomials, QS polynomials, and degenerated sigmoid (DS) polynomials.

Definition 6. *If h is a non-negative integer and $|q| < 1$, the DQS polynomials $\mathscr{S}_{n,q}(\varphi : h)$ are defined as follows:*

$$\sum_{\kappa=0}^{\infty} \mathscr{S}_{\kappa,q}(\varphi : h)\frac{\mu^{\kappa}}{[\kappa]_q!} = \frac{2}{e_{q,h}(-1:\mu)+1}e_{q,h}(\varphi:\mu).$$

For $\varphi = 0$, we note that

$$\sum_{\kappa=0}^{\infty} \mathscr{S}_{\kappa,q}(h)\frac{\mu^{\kappa}}{[\kappa]_q!} = \frac{2}{e_{q,h}(-1:\mu)+1},$$

and we call $\mathscr{S}_{n,q}(h)$ the DQS numbers.

By modifying the conditions of q and h in Definition 6, we can see that there are several types of sigmoid numbers and polynomials. Let $h \to 0$, $q \to 1$ in the DQS polynomials. Then, we have the following sigmoid numbers $\mathcal{S}_{\kappa}$ and polynomials $\mathcal{S}_{\kappa}(\varphi)$:

$$\sum_{\kappa=0}^{\infty} \mathcal{S}_{\kappa}\frac{\mu^{\kappa}}{\kappa!} = \frac{2}{e^{-\mu}+1}, \qquad \sum_{\kappa=0}^{\infty} \mathcal{S}_{\kappa}(\varphi)\frac{\mu^{\kappa}}{\kappa!} = \frac{2}{e^{-\mu}+1}e^{\mu\varphi}.$$

Putting $h \to 0$ in the DQS polynomials, we find the following QS numbers $\mathcal{S}_{\kappa,q}$ and polynomials $\mathcal{S}_{n,q}(\varphi)$:

$$\sum_{\kappa=0}^{\infty} \mathcal{S}_{\kappa,q}\frac{\mu^{\kappa}}{[\kappa]_q!} = \frac{2}{e_q(-\mu)+1}, \qquad \sum_{\kappa=0}^{\infty} \mathcal{S}_{\kappa,q}(\varphi)\frac{\mu^{\kappa}}{[\kappa]_q!} = \frac{2}{e_q(-\mu)+1}e_q(\mu\varphi).$$

Furthermore, for $q \to 1$ in the DQS polynomials, we define the DS numbers $\mathscr{S}_{\kappa}(h)$ and polynomials $\mathscr{S}_{\kappa}(\varphi : h)$ as follows:

$$\sum_{\kappa=0}^{\infty} \mathscr{S}_{\kappa}(h)\frac{\mu^{\kappa}}{\kappa!} = \frac{2}{e_h(-\mu)+1}, \qquad \sum_{\kappa=0}^{\infty} \mathscr{S}_{\kappa}(\varphi : h)\frac{\mu^{\kappa}}{\kappa!} = \frac{2}{e_h(-\mu)+1}e_h(\mu\varphi),$$

where $\mathscr{S}_n(h) = \mathscr{S}_n(0 : h)$.

Theorem 1. *For $h \in \mathbb{N}$ and $|q| < 1$, we obtain*

$$D_{q,h}\mathscr{S}_{\kappa,q}(\varphi : h) = [\kappa]_q \mathscr{S}_{\kappa-1,q}(\varphi : h).$$

Proof. By using the DQS numbers and Cauchy product in Definition 6, we find that

$$\begin{aligned}\sum_{\kappa=0}^{\infty} \mathscr{S}_{\kappa,q}(\varphi : h)\frac{\mu^{\kappa}}{[\kappa]_q!} &= \sum_{\kappa=0}^{\infty} \mathscr{S}_{\kappa,q}(h)\frac{\mu^{\kappa}}{[\kappa]_q!}\sum_{\kappa=0}^{\infty} (\boldsymbol{\varphi})_{q,h}^{\kappa}\frac{\mu^{\kappa}}{[\kappa]_q!} \\ &= \sum_{\kappa=0}^{\infty}\left(\sum_{\rho=0}^{\kappa}\begin{bmatrix}\kappa\\ \rho\end{bmatrix}_q (\boldsymbol{\varphi})_{q,h}^{\kappa-\rho}\mathscr{S}_{\rho,q}(h)\right)\frac{\mu^{\kappa}}{[\kappa]_q!}.\end{aligned}$$

By comparing the left and right sides of the equation above, one can find the relationship between the DQS number and the polynomial, as shown in (1).

$$\mathscr{S}_{\kappa,q}(\varphi:h)=\sum_{\rho=0}^{\kappa}\begin{bmatrix}\kappa\\ \rho\end{bmatrix}_q(\boldsymbol{\varphi})_{q,h}^{\kappa-\rho}\mathscr{S}_{\rho,q}(h). \tag{1}$$

By applying Definition 2 in Equation (1), we obtain the following equation:

$$D_{q,h}\mathscr{S}_{\kappa,q}(\varphi:h)=\sum_{\rho=0}^{\kappa}\begin{bmatrix}\kappa\\ \rho\end{bmatrix}_q[\kappa-\rho]_q(\boldsymbol{\varphi})_{q,h}^{\kappa-\rho-1}\mathscr{S}_{\rho,q}(h).$$

Applying Equation (1) to the above equation gives us the desired result. □

Corollary 1. *Consider that ρ is a non-negative integer in Theorem 1. Then, the following equation holds:*

$$\mathscr{S}_{\kappa-\rho,q}(\varphi:h)=\frac{[\kappa-\rho]_q!}{[\kappa]_q!}D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(\varphi:h).$$

Corollary 2. *We obtain the following results when we use the following conditions in Theorem 1:*

(i) *Setting $q\to 1$, we have*

$$D_h\mathscr{S}_{\kappa}(\varphi:h)=\kappa\mathscr{S}_{\kappa-1}(\varphi:h),\qquad \mathscr{S}_{\kappa-\rho}(\varphi:h)=\frac{(\kappa-\rho)!}{\kappa!}D_h^{(\rho)}\mathscr{S}_{\kappa}(\varphi:h),$$

where D_h is the h-derivative, and $\mathscr{S}_{\kappa}(\varphi:h)$ represents the DS polynomials.

(ii) *Setting $h\to 0$, we have*

$$D_q\mathcal{S}_{\kappa,q}(\varphi)=[\kappa]_q\mathcal{S}_{\kappa-1,q}(\varphi),\qquad \mathcal{S}_{\kappa-\rho,q}(\varphi)=\frac{[\kappa-\rho]_q!}{[\kappa]_q!}D_q^{(\rho)}\mathcal{S}_{\kappa,q}(\varphi),$$

where D_q is the q-derivative, and $\mathcal{S}_{\kappa,q}(\varphi)$ represents the QS polynomials.

