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# Existence of Solutions for Coupled System of Sequential Liouville–Caputo-Type Fractional Integrodifferential Equations

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**Abstract:** The present investigation aims to establish the existence and uniqueness of solutions for a system containing sequential fractional differential equations. Furthermore, boundary conditions that include the Riemann–Liouville fractional integral are taken into consideration. The existence of unknown functions, fractional derivatives, and fractional integrals at lower orders are necessary for the nonlinearity to exist. In order to provide proofs for the results presented in this study, the Leray–Schauder alternative and the Banach fixed-point theorem are utilised. Finally, examples are used to support the main results.

**Keywords:** fractional integrodifferential equations; sequential derivatives; Liouville–Caputo derivatives; Riemann–Liouville integrals; existence; uniqueness; fixed point



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## 1. Introduction

In this article, we examine the system of sequential derivative-containing nonlinear Liouville–Caputo fractional integrodifferential equations (LCFIE):

$$\begin{cases} ({}^{c}\mathcal{D}^{\varsigma_1} + \lambda_1 {}^{c}\mathcal{D}^{\varsigma_1 - 1})\mathcal{G}(\tau) = \varphi_1(\tau, \mathcal{G}(\tau), \mathcal{J}(\tau), {}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau), \mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau)), \quad \tau \in (0,1), \\ ({}^{c}\mathcal{D}^{\varsigma_2} + \chi_1 {}^{c}\mathcal{D}^{\varsigma_2 - 1})\mathcal{J}(\tau) = \phi_1(\tau, \mathcal{G}(\tau), \mathcal{J}(\tau), {}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau), \mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau)), \quad \tau \in (0,1), \end{cases} \tag{1}$$

and it is enhanced by coupled Riemann–Liouville integral (RLI) boundary conditions:

$$\begin{cases} \mathcal{G}(0) = 0, \ \mathcal{G}'(0) = 0, \ \mathcal{G}'(1) = 0, \\ \mathcal{G}(1) = \int_0^{\eta} \frac{(\eta - v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})} \mathcal{G}(v)dv + \int_0^{\zeta} \frac{(\zeta - v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})} \mathcal{J}(v)dv, \\ \mathcal{J}(0) = 0, \ \mathcal{J}'(0) = 0, \ \mathcal{J}'(1) = 0, \\ \mathcal{J}(1) = \int_0^{\eta} \frac{(\eta - v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})} \mathcal{G}(v)dv + \int_0^{\zeta} \frac{(\zeta - v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})} \mathcal{J}(v)dv. \end{cases} \tag{2}$$

Here, $\varsigma_1, \varsigma_2 \in (3,4], \lambda_1, \chi_1 > 0, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2 > 0$, ${}^{c}\mathcal{D}^{\kappa}$ represents the Liouville–Caputo fractional derivative (LCFD) of order $\kappa$ (for $\kappa = \varsigma_1, \varsigma_2, \varsigma_2 - 1, \varsigma_1 - 1, \mathfrak{p}_1, \mathfrak{p}_2$), $\varphi_1, \phi_1 : [0,1] \times \mathcal{R}_e^4 \to \mathcal{R}_e$ are continuous functions, and $\mathcal{I}_{0+}^{\nu}$ denotes the fractional RLI of order $\nu$ (for $\nu = \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{v}, \mathfrak{z}$). Given certain assumptions on the functions $\varphi_1$ and $\phi_1$, we aim to establish the existence of at least one solution for Problems (1) and (2).

In the subsequent section, we will provide an overview of some scholarly articles that are relevant to the issue at hand. The concept of SFD $\mathcal{D}_{a+}^{\kappa\varsigma}$, where $\kappa$ is a positive integer, is introduced in the monograph authored by Miller and Ross [1] on page 209. The publications [2,3] provide an explanation of the connection between sequential fractional derivatives (SFDs) and non-Riemann–Liouville SFDs. The authors in Reference [2] studied

the existence of minimal, maximal, and unique solutions to an initial value problem (IVP) involving the RLSFDs $\mathcal{D}_{0+}^{2\varsigma}$, $\varsigma \in (0,1]$, where an is a value within the interval (0, 1]. The researchers employed the upper and lower solutions approach in conjunction with the accompanying monotone iterative method. In their study, the authors in Reference [3] investigated the existence of a solution for a periodic boundary value problem (BVP). Specifically, they examined a problem involving an RLSFD $\mathcal{D}_{0+}^{2\varsigma}$, $\varsigma \in (0,1]$. To analyse this problem, the writers employed the upper and lower solutions approach along with the fixed-point theorem of Schauder's.

In Reference [4], the author demonstrated the existence of solutions for a nonlinear impulsive fractional differential equation (FDE) with RLSFD. These solutions were subject to periodic boundary conditions and were obtained using the monotone iterative method. The nonexistence of solutions in the function space $\mathcal{L}^p((1,\infty),\mathcal{R}_e)$ for an IVP that includes linear sequential fractional differential equations (SFDEs) with a Riemann–Liouville derivative and a classical first-order derivative is examined in Reference [5]. In Reference [6], Klimek demonstrated the existence and uniqueness of the solution for a certain category of nonlinear Hadamard SFDEs. This was achieved through the application of the contraction principle, with the inclusion of a set of beginning conditions that incorporate fractional derivatives. In our study, the term "sequential" refers to the property of the operator $(\mathcal{D}^{\varsigma} + \lambda\mathcal{D}^{\varsigma-1})$, being expressible as a composition of the operators $\mathcal{D}^{\varsigma-1}(\mathcal{D}+\lambda)$, where $\mathcal{D}$ represents the ordinary derivative.

Ahmad and Nieto [7] introduced this particular operator in their study on the existence and uniqueness of solutions for the Caputo SFDE. In Reference [8], the authors demonstrated the existence of solutions for the sequential integrodifferential problem through the application of methods derived from fixed-point theory. The authors in Reference [9] conducted a study on the presence of solutions for the sequential Caputo FDE, incorporating boundary conditions that encompass a fractional integral of Riemann–Liouville nature. In their study, the authors of Reference [10] demonstrated the existence of solutions for a sequential fractional differential inclusion of the Caputo type. The boundary conditions of this inclusion encompass a fractional RLI. The authors of Reference [11] derived multiple results pertaining to the existence and uniqueness of the SFDE of the Caputo type. The authors of Reference [12] demonstrated the existence of solutions for SFDEs with nonlocal boundary conditions. The authors of Reference [13] derived existence results for solutions of the Caputo fractional sequential integrodifferential equation and inclusion. The existence of coupled systems of FDEs is prevalent in various practical applications, particularly in the field of biosciences (see [14] and its associated literature). In the following discussion, we will outline several fractional systems that are relevant to the topic at hand, denoted as (1) and (2). The authors of [14] employed the Banach contraction mapping concept and the Leray–Schauder alternative to establish the existence and uniqueness of solutions for the nonlinear system of Caputo-type SFDEs. The authors of Reference [15] presented a study where they established the existence of solutions for a system of nonlinear coupled differential equations and inclusions. These equations involve Caputo-type sequential derivatives of fractional order. The authors employed techniques developed from fixed-point theory to achieve this result. The work done by the authors of Reference [16] focused on investigating the presence and stability of a tripled system of SFDEs while considering multipoint boundary conditions. The authors of Reference [17] presented a study whereby they established the presence and durability of solutions for three nonlinear SFDEs with nonlocal boundary conditions. The authors of the cited Reference [18] obtained conclusions regarding the existence of solutions for a coupled system of nonlinear differential equations and inclusions that incorporate SFD.

The authors of the aforementioned study [14] successfully derived existence results for the solutions of a system consisting of Caputo FDEs,

$$\begin{cases} ({}^{c}\mathcal{D}^{\varsigma_1} + \lambda_1 {}^{c}\mathcal{D}^{\varsigma_1 - 1})\mathcal{G}(\tau) = \mathfrak{f}(\tau, \mathcal{G}(\tau), \mathcal{J}(\tau)), \ \tau \in [0,1], \\ ({}^{c}\mathcal{D}^{\varsigma_2} + \chi_1 {}^{c}\mathcal{D}^{\varsigma_2 - 1})\mathcal{J}(\tau) = \mathfrak{g}(\tau, \mathcal{G}(\tau), \mathcal{J}(\tau)), \ \tau \in [0,1], \\ \mathcal{G}(0) = \mathcal{G}'(0) = 0, \mathcal{G}(\zeta) = a\mathcal{I}^{\beta}\mathcal{G}(\eta), \\ \mathcal{J}(0) = \mathcal{J}'(0) = 0, \mathcal{J}(z) = b\mathcal{I}^{\gamma}\mathcal{J}(\theta), \end{cases} \tag{3}$$

where $\varsigma_1, \varsigma_2 \in (2,3]$, $\lambda_1, \chi_1 > 0$, $\beta, \gamma > 0$, $0 < \theta < z < 1$, and $0 < \eta < \zeta < 1$. It can be observed that Boundary Conditions (3) indicate that the value of $\mathcal{G}$ at the point $\zeta$ is alone determined by the function $\mathcal{J}$, while the value of $\mathcal{J}$ at the point $z$ is only determined by the function $\mathcal{G}$. The dependence of $\mathcal{G}(1)$ and $\mathcal{J}(1)$ on the functions $\mathcal{G}$ and $\mathcal{J}$ is evident in our Boundary Conditions (2). Furthermore, inside our System (1), the nonlinearities $\varphi_1$ and $\varphi_2$ exhibit a dependency on certain differential and integral terms, but in (3), no such dependency is found.

The authors of Reference [19] established the existence and uniqueness of solutions for a system of Hadamard-type SFDEs, including nonlocal coupled strip conditions

$$\begin{cases} ({}^{H}\mathcal{D}^{\varsigma_1} + \lambda_1 {}^{H}\mathcal{D}^{\varsigma_1 - 1})\mathcal{G}(\tau) = \mathfrak{f}(\tau, \mathcal{G}(\tau), \mathcal{J}(\tau), {}^{H}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau)), \ \tau \in [0,1], \\ ({}^{H}\mathcal{D}^{\varsigma_2} + \chi_1 {}^{H}\mathcal{D}^{\varsigma_2 - 1})\mathcal{J}(\tau) = \mathfrak{g}(\tau, \mathcal{G}(\tau), {}^{H}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau), \mathcal{J}(\tau)), \ \tau \in [0,1], \\ \mathcal{G}(1) = 0, \mathcal{G}(e) = {}^{H}\mathcal{I}^{\gamma}\mathcal{G}(\eta), \\ \mathcal{J}(1) = 0, \mathcal{J}(e) = {}^{H}\mathcal{I}^{\beta}\mathcal{J}(\zeta), \end{cases} \tag{4}$$

where $\lambda_1, \chi_1 > 0$, $\varsigma_1, \varsigma_2 \in (1,2]$, $\mathfrak{p}_1, \mathfrak{p}_2 \in (0,1)$, $\gamma > 0$, $\eta, \zeta \in (1,e)$, and $\beta > 0$. It is clear from the Boundary Conditions (4) that the function $\mathcal{J}$ alone determines the value of $\mathcal{G}$ at the point $e$, whereas the function $\mathcal{G}$ solely determines the value of $\mathcal{J}$ at the point $e$. The dependence of $\mathcal{G}(1)$ and $\mathcal{J}(1)$ on the functions $\mathcal{G}$ and $\mathcal{J}$ is evident in our Boundary Conditions (2). Furthermore, inside our System (1), the nonlinearities $\varphi_1$ and $\varphi_2$ exhibit a dependency on certain differential and integral terms, but in (4), only differential terms are found. Moreover, within our computational Systems (1) and (2), we employ LCFDs and RLIs. However, in the context of Problem (4), Hadamard fractional derivatives and integrals are employed.

Subramanian et al. [20] conducted an analysis on the existence results for a system of coupled higher-order fractional integro-differential equations. These equations were subject to nonlocal integral and multi-point boundary conditions, which were dependent on lower-order fractional derivatives and integrals:

$$\begin{cases} {}^{c}\mathcal{D}^{\varsigma_1}\mathcal{G}(\tau) = \mathfrak{f}(\tau, \mathcal{G}(\tau), \mathcal{J}(\tau), {}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau), \mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau)), \ \tau \in [0,T], \\ {}^{c}\mathcal{D}^{\varsigma_2}\mathcal{J}(\tau) = \mathfrak{g}(\tau, \mathcal{G}(\tau), {}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau), \mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau), \mathcal{J}(\tau)), \ \tau \in [0,T], \\ \mathcal{G}(0) = \phi_1(\mathcal{J}), \ \mathcal{G}'(0) = \epsilon_1 \int_0^{\nu_1} \mathcal{J}'(\theta)d\theta, \ \mathcal{G}''(0) = 0, \cdots, \ \mathcal{G}^{n-2}(0) = 0, \\ \mathcal{G}(T) = \lambda_1 \int_0^{\delta_1} \mathcal{J}(\theta)d\theta + \mu_1 \sum_{j=1}^{k-2} \varpi_j \mathcal{J}(\vartheta_j), \\ \mathcal{J}(0) = \phi_2(\mathcal{G}), \ \mathcal{J}'(0) = \epsilon_2 \int_0^{\nu_2} \mathcal{G}'(\theta)d\theta, \ \mathcal{J}''(0) = 0, \cdots, \ \mathcal{J}^{n-2}(0) = 0, \\ \mathcal{J}(T) = \lambda_2 \int_0^{\delta_2} \mathcal{G}(\theta)d\theta + \mu_2 \sum_{j=1}^{k-2} \omega_j \mathcal{G}(\varphi_j). \end{cases} \tag{5}$$

The nonlinearity is dependent on both the unknown functions and their fractional derivatives and integrals at a lower level. The consequence of existence is derived by the use of the Leray–Schauder alternative, whilst the result of uniqueness is established by employing the concept of a Banach contraction mapping.

The authors conducted a study [21] to examine the system of sequential Caputo fractional integrodifferential equations

$$\begin{cases} ({}^{c}\mathcal{D}^{\varsigma_1}+\lambda_1{}^{c}\mathcal{D}^{\varsigma_1-1})\mathcal{G}(\tau)=\mathfrak{f}(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{p}_1}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{p}_2}\mathcal{J}(\tau)),\ \tau\in(0,1),\\ ({}^{c}\mathcal{D}^{\varsigma_2}+\chi_1{}^{c}\mathcal{D}^{\varsigma_2-1})\mathcal{J}(\tau)=\mathfrak{g}(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{J}(\tau)),\ \tau\in(0,1),\\ \mathcal{G}(0)=\mathcal{G}'(0)=0,\mathcal{G}'(1)=0,\mathcal{G}(1)=\int_0^1\mathcal{G}(s)d\mathcal{H}_1(s)+\int_0^1\mathcal{J}(s)d\mathcal{H}_2(s),\\ \mathcal{J}(0)=\mathcal{J}'(0)=0,\mathcal{J}'(1)=0,\mathcal{J}(1)=\int_0^1\mathcal{G}(s)d\mathcal{K}_1(s)+\int_0^1\mathcal{J}(s)d\mathcal{K}_2(s), \end{cases} \tag{6}$$

where $\varsigma_1,\varsigma_2\in(3,4]$, $\lambda_1,\chi_1>0$, and $\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2>0$. In the last conditions of (6), they have the Riemann–Stieltjes integrals (RSIs) with bounded variation functions $\mathcal{H}_1,\mathcal{K}_1,\mathcal{H}_2,\mathcal{K}_2$. Additionally, the nonlinearities $\varphi_1$ and $\varphi_2$ in our System (1) depend on certain differential and integral terms, but in (6), there are only integral terms. Moreover, within our computational Systems (1) and (2), we employ LCFDs and RLIs. However, in the context of Problem (6), CFDs and RSIs are employed. Another important difference between these two problems is given by the following conditions: the value of unknown functions with the right end point 1 is proportional to the sum of RLIs of unknown functions with different strip lengths $(0,\eta)$, $(0,\zeta)$ in (2), but in (6), the value of unknown functions with the right end point 1 is proportional to the sum of RSIs of unknown functions with the same strip lengths $(0,1)$.

The structure of the paper is organised in the following manner. In the second section, we analyse a linear fractional BVP that is connected to our Problems (1) and (2). Section 3 focuses on our primary results regarding Equations (1) and (2), while Section 4 provides two illustrative instances that demonstrate the results of our research. At last, the results of our research are presented in Section 5 of this study.

## 2. Preliminaries

First, we outline some basic concepts of fractional calculus.

