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Qualitative Analysis of RLC Circuit Described by Hilfer Derivative with Numerical Treatment Using the Lagrange Polynomial Method

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Abstract: This paper delves into an examination of the existence, uniqueness, and stability properties of a non-local integro-differential equation featuring the Hilfer fractional derivative with order $\omega \in (1,2)$ for the RLC model. Based on Schaefer's fixed point theorem and Banach's contraction principle, the existence and uniqueness results are established. Furthermore, Ulam–Hyers and Ulam–Hyers–Rassias stability results for the boundary value problem of the RLC model are discussed. To showcase the practicality and efficacy of our theoretical findings, a two-step Lagrange polynomial interpolation method is applied to solve some numerical examples.

Keywords: fixed point theorem; fractional order integro-differential RLC circuit; Hilfer fractional derivative; non-local boundary conditions

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1. Introduction

Numerous studies in science and engineering have shown the importance of mathematical modeling and numerical simulations. Fractional-order modeling is one of the well-researched areas which has provided scientists with a useful technique for the generalization of classical results. The development of fractional order operators includes both local and non-local kernels, and singular and non-singular kernels are of great interest as well in the community of researchers. There have been some fascinating new studies examining these factors; see [1–5].

Hilfer [3] proposed an extended form of Riemann–Liouville (R-L) and Caputo fractional derivatives, called the "Hilfer fractional derivative", which allows one to interpolate with another; see [6–14].

Fractional derivatives offer numerous advantages when compared to their conventional counterparts. To begin with, they encompass memory, a fundamental characteristic in non-integer type differential equations. This quality renders fractional derivatives more effective in precisely characterizing physical systems in contrast to classical derivatives [15–18]. Furthermore, fractional derivatives play a pivotal role in enabling the exploration of various diffusion phenomena, encompassing superdiffusion, hyperdiffusion, and ballistic diffusion. This presents a fertile area of research for those intrigued by these phenomena [19].

The introduction of fractional derivatives resulted in the development of several types of novel mathematical models, particularly in the discipline of electrical circuits, which can be found in [20–22]. Some results have emerged for the investigation of fractional modeling of RL and RC circuits; hence, we refer the reader to [23–27]. The fractional LC circuit, first introduced in [25], extends to the repertory of available models. The exploration of both numerical and analytical solutions that provide the foundation of these investigations are essential to the investigation of fractional electrical circuits [26,27].

In [25], the AB derivative is employed to explore the numerical solutions for fractional RL and RC circuits. In [19], Aguilar et al. proposed the solutions for non-integer order electrical RC, LC, and RL circuits using the Mittag–Leffler fractional derivative. In [26], Rawdan et al. discussed fractional-order RL and LC circuits, suggesting a comparative analysis with conventional electrical circuits. Additionally, in [28], Aguilar et al. introduced fractional electrical circuits characterized by a non-integer derivative with a regular Kernel.

Integer order integro-differential equations find applications in various domains of science and engineering, including circuit analysis. According to Kirchhoff's second law, the total voltage drop across a closed loop is equal to the applied voltage, denoted as $\mathbb{E}(t)$. This principle essentially stems from the law of energy conservation. Consequently, an RLC circuit equation has the form

$$\mathbb{L}\frac{d}{dt}\mathbb{I}(t) + \mathbb{R}\mathbb{I}(t) + \frac{1}{\mathbb{C}}\int_0^t \mathbb{I}(s)ds = \mathbb{E}(t).$$

The RLC circuit serves as a fundamental component in the assembly of more intricate electrical circuits and networks. Illustrated in Figure 1, it comprises a resistor with a resistance of $\mathbb{R}$ ohms, an inductor with an inductance of $\mathbb{L}$ henries, and a capacitor with a capacitance of $\mathbb{C}$ farads, all arranged in series with an electromotive force source (like a battery or a generator) providing a voltage of $\mathbb{E}(t)$ volts at time t.

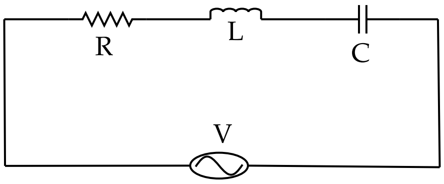


Figure 1. Diagram of a series RLC circuit.

In [29] U. Arshad et al. investigated the fractional order RLC derivative using three numerical methodologies of the system:

$$\begin{aligned} D^{2\alpha}\mathbb{I}(t) + \frac{1}{\mathbb{L}\mathbb{C}}\mathbb{I}(t) &= \frac{\mathbb{E}(t)}{\mathbb{L}} \\ D^{\alpha}\mathbb{V}(t) + \frac{1}{\mathbb{C}\mathbb{R}}\mathbb{V}(t) &= \frac{\mathbb{E}(t)}{\mathbb{R}} \\ D^{\alpha}\mathbb{I}(t) + \frac{\mathbb{R}}{\mathbb{L}}\mathbb{I}(t) &= \frac{\mathbb{E}(t)}{\mathbb{L}}. \end{aligned} \tag{1}$$

In [30], Malarvizhi et al. discussed the transient analysis of an RLC circuit in the RK4 order method. In [24], Gomez-Aguilar et al. studied the electrical circuits RC and RL for the Atangana–Beleanu–Caputo (ABC) fractional bi-order system:

$$^{ABC}D^{\beta}\mathbb{V}(t) = \delta\mathbb{E}(t) - \delta\mathbb{V}. \tag{2}$$

Inspired by the above mentioned work, we are interested in studying the existence and uniqueness of solutions and the Ulam stability analysis for the following Hilfer fractional differential equation for the RLC circuit model with non-local boundary conditions:

$$\begin{cases} D^{\omega,\tau}\mathbb{I}(t) = \frac{\mathbb{E}(t)}{\mathbb{L}} - \frac{\mathbb{R}}{\mathbb{L}}\mathbb{I}(t) - \frac{1}{\mathbb{C}\mathbb{L}}\int_0^t \mathbb{I}(s)ds, \quad t \in J = [a,b], \\ y(a) = 0, \quad y(b) = \sum_{j=1}^{k} \varrho_j I^{\nu_j} y(\zeta_j), \quad \nu_j > 0, \varrho_j \in \mathbb{R}, \zeta_j \in J. \end{cases} \tag{3}$$

The primary contribution of this endeavor can be outlined as follows:

1. The existence, uniqueness, and stability of the solution of the Hilfer fractional integro-differential equation for the RLC circuit model has been investigated via the fixed point approach.
2. We apply a novel hypothesis to verify the existence, uniqueness, and Ulam–Hyers stability of the solution to the RLC circuit Equation (3). Additionally, we illustrate numerical results using the two step Lagrangian polynomial approach, in order to validate the theoretical outcomes.