Theorem 2. *The (q,h)-difference equation*

$$\begin{aligned}&\frac{(\boldsymbol{-1})_{q,h}^{\kappa}}{[\kappa]_q!}D_{q,h}^{(\kappa)}\mathscr{S}_{\kappa,q}(\varphi:h)+\frac{(\boldsymbol{-1})_{q,h}^{\kappa-1}}{[\kappa-1]_q!}D_{q,h}^{(\kappa-1)}\mathscr{S}_{\kappa,q}(\varphi:h)+\frac{(\boldsymbol{-1})_{q,h}^{\kappa-2}}{[\kappa-2]_q!}D_{q,h}^{(\kappa-2)}\mathscr{S}_{\kappa,q}(\varphi:h)\\&+\cdots+\frac{(\boldsymbol{-1})_{q,h}^{2}}{[2]_q!}D_{q,h}^{(2)}\mathscr{S}_{\kappa,q}(\varphi:h)+(\boldsymbol{-1})_{q,h}^{1}D_{q,h}^{(1)}\mathscr{S}_{\kappa,q}(\varphi:h)\\&+2\mathscr{S}_{\kappa,q}(\varphi:h)-2(\boldsymbol{\varphi})_{q,h}^{\kappa}=0\end{aligned}$$

has a solution with DQS polynomials.

Proof. Consider that $e_{q,h}(-1:\mu)\neq -1$ in the definition of DQS polynomials. Then, we have

$$\sum_{\kappa=0}^{\infty}\mathscr{S}_{\kappa,q}(\varphi:h)\frac{\mu^{\kappa}}{[\kappa]_{q,h}!}\Big(e_{q,h}(-1:\mu)+1\Big)=2e_{q,h}(\varphi:\mu). \tag{2}$$

The left and right sides of Equation (2) can be changed as follows:

$$\begin{aligned}&\sum_{\kappa=0}^{\infty}\mathscr{S}_{\kappa,q}(\varphi:h)\frac{\mu^{\kappa}}{[\kappa]_q!}\Big(e_{q,h}(-1:\mu)+1\Big)\\&=\sum_{\kappa=0}^{\infty}\left(\sum_{\rho=0}^{\kappa}\begin{bmatrix}\kappa\\ \rho\end{bmatrix}_q(\boldsymbol{-1})_{q,h}^{\rho}\mathscr{S}_{\kappa-\rho,q,h}(\varphi)+\mathscr{S}_{\kappa,q}(\varphi:h)\right)\frac{\mu^{\kappa}}{[\kappa]_q!}\end{aligned}$$

and

$$2e_{q,h}(\varphi:\mu)=2\sum_{\kappa=0}^{\infty}(\boldsymbol{\varphi})_{q,h}^{\kappa}\frac{\mu^{\kappa}}{[\kappa]_q!}.$$

From the above equations, we can obtain

$$\sum_{\rho=0}^{\kappa}\begin{bmatrix}\kappa\\ \rho\end{bmatrix}_q(-\mathbf{1})_{q,h}^{\rho}\mathscr{S}_{\kappa-\rho,q}(\varphi:h)+\mathscr{S}_{\kappa,q}(\varphi:h)=2(\boldsymbol{\varphi})_{q,h}^{\kappa}. \tag{3}$$

Applying Corollary 1 to Equation (3), we obtain

$$\sum_{\rho=0}^{\kappa}\frac{(-\mathbf{1})_{q,h}^{\rho}}{[\rho]_q!}D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(\varphi:h)+\mathscr{S}_{\kappa,q}(\varphi:h)-2(\boldsymbol{\varphi})_{q,h}^{\kappa}=0,$$

which is the desired result. □

Corollary 3. *Setting $q\to1$ in Theorem 2, it holds that*

$$\begin{aligned}&\frac{(-\mathbf{1})_h^{\kappa}}{\kappa!}D_h^{(\kappa)}\mathscr{S}_{\kappa}(\varphi:h)+\frac{(-\mathbf{1})_h^{\kappa-1}}{(\kappa-1)!}D_h^{(\kappa-1)}\mathscr{S}_{\kappa}(\varphi:h)+\frac{(-\mathbf{1})_h^{\kappa-2}}{(\kappa-1)!}D_h^{(\kappa-2)}\mathscr{S}_{\kappa}(\varphi:h)\\&+\cdots+\frac{(-\mathbf{1})_h^{2}}{2!}D_h^{(2)}\mathscr{S}_{\kappa}(\varphi:h)+(-\mathbf{1})_hD_h^{(1)}\mathscr{S}_{\kappa}(\varphi:h)+2\mathscr{S}_{\kappa}(\varphi:h)-2(\boldsymbol{\varphi})_h^{\kappa}=0,\end{aligned}$$

where D_h is the h-derivative, and $\mathscr{S}_{\kappa}(\varphi:h)$ represents the DS polynomials.

Corollary 4. *Let $h\to0$ in Theorem 2. Then, it holds that*

$$\begin{aligned}&\frac{(-1)^{\kappa}}{[\kappa]_q!}D_q^{(\kappa)}\mathcal{S}_{\kappa,q}(\varphi)+\frac{(-1)^{\kappa-1}}{[\kappa-1]_q!}D_q^{(\kappa-1)}\mathcal{S}_{\kappa,q}(\varphi)+\frac{1}{[\kappa-2]_q!}D_q^{(\kappa-2)}\mathcal{S}_{\kappa,q}(\varphi)+\cdots\\&+\frac{1}{[2]_q!}D_q^{(2)}\mathcal{S}_{\kappa,q}(\varphi)-D_q^{(1)}\mathcal{S}_{\kappa,q}(\varphi)+2\mathcal{S}_{\kappa,q}(\varphi)-2\varphi^{\kappa}=0.\end{aligned}$$

where D_q is the q-derivative, and $\mathcal{S}_{\kappa,q}(\varphi)$ represents the QS polynomials.

Theorem 3. *DQS polynomials are a solution of a (q,h)-difference equation of higher order:*

$$\begin{aligned}&\frac{\mathscr{S}_{\kappa,q}(-1:h)+\mathscr{S}_{\kappa,q}(h)}{[\kappa]_q!}D_{q,h}^{(\kappa)}\mathscr{S}_{\kappa,q}(\varphi:h)\\&+\frac{\mathscr{S}_{\kappa-1,q}(-1:h)+\mathscr{S}_{\kappa-1,q}(h)}{[\kappa-1]_q!}D_{q,h}^{(\kappa-1)}\mathscr{S}_{\kappa,q}(\varphi:h)+\cdots\\&+\frac{\mathscr{S}_{2,q}(-1:h)+\mathscr{S}_{2,q}(h)}{[2]_q!}D_{q,h}^{(2)}\mathscr{S}_{\kappa,q}(\varphi:h)+(\mathscr{S}_{1,q}(-1:h)+\mathscr{S}_{1,q}(h))D_{q,h}^{(1)}\mathscr{S}_{\kappa,q}(\varphi:h)\\&+\big(\mathscr{S}_{0,q}(-1:h)+\mathscr{S}_{0,q}(h)-2\big)\mathscr{S}_{\kappa,q}(\varphi:h)=0.\end{aligned}$$