**Definition 1** ([22]). *The fractional order of RLI $\varsigma\in\mathcal{R}_e(\varsigma>0)$ for a locally integrable, real-valued function $\mathcal{X}$ on $\infty\le\mathfrak{a}\le\mathfrak{b}+\infty$, denoted by $\mathcal{I}_{\mathfrak{a}}^{\varsigma}(\mathfrak{s})$, is defined by*

$$\mathcal{I}_{\mathfrak{a}}^{\varsigma}\mathcal{X}(\mathfrak{z})=\int_{\mathfrak{a}}^{\mathfrak{z}}\frac{(\mathfrak{z}-\mathfrak{s})^{\varsigma-1}}{\Gamma(\varsigma)}\mathcal{X}(\mathfrak{s})d\mathfrak{s},$$

*and the Gamma function is represented here by $\Gamma(\cdot)$.*

**Definition 2** ([23]). *The fractional order of the Caputo-type $\varsigma$ for an $(r-1)$- times absolutely continuous function $\mathcal{X}:[\mathfrak{a},\infty)\to\mathcal{R}_e$ is defined as*

$${}^{c}\mathcal{D}^{\varsigma}\mathcal{X}(\mathfrak{z})=\int_{\mathfrak{a}}^{\mathfrak{z}}\frac{(\mathfrak{z}-\mathfrak{s})^{r-\varsigma-1}}{\Gamma(r-\varsigma)}\mathcal{X}^{(r)}(\mathfrak{s})d\mathfrak{s},\ r-1<\varsigma<r,\ r=[\varsigma]+1,$$

*where the integral portion of the real number $\varsigma$ is denoted by $[\varsigma]$.*

## 3. Auxiliary Results

This section focuses on the investigation of linear FDEs.

$$\begin{cases} ({}^{c}\mathcal{D}^{\varsigma_1}+\lambda_1{}^{c}\mathcal{D}^{\varsigma_1-1})\mathcal{G}(\tau)=\mathfrak{h}_1(\tau),\ \ \tau\in(0,1),\\ ({}^{c}\mathcal{D}^{\varsigma_2}+\chi_1{}^{c}\mathcal{D}^{\varsigma_2-1})\mathcal{J}(\tau)=\mathfrak{h}_2(\tau),\ \ \tau\in(0,1), \end{cases} \tag{7}$$

which are supplemented with boundary condition (2), with $\mathfrak{h}_1,\mathfrak{h}_2\in[0,1]$. We denote these by

$$\Omega_1 = \frac{1}{\lambda_1}(1-e^{-\lambda_1}),\ \Omega_2 = \frac{1}{\lambda_1^2}(2\lambda_1 - 2 + e^{-\lambda_1}),$$
$$\Omega_3 = \frac{1}{\chi_1}(1-e^{-\chi_1}),\ \Omega_4 = \frac{1}{{\chi_1}^2}(2\chi_1 - 2 + e^{-\chi_1}),$$
$$\Omega_5 = \frac{1}{\lambda_1^2}[(\lambda_1 - 1 + e^{-\lambda_1}) - \int_0^\eta \frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}(\lambda_1 v - 1 + e^{-\lambda_1 v})dv],$$
$$\Omega_6 = \frac{1}{\lambda_1^3}[(\lambda_1^2 - 2\lambda_1 + 2 - 2e^{-\lambda_1}) - \int_0^\eta \frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}(\lambda_1^2 v^2 - 2\lambda_1 v + 2 - 2e^{\lambda_1 v})dv],$$
$$\Omega_7 = \frac{1}{{\chi_1}^2}\int_0^\zeta \frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}(\chi_1 v - 1 + e^{-\chi_1 v})dv, \tag{8}$$
$$\Omega_8 = \frac{1}{{\chi_1}^3}\int_0^\zeta \frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}({\chi_1}^2 v^2 - 2\chi_1 v + 2 - 2e^{-\chi_1 v})dv,$$
$$\Omega_9 = \frac{1}{\lambda_1^2}\int_0^\eta \frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}(\lambda_1 v - 1 + e^{-\lambda_1 v})dv,$$
$$\Omega_{10} = \frac{1}{\lambda_1^3}\int_0^\eta \frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}(\lambda_1^2 v^2 - 2\lambda_1 v + 2 - 2e^{\lambda_1 v})dv,$$
$$\Omega_{11} = \frac{1}{{\chi_1}^2}(\mu - 1 + e^{-\mu}) - \frac{1}{{\chi_1}^2}\int_0^\zeta \frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}(\chi_1 v - 1 + e^{-\chi_1 v})dv,$$
$$\Omega_{12} = \frac{1}{{\chi_1}^3}({\chi_1}^2 - 2\chi_1 + 2 - 2e^{-\mu}) - \frac{1}{{\chi_1}^3}\int_0^\zeta \frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}({\chi_1}^2 v^2 - 2\chi_1 v + 2 - 2e^{-\chi_1 v})dv,$$

and

$$\begin{aligned}
\mathcal{I}_1 =& \lambda_1 \int_0^1 e^{-\lambda_1(1-v)}\left(\int_0^v \frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\varepsilon)d\varepsilon\right)dv - \int_0^1 \frac{(1-v)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(v)dv,\\
\mathcal{I}_2 =& \chi_1 \int_0^1 e^{-\chi_1(1-v)}\left(\int_0^v \frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\varepsilon)d\varepsilon\right)dv - \int_0^1 \frac{(1-v)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(v)dv,\\
\mathcal{I}_3 =& \int_0^\eta \frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\left(\int_0^v e^{-\lambda_1(v-\varepsilon)}\left(\int_0^\varepsilon \frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\mathfrak{m})d\mathfrak{m}\right)d\varepsilon\right)dv\\
&+ \int_0^\zeta \frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\left(\int_0^v e^{-\chi_1(v-\varepsilon)}\left(\int_0^\varepsilon \frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\mathfrak{m})d\mathfrak{m}\right)d\varepsilon\right)dv\\
&- \int_0^1 e^{-\lambda_1(1-v)}\left(\int_0^v \frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\varepsilon)d\varepsilon\right)dv,\\
\mathcal{I}_4 =& \int_0^\eta \frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\left(\int_0^v e^{-\lambda_1(v-\varepsilon)}\left(\int_0^\varepsilon \frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\mathfrak{m})d\mathfrak{m}\right)d\varepsilon\right)dv\\
&+ \int_0^\zeta \frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\left(\int_0^v e^{-\chi_1(v-\varepsilon)}\left(\int_0^\varepsilon \frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\mathfrak{m})d\mathfrak{m}\right)d\varepsilon\right)dv\\
&- \int_0^1 e^{-\chi_1(1-v)}\left(\int_0^v \frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\varepsilon)d\varepsilon\right)dv,\\
\Delta_1 =& (\Omega_2\Omega_5 - \Omega_1\Omega_6)(\Omega_4\Omega_{11} - \Omega_3\Omega_{12}) - (\Omega_2\Omega_9 - \Omega_1\Omega_{10})(\Omega_4\Omega_7 - \Omega_3\Omega_8),\\
\Delta =& \Omega_2\Omega_4\Delta_1.
\end{aligned} \tag{9}$$

**Lemma 1.** *If $\Delta_1 \neq 0$, then the solution $(\mathcal{G}, \mathcal{J}) \in (\mathcal{C}^4[0,1])^2$ of the BVP*

$$\begin{cases} ({}^c\mathcal{D}^{\varsigma_1} + \lambda_1\,{}^c\mathcal{D}^{\varsigma_1-1})\mathcal{G}(\tau) = \mathfrak{h}_1(\tau), \ \ \tau \in (0,1),\\ ({}^c\mathcal{D}^{\varsigma_2} + \chi_1\,{}^c\mathcal{D}^{\varsigma_2-1})\mathcal{J}(\tau) = \mathfrak{h}_2(\tau), \ \ \tau \in (0,1), \end{cases} \tag{10}$$

*supplemented with the boundary conditions*

$$\begin{cases} \mathcal{G}(0)=0,\ \ \mathcal{G}'(0)=0,\ \ \mathcal{G}'(1)=0, \\ \mathcal{G}(1)=\int_0^{\eta}\frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\mathcal{G}(v)dv+\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\mathcal{J}(v)dv, \\ \mathcal{J}(0)=0,\ \ \mathcal{J}'(0)=0,\ \ \mathcal{J}'(1)=0, \\ \mathcal{J}(1)=\int_0^{\eta}\frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\mathcal{G}(v)dv+\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\mathcal{J}(v)dv, \end{cases} \tag{11}$$

*is given by*

$$\begin{cases} \mathcal{G}(\tau)=\sum_{i=1}^{4}\Phi_i(\tau)\mathcal{I}_i+\int_0^{\tau}e^{-\lambda_1(\tau-v)}\left(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\varepsilon)d\varepsilon\right)dv, \\ \mathcal{J}(\tau)=\sum_{i=1}^{4}\Psi_i(\tau)\mathcal{I}_i+\int_0^{\tau}e^{-\chi_1(\tau-v)}\left(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\varepsilon)d\varepsilon\right)dv, \end{cases} \tag{12}$$

*where*

$$\begin{cases} \Phi_i(\tau)=\frac{1}{\lambda_1^2}(\Lambda_i)(\lambda_1\tau-1+e^{-\lambda_1\tau})+\frac{1}{\lambda_1^3}(\Theta_i)(\lambda_1^2\tau^2-2\lambda_1\tau+2-2e^{-\lambda_1\tau}),\ \ i=1,2,3,4. \\ \Psi_i(\tau)=\frac{1}{{\chi_1}^2}(\Xi_i)(\chi_1\tau-1+e^{-\chi_1\tau})+\frac{1}{{\chi_1}^3}(\mathrm{Y}_i)({\chi_1}^2\tau^2-2\chi_1\tau+2-2e^{-\chi_1\tau}),\ \ i=1,2,3,4. \end{cases} \tag{13}$$

**Proof.** The system denoted as (10) can be expressed in the following manner:

$$\begin{cases} ({}^c\mathcal{D}^{\varsigma_1}+\lambda_1{}^c\mathcal{D}^{\varsigma_1-1})\mathcal{G}(\tau)=\mathfrak{h}_1(\tau),\ \ \tau\in(0,1), \\ ({}^c\mathcal{D}^{\varsigma_2}+\chi_1{}^c\mathcal{D}^{\varsigma_2-1})\mathcal{J}(\tau)=\mathfrak{h}_2(\tau),\ \ \tau\in(0,1). \end{cases} \tag{14}$$

The general solutions of system (14) are

$$\begin{aligned} \mathcal{G}(\tau)=&\mathfrak{c}_0e^{-\lambda_1\tau}+\frac{\mathfrak{c}_1}{\lambda_1}(1-e^{-\lambda_1\tau})+\frac{\mathfrak{c}_2}{\lambda_1^2}(\lambda_1\tau-1+e^{-\lambda_1\tau})+\frac{\mathfrak{c}_3}{\lambda_1^3}(\lambda_1^2\tau^2-2\lambda_1\tau+2-2e^{-\lambda_1\tau}) \\ &+\int_0^{\tau}e^{-\lambda_1(\tau-v)}\left(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\varepsilon)d\varepsilon\right)dv, \\ \mathcal{J}(\tau)=&\mathfrak{d}_0e^{-\chi_1\tau}+\frac{\mathfrak{d}_1}{\chi_1}(1-e^{-\chi_1\tau})+\frac{\mathfrak{d}_2}{{\chi_1}^2}(\chi_1\tau-1+e^{-\chi_1\tau})+\frac{\mathfrak{d}_3}{{\chi_1}^3}({\chi_1}^2\tau^2-2\chi_1\tau+2-2e^{-\chi_1\tau}) \\ &+\int_0^{\tau}e^{-\chi_1(\tau-v)}\left(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\varepsilon)d\varepsilon\right)dv. \end{aligned}$$

Using the BCs $\mathcal{G}(0)=\mathcal{G}'(0)=0, \mathcal{J}(0)=\mathcal{J}'(0)=0$ from (11), we derive that $\mathfrak{c}_0=\mathfrak{c}_1=0$ and $\mathfrak{d}_0=\mathfrak{d}_1=0$. Hence, we infer

$$\begin{cases} \mathcal{G}(\tau)= & \frac{\mathfrak{c}_2}{\lambda_1^2}(\lambda_1\tau-1+e^{-\lambda_1\tau})+\frac{\mathfrak{c}_3}{\lambda_1^3}(\lambda_1^2\tau^2-2\lambda_1\tau+2-2e^{-\lambda_1\tau}) \\ & +\int_0^{\tau}e^{-\lambda_1(\tau-v)}\left(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\varepsilon)d\varepsilon\right)dv, \\ \mathcal{J}(\tau)= & \frac{\mathfrak{d}_2}{{\chi_1}^2}(\chi_1\tau-1+e^{-\chi_1\tau})+\frac{\mathfrak{d}_3}{{\chi_1}^3}({\chi_1}^2\tau^2-2\chi_1\tau+2-2e^{-\chi_1\tau}) \\ & +\int_0^{\tau}e^{-\chi_1(\tau-v)}\left(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\varepsilon)d\varepsilon\right)dv. \end{cases} \tag{15}$$

The following is determined by differentiating system (15):

$$\begin{cases} \mathcal{G}'(\tau) = & \frac{\mathfrak{c}_2}{\lambda_1^2}(\lambda_1 - \lambda_1 e^{-\lambda_1\tau}) + \frac{\mathfrak{c}_3}{\lambda_1^3}(2\lambda_1^2\tau - 2\lambda_1 + 2e^{-\lambda_1\tau}) \\ & -\lambda_1 \int_0^\tau e^{-\lambda_1(\tau-\upsilon)}\left(\int_0^\upsilon \frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\varepsilon)d\varepsilon\right)d\upsilon + \int_0^\tau \frac{(\tau-\upsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\upsilon)d\upsilon, \\ \mathcal{J}'(\tau) = & \frac{\mathfrak{d}_2}{{\chi_1}^2}(\chi_1 - \chi_1 e^{-\chi_1\tau}) + \frac{\mathfrak{d}_3}{{\chi_1}^3}(2{\chi_1}^2\tau - 2\chi_1 + 2\chi_1 e^{-\chi_1\tau}) \\ & -\chi_1 \int_0^\tau e^{-\chi_1(\tau-\upsilon)}\left(\int_0^\upsilon \frac{(\upsilon-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\varepsilon)d\varepsilon\right)d\upsilon + \int_0^\tau \frac{(\tau-\upsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\upsilon)d\upsilon. \end{cases} \tag{16}$$

We acquire the following expressions by demanding the constraints, namely $\mathcal{G}'(1) = \mathcal{J}'(1) = 0$, as

$$\begin{cases} \frac{\mathfrak{c}_2}{\lambda_1}(1 - e^{-\lambda_1}) + \frac{\mathfrak{c}_3}{\lambda_1^2}(2\lambda_1^2 - 2 + 2e^{-\lambda_1}) = & \lambda_1 \int_0^1 e^{-\lambda_1(1-\upsilon)}\left(\int_0^\upsilon \frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\varepsilon)d\varepsilon\right)d\upsilon \\ & -\int_0^1 \frac{(1-\upsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\upsilon)d\upsilon, \\ \frac{\mathfrak{d}_2}{\chi_1}(1 - e^{-\chi_1}) + \frac{\mathfrak{d}_3}{{\chi_1}^2}(2\chi_1 - 2 + 2\chi_1 e^{-\chi_1}) = & \chi_1 \int_0^1 e^{-\chi_1(1-\upsilon)}\left(\int_0^\upsilon \frac{(\upsilon-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\varepsilon)d\varepsilon\right)d\upsilon \\ & -\int_0^1 \frac{(\tau-\upsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\upsilon)d\upsilon. \end{cases} \tag{17}$$

