fractal and fractional

Article

Maximum Principle for Variable-Order Fractional Conformable Differential Equation with a Generalized Tempered Fractional Laplace Operator

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Abstract: In this paper, we investigate properties of solutions to a space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operatorby using the maximum principle. We first establish some new important fractional various-order conformable inequalities. With these inequalities, we prove a new maximum principle with space-time fractional variable-order conformable derivatives and a generalized tempered fractional Laplace operator. Moreover, we discuss some results about comparison principles and properties of solutions for a family of space-time fractional variable-order conformable nonlinear differential equations with a generalized tempered fractional Laplace operator by maximum principle.

Keywords: maximum principle; fractional variable-order conformable derivative; generalized tempered fractional laplace operator; uniqueness and continuous dependence

Citation: Guan, T.; Zhang, L. Maximum Principle for Variable-Order Fractional Conformable Differential Equation with a Generalized Tempered Fractional Laplace Operator. *Fractal Fract.* **2023**, 7, 798. https://doi.org/10.3390/fractalfract7110798

Academic Editor: Carlo Cattani

Received: 22 August 2023
Revised: 19 October 2023
Accepted: 30 October 2023
Published: 1 November 2023

1. Introduction

Owing to fractional calculus linkage with memory, fractional differential equations have been applied successively to the modeling of physical, chemical, engineer and economics processes. Examples include fluctuations of the external pressure fields in the anomalous diffusion model [1], biological population model [2], process of geographical data [3], the complex dynamics of financial processes [4], etc.

Maximum principle is a useful tool to study fractional partial differential equations (FPDE). By using maximum principle, some important properties of solution without specific expression for FPDE can be indirectly or directly produced. Luchko [5] formulated a maximum principle for a FPDE in an explicit form in 2009. In 2016, Liu, Zeng and Bai [6] proved the maximum principle for FPDE with a space-time multi-term Riesz–Caputo variable-order derivative. They also discussed the uniqueness of solutions for FPDE with space-time multi-term Riesz–Caputo variable-order derivative and continuous dependence of solutions for IBVP. In 2020, Zeng et al. [7] established the space-time multi-term fractional variable-order maximum principles. Applying the maximum principle, they investigated the generalized time-fractional variable-order Caputo diffusion equations and fractional variable-order Riesz–Caputo diffusion equations. For other new developments of the maximum principle, the reader can refer to [8–16] and the references therein.

In 2018, Deng, Li, Tian and Zhang [17] gave the mathematic definition of the tempered fractional Laplace operator. In 2018, Sun, Nie and Deng [18] advanced the finite difference discretization for the tempered fractional Laplace operator by the weighted trapezoidal rule and bilinear interpolation. On this basis, Zhang et al. [19] proposed a new type of generalized tempered fractional p-Laplace operator in 2020. Zhang, Deng and Fan [20] established the finite difference schemes for the tempered fractional Laplacian equation on the generalized Dirichlet type boundary condition. Using the direct method of moving planes, Wang et al. [21] studied parabolic equation with the tempered fractional Laplacian and logarithmic nonlinearity. Zhang, Deng and Karniadakis [22] presented new computational methods for the tempered fractional Laplacian equation on the homogeneous and nonhomogeneous generalized Dirichlet type boundary conditions. Other new developments of the tempered fractional Laplace operator can be found in [23–31]. The conception and properties of fractional conformable Caputo and Riemann–Liouville derivatives were formulated by Jarad et al. [32]. However, there are few studies on the maximum principle and its application to fractional various-order conformable Caputo derivatives. In fact, the variable-order operator has been applied successively to complex diffusion modeling, such as the processing of geographical data [3], signature verification [33], financial processes [4], etc. In addition, the studies on fractional conformable derivatives did not mention a generalized tempered fractional Laplace operator.

Intrigued by past works, in this paper we investigate the following space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator on $(c,d) \times [T,T_1]$:

$$\begin{aligned}\left({}^{C\beta(\vartheta,\theta)}_{T}D_{\theta}^{\epsilon(\vartheta,\theta)}\right)w(\vartheta,\theta) - \left[\left({}^{C\sigma(\vartheta,\theta)}_{c}D_{\vartheta}^{\epsilon(\vartheta,\theta)}\right)w(\vartheta,\theta) + \left({}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)}\right)w(\vartheta,\theta)\right] \\ + (-\Delta - \lambda_f)^{\frac{\alpha}{2}}w(\vartheta,\theta) - e(\vartheta,\theta)w(\vartheta,\theta) = F(\vartheta,\theta,w).\end{aligned} \tag{1}$$

Here, ${}^{C\beta(\vartheta,\theta)}_{T}D_{\theta}^{\epsilon(\vartheta,\theta)}$ is left fractional variable-order conformable Caputo derivative with respect to the variable θ of order $0 < \beta(\vartheta,\theta) < 1$. ${}^{C\sigma(\vartheta,\theta)}_{c}D_{\vartheta}^{\epsilon(\vartheta,\theta)}$ and ${}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)}$ are left and right fractional variable-order conformable Caputo derivatives (LFVCCD and RFVCCD) to the variable ϑ of order $1 < \sigma(\vartheta,\theta) < 2$, respectively. $(-\Delta - \lambda_f)^{\frac{\alpha}{2}}$ is a generalized tempered fractional Laplace operator and $e(\vartheta,\theta)$ is a continuous function.

In this paper, we focus our attention on the maximum principle for Equation (1). We emphasize that the introduction of variable-order derivatives and generalized tempered fractional Laplace operator bring the main difficulties to prove our main result, see Theorem 1. To handle these difficulties, we first propose the fractional variable-order conformable derivative and extend the constant-order derivative to the variable-order derivative. Then, we prove the extreme principles of fractional variable-order conformable derivative (see Lemmas 1 and 2). Finally, we prove the maximum principle of a space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator (see Theorem 1). The main result can be stated as follows.

Lemma 1. *Let* $0 < \epsilon(\vartheta,\theta) < 1, 0 < \beta(\vartheta,\theta) < 1, \forall\theta \in [T,T_1]$. *If* $f \in C^1_{\epsilon,T}([T,T_1])$, $\theta_0 \in (T,T_1]$ *is its maximum, then the inequality*

$$\left({}^{C\beta(\vartheta,\theta)}_{T}D_{\theta}^{\epsilon(\vartheta,\theta)}f\right)(\theta_0) \geq \frac{1}{\Gamma(1-\beta(\vartheta,\theta_0))}\left(\frac{(\theta_0 - T)^{\epsilon(\vartheta,\theta_0)}}{\epsilon(\vartheta,\theta_0)}\right)^{-\beta(\vartheta,\theta_0)}(f(\theta_0) - f(T)) \geq 0 \tag{2}$$

holds.