Proof. Definition 6 can be considered as follows:

$$\begin{aligned}&\sum_{\kappa=0}^{\infty}\mathscr{S}_{\kappa,q}(\varphi:h)\frac{\mu^{\kappa}}{[\kappa]_q!}\\&=\frac{2}{e_{q,h}(-1:\mu)+1}e_{q,h}(\varphi:\mu)\\&=\frac{1}{2}\left(\frac{2}{e_{q,h}(-1:\mu)+1}e_{q,h}(-1:\mu)+\frac{2}{e_{q,h}(-1:\mu)+1}\right)\frac{2}{e_{q,h}(-1:\mu)+1}e_{q,h}(\varphi:\mu).\end{aligned}$$

By using the definitions of DQS numbers and polynomials in the above equation and the Cauchy product, we find the following relationship:

$$2\sum_{\kappa=0}^{\infty}\mathscr{S}_{\kappa,q}(\varphi:h)\frac{\mu^{\kappa}}{[\kappa]_q!}=\sum_{\kappa=0}^{\infty}\left(\sum_{\rho=0}^{\kappa}\begin{bmatrix}\kappa\\ \rho\end{bmatrix}_q\Big(\mathscr{S}_{\rho,q,h}(1)+\mathscr{S}_{\rho,q,h}\Big)\mathscr{S}_{\kappa-\rho,q,h}(\varphi)\right)\frac{\mu^{\kappa}}{[\kappa]_q!}.$$

Comparing the coefficients on both sides, the above equation is changed into Equation (4) as follows:

$$\sum_{\rho=0}^{\kappa}\begin{bmatrix}\kappa\\ \rho\end{bmatrix}_q(\mathscr{S}_{\rho,q}(-1:h)+\mathscr{S}_{\rho,q}(h))\mathscr{S}_{\kappa-\rho,q}(\varphi:h)-2\mathscr{S}_{\kappa,q}(\varphi:h)=0. \tag{4}$$

Using $D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(\varphi:h)$ to replace $\mathscr{S}_{\kappa-\rho,q}(\varphi:h)$ in Equation (4), we obtain

$$\sum_{\rho=0}^{\kappa}\frac{(\mathscr{S}_{\rho,q}(-1:h)+\mathscr{S}_{\rho,q}(h))}{[\rho]_q!}D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(\varphi:h)-2\mathscr{S}_{\kappa,q}(\varphi:h)=0.$$

By expanding the equation above, we finish the proof of Theorem 3. □

Corollary 5. *If we set $h\to 0$ in Theorem 3, we can obtain the difference equation related to the QS polynomials $\mathcal{S}_{\kappa,q}(\varphi)$ as follows:*

$$\begin{aligned}&\frac{\mathcal{S}_{\kappa,q}(1)+\mathcal{S}_{\kappa,q}}{[\kappa]_q!}D_q^{(\kappa)}\mathcal{S}_{\kappa,q}(\varphi)+\frac{\mathcal{S}_{\kappa-1,q}(1)+\mathcal{S}_{\kappa-1,q}}{[\kappa-1]_q!}D_q^{(\kappa-1)}\mathcal{S}_{\kappa,q}(\varphi)+\cdots\\&+\frac{\mathcal{S}_{2,q}(1)+\mathcal{S}_{2,q}}{[2]_q!}D_q^{(2)}\mathcal{S}_{\kappa,q}(\varphi)+(\mathcal{S}_{1,q}(1)+\mathcal{S}_{1,q})D_q^{(1)}\mathcal{S}_{\kappa,q}(\varphi)\\&+\big(\mathcal{S}_{0,q,}(1)+\mathcal{S}_{0,q}-2\big)\mathcal{S}_{\kappa,q}(\varphi)=0.\end{aligned}$$

Corollary 6. *If we use $q\to 1$ in Theorem 3, we can obtain the difference equation related to the DS polynomials $\mathscr{S}_{\kappa}(\varphi:h)$ as follows:*

$$\begin{aligned}&\frac{\mathscr{S}_{\kappa}(-1:h)+\mathscr{S}_{\kappa}(h)}{\kappa!}D_h^{(\kappa)}\mathscr{S}_{\kappa}(\varphi:h)+\frac{\mathscr{S}_{\kappa-1}(-1:h)+\mathscr{S}_{\kappa-1}(h)}{(\kappa-1)!}D_h^{(\kappa-1)}\mathscr{S}_{\kappa}(\varphi:h)+\cdots\\&+\frac{\mathscr{S}_{2}(-1:h)+\mathscr{S}_{2}(h)}{2!}D_h^{(2)}\mathcal{S}_{\kappa}(\varphi:h)+(\mathscr{S}_{1}(-1:h)+\mathscr{S}_{1}(h))D_h^{(1)}\mathscr{S}_{\kappa}(\varphi:h)\\&+(\mathscr{S}_{0}(-1:h)+\mathscr{S}_{0}(h)-2)\mathscr{S}_{\kappa}(\varphi:h)=0.\end{aligned}$$

Theorem 4. *DQS polynomials are solutions of the following higher-order (q,h)-difference equation:*

$$\begin{aligned}&\frac{q^{\kappa}(\mathscr{S}_{\kappa,q}(-1:q^{-1}h)+\mathscr{S}_{\kappa,q}(q^{-1}h))}{[\kappa]_q!}D_{q,h}^{(\kappa)}\mathscr{S}_{\kappa,q}(\varphi:h)\\&+\frac{q^{\kappa-1}(\mathscr{S}_{\kappa-1,q}(-1:q^{-1}h)+\mathscr{S}_{\kappa-1,q}(q^{-1}h))}{[\kappa-1]_q!}D_{q,h}^{(\kappa-1)}\mathscr{S}_{\kappa,q}(\varphi:h)+\cdots\\&+\frac{q^{2}(\mathscr{S}_{2,q}(-1:q^{-1}h)+\mathcal{S}_{2,q}(q^{-1}h))}{[2]_q!}D_{q,h}^{(2)}\mathscr{S}_{\kappa,q}(\varphi:h)\\&+q(\mathscr{S}_{1,q}(-1:q^{-1}h)+\mathscr{S}_{1,q}(q^{-1}h))D_{q,h}^{(1)}\mathscr{S}_{\kappa,q}(\varphi:h)\\&+\Big(\mathscr{S}_{0,q}(-1:q^{-1}h)+\mathscr{S}_{0,q}(q^{-1}h)-2\Big)\mathscr{S}_{\kappa,q}(\varphi:h)=0.\end{aligned}$$

Proof. To obtain Theorem 4, we check one of the properties of $e_{q,h}(\varphi:\mu)$ as follows:

$$\begin{aligned} e_{q,h}(q\varphi:\mu) &= \sum_{\kappa=0}^{\infty} q\varphi(q\varphi-h)(q\varphi-[2]_q h)(q\varphi-[3]_q h)\cdots(q\varphi-[\kappa-1]_q h)\frac{\mu^\kappa}{[\kappa]_q!} \\ &= e_{q,q^{-1}h}(\varphi:q\mu). \end{aligned} \tag{5}$$