Now, using the last boundary conditions from (2), namely $\mathcal{G}(1) = \int_0^\eta \frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\mathcal{G}(\upsilon)d\upsilon + \int_0^\zeta \frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\mathcal{J}(\upsilon)d\upsilon$ and $\mathcal{J}(1) = \int_0^\eta \frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\mathcal{G}(\upsilon)d\upsilon + \int_0^\zeta \frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\mathcal{J}(\upsilon)d\upsilon$, we deduce

$$\begin{aligned} & \mathfrak{c}_2\left[\frac{1}{\lambda_1^2}[(\lambda_1 - 1 + e^{-\lambda_1}) - \int_0^\eta \frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}(\lambda_1\upsilon - 1 + e^{-\lambda_1\upsilon})d\upsilon]\right] \\ & + \mathfrak{c}_3\left[\frac{1}{\lambda_1^3}[(\lambda_1^2 - 2\lambda_1 + 2 - 2e^{-\lambda_1}) - \int_0^\eta \frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}(\lambda_1^2\upsilon^2 - 2\lambda_1\upsilon + 2 - 2e^{\lambda_1\upsilon})d\upsilon]\right] \\ & - \mathfrak{d}_2\left[\frac{1}{{\chi_1}^2}\int_0^\zeta \frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}(\chi_1\upsilon - 1 + e^{-\chi_1\upsilon})d\upsilon\right] \\ & + \mathfrak{d}_3\left[\frac{1}{{\chi_1}^3}\int_0^\zeta \frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}({\chi_1}^2\upsilon^2 - 2\chi_1\upsilon + 2 - 2e^{-\chi_1\upsilon})d\upsilon\right], \\ = & \int_0^\eta \frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\left(\int_0^\upsilon e^{-\lambda_1(\upsilon-\varepsilon)}\left(\int_0^\varepsilon \frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\mathfrak{m})d\mathfrak{m}\right)d\varepsilon\right)d\upsilon \\ & + \int_0^\zeta \frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\left(\int_0^\upsilon e^{-\chi_1(\upsilon-\varepsilon)}\left(\int_0^\varepsilon \frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\mathfrak{m})d\mathfrak{m}\right)d\varepsilon\right)d\upsilon \\ & - \int_0^1 e^{-\lambda_1(1-\upsilon)}\left(\int_0^\upsilon \frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\varepsilon)d\varepsilon\right)d\upsilon, \end{aligned}$$

and

$$\begin{aligned} & -\mathfrak{c}_2\left[\frac{1}{\lambda_1^2}\int_0^\eta \frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}(\lambda_1\upsilon - 1 + e^{-\lambda_1\upsilon})d\upsilon\right] \\ & -\mathfrak{c}_3\left[\frac{1}{\lambda_1^3}\int_0^\eta \frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}(\lambda_1^2\upsilon^2 - 2\lambda_1\upsilon + 2 - 2e^{\lambda_1\upsilon})d\upsilon\right] \end{aligned} \tag{18}$$

$$
\begin{aligned}
&+\mathfrak{d}_2\left[\frac{1}{{\chi_1}^2}(\mu-1+e^{-\mu})-\frac{1}{{\chi_1}^2}\int_0^1(\chi_1 v-1+e^{-\chi_1 v})dv\right]\\
&+\mathfrak{d}_3\left[\frac{1}{{\chi_1}^3}({\chi_1}^2-2\chi_1+2-2e^{-\mu})-\frac{1}{{\chi_1}^3}\int_0^\zeta\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}({\chi_1}^2v^2-2\chi_1 v+2-2e^{-\chi_1 v})dv\right],\\
=&\int_0^\eta\frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\left(\int_0^v e^{-\lambda_1(v-\varepsilon)}\left(\int_0^\varepsilon\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathfrak{h}_1(\mathfrak{m})d\mathfrak{m}\right)d\varepsilon\right)dv\\
&+\int_0^\zeta\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\left(\int_0^v e^{-\chi_1(v-\varepsilon)}\left(\int_0^\varepsilon\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\mathfrak{m})d\mathfrak{m}\right)d\varepsilon\right)dv\\
&-\int_0^1 e^{-\chi_1(1-v)}\left(\int_0^v\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathfrak{h}_2(\varepsilon)d\varepsilon\right)dv.
\end{aligned}\tag{19}
$$

Therefore, by (8), (9), (17), and (18), we find the system in the unknowns $\mathfrak{c}_2$, $\mathfrak{c}_3$, $\mathfrak{d}_2$ and $\mathfrak{d}_3$:

$$
\begin{cases}
\Omega_1\mathfrak{c}_2+\Omega_2\mathfrak{c}_3=\mathcal{I}_1,\\
\Omega_3\mathfrak{d}_2+\Omega_4\mathfrak{d}_3=\mathcal{I}_2,\\
\Omega_5\mathfrak{c}_2+\Omega_6\mathfrak{c}_3-\Omega_7\mathfrak{d}_2-\Omega_8\mathfrak{c}_3=\mathcal{I}_3,\\
-\Omega_9\mathfrak{c}_2-\Omega_{10}\mathfrak{c}_3+\Omega_{11}\mathfrak{d}_2+\Omega_{12}\mathfrak{c}_3=\mathcal{I}_4.
\end{cases}\tag{20}
$$

By solving the first two equations of (20), we find $\mathfrak{c}_3=\frac{\mathcal{I}_1-\Omega_1\mathfrak{c}_2}{\Omega_2}$ and $\mathfrak{d}_3=\frac{\mathcal{I}_2-\Omega_3\mathfrak{d}_2}{\Omega_2}$, $(\Omega_2,\Omega_4>0)$. Plugging these values of $\mathfrak{c}_3$ and $\mathfrak{d}_3$ into the last two equations of (20), we arrive at the system that is given below, including the unknown parameters $\mathfrak{c}_2$ and $\mathfrak{d}_2$:

$$
\begin{cases}
\mathfrak{c}_2\Omega_4(\Omega_2\Omega_5-\Omega_1\Omega_6)-\mathfrak{d}_2\Omega_2(\Omega_4\Omega_7-\Omega_3\Omega_8)=\Omega_2\Omega_4\mathcal{I}_3-\Omega_4\Omega_6\mathcal{I}_1+\Omega_2\Omega_8\mathcal{I}_2,\\
-\mathfrak{c}_2\Omega_4(\Omega_4\Omega_9-\Omega_1\Omega_{10})+\mathfrak{d}_2\Omega_2(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})=\Omega_2\Omega_4\mathcal{I}_4+\Omega_4\Omega_{10}\mathcal{I}_1-\Omega_2\Omega_{12}\mathcal{I}_2.
\end{cases}\tag{21}
$$

The determinant of system (21) is denoted as $\Delta=\Omega_2\Omega_4\Delta_1$, with $\Delta_1$ being determined by (9). According to the assumption made in Lemma 1, $\Delta_1\neq 0$, thus resulting in $\Delta=0$. Consequently, the solution to system (21) is as follows:

$$
\begin{aligned}
\mathfrak{c}_2=&\frac{\Omega_2}{\Delta}\{\mathcal{I}_1[-\Omega_4\Omega_6(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})+\Omega_4\Omega_{10}(\Omega_4\Omega_7-\Omega_3\Omega_8)]\\
&+\mathcal{I}_2[-\Omega_2\Omega_8(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})+\Omega_2\Omega_{12}(\Omega_4\Omega_7-\Omega_3\Omega_8)]\\
&+\mathcal{I}_3\Omega_2\Omega_4(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})+\mathcal{I}_4\Omega_2\Omega_4(\Omega_4\Omega_7-\Omega_3\Omega_8)\},\\
&=\Lambda_1\mathcal{I}_1+\Lambda_2\mathcal{I}_2+\Lambda_3\mathcal{I}_3+\Lambda_4\mathcal{I}_4,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{d}_2=&\frac{\Omega_4}{\Delta}\{\mathcal{I}_1[-\Omega_4\Omega_{10}(\Omega_2\Omega_5-\Omega_1\Omega_6)-\Omega_4\Omega_6(\Omega_2\Omega_9-\Omega_1\Omega_{10})]\\
&+\mathcal{I}_2[-\Omega_2\Omega_{12}(\Omega_2\Omega_5-\Omega_1\Omega_6)+\Omega_2\Omega_8(\Omega_2\Omega_9-\Omega_1\Omega_{10})]\\
&+\mathcal{I}_3\Omega_2\Omega_4(\Omega_2\Omega_9-\Omega_1\Omega_{10})+\mathcal{I}_4\Omega_2\Omega_4(\Omega_2\Omega_5-\Omega_1\Omega_6)\},\\
&=\Xi_1\mathcal{I}_1+\Xi_2\mathcal{I}_2+\Xi_3\mathcal{I}_3+\Xi_4\mathcal{I}_4.
\end{aligned}
$$

Hence, we can determine the values of the constants $\mathfrak{c}_3$ and $\mathfrak{d}_3$ as

$$
\begin{aligned}
\mathfrak{c}_3=&\frac{1}{\Omega_2\Delta}\{\mathcal{I}_1\Omega_2\Omega_4[(\Omega_2\Omega_5-\Omega_1\Omega_6)(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})-(\Omega_2\Omega_9-\Omega_1\Omega_{10})(\Omega_4\Omega_7-\Omega_3\Omega_8)]\\
&-\mathcal{I}_1\Omega_1\Omega_2[\Omega_4\Omega_6(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})+\Omega_4\Omega_{10}(\Omega_4\Omega_7-\Omega_3\Omega_8)]\\
&-\mathcal{I}_2\Omega_1\Omega_2[\Omega_2\Omega_8(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})+\Omega_2\Omega_{12}(\Omega_4\Omega_7-\Omega_3\Omega_8)]\\
&-\mathcal{I}_3\Omega_1\Omega_2^2\Omega_4(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})+\mathcal{I}_4\Omega_1\Omega_2^2\Omega_{12}(\Omega_4\Omega_7-\Omega_3\Omega_8)\},\\
=&\Theta_1\mathcal{I}_1+\Theta_2\mathcal{I}_2+\Theta_3\mathcal{I}_3+\Theta_4\mathcal{I}_4,
\end{aligned}
$$

$$\begin{aligned}
\mathfrak{d}_3 =& \frac{1}{\Omega_4\Delta}\{-\mathcal{I}_1\Omega_3\Omega_4[\Omega_4\Omega_{10}(\Omega_2\Omega_5-\Omega_1\Omega_6)-\Omega_4\Omega_6(\Omega_2\Omega_9-\Omega_1\Omega_{10})]\\
&-\mathcal{I}_2\Omega_2\Omega_4[(\Omega_2\Omega_5-\Omega_1\Omega_6)(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})-(\Omega_2\Omega_9-\Omega_1\Omega_{10})(\Omega_4\Omega_7-\Omega_3\Omega_8)]\\
&-\mathcal{I}_2\Omega_3\Omega_4[-\Omega_2\Omega_{12}(\Omega_2\Omega_5-\Omega_1\Omega_6)+\Omega_2\Omega_8(\Omega_2\Omega_9-\Omega_1\Omega_{10})]\\
&-\mathcal{I}_3\Omega_2\Omega_4^2\Omega_3(\Omega_2\Omega_9-\Omega_1\Omega_{10})+\mathcal{I}_4\Omega_2\Omega_4^2\Omega_3(\Omega_2\Omega_5-\Omega_1\Omega_6)\},\\
=&\Upsilon_1\mathcal{I}_1+\Upsilon_2\mathcal{I}_2+\Upsilon_3\mathcal{I}_3+\Upsilon_4\mathcal{I}_4,
\end{aligned}$$

$$\begin{aligned}
&\Lambda_1=\frac{1}{\Delta_1}[-\Omega_6(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})+\Omega_{10}(\Omega_4\Omega_7-\Omega_3\Omega_8)],\\
&\Lambda_2=\frac{\Omega_2}{\Delta_1}(\Omega_8\Omega_{11}-\Omega_7\Omega_{12}),\Lambda_3=\frac{\Omega_2}{\Delta_1}(\Omega_4\Omega_{11}-\Omega_3\Omega_{12}),\ \ \Lambda_4=\frac{\Omega_2}{\Delta_1}(\Omega_4\Omega_7-\Omega_3\Omega_8),\\
&\Xi_1=\frac{\Omega_4}{\Delta_1}(\Omega_5\Omega_{10}-\Omega_6\Omega_9),\ \ \Xi_2=\frac{1}{\Delta_1}[-\Omega_{12}(\Omega_2\Omega_5-\Omega_1\Omega_6)+\Omega_8(\Omega_2\Omega_9-\Omega_1\Omega_{10})],\\
&\Xi_3=\frac{\Omega_4}{\Delta_1}(\Omega_2\Omega_9-\Omega_1\Omega_{10}),\ \ \Xi_4=\frac{\Omega_4}{\Delta_1}(\Omega_2\Omega_5-\Omega_1\Omega_6),\\
&\Theta_1=\frac{1}{\Delta_1}[-\Omega_5(\Omega_4\Omega_{11}-\Omega_3\Omega_{12})+\Omega_9(\Omega_4\Omega_7-\Omega_3\Omega_8)],\\
&\Theta_2=\frac{\Omega_1}{\Delta_1}(-\Omega_8\Omega_{11}+\Omega_7\Omega_{12}),\Theta_3=\frac{\Omega_1}{\Delta_1}(\Omega_3\Omega_{12}-\Omega_4\Omega_{11}),\ \ \Theta_4=\frac{\Omega_1}{\Delta_1}(\Omega_3\Omega_8-\Omega_4\Omega_7),\\
&\Upsilon_1=\frac{\Omega_3}{\Delta_1}(\Omega_6\Omega_9-\Omega_5\Omega_{10}),\ \ \Upsilon_2=\frac{1}{\Delta_1}[\Omega_{11}(\Omega_2\Omega_5-\Omega_1\Omega_6)+\Omega_7(\Omega_2\Omega_9-\Omega_1\Omega_{10})],\\
&\Upsilon_3=\frac{\Omega_3}{\Delta_1}(\Omega_1\Omega_{10}-\Omega_2\Omega_9),\ \ \Upsilon_4=\frac{\Omega_3}{\Delta_1}(\Omega_1\Omega_6-\Omega_2\Omega_5),
\end{aligned}\tag{22}$$

where $\Lambda_1, \Xi_i, \Theta_i, \Upsilon_i, i = 1, \cdots, 4$ are given by (22). Upon substituting the aforementioned constants $\mathfrak{c}_2$, $\mathfrak{d}_2$, $\mathfrak{c}_3$, and $\mathfrak{d}_3$ into System (15), we can determine the solution to Problems (2)–(7). Conversely, the reverse of this result can be obtained through straightforward computations. □

Next, we introduce the Leray–Schauder alternative, which will be employed in the proof of the existence of solutions for Problems (1) and (2).

**Theorem 1.** *Let $\mathcal{E}$ be a Banach space and $\Psi : \mathcal{E} \to \mathcal{E}$ be a completely continuous operator. Let $\varphi_1 = \{\mathfrak{l} \in \mathcal{E}, \mathfrak{l} = \nu\Psi(\mathfrak{l})$ for some $0 < \nu < 1\}$. Then, either the set $\varphi_1$ is unbounded or $\Psi$ has at least one fixed point.*

## 4. Main Results

We consider the space $\mathcal{U} = \{\mathcal{G} \in \mathcal{C}[0,1], {}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G} \in \mathcal{C}[0,1]\}$ and $\mathcal{Q} = \{\mathcal{J} \in \mathcal{C}[0,1], {}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J} \in \mathcal{C}[0,1]\}$ to be equipped, respectively, with the norms $||\mathcal{G}||_{\mathcal{U}} = ||\mathcal{G}|| + ||{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}||$ and $||\mathcal{J}||_{\mathcal{Q}} = ||\mathcal{J}|| + ||{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}||$, where $||\cdot||$ is the sup norm; that is, $||\mathfrak{w}|| = \sup_{\tau\in[0,1]} |\mathfrak{w}(\tau)|$ for $\mathfrak{w} \in \mathcal{C}[0,1]$. The spaces $(\mathcal{U}, ||\cdot||_{\mathcal{U}})$ and $(\mathcal{Q}, ||\cdot||_{\mathcal{Q}})$ both constitute Banach spaces, and when we consider the product space $\mathcal{U} \times \mathcal{Q}$ equipped with the norm $||(\mathcal{G}, \mathcal{J})||_{\mathcal{U}\times\mathcal{Q}} = ||\mathcal{G}||_{\mathcal{U}} + ||\mathcal{J}||_{\mathcal{Q}}$, it also forms a Banach space.