Lemma 2. *Let* $0 < \epsilon(\vartheta,\theta) < 1, 1 < \sigma(\vartheta,\theta) < 2, \forall\vartheta \in [c,d]$. *If f attains its maximum value at* $\vartheta_0 \in (T,T_1]$, *then*

(1) if $f \in C^2_{\epsilon,c}([c,d])$

$$\left({}^{C\sigma(\vartheta,\theta)}_{c}D_{\vartheta}^{\epsilon(\vartheta,\theta)}f\right)(\vartheta_0) \leq -\frac{\sigma(\vartheta_0,\theta)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(\vartheta_0 - c)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}(f(\vartheta_0) - f(c)) \leq 0 \tag{3}$$

(2) if $f \in C^2_{\epsilon,d}([c,d])$

$$\left({}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)} f\right)(\vartheta_0) \le -\frac{\sigma(\vartheta_0,t)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(d-\vartheta_0)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}(f(\vartheta_0)-f(d)) \le 0 \tag{4}$$

hold.

Theorem 1. *(Maximum principle) Assume $F(\vartheta,\theta,w) \le 0$ and $e(\vartheta,\theta) \le 0$, $\forall(\vartheta,\theta) \in U$. If $w \in H(\bar{U})$ satisfies the space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator (18), then*

$$w(\vartheta,\theta) \le \max\{\max_{\vartheta\in[c,d]}\phi(\vartheta), \max_{\theta\in[T,T_1]} g_1(\theta), \max_{\theta\in[T,T_1]} g_2(\theta), 0\}, \quad \forall(\vartheta,\theta)\in\bar{U} \tag{5}$$

holds.

The remainder of this paper is as follows: Some definitions are given in Section 2. The main results are derived and proved in Section 3. In Section 4, the maximum principles are utilized to gain the comparison principle, the uniqueness and continuous dependence of solution of space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator.

2. Some Definitions

In this section, the definitions of fractional variable-order conformable Caputo derivatives and generalized tempered fractional p-Laplace operator are given.

First, we shall give the definitions of fractional variable-order conformable Caputo derivatives.

Definition 1. *Let $\epsilon : [c,d]\times[T,T_1] \to \mathbb{R}_+ = (0,\infty)$ and $\sigma : [c,d]\times[T,T_1] \to (m-1,m)$.*

(1) If $f \in C^m_{\epsilon,c}([c,d])$ with $m \in \mathbb{N}$, the definition of LFVCCD on variable-order $\sigma(\vartheta,\theta)$ is

$$\begin{aligned}&{}^{C\sigma(\vartheta,\theta)}_{c}D_{\vartheta}^{\epsilon(\vartheta,\theta)} f(\vartheta)\\ &= \frac{1}{\Gamma(m-\sigma(\vartheta,\theta))}\int_c^{\vartheta}\left(\frac{(\vartheta-c)^{\epsilon(\vartheta,\theta)}-(s-c)^{\epsilon(\vartheta,\theta)}}{\epsilon(\vartheta,\theta)}\right)^{m-\sigma(\vartheta,\theta)-1}\frac{{}^m_c T^{\epsilon(\vartheta,\theta)} f(s)}{(s-c)^{1-\epsilon(\vartheta,\theta)}}ds.\end{aligned} \tag{6}$$

(2) If $f \in C^m_{\epsilon,d}([c,d])$ with $m \in \mathbb{N}$, the definition of RFVCCD on variable-order $\sigma(\vartheta,\theta)$ is

$$\begin{aligned}&{}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)} f(\vartheta)\\ &= \frac{(-1)^m}{\Gamma(m-\sigma(\vartheta,\theta))}\int_{\vartheta}^{d}\left(\frac{(d-\vartheta)^{\epsilon(\vartheta,\theta)}-(d-s)^{\epsilon(\vartheta,\theta)}}{\epsilon(\vartheta,\theta)}\right)^{m-\sigma(\vartheta,\theta)-1}\frac{{}^m T_d^{\epsilon(\vartheta,\theta)} f(s)}{(d-s)^{1-\epsilon(\vartheta,\theta)}}ds.\end{aligned} \tag{7}$$

with $m = [\sigma(\vartheta,\theta)]+1$, $[\sigma(\vartheta,\theta)]$ is the biggest integer of no more than $\sigma(\vartheta,\theta)$, ${}_cT^{\epsilon(\vartheta,\theta)} f(\theta) = (\theta-c)^{1-\epsilon(\vartheta,\theta)} f'(\theta)$, $T_d^{\epsilon(\vartheta,\theta)} f(\theta) = (d-\theta)^{1-\epsilon(\vartheta,\theta)} f'(\theta)$, ${}^m_c T^{\epsilon(\vartheta,\theta)} = \underbrace{{}_cT^{\epsilon(\vartheta,\theta)}\,{}_cT^{\epsilon(\vartheta,\theta)}\cdots{}_cT^{\epsilon(\vartheta,\theta)}}_{m\ times}$, ${}^m T_d^{\epsilon(\vartheta,\theta)} = \underbrace{T_d^{\epsilon(\vartheta,\theta)} T_d^{\epsilon(\vartheta,\theta)}\cdots T_d^{\epsilon(\vartheta,\theta)}}_{m\ times}$, $C^m_{\epsilon,c}[c,d] = \{f : [c,d] \to \mathbb{R} |\ {}^{m-1}T_d^{\epsilon} f \in {}_{\epsilon}I[c,d]\}$ and $C^m_{\epsilon,c}[c,d] = \{f : [c,d] \to \mathbb{R} |\ {}^{m-1}_{c}T^{\epsilon} f \in I_{\epsilon}[c,d]\}$ ($I_{\epsilon}[c,d]$ and ${}_{\epsilon}I[c,d]$ are defined in Definition 3.1 in [34]).

Remark 1. *If variable-order $\sigma(\vartheta,\theta) = \beta$(constant) in the Definition 1, the LFVCCD became left fractional conformable Caputo of order β [32], i.e.,*

$$ {}^{C\beta}_{T}D^{\epsilon}_{\theta}f(\theta) = \frac{1}{\Gamma(n-\beta)}\int_{T}^{\theta}\left(\frac{(\theta-T)^{\epsilon}-(s-T)^{\epsilon}}{\epsilon}\right)^{n-\beta-1}\frac{{}^{n}_{T}T^{\epsilon}f(s)}{(s-T)^{1-\epsilon}}ds. $$

If variable-order $\sigma(\vartheta,\theta)=\sigma(constant)$ in the Definition 1, the LFVCCD and RFVCCD became left and right fractional conformable Caputo derivatives of order σ [32], i.e.,

$$ {}^{C\sigma}_{c}D^{\epsilon}_{\vartheta}f(\vartheta) = \frac{1}{\Gamma(m-\sigma)}\int_{c}^{\vartheta}\left(\frac{(\vartheta-c)^{\epsilon}-(s-c)^{\epsilon}}{\epsilon}\right)^{m-\sigma-1}\frac{{}^{m}_{c}T^{\epsilon}f(s)}{(s-c)^{1-\epsilon}}ds, $$

and

$$ {}^{C\sigma}D^{\epsilon}_{d,\vartheta}f(\vartheta) = \frac{(-1)^{m}}{\Gamma(m-\sigma)}\int_{\vartheta}^{d}\left(\frac{(d-\vartheta)^{\epsilon}-(d-s)^{\epsilon}}{\epsilon}\right)^{m-\sigma-1}\frac{{}^{m}T^{\epsilon}_{d}f(s)}{(d-s)^{1-\epsilon}}ds, $$

respectively.