Consider Equation (5) in $e_{q,h}(q\varphi:\mu)$ from Definition 6. Then, we obtain

$$\begin{aligned} \sum_{\kappa=0}^{\infty}\mathscr{S}_{\kappa,q}(q\varphi:h)\frac{\mu^\kappa}{[\kappa]_q!} &= \frac{2}{e_{q,h}(-1:\mu)+1}e_{q,h}(q\varphi:\mu) \\ &= \frac{1}{2}\left(\frac{2}{e_{q,q^{-1}h}(-1:q\mu)+1}e_{q,q^{-1}h}(-1:q\mu)+\frac{2}{e_{q,q^{-1}h}(-1:q\mu)+1}\right) \\ &\quad\times\frac{2}{e_{q,h}(-1:\mu)+1}e_{q,h}(q\varphi:\mu). \end{aligned}$$

From $\mathscr{S}_{\kappa,q}(\varphi:h)$, we have the following relation:

$$\begin{aligned} &2\sum_{\kappa=0}^{\infty}\mathscr{S}_{\kappa,q}(q\varphi:h)\frac{\mu^\kappa}{[\kappa]_q!} \\ &=\sum_{\kappa=0}^{\infty}\left(\sum_{\rho=0}^{\kappa}\begin{bmatrix}\kappa\\ \rho\end{bmatrix}_q q^\rho\Big(\mathscr{S}_{\rho,q}(-1:q^{-1}h)+\mathscr{S}_{\rho,q}(q^{-1}h)\Big)\mathscr{S}_{\kappa-\rho,q}(q\varphi:h)\right)\frac{\mu^\kappa}{[\kappa]_q!}. \end{aligned}$$

From the above equation, we find that

$$\sum_{\rho=0}^{\kappa}\begin{bmatrix}\kappa\\ \rho\end{bmatrix}_q q^\rho\Big(\mathscr{S}_{\rho,q}(-1:q^{-1}h)+\mathscr{S}_{\rho,q}(q^{-1}h)\Big)\mathscr{S}_{\kappa-\rho,q}(q\varphi:h)-2\mathscr{S}_{\kappa,q}(q\varphi:h)=0. \tag{6}$$

Substituting $q\varphi$ for φ in Corollary 1, we note that

$$\mathscr{S}_{\kappa-\rho,q}(q\varphi:h)=\frac{[\kappa-\rho]_q!}{[\kappa]_q!}D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(q\varphi:h). \tag{7}$$

By applying Equation (7) in Equation (6), we have

$$\sum_{\rho=0}^{\kappa}\frac{q^\rho\left(\mathscr{S}_{\rho,q}(-1:q^{-1}h)+\mathscr{S}_{\rho,q}(q^{-1}h)\right)}{[\rho]_q!}D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(q\varphi:h)-2\mathscr{S}_{\kappa,q}(q\varphi:h)=0,$$

which has the required result. □

Corollary 7. *Setting $h\to 0$ in Theorem 4, we obtain the difference equation of the QS polynomials $\mathcal{S}_{\kappa,q}(\varphi)$ with the q-derivative D_q.*

$$\begin{aligned} &\frac{q^\kappa\left(\mathcal{S}_{\kappa,q}(1)+\mathcal{S}_{\kappa,q}\right)}{[\kappa]_q!}D_q^{(\kappa)}\mathcal{S}_{\kappa,q}(\varphi)+\frac{q^{\kappa-1}\left(\mathcal{S}_{\kappa-1,q}(1)+\mathcal{S}_{\kappa-1,q}\right)}{[\kappa-1]_q!}D_q^{(\kappa-1)}\mathcal{S}_{\kappa,q}(\varphi)+\cdots \\ &+\frac{q^2\left(\mathcal{S}_{2,q}(1)+\mathcal{S}_{2,q}\right)}{[2]_q!}D_q^{(2)}\mathcal{S}_{\kappa,q}(\varphi)+q(\mathcal{S}_{1,q}(1)+\mathcal{S}_{1,q})D_q^{(1)}\mathcal{S}_{\kappa,q}(\varphi) \\ &+\left(\mathcal{S}_{0,q}(1)+\mathcal{S}_{0,q}-2\right)\mathcal{S}_{\kappa,q}(\varphi)=0. \end{aligned}$$

Theorem 5. *Let $|q| < 1$ with $\gamma, \delta \neq 0$. Then, we find a basic symmetry property of the (q,h)-difference equation:*

$$
\begin{aligned}
&\frac{\delta^{\kappa}\mathscr{S}_{\kappa,q}(\gamma\chi : \delta^{-1}h)}{[\kappa]_q!} D_{q,h}^{(\kappa)} \mathscr{S}_{\kappa,q}(\delta\varphi : \gamma^{-1}h) \\
&+ \frac{\delta^{\kappa-1}\gamma\mathscr{S}_{\kappa-1,q}(\gamma\chi : \delta^{-1}h)}{[\kappa-1]_q!} D_{q,h}^{(\kappa-1)} \mathscr{S}_{\kappa,q}(\delta\varphi : \gamma^{-1}h) \\
&+ \cdots + \frac{\delta^{2}\gamma^{\kappa-2}\mathscr{S}_{2,q}(\gamma\chi : \delta^{-1}h)}{[2]_q!} D_{q,h}^{(2)} \mathscr{S}_{\kappa,q}(\delta\varphi : \gamma^{-1}h) \\
&+ \delta\gamma^{\kappa-1}\mathscr{S}_{1,q}(\gamma\chi : \delta^{-1}h) D_{q,h}^{(1)} \mathscr{S}_{\kappa,q}(\delta\varphi : \gamma^{-1}h) + \gamma^{\kappa}\mathscr{S}_{0,q}(\gamma\chi : \delta^{-1}h)\mathscr{S}_{\kappa,q}(\delta\varphi : \gamma^{-1}h) \\
&= \frac{\gamma^{\kappa}\mathscr{S}_{\kappa,q}(\delta\chi : \gamma^{-1}h)}{[\kappa]_q!} D_{q,h}^{(\kappa)} \mathscr{S}_{\kappa,q}(\gamma\varphi : \delta^{-1}h) \\
&+ \frac{\gamma^{\kappa-1}\delta\mathscr{S}_{\kappa-1,q}(\delta\chi : \gamma^{-1}h)}{[\kappa-1]_q!} D_{q,h}^{(\kappa-1)} \mathscr{S}_{\kappa,q}(\gamma\varphi : \delta^{-1}h) \\
&+ \cdots + \frac{\gamma^{2}\delta^{\kappa-2}\mathscr{S}_{2,q}(\delta\chi : \gamma^{-1}h)}{[2]_q!} D_{q,h}^{(2)} \mathscr{S}_{\kappa,q}(\gamma\varphi : \delta^{-1}h) \\
&+ \gamma\delta^{\kappa-1}\mathscr{S}_{1,q}(\delta\chi : \gamma^{-1}h) D_{q,h}^{(1)} \mathscr{S}_{\kappa,q}(\gamma\varphi : \delta^{-1}h) + \delta^{\kappa}\mathscr{S}_{0,q}(\delta\chi : \gamma^{-1}h)\mathscr{S}_{\kappa,q}(\gamma\varphi : \delta^{-1}h).
\end{aligned}
$$