Implementing Lemma (1), we introduce the operator $\mathcal{T} : \mathcal{U} \times \mathcal{Q} \to \mathcal{U} \times \mathcal{Q}$ defined by $\mathcal{T}(\mathcal{G}, \mathcal{J}) = (\mathcal{T}_1(\mathcal{G}, \mathcal{J}), \mathcal{T}_2(\mathcal{G}, \mathcal{J}))$ for $(\mathcal{G}, \mathcal{J}) \in \mathcal{U} \times \mathcal{Q}$, where the operators $\mathcal{T}_1 : \mathcal{U} \times \mathcal{Q} \to \mathcal{U}$ and $\mathcal{T}_2 : \mathcal{U} \times \mathcal{Q} \to \mathcal{Q}$ are given by

$$\begin{aligned}
&\mathcal{T}_1(\mathcal{G}, \mathcal{J})(\tau)\\
=&\Phi_1(\tau)\left[\lambda_1\int_0^1 e^{-\lambda_1(1-\upsilon)}\left(\int_0^\upsilon \frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\varepsilon)d\varepsilon\right)d\upsilon\right.\\
&\left.-\int_0^1\frac{(1-\upsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\upsilon)d\upsilon\right]\\
&+\Phi_2(\tau)\left[\chi_1\int_0^1 e^{-\chi_1(1-\upsilon)}\left(\int_0^\upsilon \frac{(\upsilon-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\varepsilon)d\varepsilon\right)d\upsilon\right.\\
&\left.-\int_0^1\frac{(1-\upsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\upsilon)d\upsilon\right]
\end{aligned}$$

$$
\begin{aligned}
&+\Phi_3(\tau)\Bigg[\int_0^{\eta}\frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^{\upsilon}e^{-\lambda_1(\upsilon-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\\
&\quad\times\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\mathfrak{m})d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&+\int_0^{\zeta}\frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{\upsilon}e^{-\chi_1(\upsilon-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\mathfrak{m})d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&-\int_0^{1}e^{-\lambda_1(1-\upsilon)}\Bigg(\int_0^{\upsilon}\frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\varepsilon)d\varepsilon\Bigg)d\upsilon\Bigg]\\
&+\Phi_4(\tau)\Bigg[\int_0^{\eta}\frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^{\upsilon}e^{-\lambda_1(\upsilon-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\\
&\quad\times\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\mathfrak{m})d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&+\int_0^{\zeta}\frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{\upsilon}e^{-\chi_1(\upsilon-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\mathfrak{m})d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&-\int_0^{1}e^{-\chi_1(1-\upsilon)}\Bigg(\int_0^{\upsilon}\frac{(\upsilon-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\varepsilon)d\varepsilon\Bigg)d\upsilon\Bigg]\\
&+\int_0^{\tau}e^{-\lambda_1(\tau-\upsilon)}\Bigg(\int_0^{\upsilon}\frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\varepsilon)d\varepsilon\Bigg)d\upsilon,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&\mathcal{T}_2(\mathcal{G},\mathcal{J})(\tau)\\
=&\Psi_1(\tau)\Bigg[\lambda_1\int_0^{1}e^{-\lambda_1(1-\upsilon)}\Bigg(\int_0^{\upsilon}\frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\varepsilon)d\varepsilon\Bigg)d\upsilon\\
&-\int_0^{1}\frac{(1-\upsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\upsilon)d\upsilon\Bigg]\\
&+\Psi_2(\tau)\Bigg[\chi_1\int_0^{1}e^{-\chi_1(1-\upsilon)}\Bigg(\int_0^{\upsilon}\frac{(\upsilon-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\varepsilon)d\varepsilon\Bigg)d\upsilon\\
&-\int_0^{1}\frac{(1-\upsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\upsilon)d\upsilon\Bigg]\\
&+\Psi_3(\tau)\Bigg[\int_0^{\eta}\frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^{\upsilon}e^{-\lambda_1(\upsilon-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\\
&\quad\times\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\mathfrak{m})d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&+\int_0^{\zeta}\frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{\upsilon}e^{-\chi_1(\upsilon-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\mathfrak{m})d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&-\int_0^{1}e^{-\lambda_1(1-\upsilon)}\Bigg(\int_0^{\upsilon}\frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\varepsilon)d\varepsilon\Bigg)d\upsilon\Bigg]\\
&+\Psi_4(\tau)\Bigg[\int_0^{\eta}\frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^{\upsilon}e^{-\lambda_1(\upsilon-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\\
&\quad\times\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))(\mathfrak{m})d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&+\int_0^{\zeta}\frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{\upsilon}e^{-\chi_1(\upsilon-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\mathfrak{m})d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&-\int_0^{1}e^{-\chi_1(1-\upsilon)}\Bigg(\int_0^{\upsilon}\frac{(\upsilon-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\varepsilon)d\varepsilon\Bigg)d\upsilon\Bigg]\\
&+\int_0^{\tau}e^{-\chi_1(\tau-\upsilon)}\Bigg(\int_0^{\upsilon}\frac{(\upsilon-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))(\varepsilon)d\varepsilon\Bigg)d\upsilon.
\end{aligned}
$$

The pair $(\mathcal{G}, \mathcal{J})$ constitutes a solution to Problems (1) and (2) when it serves as a fixed point of the operator $\mathcal{T}$. Let us now outline the assumptions that have been employed in this section.

1. $[\mathcal{H}_1]$ The functions $\varphi_1, \phi_1$, defined on the domain $[0,1] \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$, are continuous. Additionally, there exist real $\mathfrak{a}_i, \mathfrak{b}_i \geq 0, i = 1,2,3,4$ and $\mathfrak{a}_0 > 0, \mathfrak{b}_0 > 0$ such that

$$|\varphi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4)| \leq \mathfrak{a}_0 + \mathfrak{a}_1|\mathfrak{l}_1| + \mathfrak{a}_2|\mathfrak{l}_2| + \mathfrak{a}_3|\mathfrak{l}_3| + \mathfrak{a}_4|\mathfrak{l}_4|,$$
$$|\phi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4)| \leq \mathfrak{b}_0 + \mathfrak{b}_1|\mathfrak{l}_1| + \mathfrak{b}_2|\mathfrak{l}_2| + \mathfrak{b}_3|\mathfrak{l}_3| + \mathfrak{b}_4|\mathfrak{l}_4|,$$

for all $\tau \in [0,1]$ and $\mathfrak{l}_i \in \mathcal{R}_e, i = 1,2,3,4$.

2. $[\mathcal{H}_2]$ The functions $\varphi_1, \phi_1 : [0,1] \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$ are continuous and non-negative constants $\mathfrak{W}_0 > 0$, $\mathfrak{K}_0 > 0$ such that

$$|\varphi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4) - \varphi_1(\tau, \mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \mathfrak{n}_4)| \leq \mathfrak{W}_0(|\mathfrak{l}_1 - \mathfrak{n}_1| + |\mathfrak{l}_2 - \mathfrak{n}_2| + |\mathfrak{l}_3 - \mathfrak{n}_3| + |\mathfrak{l}_4 - \mathfrak{n}_4|),$$
$$|\phi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4) - \phi_1(\tau, \mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \mathfrak{n}_4)| \leq \mathfrak{K}_0(|\mathfrak{l}_1 - \mathfrak{n}_1| + |\mathfrak{l}_2 - \mathfrak{n}_2| + |\mathfrak{l}_3 - \mathfrak{n}_3| + |\mathfrak{l}_4 - \mathfrak{n}_4|),$$

for all $\tau \in [0,1]$ and $\mathfrak{l}_i, \mathfrak{n}_i \in \mathcal{R}_e$, $i = 1,2,3,4$.

$$\begin{cases} \Phi_i'(\tau) = \frac{1}{\lambda_1^2}(\Lambda_i)(1 - \lambda_1 e^{-\lambda_1}) + \frac{1}{\lambda_1^3}(\Theta_i)(2\lambda_1 - 2 + 2e^{-\lambda_1}), \ \ i = 1,2,3,4. \\ \Psi_i'(\tau) = \frac{1}{\lambda_1^2}(\Xi_i)(1 - \chi_1 e^{-\chi_1}) + \frac{1}{\lambda_1^3}(\Upsilon_i)(2\chi_1 - 2 + 2e^{-\chi_1}), \ \ i = 1,2,3,4. \end{cases} \tag{24}$$

We $\hat{\Phi}_i = \sup_{\tau \in [0,1]} |\Phi_i(\tau)|$, $\hat{\Psi}_i = \sup_{\tau \in [0,1]} |\Psi_i(\tau)|$ for $i = 1,2,3,4$.

$$\begin{aligned}
\mathcal{U}_1 =& \hat{\Phi}_1 \frac{(2 - e^{-\lambda_1})}{\Gamma(\varsigma_1)} + \hat{\Phi}_3 \left[ \frac{(1 - e^{-\lambda_1})}{\lambda_1 \Gamma(\varsigma_1)} + \frac{\eta^{\varsigma_1 + \mathfrak{v} - 1}}{\lambda_1^2 \Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1 \eta + e^{-\lambda_1 \eta} - 1) \right] \\
& + \hat{\Phi}_4 \left( \frac{\eta^{\varsigma_1 + \mathfrak{v} - 1}}{\lambda_1^2 \Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1 \eta + e^{-\lambda_1 \eta} - 1) \right) + \frac{(1 - e^{-\lambda_1})}{\lambda_1 \Gamma(\varsigma_1)}, \\
\mathcal{U}_2 =& \hat{\Phi}_2 \frac{(2 - e^{-\chi_1})}{\Gamma(\varsigma_2)} + \hat{\Phi}_3 \left( \frac{\zeta^{\varsigma_2 + \mathfrak{z} - 1}}{{\chi_1}^2 \Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1 \zeta + e^{-\chi_1 \zeta} - 1) \right) \\
& + \hat{\Phi}_4 \left[ \frac{(1 - e^{-\chi_1})}{\chi_1 \Gamma(\varsigma_2)} + \frac{\zeta^{\varsigma_2 + \mathfrak{z} - 1}}{{\chi_1}^2 \Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1 \zeta + e^{-\chi_1 \zeta} - 1) \right], \\
\mathcal{A}_1 =& \hat{\Psi}_1 \frac{(2 - e^{-\lambda_1})}{\Gamma(\varsigma_1)} + \hat{\Psi}_3 \left[ \frac{(1 - e^{-\lambda_1})}{\lambda_1 \Gamma(\varsigma_1)} + \frac{\eta^{\varsigma_1 + \mathfrak{v} - 1}}{\lambda_1^2 \Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1 \eta + e^{-\lambda_1 \eta} - 1) \right] \\
& + \hat{\Psi}_4 \left( \frac{\eta^{\varsigma_1 + \mathfrak{v} - 1}}{\lambda_1^2 \Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1 \eta + e^{-\lambda_1 \eta} - 1) \right), \\
\mathcal{A}_2 =& \hat{\Psi}_2 \frac{(2 - e^{-\chi_1})}{\Gamma(\varsigma_2)} + \hat{\Psi}_3 \left( \frac{\zeta^{\varsigma_2 + \mathfrak{z} - 1}}{{\chi_1}^2 \Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1 \zeta + e^{-\chi_1 \zeta} - 1) \right) \\
& + \hat{\Psi}_4 \left[ \frac{(1 - e^{-\chi_1})}{\chi_1 \Gamma(\varsigma_2)} + \frac{\zeta^{\varsigma_2 + \mathfrak{z} - 1}}{{\chi_1}^2 \Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1 \zeta + e^{-\chi_1 \zeta} - 1) \right] + \frac{(1 - e^{-\chi_1})}{\chi_1 \Gamma(\varsigma_2)}, \\
\mathcal{U}_1^* =& \hat{\Phi}_1' \frac{(2 - e^{-\lambda_1})}{\Gamma(\varsigma_1)} + \hat{\Phi}_3' \left[ \frac{(1 - e^{-\lambda_1})}{\lambda_1 \Gamma(\varsigma_1)} + \frac{\eta^{\varsigma_1 + \mathfrak{v} - 1}}{\lambda_1^2 \Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1 \eta + e^{-\lambda_1 \eta} - 1) \right] \\
& + \hat{\Phi}_4' \left( \frac{\eta^{\varsigma_1 + \mathfrak{v} - 1}}{\lambda_1^2 \Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1 \eta + e^{-\lambda_1 \eta} - 1) \right) + \frac{(2 - e^{-\lambda_1})}{\Gamma(\varsigma_1)}, \\
\mathcal{U}_2^* =& \hat{\Phi}_2' \frac{(2 - e^{-\chi_1})}{\Gamma(\varsigma_2)} + \hat{\Phi}_3' \left( \frac{\zeta^{\varsigma_2 + \mathfrak{z} - 1}}{{\chi_1}^2 \Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1 \zeta + e^{-\chi_1 \zeta} - 1) \right)
\end{aligned} \tag{25}$$

$$
\begin{aligned}
&+\hat{\Phi}_4'\left[\frac{(1-e^{-\chi_1})}{\chi_1\Gamma(\varsigma_2)}+\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\right],\\
\mathcal{A}_1^* =&\hat{\Psi}_1'\frac{(2-e^{-\lambda_1})}{\Gamma(\varsigma_1)}+\hat{\Psi}_3'\left[\frac{(1-e^{-\lambda_1})}{\lambda_1\Gamma(\varsigma_1)}+\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\right]\\
&+\hat{\Psi}_4'\left(\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\right),\\
\mathcal{A}_2^* =&\hat{\Psi}_2'\frac{(2-e^{-\chi_1})}{\Gamma(\varsigma_2)}+\hat{\Psi}_3'\left(\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\right)\\
&+\hat{\Psi}_4'\left[\frac{(1-e^{-\chi_1})}{\chi_1\Gamma(\varsigma_2)}+\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\right]+\frac{(2-e^{-\chi_1})}{\Gamma(\varsigma_2)},\\
\mathcal{N}_1 =&\mathcal{U}_1+\mathcal{A}_1+\frac{\mathcal{U}_1^*}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{A}_1^*}{\Gamma(2-\mathfrak{p}_1)},\\
\mathcal{N}_2 =&\mathcal{U}_2+\mathcal{A}_2+\frac{\mathcal{U}_2^*}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{A}_2^*}{\Gamma(2-\mathfrak{p}_1)},\\
\mathcal{N}_3 =&\mathfrak{a}_0\mathcal{N}_1+\mathfrak{b}_0\mathcal{N}_2,\\
\mathcal{N}_4 =&\mathfrak{a}_1\mathcal{N}_1+\left(\mathfrak{b}_1+\mathfrak{b}_2+\frac{\mathfrak{b}_3}{\Gamma(\mathfrak{q}_2+1)}\right)\mathcal{N}_2,\\
\mathcal{N}_5 =&\left(\mathfrak{a}_2+\mathfrak{a}_3+\frac{\mathfrak{a}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\mathcal{N}_1+\mathfrak{b}_4\mathcal{N}_2,\\
\varrho_1 =&1+\frac{1}{\Gamma(1+\mathfrak{q}_1)},\varrho_2=1+\frac{1}{\Gamma(1+\mathfrak{q}_2)}.
\end{aligned}
$$

**Theorem 2.** *Assume that $\mathcal{H}_1$ holds. If*

$$\max\{\mathcal{N}_4,\mathcal{N}_5\}<1, \tag{26}$$

*then the BVPs (1) and (2) have at least one solution $(\mathcal{G}(\tau),\mathcal{J}(\tau))$, $\tau\in[0,1]$.*

**Proof.** We begin by establishing that the operator $\mathcal{T}:\mathcal{U}\times\mathcal{Q}\to\mathcal{U}\times\mathcal{Q}$ is a completely continuous (c.c.) operator. The continuity of the functions $\varphi_1$ and $\phi_1$ allows us to infer that the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous as well. Consequently, $\mathcal{T}$ emerges as a continuous operator.

We proceed to demonstrate that $\mathcal{T}$ is uniformly bounded. Suppose we have a bounded set $\Pi\subset\mathcal{U}\times\mathcal{Q}$. In this case, there exist positive constants $\mathcal{L}_1$ and $\mathcal{L}_2$ such that

$$
\begin{aligned}
|\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))|\le\mathcal{L}_1,\\
|\phi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau))|\le\mathcal{L}_2.
\end{aligned}
$$

for all $(\mathcal{G},\mathcal{J})\in\Pi$ and $\tau\in[0,1]$.