Very recently, Zhang, Hou, Ahmad and Wang [19] proposed a new type of generalized tempered fractional p-Laplace operator defined by

$$ (-\Delta-\lambda_{f})^{s}_{p}\phi(\vartheta) = C_{n,sp}P.V.\int_{\mathbb{R}^{n}}\frac{|\phi(\vartheta)-\phi(y)|^{p-2}[\phi(\vartheta)-\phi(y)]}{e^{\lambda f(|\vartheta-y|)}|\vartheta-y|^{n+sp}}dy. $$

When $p=2$ and f is an identity map, the above-mentioned generalized tempered fractional p-Laplace operator becomes the tempered fractional Laplace operator $(-\Delta-\lambda_{f})^{\frac{\alpha}{2}}$. When $\phi\in C^{1,1}_{loc}\cap L^{\alpha}$, the tempered fractional Laplace operator defined by

$$ (-\Delta-\lambda_{f})^{\frac{\alpha}{2}}\phi(\vartheta) = C_{n,\alpha}P.V.\int_{\mathbb{R}}\frac{\phi(\vartheta)-\phi(y)}{e^{\lambda f(|\vartheta-y|)}|\vartheta-y|^{1+\alpha}}dy, \tag{8} $$

with $\alpha\in(0,2)$, $C_{n,\alpha}=\frac{\Gamma(\frac{1}{2})}{2\pi^{1/2}|\Gamma(-\alpha)|}$, $P.V.$ refers to the Cauchy principal value, λ is a sufficiently small positive number, f is nondecreasing with respect to $|\vartheta-y|$ and

$$ L^{\alpha} = \left\{\phi\in L^{1}_{loc}\,\middle|\,\int_{\mathbb{R}}\frac{|1+\phi(y)|}{1+|y|^{1+\alpha}}dy<\infty\right\}. \tag{9} $$

3. Main Result

In this part, the extreme principles of these variable-order derivatives and the maximum principles of Equation (18) are established and proved.

Next, we will establish some extremum principles of LFVCCD and RFVCCD to prove our maximum principle.

Proof of Theorem 1. Let

$$ h(\theta) = f(\theta_0)-f(\theta)\geq 0,\quad \theta\in[T,T_1]. \tag{10} $$

Obviously,
(1) $h(\theta)\in C^{1}_{\epsilon,T}([T,T_1]), h(\theta)\geq 0, \theta\in[T,T_1]$;
(2) $h(\theta_0)=h'(\theta_0)=0$;
(3) $\left({}^{C\beta(\vartheta,\theta)}_{T}D^{\epsilon(\vartheta,\theta)}_{\theta}h\right)(\theta) = -\left({}^{C\beta(\vartheta,\theta)}_{T}D^{\epsilon(\vartheta,\theta)}_{\theta}f\right)(\theta)$.

By calculation, we notice that

$$\begin{aligned}
&\left({}^{C\beta(\vartheta,\theta)}_{T}D_{\theta}^{\epsilon(\vartheta,\theta)}h\right)(\theta_0) \\
&= \frac{1}{\Gamma(1-\beta(\vartheta,\theta_0))}\left(\frac{(\theta_0-T)^{\epsilon(\vartheta,\theta_0)}-(\zeta-T)^{\epsilon(\vartheta,\theta_0)}}{\epsilon(\vartheta,\theta_0)}\right)^{-\beta(\vartheta,\theta_0)}h(\zeta)\Bigg|_T^{\theta_0} \\
&-\frac{\beta(\vartheta,\theta_0)}{\Gamma(1-\beta(\vartheta,\theta_0))}\int_T^{\theta_0}\left(\frac{(\theta_0-T)^{\epsilon(\vartheta,\theta_0)}-(\zeta-T)^{\epsilon(\vartheta,\theta_0)}}{\epsilon(\vartheta,\theta_0)}\right)^{-\beta(\vartheta,\theta_0)-1} \\
&(\zeta-T)^{\epsilon(\vartheta,\theta_0)-1}h(\zeta)d\zeta. \\
&= -\frac{1}{\Gamma(1-\beta(\vartheta,\theta_0))}\left(\frac{(\theta_0-T)^{\epsilon(\vartheta,\theta_0)}}{\epsilon(\vartheta,\theta_0)}\right)^{-\beta(\vartheta,\theta_0)}h(T) \\
&-\frac{\beta(\vartheta,\theta_0)}{\Gamma(1-\beta(\vartheta,\theta_0))}\int_T^{\theta_0}\left(\frac{(\theta_0-T)^{\epsilon(\vartheta,\theta_0)}-(\zeta-T)^{\epsilon(\vartheta,\theta_0)}}{\epsilon(\vartheta,\theta_0)}\right)^{-\beta(\vartheta,\theta_0)-1} \\
&(\zeta-T)^{\epsilon(\vartheta,\theta_0)-1}h(\zeta)d\zeta \\
&\le -\frac{1}{\Gamma(1-\beta(\vartheta,\theta_0))}\left(\frac{(\theta_0-T)^{\epsilon(\vartheta,\theta_0)}}{\epsilon(\vartheta,\theta_0)}\right)^{-\beta(\vartheta,\theta_0)}h(T) \\
&\le 0.
\end{aligned} \tag{11}$$

We obtain

$$\left({}^{C\beta(\vartheta,\theta)}_{T}D_{\theta}^{\epsilon(\vartheta,\theta)}f\right)(\theta_0) \ge \frac{1}{\Gamma(1-\beta(\vartheta,\theta_0))}\left(\frac{(\theta_0-T)^{\epsilon(\vartheta,\theta_0)}}{\epsilon(\vartheta,\theta_0)}\right)^{-\beta(\vartheta,\theta_0)}(f(\theta_0)-f(T)) \ge 0.$$