The paper is structured as follows. In Section 2, we introduce various definitions and preliminaries. The existence and uniqueness results for the Hilfer fractional boundary value problem for the RLC model are discussed in Section 3. The Ulam-type stability results are studied in Section 4. Some numerical examples are listed in Section 5. We end with Section 6 containing the conclusions.

2. Auxiliary Results

In this section, we recall some important preliminaries that are related to our analysis.

Let $Y = C[J,\mathbb{R}]$ be the space of all continuous function form J into $\mathbb{R}$ with norm $||v|| = \max\{|v(t)|,\ t \in J\}$. Obviously, Y is a Banach space under this norm, and hence, the product is also a Banach space with norm $||(v,w)|| = ||v|| + ||w||$.

Definition 1 ([10] Caputo fractional derivative)**.** *The Caputo derivative of order q for the function $g : J \to \mathbb{R}$ is defined as:*

$$ {}^{c}D^{\omega} g(x) = \frac{1}{\Gamma(p-\omega)} \int_a^x \frac{g^{(p)}(s)}{(x-s)^{\omega+1-p}}\,ds = I^{p-\omega} g^{(p)}(x), \quad x > 0,\ p-1 < \omega < p. $$

Definition 2 ([10] Riemann–Liouville fractional integral)**.** *The R-L fractional integral of order $\omega > 0$ of the function g is defined as:*

$$ I_a^{\omega} g(x) = \frac{1}{\Gamma(p-\omega)} \int_a^x \frac{g(s)}{(x-s)^{p-\omega-1}}\,ds, \quad p-1 < \omega < p. $$

Definition 3 ([10] Riemann–Liouville fractional derivative)**.** *The R-L fractional derivative of order $\omega > 0$ of a continuous function g is defined as:*

$$ \begin{aligned} {}^{RL}D^{\omega} g(t) &= D^p I^{p-\omega} g(t) \\ &= \frac{1}{\Gamma(p-\omega)} \left(\frac{d^p}{dt^p}\right) \int_a^t \frac{g(s)}{(t-s)^{p-\omega-1}}\,ds, \quad p-1 < \omega < p. \end{aligned} $$

By a new theory of the fractional derivative which had been proposed by Hilfer [3], the generalized R-L fractional derivative of a continuous function g is defined as:

Definition 4 ([10] Hilfer fractional derivative)**.** *The generalized R-L fractional derivative of order ω and parameter τ of a function g is described as:*

$$ {}^{H}D^{\omega,\tau} g(x) = I^{\tau(p-\omega)} D^p I^{(1-\tau)(p-\omega)} g(x), $$

where $\omega \in (p-1,p)$, $\tau \in [0,1]$, $x > a, D = \frac{d}{dx}$.

Remark 1 ([10])**.** *From Definition 4, we observe that:*

1. *The operator ${}^{H}D^{\omega,\tau}$ can be written as*

$$ {}^{H}D^{\omega,\tau} = I^{\tau(1-\omega)} D I^{(1-\gamma)} = I^{\tau(1-\omega)} D^{\gamma}, \qquad \gamma = \omega + \tau - \tau\omega. $$

2. *The Hilfer fractional derivative can be interpolated between the R-L fractional derivative* $(\tau = 0)$ *and the Caputo fractional derivative* $(\tau = 1)$ *as:*

$$^{H}D^{\omega,\tau} = \begin{cases} DI^{(1-\omega)} = D^{\omega}, & \text{if } \tau = 0; \\ I^{(1-\omega)}D = {}^{C}D^{\omega}, & \text{if } \tau = 1. \end{cases}$$

Lemma 1 ([10]). *If* $1 < \omega \leq 2$, *then,*

$$I^{\omega}(D^{\omega}g)(t) = g(t) - \frac{(I^{1-\omega}g)(a)}{\Gamma(\omega)}(t-a)^{\omega-1} - \frac{(I^{2-\omega}g)(a)}{\Gamma(\omega-1)}(t-a)^{\omega-2}. \tag{4}$$

3. Main Results

Here, we introduce some assumptions for the following sequels.

$(A1)$ The function $g : J \times Y \times Y \to Y$ is completely continuous and there exists a function $\mu \in L^{1}(J, \mathbb{R})$ such that:

$$|g(t,x,y)| \leq \mu(t), \quad \forall t \in J,\ x,y \in Y.$$

$(A2)$ The function g is continuous and there exist constants $\mathtt{L}_1, \mathtt{L}_2 > 0$ such that:

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq \mathtt{L}_1|x_1 - x_2| + \mathtt{L}_2|y_1 - y_2|, \quad \forall t \in J,\ x_i, y_i \in Y,\ i = 1,2.$$

$(A3)$ The function f is continuous and there exists a constant $\mathtt{M} > 0$ such that:

$$|f(t,s,x_1) - f(t,s,x_2)| \leq \mathtt{M}|x_1 - x_2|, \quad \forall t \in J,\ x_i \in Y,\ i = 1,2.$$

Problem Formulation

Let us consider the general structure of the Hilfer fractional order RLC circuit integro-differential equation with nonlocal boundary conditions:

$$^{H}D^{\omega,\tau}y(t) = g(t, y(t), H(y(s))), \quad t \in J, \tag{5}$$

$$y(a) = 0, \quad y(b) = \sum_{j=1}^{k} \varrho_j I^{\nu_j} y(\zeta_j), \quad \nu_j > 0, \varrho_j \in \mathbb{R}, \zeta_j \in J, \tag{6}$$

where $^{H}D^{\omega,\tau}$ is the Hilfer fractional derivative of order $\omega \in (1,2)$, and parameter $\tau \in [0,1]$, I^{ν_j} is the R-L fractional integral of order $\nu_j > 0, \zeta_j \in [a,b], a \geq 0$, $\varrho_j \in \mathbb{R}$, $j = 1, \ldots, k$, $g(t, y(t), \int_a^t f(t,s,y(s)ds) = \frac{\mathbb{E}(t)}{\mathbb{L}} - \frac{\mathbb{R}}{\mathbb{L}}\mathbb{I}(t) - \frac{1}{\mathbb{CL}}\int_0^t \mathbb{I}(s)ds$, and $H(y(s)) = \int_a^t f(t,s,y(s))ds$. Using some fixed point theorems, the existence and uniqueness results are established. For (5) and (6), we employ Banach's fixed point theorem and Schaefer's fixed point theorem for uniqueness and existence results.