Then, for any $(\mathcal{G},\mathcal{J})\in\Pi$ and $\tau\in[0,1]$, we obtain

$$
\begin{aligned}
&|\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau)|\\
=&|\Phi_1(\tau)|\left|\left[\lambda_1\int_0^1 e^{-\lambda_1(1-v)}\left(\int_0^v\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon))|d\varepsilon\right)dv\right.\right.\\
&\left.\left.+\int_0^1\frac{(1-v)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(v,\mathcal{G}(v),\mathcal{J}(v),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(v),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(v))|dv\right]\right.\\
&+|\Phi_2(\tau)|\left[\chi_1\int_0^1 e^{-\chi_1(1-v)}\left(\int_0^v\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}\varepsilon)|d\varepsilon\right)dv\right.\\
&\left.+\int_0^1\frac{(1-v)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(v,\mathcal{G}(v),\mathcal{J}(v),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(v),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(v))|dv\right]
\end{aligned}
$$

$$
\begin{aligned}
&+|\Phi_3(\tau)|\Bigg[\int_0^{\eta}\frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^{v}e^{-\lambda_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\\
&\quad\times|\varphi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\\
&+\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{v}e^{-\chi_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\\
&+\int_0^{1}e^{-\lambda_1(1-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon))|d\varepsilon\Bigg)dv\Bigg]\\
&+|\Phi_4(\tau)|\Bigg[\int_0^{\eta}\frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^{v}e^{-\lambda_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\\
&\quad|\varphi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\\
&+\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{v}e^{-\chi_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\\
&+\int_0^{1}e^{-\chi_1(1-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\varepsilon))|d\varepsilon\Bigg)dv\Bigg]\Bigg|\\
&+\int_0^{v}e^{-\lambda_1(v-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(v,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon))|d\varepsilon\Bigg)dv,\\
\leq&\mathcal{L}_1\Bigg\{\hat{\Phi}_1\frac{(2-e^{-\lambda_1})}{\Gamma(\varsigma_1)}+\hat{\Phi}_3\Bigg[\frac{(1-e^{-\lambda_1})}{\lambda_1\Gamma(\varsigma_1)}+\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\Bigg]\\
&+\hat{\Phi}_4\Bigg(\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\Bigg)\Bigg\}\\
&+\mathcal{L}_2\Bigg\{\hat{\Phi}_2\frac{(2-e^{-\chi_1})}{\Gamma(\varsigma_2)}+\hat{\Phi}_3\Bigg(\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\Bigg)\\
&+\hat{\Phi}_4\Bigg[\frac{(1-e^{-\chi_1})}{\chi_1\Gamma(\varsigma_2)}+\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\Bigg]\Bigg\}\\
&+\mathcal{L}_1\frac{(1-e^{-\lambda_1})}{\lambda_1\Gamma(\varsigma_1)}=\mathcal{L}_1\mathcal{U}_1+\mathcal{L}_2\mathcal{U}_2.
\end{aligned}
\tag{27}
$$

Then, $||\mathcal{T}_1(\mathcal{G},\mathcal{J})||\leq\mathcal{L}_1\mathcal{U}_1+\mathcal{L}_2\mathcal{U}_2$, for all $(\mathcal{G},\mathcal{J})\in\Pi$.
By examining the definition of $\mathcal{T}_1(\mathcal{G},\mathcal{J})$, we can observe that

$$
\begin{aligned}
&|\mathcal{T}_1'(\mathcal{G},\mathcal{J})(\tau)|\\
=&|\Phi_1'(\tau)|\Bigg|\Bigg[\lambda_1\int_0^{1}e^{-\lambda_1(1-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon))|d\varepsilon\Bigg)dv\\
&+\int_0^{1}\frac{(1-v)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(v,\mathcal{G}(v),\mathcal{J}(v),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(v),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(v))|dv\Bigg]\\
&+|\Phi_2'(\tau)|\Bigg[\chi_1\int_0^{1}e^{-\chi_1(1-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}\varepsilon)|d\varepsilon\Bigg)dv\\
&+\int_0^{1}\frac{(1-v)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(v,\mathcal{G}(v),\mathcal{J}(v),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(v),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(v))|dv\Bigg]\\
&+|\Phi_3'(\tau)|\Bigg[\int_0^{\eta}\frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^{v}e^{-\lambda_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}
\end{aligned}
$$

$$
\begin{aligned}
&|\varphi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\\
&+\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{v}e^{-\chi_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\\
&+\int_0^{1}e^{-\lambda_1(1-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon))|d\varepsilon\Bigg)dv\Bigg] \qquad (28)\\
&+|\Phi_4'(\tau)|\Bigg[\int_0^{\eta}\frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^{v}e^{-\lambda_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\\
&\quad|\varphi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\\
&+\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{v}e^{-\chi_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\\
&+\int_0^{1}e^{-\chi_1(1-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\varepsilon))|d\varepsilon\Bigg)dv\Bigg]\Bigg|\\
&+\int_0^{v}e^{-\lambda_1(v-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(v,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon))|d\varepsilon\Bigg)dv,\\
\leq&\mathcal{L}_1\Bigg\{\hat{\Phi}_1'\frac{(2-e^{-\lambda_1})}{\Gamma(\varsigma_1)}+\hat{\Phi}_3'\Bigg[\frac{(1-e^{-\lambda_1})}{\lambda_1\Gamma(\varsigma_1)}+\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\Bigg]\\
&+\hat{\Phi}_4'\Bigg(\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\Bigg)\Bigg\}\\
&+\mathcal{L}_2\Bigg\{\hat{\Phi}_2'\frac{(2-e^{-\chi_1})}{\Gamma(\varsigma_2)}+\hat{\Phi}_3'\Bigg(\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\Bigg)\\
&+\hat{\Phi}_4'\Bigg[\frac{(1-e^{-\chi_1})}{\chi_1\Gamma(\varsigma_2)}+\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\Bigg]\Bigg\}\\
&+\mathcal{L}_1\frac{(1-e^{-\lambda_1})}{\lambda_1\Gamma(\varsigma_1)}=\mathcal{L}_1\mathcal{U}_1^{*}+\mathcal{L}_2\mathcal{U}_2^{*}.
\end{aligned}
$$

By considering the definition of the Caputo fractional derivative of order $\mathfrak{p}_2\in(0,1)$, we can infer that

$$
\begin{aligned}
|{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau)|&\leq\int_0^{\tau}\frac{(\tau-v)^{-\mathfrak{p}_2}}{\Gamma(1-\mathfrak{p}_2)}|\mathcal{T}_1'(\mathcal{G},\mathcal{J})|dv\leq(\mathcal{L}_1\mathcal{U}_1^{*}+\mathcal{L}_2\mathcal{U}_2^{*})\int_0^{\tau}\frac{(\tau-v)^{-\mathfrak{p}_2}}{\Gamma(1-\mathfrak{p}_2)}dv,\\
&\leq\frac{(\mathcal{L}_1\mathcal{U}_1^{*}+\mathcal{L}_2\mathcal{U}_2^{*})}{\Gamma(2-\mathfrak{p}_2)},\ \ \forall\,\tau\in[0,1],
\end{aligned}
$$

from which we obtain

$$
||{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau)||\leq\frac{(\mathcal{L}_1\mathcal{U}_1^{*}+\mathcal{L}_2\mathcal{U}_2^{*})}{\Gamma(2-\mathfrak{p}_2)},\ \ \forall\,\tau\in[0,1].
$$

Hence, we conclude

$$
\begin{aligned}
||\mathcal{T}_1(\mathcal{G},\mathcal{J})||_{\mathcal{U}}=&||\mathcal{T}_1(\mathcal{G},\mathcal{J})||+||{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G},\mathcal{J})||,\\
\leq&\mathcal{L}_1\mathcal{U}_1+\mathcal{L}_2\mathcal{U}_2+\frac{(\mathcal{L}_1\mathcal{U}_1^{*}+\mathcal{L}_2\mathcal{U}_2^{*})}{\Gamma(2-\mathfrak{p}_2)}. \qquad (29)
\end{aligned}
$$

In the same way, we obtain

$$\begin{aligned}|\mathcal{Y}_2(\mathcal{G},\mathcal{J})(\tau)| \leq & \mathcal{A}_1\Bigg\{\hat{\Psi}_1\frac{(2-e^{-\lambda_1})}{\Gamma(\varsigma_1)}+\hat{\Psi}_3\Bigg[\frac{(1-e^{-\lambda_1})}{\lambda_1\Gamma(\varsigma_1)}+\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\Bigg]\\ &+\hat{\Psi}_4\Bigg(\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\Bigg)\Bigg\}\\ &+\mathcal{A}_2\Bigg\{\hat{\Psi}_2\frac{(2-e^{-\chi_1})}{\Gamma(\varsigma_2)}+\hat{\Psi}_3\Bigg(\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\Bigg)\\ &+\hat{\Psi}_4\Bigg[\frac{(1-e^{-\chi_1})}{\chi_1\Gamma(\varsigma_2)}+\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\Bigg]\Bigg\}\\ &+\mathcal{A}_2\frac{(1-e^{-\chi_1})}{\chi_1\Gamma(\varsigma_2)}=\mathcal{L}_1\mathcal{A}_1+\mathcal{L}_2\mathcal{A}_2,\end{aligned}$$

$$\begin{aligned}|\mathcal{Y}_2'(\mathcal{G},\mathcal{J})(\tau)| \leq & \mathcal{A}_1\Bigg\{\hat{\Psi}_1'\frac{(2-e^{-\lambda_1})}{\Gamma(\varsigma_1)}+\hat{\Psi}_3'\Bigg[\frac{(1-e^{-\lambda_1})}{\lambda_1\Gamma(\varsigma_1)}+\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\Bigg]\\ &+\hat{\Psi}_4'\Bigg(\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\Bigg)\Bigg\}\\ &+\mathcal{A}_2\Bigg\{\hat{\Psi}_2'\frac{(2-e^{-\chi_1})}{\Gamma(\varsigma_2)}+\hat{\Psi}_3'\Bigg(\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\Bigg)\\ &+\hat{\Psi}_4'\Bigg[\frac{(1-e^{-\chi_1})}{\chi_1\Gamma(\varsigma_2)}+\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\Bigg]\Bigg\}\\ &+\mathcal{A}_2\frac{(1-e^{-\chi_1})}{\chi_1\Gamma(\varsigma_2)}=\mathcal{L}_1\mathcal{A}_1^*+\mathcal{L}_2\mathcal{A}_2^*,\end{aligned}$$

$$|{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{T}_2(\mathcal{G},\mathcal{J})(\tau)| \leq \frac{(\mathcal{L}_1\mathcal{A}_1^*+\mathcal{L}_2\mathcal{A}_2^*)}{\Gamma(2-\mathfrak{p}_1)},\ \ \forall\,\tau\in[0,1].$$

Therefore, we conclude

$$\begin{aligned}||\mathcal{T}_2(\mathcal{G},\mathcal{J})||_{\mathcal{Q}} =& ||\mathcal{T}_2(\mathcal{G},\mathcal{J})|| + ||{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{T}_1(\mathcal{G},\mathcal{J})||,\\ \leq& \mathcal{L}_1\mathcal{A}_1+\mathcal{L}_2\mathcal{A}_2+\frac{(\mathcal{L}_1\mathcal{A}_1^*+\mathcal{L}_2\mathcal{A}_2^*)}{\Gamma(2-\mathfrak{p}_1)}.\end{aligned} \tag{30}$$

Based on Inequalities (29) and (30), it can be concluded that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are uniformly bounded. This, in turn, signifies that the operator $\mathcal{T}$ is uniformly bounded. Next, we will demonstrate that $\mathcal{T}$ is equi-continuous. Suppose we have $\tau_1,\tau_2\in[0,1]$, with $\tau_1<\tau_2$. In this case, we can establish

$$\begin{aligned}&|\mathcal{T}_2(\mathcal{G},\mathcal{J})(\tau_2)-\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau_1)|\\ \leq&|\Phi_1(\tau_2)-\Phi_1(\tau_1)|\Bigg|\Bigg[\lambda_1\int_0^1 e^{-\lambda_1(1-\upsilon)}\Bigg(\int_0^\upsilon\frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon))|d\varepsilon\Bigg)d\upsilon\\ &+\int_0^1\frac{(1-\upsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\upsilon,\mathcal{G}(\upsilon),\mathcal{J}(\upsilon),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\upsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\upsilon))|d\upsilon\Bigg]\Bigg|\\ &+|\Phi_2(\tau_2)-\Phi_2(\tau_1)|\Bigg|\Bigg[\chi_1\int_0^1 e^{-\chi_1(1-\upsilon)}\Bigg(\int_0^\upsilon\frac{(\upsilon-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}\varepsilon)|d\varepsilon\Bigg)d\upsilon\\ &+\int_0^1\frac{(1-\upsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\upsilon,\mathcal{G}(\upsilon),\mathcal{J}(\upsilon),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\upsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\upsilon))|d\upsilon\Bigg]\Bigg|\\ &+|\Phi_3(\tau_2)-\Phi_3(\tau_1)|\Bigg[\Bigg|\int_0^\eta\frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^\upsilon e^{-\lambda_1(\upsilon-\varepsilon)}\Bigg(\int_0^\varepsilon\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\end{aligned}$$

$$\times |\varphi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\Bigg|$$
$$+\left|\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\left(\int_0^{v}e^{-\chi_1(v-\varepsilon)}\left(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\mathfrak{m}))|d\mathfrak{m}\right)d\varepsilon\right)dv\right|$$
$$+\left|\int_0^{1}e^{-\lambda_1(1-v)}\left(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon))|d\varepsilon\right)dv\right|\Bigg]$$
$$+|\Phi_4(\tau_2)-\Phi_4(\tau_1)|\Bigg[\left|\int_0^{\eta}\frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\left(\int_0^{v}e^{-\lambda_1(v-\varepsilon)}\left(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\right.\right.\right.$$
$$\times |\varphi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\Bigg|$$
$$+\left|\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\left(\int_0^{v}e^{-\chi_1(v-\varepsilon)}\left(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\mathfrak{m},\mathcal{G}(\mathfrak{m}),\mathcal{J}(\mathfrak{m}),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\mathfrak{m}))|d\mathfrak{m}\right)d\varepsilon\right)dv\right|$$
$$+\left|\int_0^{1}e^{-\chi_1(1-v)}\left(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\varepsilon,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\varepsilon))|d\varepsilon\right)dv\right|\Bigg]$$
$$+\left|\int_0^{\tau}e^{-\lambda_1(\tau-v)}\left(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(v,\mathcal{G}(\varepsilon),\mathcal{J}(\varepsilon),{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon))|d\varepsilon\right)dv\right|,$$

and

$$\leq\mathcal{L}_1\frac{1}{\lambda_1\Gamma(\varsigma_1)}\Big((1-e^{-\lambda_1\tau_1})(1-e^{-\lambda_1(\tau_2-\tau_1)})\Big)+\mathcal{L}_1\frac{1}{\lambda_1}(1-e^{-\lambda_1(\tau_2-\tau_1)})$$
$$+\mathcal{L}_1\Bigg\{\hat{\Phi}_1(\tau_2)-\hat{\Phi}_1(\tau_1)\frac{(2-e^{-\lambda_1})}{\Gamma(\varsigma_1)}+\hat{\Phi}_3(\tau_2)-\hat{\Phi}_3(\tau_1)\left[\frac{(1-e^{-\lambda_1})}{\lambda_1\Gamma(\varsigma_1)}+\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\right]$$
$$+\hat{\Phi}_4(\tau)_2-\hat{\Phi}_4(\tau)_1\left(\frac{\eta^{\varsigma_1+\mathfrak{v}-1}}{\lambda_1^2\Gamma(\varsigma_1)\Gamma(\mathfrak{v})}(\lambda_1\eta+e^{-\lambda_1\eta}-1)\right)\Bigg\}$$
$$+\mathcal{L}_2\Bigg\{\hat{\Phi}_2(\tau_2)-\hat{\Phi}_2(\tau_1)\frac{(2-e^{-\chi_1})}{\Gamma(\varsigma_2)}+\hat{\Phi}_3(\tau_2)-\hat{\Phi}_3(\tau_1)\left(\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\right)$$
$$+\hat{\Phi}_4(\tau_2)-\hat{\Phi}_4(\tau_1)\left[\frac{(1-e^{-\chi_1})}{\chi_1\Gamma(\varsigma_2)}+\frac{\zeta^{\varsigma_2+\mathfrak{z}-1}}{{\chi_1}^2\Gamma(\varsigma_2)\Gamma(\mathfrak{z})}(\chi_1\zeta+e^{-\chi_1\zeta}-1)\right]\Bigg\}.$$

This is because

$$\Phi_i(\tau_2)-\Phi_i(\tau_2)=\left|\frac{1}{\lambda_1^2}(\Lambda_i)(\lambda_1\tau_2-1+e^{-\lambda_1\tau_2})+\frac{1}{\lambda_1^3}(\Theta_i)(\lambda_1^2\tau_2^2-2\lambda_1\tau_2+2-2e^{-\lambda_1\tau_2})\right.$$
$$\left.-\frac{1}{\lambda_1^2}(\Lambda_i)(\lambda_1\tau_1-1+e^{-\lambda_1\tau_1})+\frac{1}{\lambda_1^3}(\Theta_i)(\lambda_1^2\tau_1^2-2\lambda_1\tau_1+2-2e^{-\lambda_1\tau_1})\right|\ i=1,2,3,4.$$
$$\leq\frac{1}{\lambda_1^2}|\Lambda_i|\Big|\lambda_1(\tau_2-\tau_1)-e^{\lambda_1\tau_1}+e^{\lambda_1\tau_2}\Big|$$
$$+\frac{1}{\lambda_1^3}|\Theta_i|\Big|\lambda_1^2(\tau_2^2(\tau_2^2-\tau_1^2)-2(e^{-\lambda_1\tau_2}-e^{-\lambda_1\tau_1}))-\lambda_1(\tau_2-\tau_1)\Big|\to 0,$$
$$\text{as } \tau_2\to\tau_1,\ i=1,2,3,4.$$

Clearly, $|\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau_2)-\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau_1)|\to 0$, as $\tau_2\to\tau_1$. Additionally, we obtain

$$|{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau_2)-{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau_1)|\leq\frac{1}{\Gamma(1-\mathfrak{p}_2)}\int_0^{\tau_1}\frac{|(\tau_1-v)^{\mathfrak{p}_2}-(\tau_2-v)^{\mathfrak{p}_2}|}{(\tau_1-v)^{\mathfrak{p}_2}(\tau_2-v)^{\mathfrak{p}_2}}|\mathcal{T}_1'(\mathcal{G},\mathcal{J})(v)|dv$$

$$+\frac{1}{\Gamma(1-\mathfrak{p}_2)}\int_{\tau_1}^{\tau_2}(\tau_2-\upsilon)^{-\mathfrak{p}_2}|\mathcal{T}_1'(\mathcal{G},\mathcal{J})(\upsilon)|d\upsilon,$$
$$=\frac{\mathcal{L}_1}{\Gamma(2-\mathfrak{p}_1)}\Big(2(\tau_2-\tau_1)^{1-\mathfrak{p}_2}-\tau_2^{1-\mathfrak{p}_2}+\tau_1)^{1-\mathfrak{p}_2}\Big)\to 0,$$

*as* $\tau_2\to\tau_1$.