□

Remark 2. *If $\epsilon(\vartheta,\theta)$ and $\beta(\vartheta,\theta)$ reduce to constants $0<\epsilon<1$ and $0<\beta<1$, Guan and Wang [15] obtained a similar result as*

$$\left({}^{C\beta}_{T}D_{\theta}^{\epsilon}f\right)(\theta_0) \ge \frac{1}{\Gamma(1-\beta))}\left(\frac{(\theta_0-T)^{\epsilon}}{\epsilon}\right)^{-\beta}(f(\theta_0)-f(T)) \ge 0.$$

Proof of Theorem 2. Let

$$h(\vartheta) = f(\vartheta_0) - f(\vartheta) \ge 0, \quad \vartheta \in [c,d]. \tag{12}$$

Obviously,
(1) $h(\vartheta) \in C^2_{\epsilon,c}([c,d]), h(\vartheta) \ge 0, \vartheta \in [c,d]$;
(2) $h(\vartheta_0) = h'(\vartheta_0) = 0$;
(3) $\left({}^{C\sigma(\vartheta,\theta)}_{c}D_{\vartheta}^{\epsilon(\vartheta,\theta)}h\right)(\vartheta) = -\left({}^{C\sigma(\vartheta,\theta)}_{c}D_{\vartheta}^{\epsilon(\vartheta,\theta)}f\right)(\vartheta)$,
$\left({}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)}h\right)(\vartheta) = -\left({}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)}f\right)(\vartheta)$.

By calculation, we notice that

$$\begin{aligned}
&\left({}^{C\sigma(\vartheta,\theta)}_{c}D^{\epsilon(\vartheta,\theta)}_{\vartheta}h\right)(\vartheta_0)\\
&=\frac{1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(\vartheta_0-c)^{\epsilon(\vartheta_0,\theta)}-(\zeta-c)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{1-\sigma(\vartheta_0,\theta)}(\zeta-c)^{1-\epsilon(\vartheta_0,\theta)}h'(\zeta)\Big|_{c}^{\vartheta_0}\\
&+\frac{1-\sigma(\vartheta_0,\theta)}{\Gamma(2-\sigma(\vartheta_0,\theta))}\int_c^{\vartheta_0}\left(\frac{(\vartheta_0-c)^{\epsilon(\vartheta_0,\theta)}-(\zeta-c)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}h'(\zeta)d\zeta.
\end{aligned}\tag{13}$$

$$\begin{aligned}
&=\frac{\sigma(\vartheta_0,\theta)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(\vartheta_0-c)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}h(c)\\
&\quad+\frac{\sigma(\vartheta_0,\theta)(\sigma(\vartheta_0,\theta)-1)}{\Gamma(2-\sigma(\vartheta_0,\theta))}\int_c^{\vartheta_0}\left(\frac{(\vartheta_0-c)^{\epsilon(\vartheta_0,\theta)}-(\zeta-c)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)-1}\\
&(\zeta-c)^{\epsilon(\vartheta_0,\theta)-1}h(\zeta)d\zeta\\
&\geq\frac{\sigma(\vartheta_0,\theta)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(\vartheta_0-c)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}h(c)\\
&\geq 0.
\end{aligned}$$

We obtain

$$\left({}^{C\sigma(\vartheta,\theta)}_{c}D^{\epsilon(\vartheta,\theta)}_{\vartheta}f\right)(\vartheta_0)\leq-\frac{\sigma(\vartheta_0,\theta)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(\vartheta_0-c)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}(f(\vartheta_0)-f(c))\leq 0.$$

$$\begin{aligned}
&\left({}^{C\sigma(\vartheta,\theta)}D^{\epsilon(\vartheta,\theta)}_{d,\vartheta}h\right)(\vartheta_0)\\
&=\frac{1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(d-\vartheta_0)^{\epsilon(\vartheta_0,\theta)}-(d-\zeta)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{1-\sigma(\vartheta_0,\theta)}(d-\zeta)^{1-\epsilon(\vartheta_0,\theta)}h'(\zeta)\Big|_{\vartheta_0}^{d}\\
&\quad-\frac{1-\sigma(\vartheta_0,\theta)}{\Gamma(2-\sigma(\vartheta_0,\theta))}\int_{\vartheta_0}^{d}\left(\frac{(d-\vartheta_0)^{\epsilon(\vartheta_0,\theta)}-(d-\zeta)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}h'(\zeta)d\zeta.\\
&=\frac{\sigma(\vartheta_0,\theta)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(d-\vartheta_0)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}h(d)\\
&\quad+\frac{\sigma(\vartheta_0,\theta)(\sigma(\vartheta_0,\theta)-1)}{\Gamma(2-\sigma(\vartheta_0,\theta))}\int_{\vartheta_0}^{d}\left(\frac{(d-\vartheta_0)^{\epsilon(\vartheta_0,\theta)}-(d-\zeta)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)-1}\\
&(d-\zeta)^{\epsilon(\vartheta_0,\theta)-1}h(\zeta)d\zeta\\
&\geq\frac{\sigma(\vartheta_0,\theta)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(d-\vartheta_0)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}h(d)\\
&\geq 0.
\end{aligned}\tag{14}$$

We obtain

$$\left({}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)}f\right)(\vartheta_0) \leq -\frac{\sigma(\vartheta_0,\theta)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(d-\vartheta_0)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}(f(\vartheta_0)-f(d)) \leq 0.$$

□

Remark 3. *If $\epsilon(\vartheta,\theta)$ and $\sigma(\vartheta,\theta)$ reduce to constants $0<\epsilon<1$ and $1<\sigma<2$, Guan and Wang [16] obtained a similar result as*

$$\left({}_{c}^{C\sigma}D_{\vartheta}^{\epsilon}f\right)(\vartheta_0) \leq -\frac{\sigma-1}{\Gamma(2-\sigma)}\left(\frac{(\vartheta_0-c)^{\epsilon}}{\epsilon}\right)^{-\sigma}(f(\vartheta_0)-f(c)) \leq 0$$

and

$$\left({}^{C\sigma}D_{d,\vartheta}^{\epsilon}f\right)(\vartheta_0) \leq -\frac{\sigma-1}{\Gamma(2-\sigma)}\left(\frac{(d-\vartheta_0)^{\epsilon}}{\epsilon}\right)^{-\sigma}(f(\vartheta_0)-f(d)) \leq 0.$$

Lemma 3. *Let $0<\epsilon(\vartheta,\theta)<1, 0<\beta(\vartheta,\theta)<1, \forall\theta\in[T,T_1]$. If $f\in C^1_{\epsilon,T}([T,T_1]), \theta_0\in(T,T_1]$ is its minimum, then the inequality*

$$\left({}_{T}^{C\beta(\vartheta,\theta)}D_{\theta}^{\epsilon(\vartheta,\theta)}f\right)(\theta_0) \leq \frac{1}{\Gamma(1-\beta(\vartheta,\theta_0))}\left(\frac{(\theta_0-T)^{\epsilon(\vartheta,\theta_0)}}{\epsilon(\vartheta,\theta_0)}\right)^{-\beta(\vartheta,\theta_0)}(f(\theta_0)-f(T)) \leq 0 \tag{15}$$

holds.