Proof. Considering $e_{q,h}(\gamma\delta\varphi : \mu) = e_{q,\gamma^{-1}h}(\delta\varphi : \gamma\mu)$, we suppose the form A as follows:

$$
A := \frac{4\gamma\delta t^2 e_{q,h}(\gamma\delta\varphi : \mu) e_{q,h}(\gamma\delta\chi : \mu)}{\left(e_{q,\gamma^{-1}h}(-1 : \gamma\mu) + 1\right)\left(e_{q,\delta^{-1}h}(-1 : \delta\mu) + 1\right)}.
$$

From form A, we can derive

$$
\begin{aligned}
A &= \frac{2\gamma}{e_{q,\gamma^{-1}h}(-1 : \gamma\mu) + 1} e_{q,h}(\gamma\delta\varphi : \mu) \frac{2\delta}{e_{q,\delta^{-1}h}(-1 : \delta\mu) + 1} e_{q,h}(\gamma\delta\chi : \mu) \\
&= \frac{2\gamma}{e_{q,\gamma^{-1}h}(-1 : \gamma\mu) + 1} e_{q,\gamma^{-1}h}(\delta\varphi : \gamma\mu) \frac{2\delta}{e_{q,\delta^{-1}h}(-1 : \delta\mu) + 1} e_{q,\delta^{-1}h}(\gamma\chi : \delta\mu) \\
&= \sum_{\kappa=0}^{\infty}\left(\sum_{\rho=0}^{\kappa} \begin{bmatrix}\kappa \\ \rho\end{bmatrix}_q \delta^{\rho}\gamma^{\kappa-\rho}\mathscr{S}_{\rho,q}(\gamma\chi : \delta^{-1}h)\mathscr{S}_{\kappa-\rho,q}(\delta\varphi : \gamma^{-1}h)\right)\frac{\mu^{\kappa}}{[\kappa]_q!},
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
A &= \frac{2\delta}{e_{q,\delta^{-1}h}(-1 : \delta t) + 1} e_{q,\delta^{-1}h}(\gamma\varphi : \delta\mu) \frac{2\gamma}{e_{q,\gamma^{-1}h}(-1 : \gamma t) + 1} e_{q,\gamma^{-1}h}(\delta\chi : \gamma\mu) \\
&= \sum_{\kappa=0}^{\infty}\left(\sum_{\rho=0}^{\kappa} \begin{bmatrix}\kappa \\ \rho\end{bmatrix}_q \gamma^{\rho}\delta^{\kappa-\rho}\mathscr{S}_{\rho,q}(\delta\chi : \gamma^{-1}h)\mathscr{S}_{\kappa-\rho,q}(\gamma\varphi : \delta^{-1}h)\right)\frac{\mu^{\kappa}}{[\kappa]_q!}.
\end{aligned} \tag{9}
$$

Comparing the coefficients of both sides in Equations (8) and (9), we find

$$
\begin{aligned}
&\sum_{\rho=0}^{\kappa} \begin{bmatrix}\kappa \\ \rho\end{bmatrix}_q \delta^{\rho}\gamma^{\kappa-\rho}\mathscr{S}_{\rho,q}(\gamma\chi : \delta^{-1}h)\mathscr{S}_{\kappa-\rho,q}(\delta\varphi : \gamma^{-1}h) \\
&= \sum_{\rho=0}^{\kappa} \begin{bmatrix}\kappa \\ \rho\end{bmatrix}_q \gamma^{\rho}\delta^{\kappa-\rho}\mathscr{S}_{\rho,q}(\delta\chi : \gamma^{-1}h)\mathscr{S}_{\kappa-\rho,q}(\gamma\varphi : \delta^{-1}h).
\end{aligned} \tag{10}
$$

From Corollary 1, we can note that

$$\begin{aligned}\mathscr{S}_{\kappa-\rho,q}(\delta\varphi,\gamma^{-1}h) &= \frac{[\kappa-\rho]_q!}{[\kappa]_q!}D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(\delta\varphi:\gamma^{-1}h),\\ \mathscr{S}_{\kappa-\rho,q}(\gamma\varphi,\delta^{-1}h) &= \frac{[\kappa-\rho]_q!}{[\kappa]_q!}D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(\gamma\varphi:\delta^{-1}h).\end{aligned} \tag{11}$$

Replacing Equation (10) with Equation (11), we have

$$\begin{aligned}&\sum_{\rho=0}^{\kappa}\frac{\delta^{\rho}\gamma^{\kappa-\rho}\mathscr{S}_{\rho,q}(\gamma\chi:\delta^{-1}h)}{[k]_q!}D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(\delta\varphi:\gamma^{-1}h)\\ &=\sum_{\rho=0}^{\kappa}\frac{\gamma^{\rho}\delta^{\kappa-\rho}\mathscr{S}_{\rho,q}(\delta\chi:\gamma^{-1}h)}{[k]_q!}D_{q,h}^{(\rho)}\mathscr{S}_{\kappa,q}(\gamma\varphi:\delta^{-1}h).\end{aligned} \tag{12}$$

By using Equation (12), we complete the proof of Theorem 5. □

Corollary 8. *From Theorem 5, we hold the following:*

(i) For $q \to 1$, this satisfies the following:

$$\begin{aligned}&\frac{\delta^{\kappa}\mathscr{S}_{\kappa}(\gamma\chi:\delta^{-1}h)}{\kappa!}D_h^{(\kappa)}\mathscr{S}_{\kappa}(\delta\varphi:\gamma^{-1}h)+\frac{\delta^{\kappa-1}\gamma\mathscr{S}_{\kappa-1}(\gamma\chi:\delta^{-1}h)}{(\kappa-1)!}D_h^{(\kappa-1)}\mathscr{S}_{\kappa}(\delta\varphi:\gamma^{-1}h)\\ &+\cdots+\frac{\delta^{2}\gamma^{\kappa-2}\mathscr{S}_{2}(\gamma\chi:\delta^{-1}h)}{2!}D_h^{(2)}\mathscr{S}_{\kappa}(\delta\varphi:\gamma^{-1}h)\\ &+\delta\gamma^{\kappa-1}\mathscr{S}_{1}(\gamma\chi:\delta^{-1}h)D_h^{(1)}\mathscr{S}_{\kappa}(\delta\varphi:\gamma^{-1}h)+\gamma^{\kappa}\mathscr{S}_{0}(\gamma\chi:\delta^{-1}h)\mathscr{S}_{\kappa}(\delta\varphi:\gamma^{-1}h)\\ &=\frac{\gamma^{\kappa}\mathscr{S}_{\kappa}(\delta\chi:\gamma^{-1}h)}{\kappa!}D_h^{(\kappa)}\mathscr{S}_{\kappa}(\gamma\varphi:\delta^{-1}h)+\frac{\gamma^{\kappa-1}\delta\mathscr{S}_{\kappa-1}(\delta\chi:\gamma^{-1}h)}{(\kappa-1)!}D_h^{(\kappa-1)}\mathscr{S}_{\kappa}(\gamma\varphi:\delta^{-1}h)\\ &+\cdots+\frac{\gamma^{2}\delta^{\kappa-2}\mathscr{S}_{2}(\delta\chi:\gamma^{-1}h)}{2!}D_h^{(2)}\mathscr{S}_{\kappa}(\gamma\varphi:\delta^{-1}h)\\ &+\gamma\delta^{\kappa-1}\mathscr{S}_{1}(\delta\chi:\gamma^{-1}h)D_h^{(1)}\mathscr{S}_{\kappa}(\gamma\varphi:\delta^{-1}h)+\delta^{\kappa}\mathscr{S}_{0}(\delta\chi:\gamma^{-1}h)\mathscr{S}_{\kappa}(\gamma\varphi:\delta^{-1}h).\end{aligned}$$