In a similar manner, we have

$$|\mathcal{T}_2(\mathcal{G},\mathcal{J})(\tau_2)-\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau_1)|\to 0 \text{ and}$$
$$|{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau_2)-{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau_1)|\to 0,\ as\ \tau_2\to\tau_1.$$

Consequently, both the operators $\mathcal{T}1$ and $\mathcal{T}2$ exhibit equi-continuous aspects, implying that $\mathcal{T}$ itself is equi-continuous.

Utilising the Ascoli–Arzela theorem, we can deduce that $\mathcal{T}$ is compact. Therefore, we can conclude that $\mathcal{T}$ is a completely continuous operator.

Now, let us establish that the set $\hat{\varphi}_1=\{(\mathcal{G},\mathcal{J})\in\mathcal{U}\times\mathcal{Q},(\mathcal{U},\mathcal{Q})=\nu\mathcal{T}(\mathcal{G},\mathcal{J}),0\le\nu\le 1\}$ is bounded. Consider an arbitrary pair $\{(\mathcal{G},\mathcal{J})\in\hat{\varphi}_1$, which implies that $\{(\mathcal{G},\mathcal{J})=\nu\mathcal{T}(\mathcal{G},\mathcal{J})$ for some $\nu\in[0,1]$. Then, for any $\tau\in[0,1]$, we have $\mathcal{G}(\tau)=\nu\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau),\mathcal{J}(\tau)=\nu\mathcal{T}_2(\mathcal{G},\mathcal{J})(\tau)$.

From these relations, it follows that $|\mathcal{G}(\tau)|\le|\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau)|$ and $|\mathcal{J}(\tau)|\le|\mathcal{T}_2(\mathcal{G},\mathcal{J})(\tau)|$ $\forall\ \tau\in[0,1]$.

Then, by $\mathcal{H}_1$, we obtain

$$\begin{aligned}
&|\mathcal{G}(\tau)|\\
\le&|\Phi_1(\tau)|\Bigg[\lambda_1\int_0^1 e^{-\lambda_1(1-\upsilon)}\Bigg(\int_0^\upsilon\frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathfrak{a}_0|+|\mathfrak{a}_1\mathcal{G}(\varepsilon)|+|\mathfrak{a}_2\mathcal{J}(\varepsilon)|+|\mathfrak{a}_3{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon)|+|\mathfrak{a}_4\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon)|d\varepsilon\Bigg)d\upsilon\\
&+\int_0^1\frac{(1-\upsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}[|\mathfrak{a}_0|+|\mathfrak{a}_1\mathcal{G}(\upsilon)|+|\mathfrak{a}_2\mathcal{J}(\upsilon)|+|\mathfrak{a}_3{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\upsilon)|+|\mathfrak{a}_4\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\upsilon)|]d\upsilon\Bigg]\\
&+|\Phi_2(\tau)|\Bigg[\chi_1\int_0^1 e^{-\chi_1(1-\upsilon)}\Bigg(\int_0^\upsilon\frac{(\upsilon-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\\
&\quad\times\Big[|\mathfrak{b}_0|+\mathfrak{b}_1\mathcal{G}(\varepsilon)|+|\mathfrak{b}_2\mathcal{J}(\tau)(\varepsilon)|+|\mathfrak{b}_3{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\varepsilon)|+\mathfrak{b}_4\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\varepsilon)|\Big]d\varepsilon\Bigg)d\upsilon\\
&+\int_0^1\frac{(1-\upsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\Big[|\mathfrak{b}_0|+\mathfrak{b}_1\mathcal{G}(\upsilon)|+|\mathfrak{b}_2\mathcal{J}(\tau)(\upsilon)|+|\mathfrak{b}_3{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\upsilon)|+\mathfrak{b}_4\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\upsilon)|\Big]d\upsilon\Bigg]\\
&+|\Phi_3(\tau)|\Bigg[\int_0^\eta\frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^\upsilon e^{-\lambda_1(\upsilon-\varepsilon)}\Bigg(\int_0^\varepsilon\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\\
&\quad\times\Big[|\mathfrak{a}_0|+|\mathfrak{a}_1\mathcal{G}(\mathfrak{m})|+|\mathfrak{a}_2\mathcal{J}(\mathfrak{m})|+|\mathfrak{a}_3{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\mathfrak{m})|+|\mathfrak{a}_4\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\mathfrak{m})|\Big]d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&+\int_0^\zeta\frac{(\zeta-\upsilon)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^\upsilon e^{-\chi_1(\upsilon-\varepsilon)}\Bigg(\int_0^\varepsilon\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\\
&\quad\times\Big[|\mathfrak{b}_0|+\mathfrak{b}_1\mathcal{G}(\mathfrak{m})|+|\mathfrak{b}_2\mathcal{J}(\tau)(\mathfrak{m})|+|\mathfrak{b}_3{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\mathfrak{m})|+\mathfrak{b}_4\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\mathfrak{m})|\Big]d\mathfrak{m}\Bigg)d\varepsilon\Bigg)d\upsilon\\
&+\int_0^1 e^{-\lambda_1(1-\upsilon)}\Bigg(\int_0^\upsilon\frac{(\upsilon-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\Big[|\mathfrak{a}_0|+|\mathfrak{a}_1\mathcal{G}(\varepsilon)|+|\mathfrak{a}_2\mathcal{J}(\varepsilon)|+|\mathfrak{a}_3{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon)|+|\mathfrak{a}_4\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon)|\Big]d\varepsilon\Bigg)d\upsilon\Bigg]\\
&+|\Phi_4(\tau)|\Bigg[\int_0^\eta\frac{(\eta-\upsilon)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\Bigg(\int_0^\upsilon e^{-\lambda_1(\upsilon-\varepsilon)}\Bigg(\int_0^\varepsilon\frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}
\end{aligned}$$

$$\times \left[|\mathfrak{a}_0| + |\mathfrak{a}_1\mathcal{G}(\mathfrak{m})| + |\mathfrak{a}_2\mathcal{J}(\mathfrak{m})| + |\mathfrak{a}_3{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\mathfrak{m})| + |\mathfrak{a}_4\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\mathfrak{m})|\right]d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv$$

$$+\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{v}e^{-\chi_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}$$

$$\times \left[|\mathfrak{b}_0| + \mathfrak{b}_1\mathcal{G}(\mathfrak{m})| + |\mathfrak{b}_2\mathcal{J}(\tau)(\mathfrak{m})| + |\mathfrak{b}_3{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\mathfrak{m})| + \mathfrak{b}_4\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\mathfrak{m})|\right]d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv$$

$$+\int_0^{1}e^{-\chi_1(1-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\left[|\mathfrak{b}_0| + \mathfrak{b}_1\mathcal{G}(\varepsilon)| + |\mathfrak{b}_2\mathcal{J}(\tau)(\varepsilon)| + |\mathfrak{b}_3{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\varepsilon)| + \mathfrak{b}_4\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\varepsilon)|\right]d\varepsilon\Bigg)dv\Bigg]$$

$$+\int_0^{\tau}e^{-\lambda_1(\tau-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\left[|\mathfrak{a}_0| + |\mathfrak{a}_1\mathcal{G}(\varepsilon)| + |\mathfrak{a}_2\mathcal{J}(\varepsilon)| + |\mathfrak{a}_3{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\varepsilon)| + |\mathfrak{a}_4\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\varepsilon)|\right]d\varepsilon\Bigg)dv,$$

which, on taking the norm for $\tau \in [0,1]$, yields

$$||\mathcal{G}|| \leq \left[\mathfrak{a}_0 + \mathfrak{a}_1||\mathcal{G}||_{\mathcal{U}} + \left(\mathfrak{a}_2 + \mathfrak{a}_3 + \frac{\mathfrak{a}_4}{\Gamma(\mathfrak{q}_1+1)}\right)||\mathcal{J}||_{\mathcal{Q}}\right]\mathcal{U}_1$$
$$+\left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{a}_4}{\Gamma(1+\mathfrak{p}_1)}\right)||\mathcal{G}||_{\mathcal{U}} + \mathfrak{b}_3||\mathcal{G}||_{\mathcal{Q}}\right]\mathcal{U}_2.$$

Similarly, we obtain

$$||\mathcal{G}'|| \leq \left[\mathfrak{a}_0 + \mathfrak{a}_1||\mathcal{G}||_{\mathcal{U}} + \left(\mathfrak{a}_2 + \mathfrak{a}_3 + \frac{\mathfrak{a}_4}{\Gamma(\mathfrak{q}_1+1)}\right)||\mathcal{J}||_{\mathcal{Q}}\right]\mathcal{U}_1^*$$
$$+\left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{a}_4}{\Gamma(1+\mathfrak{p}_1)}\right)||\mathcal{G}||_{\mathcal{U}} + \mathfrak{b}_3||\mathcal{G}||_{\mathcal{Q}}\right]\mathcal{U}_2^*.$$

This implies that

$$||{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}|| \leq \frac{1}{\Gamma(2-\mathfrak{p}_2)}\Bigg\{\left[\mathfrak{a}_0 + \mathfrak{a}_1||\mathcal{G}||_{\mathcal{U}} + \left(\mathfrak{a}_2 + \mathfrak{a}_3 + \frac{\mathfrak{a}_4}{\Gamma(\mathfrak{q}_1+1)}\right)||\mathcal{J}||_{\mathcal{Q}}\right]\mathcal{U}_1^*$$
$$+\left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{a}_4}{\Gamma(1+\mathfrak{p}_1)}\right)||\mathcal{G}||_{\mathcal{U}} + \mathfrak{b}_3||\mathcal{G}||_{\mathcal{Q}}\right]\mathcal{U}_2^*\Bigg\}. \tag{31}$$

Thus, we have

$$\begin{aligned}||\mathcal{G}||_{\mathcal{U}} =& ||\mathcal{G}|| + ||{}^{c}\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}|| \\ \leq& \left[\mathfrak{a}_0 + \mathfrak{a}_1||\mathcal{G}||_{\mathcal{U}} + \left(\mathfrak{a}_2 + \mathfrak{a}_3 + \frac{\mathfrak{a}_4}{\Gamma(\mathfrak{q}_1+1)}\right)||\mathcal{J}||_{\mathcal{Q}}\right]\mathcal{U}_1 \\ &+\left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{a}_4}{\Gamma(1+\mathfrak{p}_1)}\right)||\mathcal{G}||_{\mathcal{U}} + \mathfrak{b}_3||\mathcal{G}||_{\mathcal{Q}}\right]\mathcal{U}_2 \\ &+\frac{1}{\Gamma(2-\mathfrak{p}_2)}\Bigg\{\left[\mathfrak{a}_0 + \mathfrak{a}_1||\mathcal{G}||_{\mathcal{U}} + \left(\mathfrak{a}_2 + \mathfrak{a}_3 + \frac{\mathfrak{a}_4}{\Gamma(\mathfrak{q}_1+1)}\right)||\mathcal{J}||_{\mathcal{Q}}\right]\mathcal{U}_1^* \\ &+\left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{a}_4}{\Gamma(1+\mathfrak{p}_1)}\right)||\mathcal{G}||_{\mathcal{U}} + \mathfrak{b}_3||\mathcal{G}||_{\mathcal{Q}}\right]\mathcal{U}_2^*\Bigg\}.\end{aligned} \tag{32}$$

Likewise, we can have

$$\begin{aligned}||\mathcal{J}||_{\mathcal{Q}} =& ||\mathcal{J}|| + ||{}^{c}\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}|| \\ \leq& \left[\mathfrak{a}_0 + \mathfrak{a}_1||\mathcal{G}||_{\mathcal{U}} + \left(\mathfrak{a}_2 + \mathfrak{a}_3 + \frac{\mathfrak{a}_4}{\Gamma(\mathfrak{q}_1+1)}\right)||\mathcal{J}||_{\mathcal{Q}}\right]\mathcal{A}_1 \\ &+\left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{a}_4}{\Gamma(1+\mathfrak{p}_1)}\right)||\mathcal{G}||_{\mathcal{U}} + \mathfrak{b}_3||\mathcal{G}||_{\mathcal{Q}}\right]\mathcal{A}_2\end{aligned} \tag{33}$$

$$+\frac{1}{\Gamma(2-\mathfrak{p}_1)}\left\{\left[\mathfrak{a}_0+\mathfrak{a}_1||\mathcal{G}||_{\mathcal{U}}+\left(\mathfrak{a}_2+\mathfrak{a}_3+\frac{\mathfrak{a}_4}{\Gamma(\mathfrak{q}_1+1)}\right)||\mathcal{J}||_{\mathcal{Q}}\right]\mathcal{A}_1^*\right.$$
$$\left.+\left[\mathfrak{b}_0+\left(\mathfrak{b}_1+\mathfrak{b}_2+\frac{\mathfrak{a}_4}{\Gamma(1+\mathfrak{p}_1)}\right)||\mathcal{G}||_{\mathcal{U}}+\mathfrak{b}_3||\mathcal{G}||_{\mathcal{Q}}\right]\mathcal{A}_2^*\right\}.$$

From (32) and (33), we find

$$\begin{aligned}
||(\mathcal{G},\mathcal{J})||_{\mathcal{U}\times\mathcal{Q}} =& ||\mathcal{G}||_{\mathcal{U}}+||\mathcal{J}||_{\mathcal{Q}}\\
\leq& ||\mathcal{G}||_{\mathcal{U}}\left\{\mathfrak{a}_1\left[\mathcal{U}_1+\mathcal{A}_1+\frac{\mathcal{U}_1^*}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{A}_1^*}{\Gamma(2-\mathfrak{p}_1)}\right]+\left(\mathfrak{b}_1+\mathfrak{b}_2+\frac{\mathfrak{b}_3}{\Gamma(\mathfrak{q}_1+1)}\right)\right.\\
&\left.+\left[\mathcal{U}_2+\mathcal{A}_2+\frac{\mathcal{U}_2^*}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{A}_2^*}{\Gamma(2-\mathfrak{p}_1)}\right]\right\}\\
&+||\mathcal{J}||_{\mathcal{Q}}\left\{\mathfrak{b}_4\left[\mathcal{U}_2+\mathcal{A}_2+\frac{\mathcal{U}_2^*}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{A}_2^*}{\Gamma(2-\mathfrak{p}_1)}\right]+\left(\mathfrak{a}_2+\mathfrak{a}_3+\frac{\mathfrak{a}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\right.\\
&\left.+\left[\mathcal{U}_1+\mathcal{A}_1+\frac{\mathcal{U}_1^*}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{A}_1^*}{\Gamma(2-\mathfrak{p}_1)}\right]\right\}\\
&+\mathfrak{a}_0\left[\mathcal{U}_1+\mathcal{A}_1+\frac{\mathcal{U}_1^*}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{A}_1^*}{\Gamma(2-\mathfrak{p}_1)}\right]+\mathfrak{b}_0\left[\mathcal{U}_2+\mathcal{A}_2+\frac{\mathcal{U}_2^*}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{A}_2^*}{\Gamma(2-\mathfrak{p}_1)}\right]\\
\leq& \mathcal{N}_3+\max\{\mathcal{N}_4+\mathcal{N}_5\}||\mathcal{G},\mathcal{J}||_{\mathcal{U}\times\mathcal{Q}}.
\end{aligned}\tag{34}$$

Therefore, by implementing the assumption $\max\mathcal{N}_4+\mathcal{N}_5<1$, we can deduce

$$||\mathcal{G},\mathcal{J}||_{\mathcal{U}\times\mathcal{Q}}\leq\frac{\mathcal{N}_3}{1-\max\{\mathcal{N}_4+\mathcal{N}_5\}}.$$

Hence, we deduce that the set $\hat{\varphi}_1$ is bounded.