Lemma 4. *Let $0<\epsilon(\vartheta,\theta)<1, 1<\sigma(\vartheta,\theta)<2, \forall\vartheta\in[c,d]$. If f attains its minimum value at $\vartheta_0\in(T,T_1]$, then*

(1) if $f\in C^2_{\epsilon,c}([c,d])$

$$\left({}_{c}^{C\sigma(\vartheta,\theta)}D_{\vartheta}^{\epsilon(\vartheta,\theta)}f\right)(\vartheta_0) \geq -\frac{\sigma(\vartheta_0,\theta)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(\vartheta_0-c)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}(f(\vartheta_0)-f(c)) \geq 0 \tag{16}$$

(2) if $f\in C^2_{\epsilon,d}([c,d])$

$$\left({}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)}f\right)(\vartheta_0) \geq -\frac{\sigma(\vartheta_0,\theta)-1}{\Gamma(2-\sigma(\vartheta_0,\theta))}\left(\frac{(d-\vartheta_0)^{\epsilon(\vartheta_0,\theta)}}{\epsilon(\vartheta_0,\theta)}\right)^{-\sigma(\vartheta_0,\theta)}(f(\vartheta_0)-f(d)) \geq 0 \tag{17}$$

hold.

Remark 4. *If $\epsilon(\vartheta,\theta)$ and $\beta(\vartheta,\theta)$ in Lemma 3 reduce to constants $0<\epsilon<1$ and $0<\beta<1$, Guan and Wang [15] obtained a similar result as*

$$\left({}_{T}^{C\beta}D_{\theta}^{\epsilon}f\right)(\theta_0) \leq \frac{1}{\Gamma(1-\beta))}\left(\frac{(\theta_0-T)^{\epsilon}}{\epsilon}\right)^{-\beta}(f(\theta_0)-f(T)) \leq 0.$$

If $\epsilon(\vartheta,\theta)$ and $\sigma(\vartheta,\theta)$ in Lemma 4 reduce to constants $0<\epsilon<1$ and $1<\sigma<2$, Guan, Wang and Xu [16] obtained a similar result as

$$\left({}_{c}^{C\sigma}D_{\vartheta}^{\epsilon}f\right)(\vartheta_0) \geq -\frac{\sigma-1}{\Gamma(2-\sigma)}\left(\frac{(\vartheta_0-c)^{\epsilon}}{\epsilon}\right)^{-\sigma}(f(\vartheta_0)-f(c)) \geq 0$$

and

$$\left({}^{C\sigma}D^{\epsilon}_{d,\vartheta}f\right)(\vartheta_0) \geq -\frac{\sigma-1}{\Gamma(2-\sigma)}\left(\frac{(d-\vartheta_0)^{\epsilon}}{\epsilon}\right)^{-\sigma}(f(\vartheta_0)-f(d)) \geq 0.$$

Now, we will discuss the following space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator on the initial-boundary-value condition

$$\begin{cases} \left({}^{C\beta(\vartheta,\theta)}_{T}D^{\epsilon(\vartheta,\theta)}_{\theta}\right)w(\vartheta,\theta) - \left[\left({}^{C\sigma(\vartheta,\theta)}_{c}D^{\epsilon(\vartheta,\theta)}_{\vartheta}\right)w(\vartheta,\theta) + \left({}^{C\sigma(\vartheta,\theta)}D^{\epsilon(\vartheta,\theta)}_{d,\vartheta}\right)w(\vartheta,\theta)\right] \\ \quad + (-\Delta-\lambda_f)^{\frac{\alpha}{2}}w(\vartheta,\theta) - e(\vartheta,\theta)w(\vartheta,\theta) = F(\vartheta,\theta,w). \qquad (\vartheta,\theta)\in U \\ w(\vartheta,T) = \phi(\vartheta), \quad \vartheta\in(c,d), \\ w(c,\theta) = g_1(\theta), \quad \theta\in[T,T_1], \\ w(d,\theta) = g_2(\theta), \quad \theta\in[T,T_1]. \end{cases} \tag{18}$$

where $e(\vartheta,\theta)$ is bounded on $[c,d]\times[T,T_1]$, $U = (c,d)\times(T,T_1)$, $\bar{U} = [c,d]\times[T,T_1]$ and $S = ([c,d]\times\{T\}\bigcup\{c\}\times[T,T_1]\bigcup\{d\}\times[T,T_1])$.

Next, we will prove the maximum and minimum principle of Equation (18). Denote

$$H(\bar{U}) = \left\{w(\vartheta,\theta)|\, w(\vartheta,\theta)\in\ C^{2,1}(U), w(\vartheta,\theta)\in C(\bar{U})\right\}. \tag{19}$$

Proof of Theorem 1 (Maximum principle). Arguing by contradiction, if (5) is false, then $w(\vartheta,\theta)$ attains its maximum at point $(\bar{\vartheta},\bar{\theta})\in U$ and

$$w(\bar{\vartheta},\bar{\theta}) > \max\left\{\max_{\vartheta\in[c,d]}\phi(\vartheta), \max_{\theta\in[T,T_1]}g_1(\theta), \max_{\theta\in[T,T_1]}g_2(\theta), 0\right\} = N > 0. \tag{20}$$

Define the auxiliary function

$$\xi(\vartheta,\theta) = w(\vartheta,\theta) + \frac{\delta}{2}\frac{T_1-(\theta-T)}{T_1}, \quad (\vartheta,\theta)\in\bar{U}.$$

where $\delta = w(\bar{\vartheta},\bar{\theta}) - N > 0$.