(ii) For $h \to 0$, this satisfies the following:

$$\begin{aligned}&\frac{\delta^{\kappa}\mathcal{S}_{\kappa,q}(\gamma\chi)}{[\kappa]_q!}D_q^{(\kappa)}\mathcal{S}_{\kappa,q}(\delta\varphi)+\frac{\delta^{\kappa-1}\gamma\mathcal{S}_{\kappa-1,q}(\gamma\chi)}{[\kappa-1]_q!}D_q^{(\kappa-1)}\mathcal{S}_{\kappa,q}(\delta\varphi)+\cdots\\ &+\frac{\delta^{2}\gamma^{\kappa-2}\mathcal{S}_{2,q}(\gamma\chi)}{[2]_q!}D_q^{(2)}\mathcal{S}_{\kappa,q}(\delta\varphi)+\delta\gamma^{\kappa-1}\mathcal{S}_{1,q}(\gamma\chi)D_q^{(1)}\mathcal{S}_{\kappa,q}(\delta\varphi)+\gamma^{\kappa}\mathcal{S}_{0,q}(\gamma\chi)\mathcal{S}_{\kappa,q}(\delta\varphi)\\ &=\frac{\gamma^{\kappa}\mathcal{S}_{\kappa,q}(\delta\chi)}{[\kappa]_q!}D_q^{(\kappa)}\mathcal{S}_{\kappa,q}(\gamma\varphi)+\frac{\gamma^{\kappa-1}\delta\mathcal{S}_{\kappa-1,q}(\delta\chi)}{[\kappa-1]_q!}D_q^{(\kappa-1)}\mathcal{S}_{\kappa,q}(\gamma\varphi)+\cdots\\ &+\frac{\gamma^{2}\delta^{\kappa-2}\mathcal{S}_{2,q}(\delta\chi)}{[2]_q!}D_q^{(2)}\mathcal{S}_{\kappa,q}(\gamma\varphi)+\gamma\delta^{\kappa-1}\mathcal{S}_{1,q}(\delta\chi)D_q^{(1)}\mathcal{S}_{\kappa,q}(\gamma\varphi)+\delta^{\kappa}\mathcal{S}_{0,q}(\delta\chi)\mathcal{S}_{\kappa,q}(\gamma\varphi).\end{aligned}$$

3. Fractal Phenomenon and Several Figures for Quartic DQS Polynomials

In this section, we examine the phenomena of the dynamics of quartic DQS polynomials in a space with a complex structure. We identify properties for DQS polynomials related to them by using the Newton method and by finding the Julia set.

By using Equation (3), the DQS polynomial $\mathscr{S}_{\kappa,q}(\varphi:h)$ can be found:

$$
\begin{aligned}
&\mathscr{S}_{0,q}(\varphi:h)=1,\\
&\mathscr{S}_{1,q}(\varphi:h)=\frac{1}{2}+\varphi,\\
&\mathscr{S}_{2,q}(\varphi:h)=\frac{1}{4}\big(-1-2h+q+2(1-2h+q)\varphi+4\varphi^2\big),\\
&\mathscr{S}_{3,q}(\varphi:h)=\frac{1}{8}(1+q)\big((1+2h)^2-(3+4h)q+q^2\big)\\
&\qquad+\frac{1}{4}\big(-1+q^3+4h^2(1+q)-4h(1+q+q^2)\big)\varphi\\
&\qquad+\frac{1}{2}\big(1+q+q^2-2h(2+q)\big)\varphi^2+\varphi^3,\\
&\mathscr{S}_{4,q}(\varphi:h)=\frac{1}{16}\big(-(1+2h-q)(1+q)(1+q+q^2)((1+2h)^2-4(1+h)q+q^2\big)\\
&\qquad-\frac{1}{8}\big((-(1+q)^2(1+(-3+q)q)(1+q^2)+8h^3(1+q)(1+q+q^2)\big)\varphi\\
&\qquad+\frac{1}{8}\big(4h^2(1+q^2)(3+q(5+3q))+2h(-1+q)(1+q^2)(3+q(5+3q))\big)\varphi\\
&\qquad+\frac{1}{4}\big(-1+q^2(-1+q+q^3)-2h(1+q^2)(3+2q(2+q))\big)\varphi^2\\
&\qquad+\frac{1}{4}\big(4h^2(3+q(4+q(3+q)))\big)\varphi^2+\frac{1}{2}\big((1+q)(1+q^2)-2h(3+q(2+q))\big)\varphi^3+\varphi^4,\\
&\cdots.
\end{aligned}
$$

Table 1 is a DQS polynomial $\mathscr{S}_{4,q}(\varphi : h)$ that is obtained as the value of q changes when h is fixed to 1. We can guess that the DQS polynomials that appear when $q = 0.9$ is considered are similar to the DS polynomials, and as the value of q becomes smaller, the properties of both the DS polynomials and the QS polynomials appear.

Table 1. Several DQS polynomials $\mathscr{S}_{4,q}(\varphi : 1)$ approximated with q and $h = 1$.