Lemma 2. *For* $g \in C(J, \mathbb{R})$, *it is a solution of the boundary value problem,*

$$^{H}D^{\omega,\tau}y(t) = g(t, y(t), Hy(s)), \quad t \in J,$$

$$y(a) = 0, \quad y(b) = \sum_{j=1}^{k} \varrho_j I^{\nu_j} y(\zeta_j), \quad \nu_j > 0, \varrho_j \in \mathbb{R}, \zeta_j \in J,$$

which satisfies the following equation:

$$y(t) = \frac{(t-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}\left(I^{\omega}g(s,y(s),Hy(s))(b) - \sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j}g(s,y(s),Hy(s))(\zeta_j)\right)$$
$$+ I^{\omega}g(s,y(s),Hy(s))(t), \quad (7)$$

where

$$\Phi = \sum_{j=1}^{k}\frac{\varrho_j(\zeta_j-a)^{\sigma-\nu_j-1}}{\Gamma(\sigma+\nu_j)} - \frac{(b-a)^{\sigma-1}}{\Gamma(\sigma)} \neq 0, \quad (8)$$

where $j = 1,2,3,...,k,\ 1 < \omega < 2,\ \sigma = \omega + \tau - \omega\tau.$

Proof. Equation (7) can be written as:

$$I^{\tau(2-\omega)}D^2 I^{(1-\tau)(2-\tau)}y(t) = g(t). \quad (9)$$

As a result of determining the ω order integral of the related inequality, we obtain

$$I^{\omega}I^{\tau(2-\omega)}D^2 I^{(1-\tau)(2-\tau)}y(t) = I^{\omega}g(t).$$

Indeed,

$$I^{\omega}I^{\tau(2-\omega)}D^2 I^{(1-\tau)(2-\tau)}y(t) = I^{\sigma}D^2 I^{2-\sigma}y(t) = I^{\sigma}\left({}^{RL}D^{\sigma}y\right)(t),$$

and therefore,

$$I^{\sigma}\left({}^{RL}D^{\sigma}y\right)(t) = I^{\omega}g(t).$$

By using Equation (4) and setting $\left[I^{2-\omega}g\right](a) = c_1,\ \left[I^{1-\omega}g\right](a) = c_2$, one has

$$y(t) = \frac{c_2}{\Gamma(\sigma)}(t-a)^{\sigma-1} + \frac{c_1}{\Gamma(\sigma-1)}(t-a)^{\sigma-2} + I^{\omega}g(t,y(t),Hy(t)). \quad (10)$$

By the condition $y(a) = 0$, we obtain $c_1 = 0$. Then, we obtain

$$y(t) = \frac{c_2}{\Gamma(\sigma)}(t-a)^{\sigma-1} + I^{\omega}g(t,y(t),Hy(t)), \quad (11)$$

and

$$\sum_{j=1}^{k}\varrho_j I^{\nu_j}y(\zeta_j) = c_2\sum_{j=1}^{k}\frac{\varrho_j(\zeta_j-a)^{\sigma+\nu_j-1}}{\Gamma(\sigma+\nu_j)} + \sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j}g(s,y(s),Hy(s))(\zeta_j). \quad (12)$$

From our condition, by using (12), one has

$$c_2\left(\sum_{j=1}^{k}\frac{\varrho_j(\zeta_j-a)^{\sigma+\nu_j-1}}{\Gamma(\sigma+\nu_j)} - \frac{(t-a)^{\sigma-1}}{\Gamma(\sigma)}\right) = I^{\omega}g(s,y(s),Hy(s))(b)$$
$$- \sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j}g(s,y(s),Hy(s))(\zeta_j), \quad (13)$$

from which we obtain

$$c_2 = \frac{1}{\Phi}\left(I^{\omega}g(s,y(s),Hy(s))(b) - \sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j}g(s,y(s),Hy(s))(\zeta_j)\right). \quad (14)$$

Substituting the value of c_1 and c_2 in (10), we obtain the solution (7). This completes the proof. □

Theorem 1. *Assume that* $(A1)$ *is verified. Then* (5) *and* (6) *admit at least one solution on* J.

Proof. Let $C = C_{1-\sigma}[J, Y]$ and define the operator $P : C \to C$ by,

$$(Py)(t) = \frac{(t-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} I^{\omega} g(s, y(s), Hy(s))(b) - \frac{(t-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} \sum_{j=1}^{k} \varrho_j I^{\omega+\nu_j} g(s, y(s), Hy(s))(\zeta) + I^{\omega} g(s, y(s), Hy(s))(t).$$

For $q > 0$, let

$$B_q = \{y |\, y \in C : \|y\| \leq q\}.$$

Step 1: P is continuous.

Let y_n be a sequence such that $y_n \to y$ in C. For each $t \in J$, one has

$$\begin{aligned}
&|(t-a)^{1-\sigma}((Py_n)(t) - (Py)(t))| \\
&= |\frac{1}{\Phi\Gamma(\sigma)}\Big[\frac{1}{\Gamma(\omega)}\int_a^t (b-s)^{\omega-1}(g(s, y_n(s), Hy_n(s)) - g(s, y(s), Hy(s)))ds \\
&- \sum_{j=1}^{k} \varrho_j \frac{1}{\Gamma(\omega+\zeta_j)}\int_a^t (\zeta_j - s)^{\omega+\zeta_j-1}(g(s, y_n(s), Hy_n(s)) - g(s, y(s), Hy(s)))ds\Big] \\
&+ \frac{(t-a)^{1-\sigma}}{\Gamma(\omega)}\int_a^t (t-s)^{\omega-1}(g(s, y_n(s), Hy_n(s)) - g(s, y(s), Hy(s)))ds| \\
&\leq \frac{1}{|\Phi|\Gamma(\sigma)}\Big[\frac{1}{\Gamma(\omega)}\int_a^t (b-s)^{\omega-1}|(g(s, y_n(s), Hy_n(s)) - g(s, y(s), Hy(s)))|ds \\
&- \sum_{j=1}^{k} |\varrho_j| \frac{1}{\Gamma(\omega+\zeta_j)}\int_a^t (\zeta_j - s)^{\omega+\zeta_j-1}|(g(s, y_n(s), Hy_n(s)) - g(s, y(s), Hy(s)))|ds\Big] \\
&+ \frac{(t-a)^{1-\sigma}}{\Gamma(\omega)}\int_a^t (t-s)^{\omega-1}|(g(s, y_n(s), Hy_n(s)) - g(s, y(s), Hy(s)))|ds, \\
&\leq \frac{1}{\|\Phi\|\Gamma(\sigma)}\Big[\frac{(b-s)^{\omega}}{\Gamma(\omega+1)}\|(g(s, y_n(s), Hy_n(s)) - g(s, y(s), Hy(s)))\|_C \\
&- \sum_{j=1}^{k} \|\varrho_j\| \frac{(\zeta_j - s)^{\omega+\zeta_j}}{\Gamma(\omega+\zeta_j+1)}\|(g(s, y_n(s), Hy_n(s)) - g(s, y(s), Hy(s)))\|_C\Big] \\
&+ \frac{(t-s)^{\omega}(t-a)^{\sigma-1}}{\Gamma(\omega+1)}\|g(s, y_n(s), Hy_n(s)) - g(s, y(s), Hy(s))\|_C.
\end{aligned}$$