From the definition of $\xi(\vartheta,\theta)$, we obtain

$$\xi(\vartheta,\theta) \leq w(\vartheta,\theta) + \frac{\delta}{2}, \ \ (\vartheta,\theta)\in\bar{U},$$

and

$$\xi(\bar{\vartheta},\bar{\theta}) > w(\bar{\vartheta},\bar{\theta}) = N+\delta \geq \delta + w(\vartheta,\theta) > \xi(\vartheta,\theta) + \frac{\delta}{2}, \quad \forall(\vartheta,\theta)\in S.$$

According to the last inequality, $\xi(\vartheta,\theta)$ cannot be attain a maximum on S. Let $\xi(\vartheta_1,\theta_1) = \max_{(\vartheta,\theta)\in\bar{U}}\xi(\vartheta,\theta)$, then

$$\xi(\vartheta_1,\theta_1) \geq \xi(\bar{\vartheta},\bar{\theta}) \geq \delta + N > \delta.$$

$$(-\Delta-\lambda_f)^{\frac{\alpha}{2}}\xi(\vartheta,\theta)\Big|_{(\vartheta_1,\theta_1)} = C_{n,\alpha}P.V.\int_{\mathbb{R}}\frac{\xi(\vartheta_1,\theta_1)-\xi(\vartheta,\theta_1)}{e^{\lambda f(|\vartheta_1-\vartheta|)}|\vartheta_1-\vartheta|^{1+\alpha}}dy \geq 0. \tag{21}$$

From the results of Lemmas 1 and 2, we obtain

$$\begin{cases} {}^{C\beta(\vartheta,\theta)}_{T}D^{\epsilon(\vartheta,\theta)}_{\theta}\xi(\vartheta,\theta)\Big|_{(\vartheta_1,\theta_1)} \geq 0, \\ {}^{C\sigma(\vartheta,\theta)}_{c}D^{\epsilon(\vartheta,\theta)}_{\vartheta}\xi(\vartheta,\theta)\Big|_{(\vartheta_1,\theta_1)} \leq 0, \\ {}^{C\sigma(\vartheta,\theta)}D^{\epsilon(\vartheta,\theta)}_{d,\vartheta}\xi(\vartheta,\theta)\Big|_{(\vartheta_1,\theta_1)} \leq 0. \end{cases} \tag{22}$$

By calculation, we obtain

$$\begin{aligned}&\left({}^{C\beta(\vartheta,\theta)}_{T}D_{\theta}^{\epsilon(\vartheta,\theta)}\right)\left(\frac{\delta}{2}\frac{T_1-(\theta-T)}{T_1}\right)\\&=-\epsilon(\vartheta,\theta)^{\beta(\vartheta,\theta)-1}(\theta-T)^{1-\epsilon(\vartheta,\theta)\beta(\vartheta,\theta)}\frac{\delta}{2T_1}\frac{\Gamma(2-\epsilon(\vartheta,\theta))}{\Gamma(3-\epsilon(\vartheta,\theta)-\beta(\vartheta,\theta))}.\end{aligned} \tag{23}$$

Applying (21)–(23), we obtain

$$\begin{aligned}&\left({}^{C\beta(\vartheta,\theta)}_{T}D_{\theta}^{\epsilon(\vartheta,\theta)}\right)w(\vartheta,\theta)\Big|_{(\vartheta_1,\theta_1)}-\left[\left({}^{C\sigma(\vartheta,\theta)}_{c}D_{\vartheta}^{\epsilon(\vartheta,\theta)}\right)w(\vartheta,\theta)+\left({}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)}\right)w(\vartheta,\theta)\right]\Big|_{(\vartheta_1,\theta_1)}\\&+(-\Delta-\lambda_f)^{\frac{\alpha}{2}}w(\vartheta,\theta)-e(\vartheta,\theta)w(\vartheta,\theta)-F(\vartheta,\theta,w)\\&\geq\epsilon(\vartheta_1,\theta_1)^{\beta(\vartheta_1,\theta_1)-1}(\theta_1-T)^{1-\epsilon(\vartheta_1,\theta_1)\beta(\vartheta_1,\theta_1)}\frac{\delta}{2T_1}\frac{\Gamma(2-\epsilon(\vartheta_1,\theta_1))}{\Gamma(3-\epsilon(\vartheta_1,\theta_1)-\beta(\vartheta_1,\theta_1))}\\&-e(\vartheta_1,\theta_1)\delta\left(1-\frac{T_1-(\theta_1-T)}{2T_1}\right)\\&>0,\end{aligned} \tag{24}$$

which is not in accordance with (18). Theorem 1 holds. □

Analogously, the following minimum principle holds.

Theorem 2. *Assume $F(\vartheta,\theta,w)\geq 0$ and $e(\vartheta,\theta)\geq 0$, $\forall(\vartheta,\theta)\in U$. If $w\in H(\bar{U})$ satisfies the space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator (18), then*

$$w(\vartheta,\theta)\geq\min\{\min_{\vartheta\in[c,d]}\phi(\vartheta),\min_{\theta\in[T,T_1]}g_1(\theta),\min_{\theta\in[T,T_1]}g_2(\theta),0\},\quad\forall(\vartheta,\theta)\in\bar{U}$$

holds.

4. Application of Maximum Principle

In this part, some results of space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator will be obtained by the maximum principle.

Theorem 3. *Let $\phi(\vartheta)\leq 0,\vartheta\in(c,d)$ and $g_1(\theta)\leq 0, g_2(\theta)\leq 0,\theta\in[T,T_1]$. If $F(\vartheta,\theta,w)\leq 0$, $e(\vartheta,\theta)\leq 0,\forall(\vartheta,\theta)\in U$ and $w(\vartheta,\theta)\in H(\bar{U})$ is a solution of (18), then*

$$w(\vartheta,\theta)\leq 0,(\vartheta,\theta)\in\bar{U}.$$

Theorem 4. *Let $\phi(\vartheta)\geq 0,\vartheta\in(c,d)$ and $g_1(\theta)\geq 0, g_2(\theta)\geq 0,\theta\in[T,T_1]$. If $F(\vartheta,\theta,w)\geq 0$, $e(\vartheta,\theta)\geq 0,\forall(\vartheta,\theta)\in U$ and $w(\vartheta,\theta)\in H(\bar{U})$ is a solution of (18), then*

$$w(\vartheta,\theta)\geq 0,(\vartheta,\theta)\in\bar{U}.$$

According to Theorem 3 and 4, the following Remark holds.

Remark 5. *Let $\phi(\vartheta)=0,\vartheta\in[a,b]$ and $g_1(\theta)=g_2(\theta)=0,\theta\in[T,T_1]$. If $F(\vartheta,\theta,w)=0$, $e(\vartheta,\theta)=0,\forall(\vartheta,\theta)\in U$ and $w(\vartheta,\theta)\in H(\bar{U})$ is a solution of (18), then*

$$w(\vartheta,\theta)=0,(\vartheta,\theta)\in\bar{U}.$$

Theorem 5. *Let $\dfrac{\partial F}{\partial w}+e(\vartheta,\theta)\leq 0,\forall(\vartheta,\theta)\in U$. Then, (18) has at most one solution $w(x,\theta)\in H(\bar{U})$.*

Proof. Let $w_1, w_2 \in H(\bar{U})$ be two solutions of (18) and

$$w(\vartheta,\theta) = w_1(\vartheta,\theta) - w_2(\vartheta,\theta), \quad (\vartheta,\theta) \in U.$$