$h=1$	$\mathscr{S}_{4,q}(\varphi:1)$
$q = 0.1$	$-1.81692 + 1.4723\varphi + 1.45165\varphi^2 - 2.6545\varphi^3 + \varphi^4$
$q = 0.5$	$-2.15332 + 2.16992\varphi + 2.16406\varphi^2 - 3.3125\varphi^3 + \varphi^4$
$q = 0.9$	$-1.76385 + 3.56006\varphi + 2.19727\varphi^2 - 3.8905\varphi^3 + \varphi^4$

From now on, we use the Newton method for the polynomials in Table 1; see [24,26].
Step 1. Consider that

$$
\begin{aligned}
\mathscr{S}_{0.1}(\varphi) &:= \mathscr{S}_{4,0.1}(\varphi:1) = -1.81692 + 1.4723\varphi + 1.45165\varphi^2 - 2.6545\varphi^3 + \varphi^4,\\
\mathscr{S}_{0.5}(\varphi) &:= \mathscr{S}_{4,0.5}(\varphi:1) = -2.15332 + 2.16992\varphi + 2.16406\varphi^2 - 3.3125\varphi^3 + \varphi^4,\\
\mathscr{S}_{0.9}(\varphi) &:= \mathscr{S}_{4,0.9}(\varphi:1) = -1.76385 + 3.56006\varphi + 2.19727\varphi^2 - 3.8905\varphi^3 + \varphi^4.
\end{aligned}
$$

Step 2. We apply the functions of Step 1 in the Newton method:

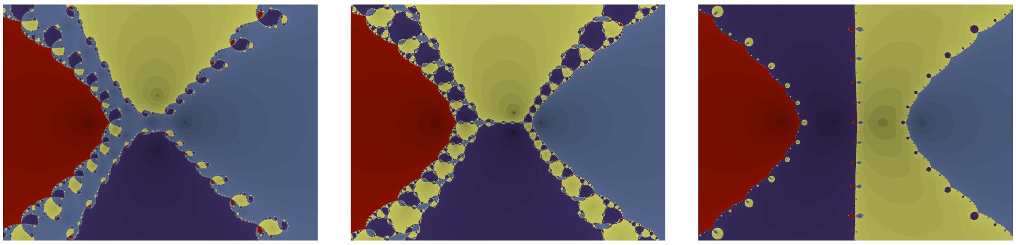
(1) Number of iterations: 50;
(2) $|\varphi - \varphi_0| < 0.01$;
(3) Range of x, y for $x + iy$: $-3 < x < 5$, $-3 < y < 3$;
(4)

$$
\varphi_{n+1} = \varphi_n - \frac{\mathscr{S}_{0.1}(\varphi_n)}{\mathscr{S}'_{0.1}(\varphi_n)},\quad \varphi_{n+1} = \varphi_n - \frac{\mathscr{S}_{0.5}(\varphi_n)}{\mathscr{S}'_{0.5}(\varphi_n)},\quad \varphi_{n+1} = \varphi_n - \frac{\mathscr{S}_{0.9}(\varphi_n)}{\mathscr{S}'_{0.9}(\varphi_n)}.
$$

We obtained Figure 1 through the process of Steps 1 and 2 by using a computer. The positions of each approximate root appearing in Figure 1 are as follows:

(a) Approximate roots of $\mathscr{S}_{0.1}(\varphi)$: $-0.835299, 0.930529 - 0.685283i, 0.930529 + 0.685283i, 1.62874$.
(b) Approximate roots of $\mathscr{S}_{0.5}(\varphi)$: $-0.838882, 1.15325 - 0.24773i, 1.15325 + 0.24773i, 1.84487$.
(c) Approximate roots of $\mathscr{S}_{0.9}(\varphi)$: $-0.875345, 0.458628, 1.65923, 2.64799$.

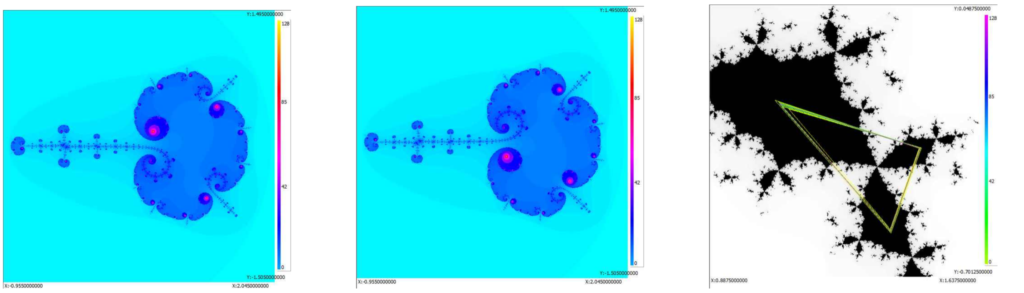
Panels (a,b) in Figure 1 show that complex numbers have approximate roots of $\mathscr{S}_{0.1}(\varphi)$ and $\mathscr{S}_{0.5}(\varphi)$, and they are displayed in yellow and ultramarine blue, respectively. In addition, red and blue colors indicate real numbers as approximate roots. Figure 1c shows that all approximate roots are real numbers: -0.875345 is red, 0.458628 is ultramarine blue, 1.65923 is yellow, and 2.64799 is blue. In panels (a–c) of Figure 1, the areas indicated by colors mean that they approach the approximate root corresponding to each color. For example, the red area in Figure 1a has an approximated fixed point of -0.835299, and if the initial value is near the approximated fixed point, it becomes an element of a basin of attraction. Even if a point in the complex plane is far from the approximated fixed point, if the initial value is in the red area, the point will converge to the approximated fixed point after infinite time passes. In Figure 1, whenever each color is about to be combined, a patch of a different color appears in between. This behavior repeats on an infinitely small scale, producing a fractal. This figure shows a Newtonian fractal, which is the result of using Newton's method, and the basin of attraction is even more interesting; see [26,27].



(**a**) $q = 0.1$ (**b**) $q = 0.5$ (**c**) $q = 0.9$

Figure 1. Basins of attraction for $\mathscr{S}_{4,q}(\varphi : h)$ in the complex plane: (**a**) $q = 0.1; h = 1$; (**b**) $q = 0.5; h = 1$; (**c**) $q = 0.9; h = 1$.

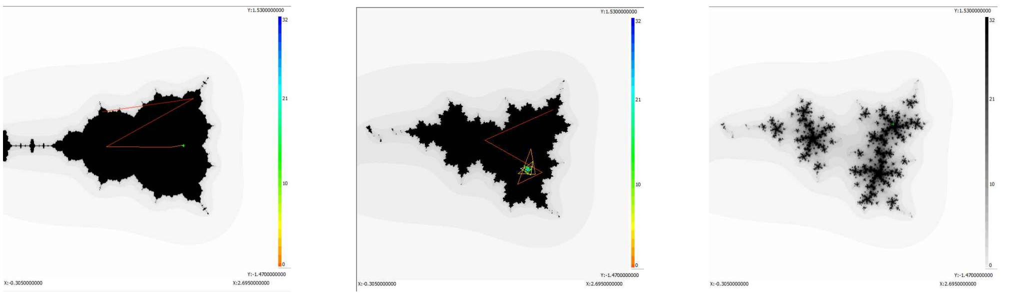
Figure 2 depicts a fractal from the Julia set of $\mathscr{S}_{0.1}(\varphi)$. Here, we assume that the function is iterated 128 times and choose a convergence radius of 2. The image centers in Figure 2a,b are $0.545 - 0.005i$. In Figure 2a, the value of the Julia offset is given as $-1.875 + 0.005i$, and in Figure 2b, the value of the Julia offset is $-1.875 - 0.005i$. Figure 2a,b does not have a fixed point, and periodic points cannot be found. We can consider symmetric properties by looking at Figure 2a,b. In Figure 2a,b, it can be seen that the range of the x-axis is $-0.955 \leq x \leq 2.045$, and the range of the y-axis is $-1.505 \leq y \leq 1.495$. In addition, it can be seen that the area that appears after 128 iterations is a dot in the red part. Figure 2c shows the shape that appears when the offset is given as $-1.315 - 0.285i$. A Julia set appears in the range of $0.8875 \leq x \leq 1.6375$, $-0.70125 \leq y \leq 0.4875$, with an image center of $1.2625 - 0.32625i$. The line that appears here shows the trace of $1.38 - 0.56375i$, which repeats up to 128 times as a three-period point.