Since the function g is continuous, then we obtain

$$\begin{aligned}
&\|(t-a)^{1-\sigma}((Py_n)(t) - (Py)(t))\| \\
&\leq \frac{1}{\|\Phi\|\Gamma(\sigma)}\Big[\frac{(b-s)^{\omega}}{\Gamma(\omega+1)}\|(g(., y_n(.), Hy_n(.)) - g(., y(.), Hy(.)))\|_C \\
&- \sum_{j=1}^{k} \|\varrho_j\| \frac{(\zeta_j - s)^{\omega+\zeta_j}}{\Gamma(\omega+\zeta_j+1)}\|(g(., y_n(.), Hy_n(.)) - g(., y(.), Hy(.)))\|_C\Big] \\
&+ \frac{(t-s)^{\omega}(t-a)^{\sigma-1}}{\Gamma(\omega+1)}\|g(., y_n(.), Hy_n(.)) - g(., y(.), Hy(.))\|_C \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, the operator P is continuous.
Step 2: $P(B_q)$ is bounded.
For each $t \in J$ and $y \in B_q$, we obtain that:

$$
\begin{aligned}
|(t-a)^{1-\sigma}(Py)(t)| \leq{} & \frac{1}{|\Phi|\Gamma(\sigma)}\left[\frac{1}{\Gamma(\omega)}\int_a^t (b-s)^{\omega-1}|g(s,y(s),Hy(s))|ds\right.\\
& \left.-\sum_{j=1}^{k}|\varrho_j|\frac{1}{\Gamma(\omega+\zeta_j)}\int_a^t(\zeta_j-s)^{\omega+\zeta_j-1}|g(s,y(s),Hy(s))|ds\right]\\
& +\frac{(t-a)^{1-\sigma}}{\Gamma(\omega)}\int_a^t (t-s)^{\omega-1}|g(s,y(s),Hy(s))|ds,\\
\leq{} & \frac{1}{|\Phi|\Gamma(\sigma)}\Big[\frac{1}{\Gamma(\omega)}\int_a^t (b-s)^{\omega-1}|\mu(s)|ds-\sum_{j=1}^{k}|\varrho_j|\frac{1}{\Gamma(\omega+\zeta_j)}\\
& \times\int_a^t(\omega\zeta_j-s)^{\omega+\zeta_j-1}|\mu(s)|ds\Big]+\frac{(t-a)^{1-\sigma}}{\Gamma(\omega)}\int_a^t(t-s)^{\omega-1}|\mu(s)|ds\\
\leq{} & \frac{\|\mu(s)\|_C}{|\Phi|\Gamma(\sigma)}\Big[\frac{(b-s)^{\omega}}{\Gamma(\omega+1)}-\sum_{j=1}^{k}|\varrho_j|\frac{(\zeta_j-s)^{\omega+\zeta_j}}{\Gamma(\omega+\zeta_j+1)}\Big]+\frac{(t-s)^{\omega}(t-a)^{1-\sigma}}{\Gamma(\omega+1)}:=\ell.
\end{aligned}
$$

Thus, $\|P(y)\| \leq \ell$.
Step 3: $P(B_q)$ is equicontinuous.
For $a \leq t_1 < t_2 \leq b,$ and $y \in B_q$, we obtain
$|(t_2-a)^{1-\sigma}(Py)(t_2)-(t_1-a)^{1-\sigma}(Py)(t_1)|$

$$
\begin{aligned}
\leq{} & \frac{1}{|\Phi|\Gamma(\sigma)}\left[\frac{1}{\Gamma(\omega)}\int_a^{t_2}(b-s)^{\omega-1}|g(s,y(s),Hy(s))|ds\right.\\
& \left.-\sum_{j=1}^{k}\frac{|\varrho_j|}{\Gamma(\omega+\zeta_j)}\int_a^{t_2}(\zeta_j-s)^{\omega+\zeta_j}|g(s,y(s),Hy(s))|ds\right]\\
& +\frac{(t_2-a)^{\sigma-1}}{\Gamma(\omega)}\int_a^{t_2}(t_2-s)^{\omega-1}|g(s,y(s),Hy(s))|ds\\
& -\frac{1}{|\Phi|\Gamma(\sigma)}\Big[\frac{1}{\Gamma(\omega)}\int_a^{t_1}(b-s)^{\omega-1}|g(s,y(s),Hy(s))|ds\\
& +\sum_{j=1}^{k}\frac{|\varrho_j|}{\Gamma(\omega+\zeta_j)}\int_a^{t_1}(\zeta_j-s)^{\omega+\zeta_j}|g(s,y(s),Hy(s))|ds\Big]\\
& -\frac{(t_1-a)^{\sigma-1}}{\Gamma(\omega)}\int_a^{t_1}(t_1-s)^{\omega-1}|g(s,y(s),Hy(s))|ds\\
\leq{} & \frac{|g(s,y(s),Hy(s))|}{|\Phi|\Gamma(\sigma)\Gamma(\omega)}\left[\int_a^{t_2}(b-s)^{\omega-1}ds-\int_a^{t_1}(b-s)^{\omega-1}ds\right]
\end{aligned}
$$