Then,

$$\begin{cases} \left({}^{C\beta(\vartheta,\theta)}_{T}D^{\epsilon(\vartheta,\theta)}_{\theta}\right)w(\vartheta,\theta) - \left[\left({}^{C\sigma(\vartheta,\theta)}_{c}D^{\epsilon(\vartheta,\theta)}_{\vartheta}\right)w(\vartheta,\theta) + \left({}^{C\sigma(\vartheta,\theta)}D^{\epsilon(\vartheta,\theta)}_{d,\vartheta}\right)w(\vartheta,\theta)\right] \\ +(-\Delta-\lambda_f)^{\frac{\alpha}{2}}w(\vartheta,\theta) - e(\vartheta,\theta)w(\vartheta,\theta) = F(\vartheta,\theta,w_1) - F(\vartheta,\theta,w_2), \quad (\vartheta,\theta) \in U, \\ w(\vartheta,T) = 0, \quad \vartheta \in (c,d), \\ w(c,\theta) = 0, \quad \theta \in [T,T_1], \\ w(d,\theta) = 0, \quad \theta \in [T,T_1]. \end{cases} \tag{25}$$

By the mean value theorem, we obtain

$$F(\vartheta,\theta,w_1) - F(\vartheta,\theta,w_2) = \frac{\partial F}{\partial w}(\tilde{w})(w_1 - w_2) \tag{26}$$

where $\tilde{w} = \rho w_1 + (1-\rho)w_2$, $0 \le \rho \le 1$.

Since $\frac{\partial F}{\partial w}(\tilde{w}) + e(\vartheta,\theta) \le 0$ and Theorem 3 , combining (25) and (26), we have

$$w(\vartheta,\theta) \le 0, \quad (\vartheta,\theta) \in \bar{U}. \tag{27}$$

Analogously, employing Theorem 3 to $-w(\vartheta,\theta)$, then

$$-w(\vartheta,\theta) \le 0, \quad (\vartheta,\theta) \in \bar{U}. \tag{28}$$

Therefore,

$$w(\vartheta,\theta) = 0, \quad (\vartheta,\theta) \in \bar{U},$$

holds. □

Theorem 6. *Let* $\frac{\partial F}{\partial w} + e(\vartheta,\theta) \le 0, \forall(\vartheta,\theta) \in U$. *If* $w_1(\vartheta,\theta)$ *and* $w_2(\vartheta,\theta)$ *are two solutions of (18) and* $F(\vartheta,\theta,w_1) \le F(\vartheta,\theta,w_2)$, *then* $w_1(\vartheta,\theta) \le w_2(\vartheta,\theta)$.

Theorem 7. *Let* $w_1(\vartheta,\theta)$ *and* $w_2(\vartheta,\theta)$ *be two solutions of (18) on the initial-boundary-value conditions*

$$\begin{cases} w_1(\vartheta,T) = \phi(\vartheta), & \vartheta \in (c,d), \\ w_1(c,\theta) = g_1(\theta), & \theta \in [T,T_1], \\ w_1(d,\theta) = g_2(\theta), & \theta \in [T,T_1], \end{cases}$$

and

$$\begin{cases} w_2(\vartheta,T) = \phi^*(\vartheta), & \vartheta \in (c,d), \\ w_2(c,\theta) = g_1^*(\theta), & \theta \in [T,T_1], \\ w_2(d,\theta) = g_2^*(\theta), & \theta \in [T,T_1], \end{cases}$$

respectively, and $e(\vartheta,\theta) \le 0$. *Then,*

$$\max_{(\vartheta,\theta)\in\bar{U}} |w_1(\vartheta,\theta) - w_2(\vartheta,\theta)| \le \{ \max_{\vartheta\in[c,d]} |\phi(\vartheta) - \phi^*(\vartheta)|, \max_{\theta\in[T,T_1]} |g_1(\theta) - g_1^*(\theta)|, \max_{\theta\in[T,T_1]} |g_2(\theta) - g_2^*(\theta)| \}$$

holds.

Proof. Let $w(\vartheta,\theta) = w_1(\vartheta,\theta) - w_2(\vartheta,\theta)$, then

$$\begin{cases} \left({}^{C\beta(\vartheta,\theta)}_{T}D^{\epsilon(\vartheta,\theta)}_{\theta}\right)w(\vartheta,\theta) - \left[\left({}^{C\sigma(\vartheta,\theta)}_{c}D^{\epsilon(\vartheta,\theta)}_{\vartheta}\right)w(\vartheta,\theta) + \left({}^{C\sigma(\vartheta,\theta)}D^{\epsilon(\vartheta,\theta)}_{d,\vartheta}\right)w(\vartheta,\theta)\right] \\ \quad + (-\Delta - \lambda_f)^{\frac{\alpha}{2}}w(\vartheta,\theta) - e(\vartheta,\theta)w(\vartheta,\theta) = 0, & (\vartheta,\theta) \in U, \\ w(\vartheta,T) = \phi(\vartheta) - \phi^*(\vartheta), & \vartheta \in (c,d), \\ w(c,\theta) = g_1(\theta) - g_1^*(\theta), & \theta \in [T,T_1], \\ w(d,\theta) = g_2(\theta) - g_2^*(\theta), & \theta \in [T,T_1]. \end{cases} \tag{29}$$

By Theorems 1 and 2, then

$$\max_{(\vartheta,\theta)\in\bar{U}} w(\vartheta,\theta) \le \max\left\{ \max_{\vartheta\in(c,d)} \phi(\vartheta) - \phi^*(\vartheta), \max_{\theta\in[T,T_1]} g_1(\theta) - g_1^*(\theta), \max_{\theta\in[T,T_1]} g_2(\theta) - g_2^*(\theta), 0\right\},$$

$$\min_{(\vartheta,\theta)\in\bar{U}} w(\vartheta,\theta) \ge \min\left\{ \min_{\vartheta\in(c,d)} \phi(\vartheta) - \phi^*(z), \min_{\theta\in[T,T_1]} g_1(\theta) - g_1^*(\theta), \min_{\theta\in[T,T_1]} g_2(\theta) - g_2^*(\theta), 0\right\},$$

From the two above inequalities, we obtain

$$\begin{aligned} \max_{(\vartheta,\theta)\in\bar{U}} |w_1(\vartheta,\theta) - w_2(\vartheta,\theta)| &= \max_{(\vartheta,\theta)\in\bar{U}} |w(\vartheta,\theta)| \\ &\le \max\left\{ \max_{\vartheta\in(c,d)} |\phi(\vartheta) - \phi^*(\vartheta)|, \max_{\theta\in[T,T_1]} |g_1(\theta) - g_1^*(\theta)|, \max_{\theta\in[T,T_1]} |g_2(\theta) - g_2^*(\theta)|\right\}. \end{aligned}$$