(**a**) (**b**) (**c**)

Figure 2. Julia sets of $\mathscr{S}_{0.1}(\varphi)$ with the following offset: (**a**) $c = -1.875 + 0.005i$; (**b**) $c = -1.875 - 0.005i$; (**c**) $c = -1.315 - 0.285i$.

In $\mathscr{S}_{0.5}(\varphi)$, we set the number of iterations to 32 and fix the range of convergence to 4. We also limit the range to $0.305 \leq x \leq 2.695$ and $-1.47 \leq y \leq 1.53$. Then, we find the Julia set, as shown in Figure 3. Here, the offsets of (a–c) are -2.5, $-2.5 - 0.3i$, and $-2.5 - 0.4i$, respectively. In Figure 3a, there are two fixed points, and they appear as green dots. This shows that $0.79 + 0.38i$ arrives at the fixed point in fewer than 10 iterations. If we change the offset of Figure 3a to the offset of Figure 3b, we find that Figure 3b is in the same range as Figure 3a. Here, it can be seen that there are three fixed points, and an attracting point can also be found. It can be seen that the point $1.865 + 0.415i$ in Figure 3b arrives at a fixed point. In Figure 3c, we can see that there is only one fixed point. However, after 64 iterations in Figure 3c, no fixed point exists.



(**a**) (**b**) (**c**)

Figure 3. Julia sets of $\mathscr{S}_{0.5}(\varphi)$ with the following offset: (**a**) $c = -2.5$; (**b**) $c = -2.5 - 0.3i$; (**c**) $c = -2.5 - 0.4i$.

For the condition in Figure 4, $\mathscr{S}_{0.9}(\varphi)$ was iterated 32 times, and the convergence radius was set to 2. Figure 4a shows the continent-shaped Julia set, and the fixed point is represented by the green dot. The offset in Figure 4a is -5.2, and when the range of the offset is $x = -5.2$, $-0.05 \leq y \leq 0$, the shape of the continent is almost similar to that in figure in (a). Figure 4b shows a Julia set filled with basilica, and the actual Julia set is defined by the boundary of the blue area. The offset in Figure 4b is -5.1, and the offset in Figure 4c is -5. Figure 4c is renormalized once based on Figure 4b, and it includes the small basilica Julia set. This renormalization can be repeated continuously to obtain an infinite basilica. In other words, since the renormalization of the basilica becomes itself, the renormalization can be repeated over and over again. The offset in Figure 4d is -4.9, and this figure is called the Feigenbaum Julia set. This figure can be considered a Julia set using Feigenbaum polynomials that can be renormalized infinite times.

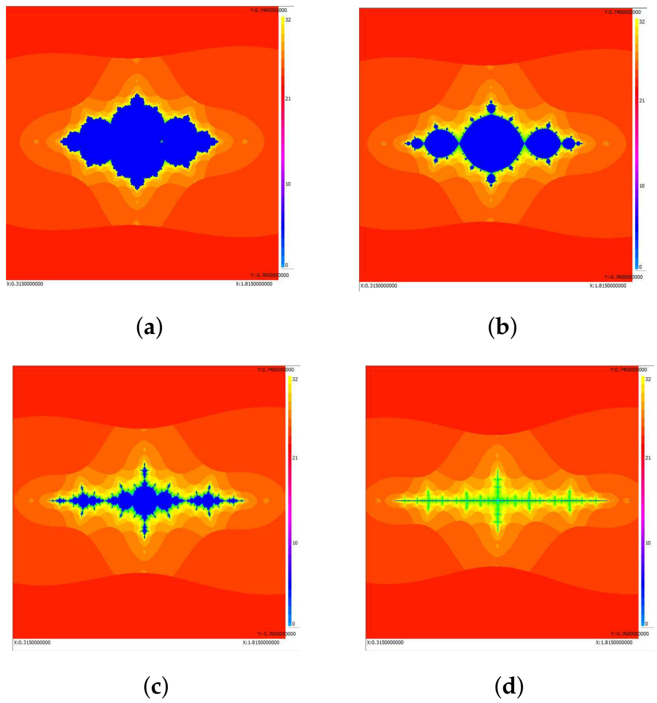


Figure 4. Julia sets of $\mathscr{S}_{0.9}(\varphi)$ with the following offset: (**a**) $c = -5.2$; (**b**) $c = -5.1$; (**c**) $c = -5$; (**d**) $c = -4.9$.

4. Conclusions

In this study, we found several difference equations with DQS polynomials as solutions. Since DQS polynomials are closely related to degenerate q-Euler (DQE) polynomials, it is expected that difference equations with the latter as solutions will be related to those found in this study. Figure 4 indicates that the horizontal-to-vertical ratio obtained with the original dynamic system does not significantly change when a small part of the dynamic system is enlarged. That is, it can be known that it is itself even if it goes through the renormalization process, and this can be regarded as a Feigenbaum polynomial that can be renormalized infinitely many times.

As a result of the numerical experiments in Section 3, we can consider the following conjectures.

Conjecture 6. *As can be seen in Figure 1, the protrusions attached to each domain of* $\mathscr{S}_q(\varphi)$ *become smaller as the value of q approaches* 1.

Conjecture 7. *As shown in Figure 2, the form of the Julia set for* $\mathscr{S}_{0.1}(\varphi)$ *has symmetric properties when the offset value is given as a complex conjugate.*

Conjecture 8. *In Figure 3, as the offset value of the Julia set of* $\mathscr{S}_{0.1}(\varphi)$ *changes from a real number to a complex number, the shape of the Julia set changes from that of a continent to the shape of dust.*

Conjecture 9. *As can be seen in Figure 4, when the offset value of the Julia set of* $\mathscr{S}_{0.9}(\varphi)$ *is a real number, the shape of the Julia set changes from the shape of a continent to the shape of an airplane. In other words, the Julia set related to* $\mathscr{S}_{0.9}(\varphi)$ *continues to undergo normalization.*

Finding the form of the Julia set in a complex plane depends on the value of the offset and does not have a certain regularity. The generalization of the approximate roots of high-order polynomials, the discovery of fractal elements using high-order polynomials, and finding the properties of the renormalization process are some of the tasks to be completed in the future.

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