$$- \sum_{j=1}^{k} \frac{\mid \varrho_j \mid |g(s,y(s),Hy(s))|}{\mid \Phi \mid \Gamma(\sigma)\Gamma(\omega+\zeta_j)} \left[\int_a^{t_2} (\zeta_j - s)^{\omega+\zeta_j-1} ds - \int_a^{t_1} (\zeta_j - s)^{\omega-1} ds \right]$$
$$+ \frac{|g(s,y(s),Hy(s))|(t_2-a)^{\sigma-1}}{\Gamma(\omega)} \int_{t_1}^{t_2} (t_2 - t_1)^{\omega-1} ds$$
$$\leq \frac{\|\mu\|_C}{\mid \Phi \mid \Gamma(\sigma)\Gamma(\omega)} \left[\int_a^{t_2} (b-s)^{\omega-1} ds - \int_a^{t_1} (b-s)^{\omega-1} ds \right]$$
$$- \sum_{j=1}^{k} \frac{\mid \varrho_j \mid \|\mu\|_C}{\mid \Phi \mid \Gamma(\sigma)\Gamma(\omega+\zeta_j)} \left[\int_a^{t_2} (\zeta_j - s)^{\omega+\zeta_j-1} ds - \int_a^{t_1} (\zeta_j - s)^{\omega-1} ds \right]$$
$$+ \frac{\|\mu\|_C (t_2-a)^{\sigma-1}}{\Gamma(\omega)} \int_{t_1}^{t_2} (t_2 - t_1)^{\omega-1} ds.$$

As $t_2 \to t_1$, the R.H.S. of the above inequality $\to 0$. Consequently, we deduce that P is completely continuous.

Step 4: The priori bounds.

We need to show that the set $\Lambda = \{y \in C : y = \varrho(P(y));\ \varrho \in (0,1)\}$ is bounded.

For this, let $y \in \Lambda$, $y = \varrho(P(y))$ for some $\varrho \in (0,1)$. Thus, for each $t \in J$, one has

$$y(t) = \varrho \Big[\frac{(t-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} I^{\omega} g(s,y(s),Hy(s))(b)$$
$$+ \frac{(t-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} \sum_{j=1}^{k} \varrho_j I^{\omega+\nu_j} g(s,y(s),Hy(s))(\zeta) + I^{\omega} g(s,y(s),Hy(s))(t) \Big].$$

This implies, by $(A2)$, that:

$$|y(t)(t-a)^{1-\sigma}| \leq |(t-a)^{1-\sigma}(Py)(t)|$$
$$\leq \frac{1}{\mid \Phi \mid \Gamma(\sigma)} \left[\frac{1}{\Gamma(\omega)} \int_a^t (b-s)^{\omega-1} |g(s,y(s),Hy(s))| ds \right.$$
$$\left. - \sum_{j=1}^{k} |\varrho_j| \frac{1}{\Gamma(\omega+\zeta_j)} \int_a^t (\zeta_j - s)^{\omega+\zeta_j-1} |g(s,y(s),Hy(s))| ds \right]$$
$$+ \frac{(t-a)^{1-\sigma}}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1} |g(s,y(s),Hy(s))| ds$$
$$\leq \frac{1}{\mid \Phi \mid \Gamma(\sigma)} \left[\frac{1}{\Gamma(\omega)} \int_a^t (b-s)^{\omega-1} |\mu(s)| ds \right.$$
$$\left. - \sum_{j=1}^{k} |\varrho_j| \frac{1}{\Gamma(\omega+\zeta_j)} \int_a^t (\zeta_j - s)^{\omega+\zeta_j-1} |\mu(s)| ds \right]$$
$$+ \frac{(t-a)^{1-\sigma}}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1} |\mu(s)| ds$$
$$\leq \frac{1}{\mid \Phi \mid \Gamma(\sigma)} \left[\frac{(b-s)^{\omega}}{\Gamma(\omega+1)} \|\mu(s)\|_C - \sum_{j=1}^{k} |\varrho_j| \frac{(\zeta_j - s)^{\omega}}{\Gamma(\omega+\zeta_j+1)} \|\mu(s)\|_C \right]$$
$$+ \frac{(t-a)^{1-\sigma}(t-s)^{\omega}}{\Gamma(\omega+1)} \|\mu(s)\|_C := \Re.$$

Thus, $\|\mu(s)\|_C \leq \Re$.

Therefore, the set Λ is bounded. Hence, we deduce that P has a fixed point that is a solution of the presumed problems (5) and (6) as an outcome of Schaefer's Fixed point theorem. □

The next theorem contains the second main result in this paper that is the uniqueness of the solution to the presumed problems (5) and (6).

Theorem 2. *Suppose that the conditions* $(A2)$ *and* $(A3)$ *are satisfied such that:*

$$(L_1 + L_2 M)\left[\frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)\Gamma(\omega+1)} + \frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)}\sum_{i=0}^{m}\frac{|\varrho_j|(\zeta_j-a)^{\omega+\zeta_j}}{\Gamma(\omega+\nu_j+1)} + \frac{(b-a)^{\omega}}{\Gamma(\omega+1)}\right] < 1. \quad (15)$$

Then, the presumed problem (5) and (6) has a unique solution on J.

Proof. We consider the operator $P : C \to C$ defined as

$$(Py)(t) = \frac{(t-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} I^{\omega} g(s, y(s), Hy(s))(b) + \frac{(t-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}\sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j} g(s, y(s), Hy(s))(\zeta)$$
$$+ I^{\omega} g(s, y(s), Hy(s))(t).$$

We shall show that P is a contraction map. Let $x, y \in C$, then one has for each $t \in J$

$$\begin{aligned}|(Py)(t) - (Px)(t)| \leq{}& \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} I^{\omega}|g(s, y(s), Hy(s)) - g(s, x(s), Hx(s))|(b)\\
&+ \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}\sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j}|g(s, y(s), Hy(s)) - g(s, x(s), Hx(s))|(\zeta)\\
&+ I^{\omega}|g(s, y(s), Hy(s)) - g(s, x(s), Hx(s))|(t)\\
&\leq (L_1 + L_2 M)|y(s) - x(s)|\left[\frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)\Gamma(\omega+1)}\right.\\
&\left.+ \frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)}\sum_{i=0}^{m}\frac{|\varrho_j|(\zeta_j-a)^{\omega+\zeta_j}}{\Gamma(\omega+\nu_j+1)} + \frac{(t-a)^{\omega}}{\Gamma(\omega+1)}\right].\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|(Py)(t) - (Px)(t)\| \leq{}& (L_1 + L_2 M)\left[\frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)\Gamma(\omega+1)}\right.\\
&\left.+ \frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)}\sum_{i=0}^{m}\frac{|\varrho_j|(\zeta_j-a)^{\omega+\zeta_j}}{\Gamma(\omega+\nu_j+1)} + \frac{(b-a)^{\omega}}{\Gamma(\omega+1)}\right]\|y - x\|.\end{aligned}$$

Hence, in view of the condition (15) and the Banach contraction principle, P has a unique fixed point. Thus, the existence of the unique solution of the presumed problems (5) and (6). □

4. Ulam Stability Results

An important part of the qualitative theory of linear and nonlinear differential equations is the stability of Ulam–Hyers (UH), originally formulated by Hyers and Ulam in 1940. Also, the study of stability analysis of Hyers–Ulam (HU) and the Ulam–Hyers–Rassias (UHS) for non-linear fractional differential equations is a hot topic of research and the study of this area has grown to be one of the most important subjects in the mathematical analysis, see [31–35]. A general view of the development of the Ulam–Hyers (UH) and the Ulam—Hyers–Rassias (UHS) stability theory for fractional differential equations can be found in [36–39].