Utilising Theorem 2, we can ascertain that the operator $\mathcal{T}$ possesses at least one fixed point, which serves as a solution to our Problems (1) and (2). With this, we conclude the proof. □

Subsequently, we will establish existence and uniqueness results for Problems (1) and (2), employing the Banach contraction mapping principle.

We introduce the concepts

$$\begin{aligned}
r_1 =& \sup_{\tau\in[0,1]}|\varphi_1(\tau,0,0,0,0)|,\ r_2=\sup_{\tau\in[0,1]}|\phi_1(\tau,0,0,0,0)|,\\
\Lambda =& \mathfrak{W}_0\rho_1\mathcal{U}_1+\mathfrak{K}_0\rho_2\mathcal{A}_1,\ \mathcal{M}=\mathfrak{W}_0\rho_1\mathcal{U}_1^*+\mathfrak{K}_0\rho_2 r_2\mathcal{A}_1^*,\\
\Lambda^* =& \mathfrak{W}_0\rho_1\mathcal{U}_2+\mathfrak{K}_0\rho_2\mathcal{A}_2,\ \mathcal{M}^*=\mathfrak{W}_0\rho_1\mathcal{U}_2^*+\mathfrak{K}_0\rho_2 r_2\mathcal{A}_2^*,\\
\mathcal{Z}_1 =& r_1\mathcal{U}_1+r_2\mathcal{A}_1,\ \ \mathcal{Z}_1^*=r_1\mathcal{U}_1^*+r_2\mathcal{A}_1^*,\hat{\mathcal{Z}}_1=r_1\mathcal{U}_2+r_2\mathcal{A}_2,\ \ \hat{\mathcal{Z}}_1^*=r_1\mathcal{U}_2^*+r_2\mathcal{A}_2^*.
\end{aligned}$$

**Theorem 3.** *Assume that $\mathcal{H}_2$ holds. Furthermore, if*

$$\left[\Lambda+\Lambda^*+\frac{\mathcal{M}}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{M}^*}{\Gamma(2-\mathfrak{p}_1)}\right]<1,\tag{35}$$

*then Problems (1) and (2) have a unique solution.*

**Proof.** We consider the positive number $r$ given by

$$r\geq\frac{\left[\mathcal{Z}_1+\mathcal{Z}_1^*+\frac{\hat{\mathcal{Z}}_1}{\Gamma(2-\mathfrak{p}_2)}+\frac{\hat{\mathcal{Z}}_1^*}{\Gamma(2-\mathfrak{p}_1)}\right]}{\left(1-\left[\Lambda+\Lambda^*+\frac{\mathcal{M}}{\Gamma(2-\mathfrak{p}_2)}+\frac{\mathcal{M}^*}{\Gamma(2-\mathfrak{p}_1)}\right]\right)}.$$

We show that $\mathcal{T}\hat{\mathfrak{B}}_r \subset \hat{\mathfrak{B}}_r$, where $\hat{\mathfrak{B}}_r = \{(\mathcal{G},\mathcal{J}) \in \mathcal{U}\times\mathcal{Q}, ||\mathcal{G},\mathcal{J}||_{\mathcal{U}\times\mathcal{Q}} < r\}$. For $\mathcal{G},\mathcal{J} \in \hat{\mathfrak{B}}_r$, we obtain

$$\begin{aligned}
&|\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))|\\
\leq&|\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau)) - \varphi_1(\tau,0,0,0,0)| + |\varphi_1(\tau,0,0,0,0)|,\\
&\leq \mathfrak{W}_0(|\mathcal{G}(\tau)| + |\mathcal{J}(\tau)| + |{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau)| + |\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau)|) + r_1,\\
&\leq \mathfrak{W}_0\left[||\mathcal{G}||_{\mathcal{U}} + ||\mathcal{J}||_{\mathcal{Q}} + \frac{1}{\Gamma(\mathfrak{q}_1+1)}||\mathcal{J}||_{\mathcal{Q}}\right] + r_1,\\
&\leq \mathfrak{W}_0(||\mathcal{G}||_{\mathcal{U}} + \rho_1||\mathcal{J}||_{\mathcal{Q}}) + r_1,\\
&\leq \mathfrak{W}_0\rho_1 r + r_1.
\end{aligned}$$

In a similar manner, we have

$$|\varphi_1(\tau,\mathcal{G}(\tau),\mathcal{J}(\tau),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau))| \leq \mathfrak{K}_0\rho_2 r + r_2.$$

Then,

$$\begin{aligned}
|\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau)| \leq&(\mathfrak{W}_0\rho_1 r + r_1)\mathcal{U}_1 + (\mathfrak{K}_0\rho_2 r + r_2)\mathcal{A}_1,\\
&= (\mathfrak{W}_0\rho_1\mathcal{U}_1 + \mathfrak{K}_0\rho_2\mathcal{A}_1) + r_2\mathcal{A}_1 + \mathcal{U}_1 r_1,\\
&= \Lambda r + \mathcal{Z}_1, \quad forall\ \tau\in[0,1].
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{T}_1'(\mathcal{G},\mathcal{J})(\tau)| \leq&(\mathfrak{W}_0\rho_1 r + r_1)\mathcal{U}_1^* + (\mathfrak{K}_0\rho_2 r + r_2)\mathcal{A}_1^*,\\
&= (\mathfrak{W}_0\rho_1\mathcal{U}_1^* + \mathfrak{K}_0\rho_2\mathcal{A}_1^*) + r_2\mathcal{A}_1^* + \mathcal{U}_1^* r_1,\\
&= \Lambda^* r + \mathcal{Z}_1^*, \quad forall\ \tau\in[0,1].
\end{aligned}$$

which gives us

$$\begin{aligned}
|{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G},\mathcal{J})(\tau)| &\leq \int_0^\tau \frac{(\tau-v)^{-\mathfrak{p}_2}}{\Gamma(1-\mathfrak{p}_2)}|\mathcal{T}_1'(\mathcal{G},\mathcal{J})(v)|dv,\\
&\leq \frac{1}{\Gamma(2-\mathfrak{p}_2)}(\mathcal{M}r + \mathcal{Z}_1^*), forall\tau\in[0,1].
\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}
||\mathcal{T}_1(\mathcal{G},\mathcal{J})||_{\mathcal{U}} =&||\mathcal{T}_1(\mathcal{G},\mathcal{J})|| + ||{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G},\mathcal{J})||,\\
&\leq \left(\Lambda + \frac{\mathcal{M}}{\Gamma(2-\mathfrak{p}_2)}\right)r + \mathcal{Z}_1 + \frac{\mathcal{Z}_1^*}{\Gamma(2-\mathfrak{p}_2)}.
\end{aligned} \tag{36}$$

In a similar manner, we obtain

$$|\mathcal{T}_2(\mathcal{G},\mathcal{J})(\tau)| \leq \Lambda^* r + \hat{\mathcal{Z}}_1,$$

$$|\mathcal{T}_2'(\mathcal{G},\mathcal{J})(\tau)| \leq \mathcal{M}^* r + \hat{\mathcal{Z}}_1{}^*,$$

$$\begin{aligned}
|{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{T}_2(\mathcal{G},\mathcal{J})(\tau)| &\leq \int_0^\tau \frac{(\tau-v)^{-\mathfrak{p}_1}}{\Gamma(1-\mathfrak{p}_1)}|\mathcal{T}_2'(\mathcal{G},\mathcal{J})(v)|dv,\\
&\leq \frac{1}{\Gamma(2-\mathfrak{p}_1)}(\mathcal{M}^* r + \hat{\mathcal{Z}}_1{}^*), forall\tau\in[0,1].
\end{aligned}$$

Then, we conclude

$$||\mathcal{T}_2(\mathcal{G},\mathcal{J})||_{\mathcal{Q}} =||\mathcal{T}_2(\mathcal{G},\mathcal{J})|| + ||{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{T}_2(\mathcal{G},\mathcal{J})||,$$

$$\leq \left(\Lambda^* + \frac{\mathcal{M}^*}{\Gamma(2-\mathfrak{p}_1)}\right) r + \hat{\mathcal{Z}}_1 + \frac{\hat{\mathcal{Z}}_1^*}{\Gamma(2-\mathfrak{p}_1)}. \tag{37}$$

By Relations (36) and (37), we deduce

$$\begin{aligned}
||\mathcal{T}(\mathcal{G},\mathcal{J})||_{\mathcal{U}\times\mathcal{Q}} &= ||\mathcal{T}_1(\mathcal{G},\mathcal{J})||_{\mathcal{U}} + ||\mathcal{T}_2(\mathcal{G},\mathcal{J})||_{\mathcal{Q}} \\
&\leq \left(\left[\Lambda + \Lambda^* + \frac{\mathcal{M}}{\Gamma(2-\mathfrak{p}_2)} + \frac{\mathcal{M}^*}{\Gamma(2-\mathfrak{p}_1)}\right]\right) r + \left[\mathcal{Z}_1 + \mathcal{Z}_1^* + \frac{\hat{\mathcal{Z}}_1}{\Gamma(2-\mathfrak{p}_2)} + \frac{\hat{\mathcal{Z}}_1^*}{\Gamma(2-\mathfrak{p}_1)}\right] \\
&= r.
\end{aligned}$$

This implies $\mathcal{T}\hat{\mathfrak{B}}_r \subset \hat{\mathfrak{B}}_r$.

Next, we prove that the operator $\mathcal{T}$ is a contraction. For $(\mathcal{G}_i, \mathcal{J}_i) \in \hat{\mathfrak{B}}_r, i = 1,2$ and for each $\tau \in [0,1]$, we have

$$\begin{aligned}
&|\mathcal{T}_1(\mathcal{G}_1,\mathcal{J}_1)(\tau) - \mathcal{T}_2(\mathcal{G}_2,\mathcal{J}_2)(\tau)| \\
=&|\Phi_1(\tau)|\left[\lambda_1 \int_0^1 e^{-\lambda_1(1-v)}\left(\int_0^v \frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\varepsilon,\mathcal{G}_1(\varepsilon),\mathcal{J}_1(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_1(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_1(\varepsilon))\right.\right. \\
&\left.-\varphi_1(\varepsilon,\mathcal{G}_2(\varepsilon),\mathcal{J}_2(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_2(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_2(\varepsilon))|d\varepsilon\right)dv \\
&\left.-\int_0^1 \frac{(1-v)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(v,\mathcal{G}_1(v),\mathcal{J}_1(v),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_1(v),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_1(v)) - \varphi_1(v,\mathcal{G}_2(v),\mathcal{J}_2(v),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_2(v),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_2(v))|dv\right] \\
&+|\Phi_2(\tau)|\left[\chi_1 \int_0^1 e^{-\chi_1(1-v)}\left(\int_0^v \frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\varepsilon,\mathcal{G}_1(\varepsilon),\mathcal{J}_1(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_1(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_1(\varepsilon))\right.\right. \\
&\left.-\phi_1(\varepsilon,\mathcal{G}_2(\varepsilon),\mathcal{J}_2(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_2(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_2(\varepsilon))|d\varepsilon\right)dv \\
&\left.-\int_0^1 \frac{(1-v)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(v,\mathcal{G}_1(v),\mathcal{J}_1(v),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_1(v),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_1(v)) - \phi_1(v,\mathcal{G}_2(v),\mathcal{J}_2(v),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_2(v),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_2(v))|dv\right] \\
&+|\Phi_3(\tau)|\left[\int_0^\eta \frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\left(\int_0^v e^{-\lambda_1(v-\varepsilon)}\left(\int_0^\varepsilon \frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\right.\right.\right. \\
&\quad\times|\varphi_1(\mathfrak{m},\mathcal{G}_1(\mathfrak{m}),\mathcal{J}_1(\mathfrak{m}),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_1(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_1(\mathfrak{m})) \\
&\left.\left.\left.-\varphi_1(\mathfrak{m},\mathcal{G}_2(\mathfrak{m}),\mathcal{J}_2(\mathfrak{m}),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_2(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_2(\mathfrak{m}))|d\mathfrak{m}\right)d\varepsilon\right)dv\right. \\
&+\int_0^\zeta \frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\left(\int_0^v e^{-\chi_1(v-\varepsilon)}\left(\int_0^\varepsilon \frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\mathfrak{m},\mathcal{G}_1(\mathfrak{m}),\mathcal{J}_1(\mathfrak{m}),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_1(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_1(\mathfrak{m}))\right.\right.\right. \\
&\left.\left.\left.-\phi_1(\mathfrak{m},\mathcal{G}_2(\mathfrak{m}),\mathcal{J}_2(\mathfrak{m}),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_2(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_2(\mathfrak{m}))|d\mathfrak{m}\right)d\varepsilon\right)dv\right. \\
&-\int_0^1 e^{-\lambda_1(1-v)}\left(\int_0^v \frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\varepsilon,\mathcal{G}_1(\varepsilon),\mathcal{J}_1(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_1(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_1(\varepsilon))\right. \\
&\left.\left.-\varphi_1(\varepsilon,\mathcal{G}_2(\varepsilon),\mathcal{J}_2(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_2(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_2(\varepsilon))|d\varepsilon\right)dv\right] \\
&+|\Phi_4(\tau)|\left[\int_0^\eta \frac{(\eta-v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\left(\int_0^v e^{-\lambda_1(v-\varepsilon)}\left(\int_0^\varepsilon \frac{(\varepsilon-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\right.\right.\right. \\
&\quad\times|\varphi_1(\mathfrak{m},\mathcal{G}_1(\mathfrak{m}),\mathcal{J}_1(\mathfrak{m}),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_1(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_1(\mathfrak{m})) \\
&\left.\left.\left.-\varphi_1(\mathfrak{m},\mathcal{G}_2(\mathfrak{m}),\mathcal{J}_2(\mathfrak{m}),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_2(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_2(\mathfrak{m}))|d\mathfrak{m}\right)d\varepsilon\right)dv\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\zeta}\frac{(\zeta-v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\Bigg(\int_0^{v}e^{-\chi_1(v-\varepsilon)}\Bigg(\int_0^{\varepsilon}\frac{(\varepsilon-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\mathfrak{m},\mathcal{G}_1(\mathfrak{m}),\mathcal{J}_1(\mathfrak{m}),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_1(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_1(\mathfrak{m}))\\
&-\phi_1(\mathfrak{m},\mathcal{G}_2(\mathfrak{m}),\mathcal{J}_2(\mathfrak{m}),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_2(\mathfrak{m}),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_2(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\varepsilon\Bigg)dv\\
&-\int_0^{1}e^{-\chi_1(1-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\phi_1(\varepsilon,\mathcal{G}_1(\varepsilon),\mathcal{J}_1(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_1(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_1(\varepsilon))\\
&-\phi_1(\varepsilon,\mathcal{G}_2(\varepsilon),\mathcal{J}_2(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_2(\varepsilon),\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_2(\varepsilon))|d\varepsilon\Bigg)dv\Bigg]\\
&+\int_0^{\tau}e^{-\lambda_1(\tau-v)}\Bigg(\int_0^{v}\frac{(v-\varepsilon)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\varphi_1(\varepsilon,\mathcal{G}_1(\varepsilon),\mathcal{J}_1(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_1(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_1(\varepsilon))\\
&-\varphi_1(\varepsilon,\mathcal{G}_2(\varepsilon),\mathcal{J}_2(\varepsilon),{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_2(\varepsilon),\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_2(\varepsilon))|d\varepsilon\Bigg)dv.
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathcal{U}_1\mathfrak{W}_0(||\mathcal{G}_1-\mathcal{G}_2||+||\mathcal{J}_1-\mathcal{J}_2||+||{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_1-{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_2||+||\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_1-\mathcal{I}^{\mathfrak{q}_1}\mathcal{J}_2||)\\
&\quad+\mathcal{A}_1\mathfrak{K}_0(||\mathcal{G}_1-\mathcal{G}_2||+||{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_1-{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_2||+||\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_1-\mathcal{I}^{\mathfrak{q}_2}\mathcal{G}_2||+||\mathcal{J}_1-\mathcal{J}_2||)\\
&\leq \mathcal{U}_1\mathfrak{W}_0(||\mathcal{G}_1-\mathcal{G}_2||+\rho_1||\mathcal{J}_1-\mathcal{J}_2||+||{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_1-{}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}_2||)\\
&\quad+\mathcal{A}_1\mathfrak{K}_0(\rho_2||\mathcal{G}_1-\mathcal{G}_2||+||{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_1-{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}_2||+||\mathcal{J}_1-\mathcal{J}_2||)\\
&\leq \Lambda(||\mathcal{G}_1-\mathcal{G}_2||_{\mathcal{U}}+||\mathcal{J}_1-\mathcal{J}_2||_{\mathcal{Q}}).
\end{aligned}
$$