□

Theorem 8 (Comparison Theorem)**.** *Let $w(\vartheta,\theta)$ be a solution of Equation (18), suppose $p_2 w + q_2(x) \le F(\vartheta,\theta,w) \le p_1 w + q_1(x)$, $p_1, p_2 > 0$. Let $w_1(\vartheta,\theta), w_2(\vartheta,\theta) \in H(\bar{U})$ satisfy*

$$\begin{cases} \left({}^{C\beta(\vartheta,\theta)}_{T}D^{\epsilon(\vartheta,\theta)}_{\theta}\right)w_1(\vartheta,\theta) - \left[\left({}^{C\sigma(\vartheta,\theta)}_{c}D^{\epsilon(\vartheta,\theta)}_{\vartheta}\right)w_1(\vartheta,\theta) + \left({}^{C\sigma(\vartheta,\theta)}D^{\epsilon(\vartheta,\theta)}_{d,\vartheta}\right)w_1(\vartheta,\theta)\right] \\ +(-\Delta - \lambda_f)^{\frac{\alpha}{2}}w_1(\vartheta,\theta) - e(\vartheta,\theta)w_1(\vartheta,\theta) = p_1 w_1(\vartheta,\theta) + q_1(x), \quad (\vartheta,\theta) \in U, \\ w_1(\vartheta,T) = \phi_{w_1}(\vartheta), \quad \vartheta \in (c,d), \\ w_1(c,\theta) = g_{1_{w_1}}(\theta), \quad \theta \in [T,T_1], \\ w_1(d,\theta) = g_{2_{w_1}}(\theta), \quad \theta \in [T,T_1], \end{cases} \tag{30}$$

and

$$\begin{cases} \left({}^{C\beta(\vartheta,\theta)}_{T}D^{\epsilon(\vartheta,\theta)}_{\theta}\right)w_2(\vartheta,\theta) - \left[\left({}^{C\sigma(\vartheta,\theta)}_{c}D^{\epsilon(\vartheta,\theta)}_{\vartheta}\right)w_2(\vartheta,\theta) + \left({}^{C\sigma(\vartheta,\theta)}D^{\epsilon(\vartheta,\theta)}_{d,\vartheta}\right)w_2(\vartheta,\theta)\right] \\ +(-\Delta - \lambda_f)^{\frac{\alpha}{2}}w_2(\vartheta,\theta) - e(\vartheta,\theta)w_2(\vartheta,\theta) = p_2 w_2(\vartheta,\theta) + q_2(x). \quad (\vartheta,\theta) \in U, \\ w_2(\vartheta,T) = \phi_{w_2}(\vartheta), \quad \vartheta \in (c,d), \\ w_2(c,\theta) = g_{1_{w_2}}(\theta), \quad \theta \in [T,T_1], \\ w_2(d,\theta) = g_{2_{w_2}}(\theta), \quad \theta \in [T,T_1]. \end{cases} \tag{31}$$

If $\phi_{w_1}(\vartheta) \ge \phi(\vartheta) \ge \phi_{w_2}(\vartheta)$, $g_{1_{w_1}}(\theta) \ge g_1(\theta) \ge g_{1_{w_2}}(\theta)$ and $g_{2_{w_1}}(\theta) \ge g_2(\theta) \ge g_{2_{w_2}}(\theta)$, then

$$w_2(\vartheta,\theta) \le w(\vartheta,\theta) \le w_1(\vartheta,\theta), (\vartheta,\theta) \in \bar{U},$$

hold.

Proof. We shall prove that $w(\vartheta,\theta) \le w_1(\vartheta,\theta)$ and by applying analogous steps one can show that $w_2(\vartheta,\theta) \le w(\vartheta,\theta)$.

Let $\tilde{w}(\vartheta,\theta) = w_1(\vartheta,\theta) - w(\vartheta,\theta)$. By (30) minus (18), we obtain

$$\begin{cases} \left({}^{C\beta(\vartheta,\theta)}_{T}D_{\theta}^{\epsilon(\vartheta,\theta)}\right)\tilde{w}(\vartheta,\theta) - \left[\left({}^{C\sigma(\vartheta,\theta)}_{c}D_{\vartheta}^{\epsilon(\vartheta,\theta)}\right)\tilde{w}(\vartheta,\theta) + \left({}^{C\sigma(\vartheta,\theta)}D_{d,\vartheta}^{\epsilon(\vartheta,\theta)}\right)\tilde{w}(\vartheta,\theta)\right] \\ +(-\Delta - \lambda_f)^{\frac{\alpha}{2}}\tilde{w}(\vartheta,\theta) - e(\vartheta,\theta)\tilde{w}(\vartheta,\theta) = p_1 w_1(\vartheta,\theta) + q_1(x) - F(\vartheta,\theta,w) \geq p_1\tilde{w}(\vartheta,\theta), \\ \tilde{w}(\vartheta,T) = \phi_{w_1}(\vartheta) - \phi(\vartheta) \geq 0, \quad \vartheta \in (c,d), \\ \tilde{w}(c,\theta) = g_{1_{w_1}}(\theta) - g_1(\theta) \geq 0, \quad \theta \in [T,T_1], \\ \tilde{w}(d,\theta) = g_{2_{w_1}}(\theta) - g_2(\theta) \geq 0, \quad \theta \in [T,T_1]. \end{cases} \tag{32}$$

Since $p_1 \geq 0$, by Theorem 3, $\tilde{w} \geq 0$, the proof is complete. □

5. Conclusions

The space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator is considered in this paper. We have given the definition of LFVCCD and RFVCCD and some extreme principles. By these extreme principles, a new maximum principle of space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator is derived. Based on the maximum principles, the comparison principle, the uniqueness and continuous dependence of the solution of space-time fractional variable-order conformable nonlinear differential equation with a generalized tempered fractional Laplace operator are proved. Abdulazeez and Modanli [35] used the modified double Laplace transform method to study the Pseudo-Hyperbolic Telegraph partial differential equation. This is an interesting analysis method that is completely different from our method. In the future, we will attempt to apply this method to study space-time fractional variable-order conformable nonlinear differential equations.

Author Contributions: Conceptualization, methodology, investigation, writing—original draft preparation, T.G.; validation, writing—review and editing, T.G. and L.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Graduate Education and Teaching Innovation Project of Shanxi, China (No.2022YJJG124) and Higher Education Science and Technology Innovation Project of Shanxi, China (No.2023L156).

Data Availability Statement: No data was used to support this study.

Acknowledgments: We would like to express our gratitude to the editor for taking time to handle the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

IBVP	Initial-boundary-value problem
FPDE	Fractional partial differential equations
LFVCCD	Left fractional variable-order conformable Caputo derivative
RFVCCD	Right fractional variable-order conformable Caputo derivative

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