Definition 5 ([11]). *Equations (5) and (6) are UH stable if there exists a real number $C_g > 0$ such that for each $\epsilon > 0$ and for each $z \in C^{\sigma}_{1-\sigma}[J]$ solution of the inequality:*

$$|D^{\omega,\tau}_{0^+} z(t) - g(t, z(t), Hz(t))| \leq \epsilon, \quad t \in J, \tag{16}$$

there exists a solution $y \in C^{\sigma}_{1-\sigma}(J)$ of Equations (5) and (6) such that:

$$| z(t) - y(t) | \leq C_g \epsilon, \quad t \in J.$$

Definition 6 ([11]). *Equations (5) and (6) are generalized UH stable if there exists $\psi_g \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\psi_g(0) = 0$, such that for a solution $z \in C^{\sigma}_{1-\sigma}(J)$ of the inequality:*

$$|D^{\omega,\tau}_{0^+} z(t) - g(t, z(t), Hz(t))| \leq \epsilon, \quad t \in J, \tag{17}$$

there exists a solution $y \in C^{\sigma}_{1-\sigma}(J)$ of Equations (5) and (6) such that:

$$| z(t) - y(t) | \leq \psi_g(\epsilon), \quad t \in J.$$

Definition 7 ([11]). *Equations (5) and (6) are UHS stable with respect to $\nu \in C(J, \mathbb{R}_+)$ if there exists a real number $c_{g,\nu} > 0$ such that for each $\epsilon > 0$ and for each $z \in C^{\sigma}_{1-\sigma}(J)$ solution of the inequality:*

$$|D^{\omega,\tau}_{0^+} z(t) - g(t, z(t), Hz(t))| \leq \epsilon \nu(t), \quad t \in J, \tag{18}$$

there exists a solution $y \in C^{\sigma}_{1-\sigma}(J)$ of Equations (5) and (6) such that:

$$| z(t) - y(t) | \leq c_{g,\nu} \epsilon \nu(t), \quad t \in J.$$

Definition 8 ([11]). *Equations (5) and (6) is generalized UHS stable with respect to $\nu \in C(J, \mathbb{R}_+)$ if there exist $C_{g,\nu} > 0$ such that for each $z \in C^{\sigma}_{1-\sigma}(J)$ solution of the inequality:*

$$|D^{\omega,\tau}_{0^+} z(t) - g(t, z(t), Hz(t))| \leq \nu(t), \quad t \in J, \tag{19}$$

there exists $y \in C^{\sigma}_{1-\sigma}(J)$ solution of Equations (5) and (6) such that:

$$| z(t) - y(t) | \leq c_{g,\nu} \nu(t), \quad t \in J.$$

Remark 2 ([11]). *A function $z \in C^{\sigma}_{1-\sigma}(J)$ is a solution of the inequality:*

$$|D^{\omega,\tau}_{0^+} z(t) - g(t, z(t), Hz(t))| \leq \epsilon, \quad t \in J,$$

if there exists a function $w \in C^{\sigma}_{1-\sigma}(J)$ such that:

1. $| w(t) | \leq \epsilon, \quad t \in J,$
2. $D^{\omega,\tau}_{0^+} z(t) = g(t, z(t), Hz(t)) + w(t), \quad t \in J.$

Remark 3 ([11]). *It is clear that:*

1. *Definition* (5) $\Rightarrow$ *Definition* (6).
2. *Definition* (7) $\Rightarrow$ *Definition* (8).

Theorem 3. *Assume that $(A1)$ and (15) are satisfied, then the presumed problems (5) and (6) is UH stable.*

Proof. Let $z \in C^{\sigma}_{1-\sigma}(J)$ be a solution of the inequality (16) and let $y \in C^{\sigma}_{1-\sigma}[a, b]$ be a unique solution of the given system:

$$
{}^{H}D^{\omega,\tau}y(t) = g(t, y(t), \int_a^t g(t, s, y(s))ds), \quad t \in J, 1 < \omega < 2, 0 \le \tau \le 1,
$$

$$
y(a) = 0, \quad y(b) = \sum_{j=1}^{k} \varrho_j I^{\nu_j} x(\zeta_j), \quad \nu_j > 0, \varrho_j \in \mathbb{R}, \zeta_j \in J,
$$

where $1 < \omega < 2$ and parameter $0 \le \tau \le 1$.

In view of Remark 2, we have

$$
\begin{aligned}
|z(t) - \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} I^{\omega} g(s, z(s), Hz(s))(b) + \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} \sum_{j=1}^{k} \varrho_j I^{\omega+\nu_j} g(s, z(s), Hz(s))(\zeta) \\
- I^{\omega} g(s, z(s), Hz(s))(t)| \le \frac{\epsilon t^{\omega}}{\Gamma(\omega+1)} \le \frac{\epsilon b^{\omega}}{\Gamma(\omega+1)}.
\end{aligned}
$$