Then, we obtain

$$|\mathcal{T}_1'(\mathcal{G}_1,\mathcal{J}_1)(\tau)-\mathcal{T}_2'(\mathcal{G}_2,\mathcal{J}_2)(\tau)|\leq\mathcal{M}(||\mathcal{G}_1-\mathcal{G}_2||_{\mathcal{U}}+||\mathcal{J}_1-\mathcal{J}_2||_{\mathcal{Q}}).$$

which gives us

$$
\begin{aligned}
|{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G}_1,\mathcal{J}_1)(\tau)-{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G}_2,\mathcal{J}_2)(\tau)|&\leq\int_0^{\tau}\frac{(\tau-v)^{-\mathfrak{p}_2}}{\Gamma(1-\mathfrak{p}_2)}|\mathcal{T}_1'(\mathcal{G}_1,\mathcal{J}_1)(v)-\mathcal{T}_1'(\mathcal{G}_2,\mathcal{J}_2)(v)|dv,\\
&\leq\frac{1}{\Gamma(2-\mathfrak{p}_2)}\mathcal{M}(||\mathcal{G}_1-\mathcal{G}_2||_{\mathcal{U}}+||\mathcal{J}_1-\mathcal{J}_2||_{\mathcal{Q}}).
\end{aligned}
$$

From the above inequalities, we conclude

$$
\begin{aligned}
&||\mathcal{T}_1(\mathcal{G}_1,\mathcal{J}_1)(\tau)-\mathcal{T}_1(\mathcal{G}_2,\mathcal{J}_2)(\tau)||_{\mathcal{U}}\\
&=||\mathcal{T}_1(\mathcal{G}_1,\mathcal{J}_1)(\tau)-\mathcal{T}_1(\mathcal{G}_2,\mathcal{J}_2)(\tau)||+||{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G}_1,\mathcal{J}_1)(\tau)-{}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{T}_1(\mathcal{G}_2,\mathcal{J}_2)(\tau)||\\
&\leq\left[\Lambda+\frac{1}{\Gamma(2-\mathfrak{p}_2)}\mathcal{M}\right](||\mathcal{G}_1-\mathcal{G}_2||_{\mathcal{U}}+||\mathcal{J}_1-\mathcal{J}_2||_{\mathcal{Q}}).
\end{aligned}
\tag{38}
$$

In a similar manner, we deduce

$$||\mathcal{T}_2(\mathcal{G}_1,\mathcal{J}_1)-\mathcal{T}_2(\mathcal{G}_2,\mathcal{J}_2)||_{\mathcal{Q}}\leq\left[\Lambda^*+\frac{1}{\Gamma(2-\mathfrak{p}_2)}\mathcal{M}^*\right](||\mathcal{G}_1-\mathcal{G}_2||_{\mathcal{U}}+||\mathcal{J}_1-\mathcal{J}_2||_{\mathcal{Q}}).\tag{39}$$

Therefore, by (38) and (39), we obtain

$$
\begin{aligned}
&||\mathcal{T}(\mathcal{G}_1,\mathcal{J}_1)-\mathcal{T}(\mathcal{G}_2,\mathcal{J}_2)||_{\mathcal{Q}}\\
&=||\mathcal{T}_1(\mathcal{G}_1,\mathcal{J}_1)-\mathcal{T}_1(\mathcal{G}_2,\mathcal{J}_2)||_{\mathcal{U}}+||\mathcal{T}_2(\mathcal{G}_1,\mathcal{J}_1)-\mathcal{T}_2(\mathcal{G}_2,\mathcal{J}_2)||_{\mathcal{Q}}\\
&\leq\left[\Lambda+\frac{1}{\Gamma(2-\mathfrak{p}_2)}\mathcal{M}\right](||\mathcal{G}_1-\mathcal{G}_2||_{\mathcal{U}}+||\mathcal{J}_1-\mathcal{J}_2||_{\mathcal{Q}})+\left[\Lambda^*+\frac{1}{\Gamma(2-\mathfrak{p}_2)}\mathcal{M}^*\right](||\mathcal{G}_1-\mathcal{G}_2||_{\mathcal{U}}+||\mathcal{J}_1-\mathcal{J}_2||_{\mathcal{Q}})\\
&\leq\left(\Lambda+\Lambda^*+\frac{1}{\Gamma(2-\mathfrak{p}_2)}\mathcal{M}+\frac{1}{\Gamma(2-\mathfrak{p}_2)}\mathcal{M}^*\right)\times(||\mathcal{G}_1-\mathcal{G}_2||_{\mathcal{U}}+||\mathcal{J}_1-\mathcal{J}_2||_{\mathcal{Q}}).
\end{aligned}
$$

Through the utilisation of Condition (35), it can be inferred that $\mathcal{T}$ qualifies as a contraction operator. Consequently, by virtue of Banach's fixed-point theorem, the operator

$\mathcal{T}$ possesses a unique fixed point, corresponding to the unique solution of Problems (1) and (2). With this, we conclude the proof. □

## 5. Example

Let $\lambda_1 = 2; \chi_1 = 3; \eta = \frac{1}{4}; v = 1; z = 1; \zeta = \frac{1}{5}; \tau = 1, \varsigma_1 = \frac{9}{2}, \varsigma_2 = \frac{7}{2}, \mathfrak{p}_1 = \frac{1}{3}$ $\mathfrak{p}_2 = \frac{1}{2}, \mathfrak{q}_1 = \frac{3}{2}, \mathfrak{q}_2 = \frac{8}{3}$. We then turn our attention to the following system of fractional differential equations:

$$\begin{cases} ({}^c\mathcal{D}^{\varsigma_1} + \lambda_1 {}^c\mathcal{D}^{\varsigma_1 - 1})\mathcal{G}(\tau) = \varphi_1(\tau, \mathcal{G}(\tau), \mathcal{J}(\tau), {}^c\mathcal{D}^{\mathfrak{p}_1}\mathcal{J}(\tau), \mathcal{I}^{\mathfrak{q}_1}\mathcal{J}(\tau)), \ \ \tau \in (0,1), \\ ({}^c\mathcal{D}^{\varsigma_2} + \chi_1 {}^c\mathcal{D}^{\varsigma_2 - 1})\mathcal{J}(\tau) = \phi_1(\tau, \mathcal{G}(\tau), \mathcal{J}(\tau), {}^c\mathcal{D}^{\mathfrak{p}_2}\mathcal{G}(\tau), \mathcal{I}^{\mathfrak{q}_2}\mathcal{G}(\tau)), \ \ \tau \in (0,1) \end{cases} \tag{40}$$

complemented by the general coupled integral boundary conditions:

$$\begin{cases} \mathcal{G}(0) = 0, \ \ \mathcal{G}'(0) = 0, \ \ \mathcal{G}'(1) = 0, \ \ \mathcal{G}(1) = \displaystyle\int_0^\eta \frac{(\eta - v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\mathcal{G}(v)dv + \int_0^\zeta \frac{(\zeta - v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\mathcal{J}(v)dv, \\ \mathcal{J}(0) = 0, \ \ \mathcal{J}'(0) = 0, \ \ \mathcal{J}'(1) = 0, \ \ \mathcal{J}(1) = \displaystyle\int_0^\eta \frac{(\eta - v)^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\mathcal{G}(v)dv + \int_0^\zeta \frac{(\zeta - v)^{\mathfrak{z}-1}}{\Gamma(\mathfrak{z})}\mathcal{J}(v)dv, \end{cases} \tag{41}$$

We have $\Omega_1 = 0.432332, \Omega_2 = 0.533834, \Omega_3 = 0.316738, \Omega_4 = 0.449976, \Omega_5 = 0.281525,$ $\Omega_6 = 0.247777, \Omega_7 = 0.00115512, \Omega_8 = 0.000118806, \Omega_9 = 0.00120999, \Omega_{10} = 0.00029549,$ $\Omega_{11} = 0.22659, \Omega_{12} = 0.181378, \Delta = 0.00192126, \Delta_1 = 0.00046151, \Lambda_1 \approx 5.74083,$ $\Lambda_2 \approx 0.0507348, \Lambda_3 \approx 17.6486, \Lambda_4 \approx 0.160883, \Xi_1 \approx 0.0507341, \Xi_2 \approx 4.07506,$ $\Xi_3 \approx 0.121363, \Xi_4 \approx 10.1098, \Theta_1 \approx -6.52253, \Theta_2 \approx 0.0410882, \Theta_3 \approx -10.017,$ $\Theta_4 \approx 0.0084369, \Upsilon_2 \approx 5.09077, \Upsilon_2 \approx 0.0357116, \Upsilon_3 \approx -0.0854273, \Upsilon_4 \approx -7.11626,$ $\Phi_1 \approx 3.03939, \Phi_2 \approx 0.00551836, \Phi_3 \approx 2.84393, \Phi_4 \approx 0.0474878, \Psi_1 \approx 0.00507333,$ $\Psi_2 \approx 0.00415164, \Psi_3 \approx 0.0121361, \Psi_4 \approx 1.01096, \mathcal{U}_1 \approx 0.344355, \mathcal{U}_2 \approx 0.00133908,$ $\mathcal{A}_1 \approx 0.000357643, \mathcal{A}_2 = 0.0966458, \mathcal{U}_1^* \approx 0.758877, \mathcal{U}_2^* \approx 0.00793721, \mathcal{A}_1^* \approx 0.0000464214,$ $\mathcal{A}_2^* = 0.59475844.$

**Example 1.** *We consider the functions*

$$|\varphi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4)| = \frac{\tau}{\tau^2 + 1}\left(3\cos\tau + \frac{1}{8}\sin(\mathfrak{l}_1 + \mathfrak{l}_2)\right) - \frac{1}{8(\tau+1)^2}\mathfrak{l}_3 + \frac{1}{10}\arctan\mathfrak{l}_4,$$
$$|\phi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4)| = \frac{\tau}{(\tau+2)^3}\left(5e^{-\tau} + \frac{\mathfrak{l}_1}{2} + 2\mathfrak{l}_2\right) - \frac{\tau}{6}\sin(\mathfrak{l}_3 + \mathfrak{l}_4),$$

*for all* $\tau \in [0,1]$, $\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4 \in \mathcal{R}_e$. *We achieve the inequalities*

$$|\varphi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4)| \leq \frac{3}{2} + \frac{1}{16}|\mathfrak{l}_1| + \frac{1}{16}|\mathfrak{l}_2| + \frac{1}{8}|\mathfrak{l}_3| + \frac{1}{10}|\mathfrak{l}_4|,$$
$$|\phi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4)| \leq \frac{5}{8} + \frac{1}{16}|\mathfrak{l}_1| + \frac{1}{4}|\mathfrak{l}_2| + \frac{1}{6}|\mathfrak{l}_3| + \frac{1}{6}|\mathfrak{l}_4|,$$

*for all* $\tau \in [0,1], \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4 \in \mathcal{R}_e$. *Hence, we arrive at* $\mathfrak{a}_0 = \frac{3}{2}, \mathfrak{a}_1 = \frac{1}{16}, \mathfrak{a}_2 = \frac{1}{16}, \mathfrak{a}_3 = \frac{1}{10},$ $\mathfrak{a}_4 = \frac{1}{10}, \mathfrak{b}_0 = \frac{5}{8}, \mathfrak{b}_1 = \frac{1}{16}, \mathfrak{b}_2 = \frac{1}{4}, \mathfrak{b}_3 = \frac{1}{6}, \mathfrak{b}_4 = \frac{1}{6}$.

*Given that* $\mathcal{N}_4 \approx 0.34571011$ *and* $\mathcal{N}_5 \approx 0.352583822$, *we can ascertain that the condition of Theorem 2,* $\max\{\mathcal{N}_4, \mathcal{N}_5\} < 1$, *is indeed met. Therefore, in accordance with Theorem 2, we can conclude that Problems (1) and (2) possess at least one solution,* $\tau \in [0,1]$.

**Example 2.** *We consider the functions*

$$\varphi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4) = \frac{\tau}{2} + \frac{1}{8(\tau+1)^2}\left(\frac{\mathfrak{l}_1}{1 + |\mathfrak{l}_1|} - \mathfrak{l}_2\right) + \frac{1}{32}\sin^2\mathfrak{l}_3 - \frac{\tau}{9}\arctan\mathfrak{l}_4,$$
$$\phi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4) = \frac{\tau^2}{\tau^3 + 1} - \frac{1}{16}\sin\mathfrak{l}_1 + \frac{1}{10}\mathfrak{l}_2 + \frac{1}{\sqrt{4 + \tau^2}}\cos\mathfrak{l}_3 - \frac{|\mathfrak{l}_4|}{6(1 + |\mathfrak{l}_4|)},$$

*for all* $\tau \in [0,1], \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4 \in \mathcal{R}_e$.

*We obtain the following inequalities*

$$|\varphi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4) - \varphi_1(\tau, \mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \mathfrak{n}_4)| \leq \frac{1}{8}(|\mathfrak{l}_1 - \mathfrak{n}_1| + |\mathfrak{l}_2 - \mathfrak{n}_2| + |\mathfrak{l}_3 - \mathfrak{n}_3| + |\mathfrak{l}_4 - \mathfrak{n}_4|)$$

$$|\phi_1(\tau, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4) - \phi_1(\tau, \mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \mathfrak{n}_4)| \leq \frac{1}{6}(|\mathfrak{l}_1 - \mathfrak{n}_1| + |\mathfrak{l}_2 - \mathfrak{n}_2| + |\mathfrak{l}_3 - \mathfrak{n}_3| + |\mathfrak{l}_4 - \mathfrak{n}_4|)$$

*for all* $\tau \in [0,1]$ *and* $\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4 \in \mathcal{R}_e$.

*Here,* $\mathfrak{K}_0 = \frac{1}{6}$ *and* $\mathfrak{M}_0 = \frac{1}{8}$. *Furthermore, we deduce* $\rho_1 \approx 0.752252$ $\rho_2 \approx 0.249238$, $\Lambda \approx 0.03243582$, $\Lambda^* \approx 0.00404732$, $\mathcal{M} \approx 0.00404732$, $\mathcal{M}^* \approx 0.0247046$ *and*

$$\left[\Lambda + \Lambda^* + \frac{\mathcal{M}}{\Gamma(2-\mathfrak{p}_2)} + \frac{\mathcal{M}^*}{\Gamma(2-\mathfrak{p}_1)}\right] \approx 0.1447348 < 1. \tag{42}$$

*Hence, all the conditions outlined in Theorem 3 are fulfilled. Consequently, in accordance with Theorem 3, we can establish that Problems (1) and (2) possess a unique solution, denoted by* $\tau \in [0,1]$.

## 6. Conclusions

The primary objective of this research is to investigate the possible existence and uniqueness of a solution to a system of fractional integrodifferential equations. Sequential Liouville–Caputo derivatives and nonlinearities comprising both integral and differential terms are present in Equation (1). Additionally, the system is supplemented with Riemann–Liouville integral boundary conditions (2). It is noted that under these circumstances, the unknown functions $\mathcal{G}$ and $\mathcal{J}$ at point 1 exhibit dependence on both $\mathcal{G}$ and $\mathcal{J}$ across the entire interval $[0,1]$. The Leray–Schauder alternative theorem and the Banach contraction mapping principle were utilised in the demonstration of our primary theorems. The results obtained in our study corresponded to specific problems when the parameters $\mathfrak{v}, \mathfrak{z}$ were established. Suppose that Problems (1) and (2) are presented in a manner that the results are obtained by setting $\mathfrak{v} = \mathfrak{z} = 1$.

$$\begin{cases} \mathcal{G}(0) = 0, \ \mathcal{G}'(0) = 0, \ \mathcal{G}'(1) = 0, \ \mathcal{G}(1) = \int_0^\eta \mathcal{G}(v)dv + \int_0^\zeta \mathcal{J}(v)dv, \\ \mathcal{J}(0) = 0, \ \mathcal{J}'(0) = 0, \ \mathcal{J}'(1) = 0, \ \mathcal{J}(1) = \int_0^\eta \mathcal{G}(v)dv + \int_0^\zeta \mathcal{J}(v)dv. \end{cases}$$

The problems mentioned above can be solved using the same approach as Problems (1) and (2), as described in the preceding section.

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