Then for each $t \in J$, we obtain

$$
\begin{aligned}
|z(t) - y(t)| \le & |z(t) - \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} I^{\omega} g(s, y(s), Hy(s))(b) \\
& + \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} \sum_{j=1}^{k} \varrho_j I^{\omega+\nu_j} g(s, y(s), Hy(s))(\zeta) - I^{\omega} g(s, y(s), Hy(s))(t)| \\
\le & |z(t) - \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} I^{\omega} g(s, z(s), Hz(s))(b) \\
& + \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} \sum_{j=1}^{k} \varrho_j I^{\omega+\nu_j} g(s, z(s), Hz(s))(\zeta) - I^{\omega} g(s, z(s), Hz(s))(t)| \\
& + |\frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} I^{\omega} \{g(s, z(s), Hz(s)) - g(s, y(s), Hy(s))\}(b) \\
& - \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)} \sum_{j=1}^{k} \varrho_j I^{\omega+\nu_j} \{g(s, z(s), Hz(s)) - g(s, y(s), Hy(s))\}(\zeta) \\
& + I^{\omega} \{g(s, z(s), Hz(s)) - g(s, y(s), Hy(s))\}(t)| \\
\le & \frac{\epsilon b^{\omega}}{\Gamma(\omega+1)} + (\mathtt{L}_1 + \mathtt{L}_2\mathtt{M}) \Bigg(\frac{(b-a)^{\omega+\sigma-1}}{|\Phi| \Gamma(\sigma)\Gamma(\omega+1)} \\
& + \frac{(b-a)^{\omega+\sigma-1}}{|\Phi| \Gamma(\sigma)} \sum_{i=0}^{m} \frac{|\varrho_j| (\zeta_j - a)^{\omega+\zeta_j}}{\Gamma(\omega+\nu_j+1)} + \frac{(b-a)^{\omega}}{\Gamma(\omega+1)} \Bigg) |z(t) - y(t)|, \\
\le & \frac{\epsilon b^{\omega}}{\Gamma(\omega+1)} + \mathtt{K}|z(t) - y(t)| \\
\le & \frac{\epsilon b^{\omega}}{(1-\mathtt{K})\Gamma(\omega+1)}.
\end{aligned}
$$

Therefore,

$$
|z(t) - y(t)| \le c_g \epsilon,
$$

where,

$$\mathtt{K} =: (\mathtt{L}_1 + \mathtt{L}_2\mathtt{M})\left(\frac{(b-a)^{\omega+\sigma-1}}{\mid \Phi \mid \Gamma(\sigma)\Gamma(\omega+1)} + \frac{(b-a)^{\omega+\sigma-1}}{\mid \Phi \mid \Gamma(\sigma)}\sum_{i=0}^{m}\frac{\mid \varrho_j \mid (\zeta_j - a)^{\omega+\zeta_j}}{\Gamma(\omega+\nu_j+1)} + \frac{(b-a)^{\omega}}{\Gamma(\omega+1)}\right).$$

This shows that (5) and (6) is UH stable. □

Theorem 4. *Assume that* $(A1)$–$(A3)$ *and* (15) *hold. Then, there exists an increasing function* $\nu \in C_{1-\sigma}[J]$ *and a real number* $\varsigma_\nu > 0$ *such that:*

$$|z(t) - y(t)| \leq \varsigma_\nu \varphi(t), \quad t \in J.$$

Then (5) *and* (6) *are UHR stable.*

Proof. Let $z \in C^{\sigma}_{1-\sigma}[a,b]$ be a solution of the inequality (18) and let $x \in C^{\sigma}_{1-\sigma}(J)$ be the unique solution of the given system:

$$\begin{aligned} &{}^{H}D^{\omega,\tau}y(t) = g(t, y(t), \int_a^t g(t,s,y(s))ds), \quad t, \\ &y(a) = 0, \quad y(b) = \sum_{j=1}^{k}\varrho_j I^{\nu_j}x(\zeta_j), \quad \nu_j > 0, \varrho_j \in \mathbb{R}, \zeta_j, \end{aligned}$$

where $1 < \omega < 2$ and parameter $0 \leq \tau \leq 1$.

By Remark 2, we have

$$\begin{aligned} |z(t) - \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}I^{\omega}g(s,z(s),Hz(s))(b) + \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}\sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j}g(s,z(s),Hz(s))(\zeta) \\ - I^{\omega}g(s,z(s),Hz(s))(t)| \leq \epsilon\varsigma_\nu\varphi(t). \end{aligned}$$

Then, for any $t \in J$, we obtain

$$\begin{aligned} |z(t) - y(t)| \leq &\left| z(t) - \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}I^{\omega}g(s,y(s),Hy(s))(b) \right. \\ &\left. + \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}\sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j}g(s,y(s),Hy(s))(\zeta) - I^{\omega}g(s,y(s),Hy(s))(t) \right| \\ \leq &\left| z(t) - \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}I^{\omega}g(s,z(s),Hz(s))(b) \right. \\ &\left. + \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}\sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j}g(s,z(s),Hz(s))(\zeta) - I^{\omega}g(s,z(s),Hz(s))(t) \right| \\ &+ \left| \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}I^{\omega}\{g(s,z(s),Hz(s)) - g(s,y(s),Hy(s))\}(b) \right. \\ &- \frac{(b-a)^{\sigma-1}}{\Phi\Gamma(\sigma)}\sum_{j=1}^{k}\varrho_j I^{\omega+\nu_j}\{g(s,z(s),Hz(s)) - g(s,y(s),Hy(s))\}(\zeta) \\ &\left. + I^{\omega}\{g(s,z(s),Hz(s)) - g(s,y(s),Hy(s))\}(t) \right| \end{aligned}$$

$$\le \epsilon\varsigma_\nu\varphi(t) + (\mathtt{L}_1+\mathtt{L}_2\mathtt{M})\left(\frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)\Gamma(\omega+1)}\right.$$
$$\left.+\frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)}\sum_{i=0}^{m}\frac{|\varrho_j|(\zeta_j-a)^{\omega+\zeta_j}}{\Gamma(\omega+\nu_j+1)}+\frac{(b-a)^{\omega}}{\Gamma(\omega+1)}\right)|z(t)-y(t)$$
$$\le \epsilon\varsigma_\nu\varphi(t) + \mathtt{K}|z(t)-y(t)|$$
$$\le \frac{\epsilon\varsigma_\nu\varphi(t)}{(1-\mathtt{K})\Gamma(\omega)}.$$

Therefore, we obtain that:

$$|z(t)-y(t)| \le c_{g,\nu}\epsilon\nu(t).$$

Hence, (5) and (6) are UHR stable. □

5. Examples

Example 1. *Consider the nonlocal BVP's by using Hilfer FIDE's of the form.*

$$\begin{cases} {}^{H}D^{\omega,\tau}y(t) = \frac{\cos^2 t}{(e^{-t+2})^2|y(t)|} + \frac{1}{2}\int\limits_0^t e^{-1/2}y(s)ds, & t\in[\frac{3}{10},\frac{13}{10}], \\ y(\frac{3}{10}) = 0, \quad y(\frac{13}{10}) = \frac{17}{50}I^{\frac{13}{15}}y(\frac{23}{50}) + \frac{21}{50}I^{\frac{37}{100}}y(\frac{91}{100}) + \frac{3}{25}I^{\frac{41}{100}}y(\frac{4}{5}). \end{cases} \tag{20}$$

where $\omega=\frac{6}{5}$, $\tau=\frac{1}{5}$, $\sigma=\frac{7}{4}$, $a=\frac{3}{10}$, $b=\frac{13}{10}$, $\varrho_1=\frac{17}{50}$, $\varrho_2=\frac{21}{50}$, $\varrho_3=\frac{3}{25}$, $\nu_1=\frac{13}{15}$, $\nu_2=\frac{37}{100}$, $\nu_3=\frac{41}{100}$, $\zeta_1=\frac{23}{50}$, $\zeta_2=\frac{91}{100}$, $\zeta_3=\frac{4}{5}$, $L_1=L_2=\frac{1}{9}$, $M=\frac{1}{8}$.

Hence, the assumptions $(A2)$ *and* $(A3)$ *hold.*

We check the condition,

$$(L_1+L_2M)\left[\frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)\Gamma(\omega+1)}+\frac{(b-a)^{\omega+\sigma-1}}{|\Phi|\Gamma(\sigma)}\sum_{i=0}^{m}\frac{|\varrho_j|(\zeta_j-a)^{\omega+\zeta_j}}{\Gamma(\omega+\nu_j+1)}+\frac{(b-a)^{\omega}}{\Gamma(\omega+1)}\right]$$
$$<1\approx 0.7347.$$

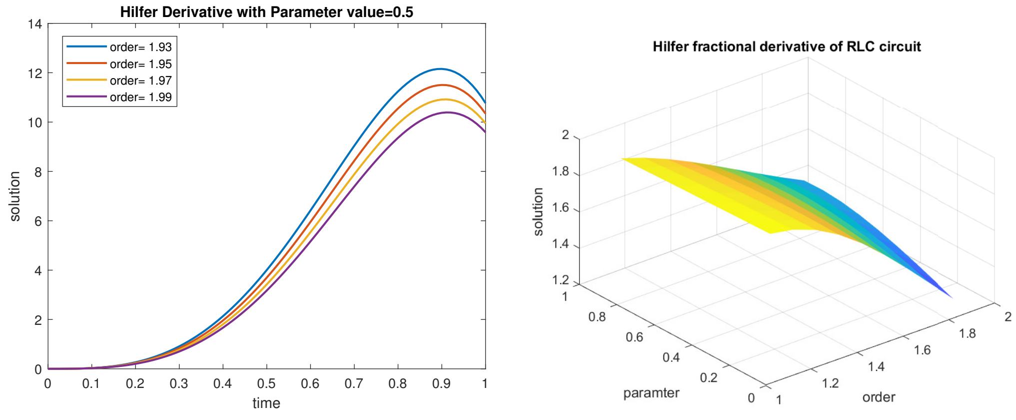
Hence, the problem (20) *has a unique solution on* $[\frac{3}{10},\frac{13}{10}]$.

Example 2. *Examine the RLC circuit equation of the Hilfer fractional differential equation of the form.*

$$\begin{cases} {}^{H}D^{\omega,\tau}\mathbb{I}(t) = \frac{\mathbb{E}_0}{\mathbb{L}} - \frac{\mathbb{R}}{\mathbb{L}}\mathbb{I}(t) - \frac{1}{\mathbb{CL}}\int\limits_0^t \mathbb{I}(s)ds, \quad t\in[0,1], \\ y(0)=0, \quad y(1) = 0.34I^{0.87}y(0.46) + 0.42I^{0.37}y(0.91) + 0.12I^{0.41}y(0.8). \end{cases} \tag{21}$$

RLC circuits are commonly used in filter design, where they can be configured as low-pass, high-pass, band-pass, or band-stop filters. These filters are crucial in signal processing, telecommunications, and audio electronics. It is used in tuned circuits, which are employed in radio receivers to select a particular frequency from a mixture of signals. This is essential for tuning in to specific radio stations. It can be used in control systems for tasks such as damping oscillations and stabilizing feedback loops.

In Figure 2a, various fractional orders fix the parameter value at 0.5 for the RLC circuit equation. Figure 2b represents the three-dimensional view of RLC with the circuit elements, $\mathbb{R}=4$, $\mathbb{I}=2$, $\mathbb{C}=5$, and $\mathbb{E}_0=10$.



(**a**) (**b**)

Figure 2. RLC circuit equation with Hilfer fractional derivative of parameters $\mathbb{R} = 4$, $\mathbb{I} = 2$, $\mathbb{C} = 5$, and $\mathbb{E}_0 = 10$. (**a**) Hilfer Fractional Derivative of RLC circuit; (**b**) 3D-view of RLC circuit.

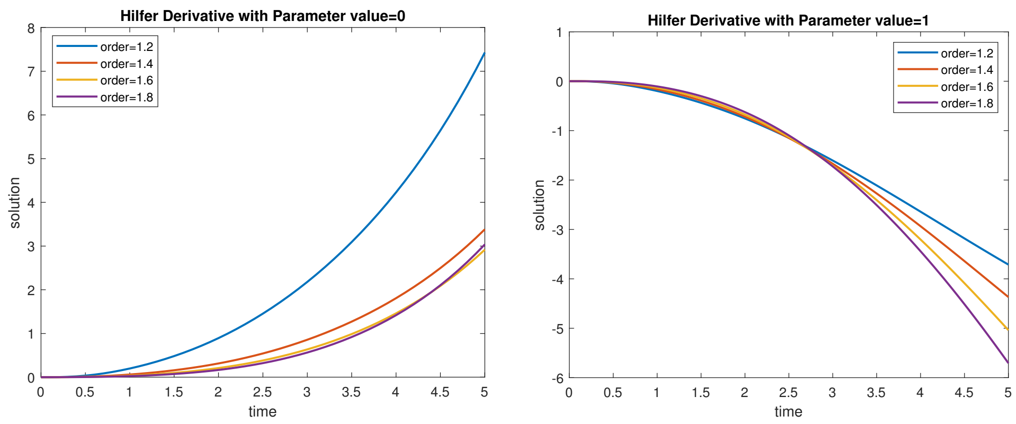
Example 3. *Consider the following integro-differential equation of the Hilfer fractional differential equation of the form.*

$$\begin{cases} {}^{H}D^{\omega,\tau}y(t) = \frac{\cos^2(t)}{(e^{-t+2})^2} + \frac{1}{2}e^{-1/2}\int\limits_0^t y(s)ds, & t \in [0,5], \\ y(0) = 0, \quad y(5) = 0.34I^{0.87}y(0.46) + 0.42I^{0.37}y(0.91) + 0.12I^{0.41}y(0.8). \end{cases} \tag{22}$$

In these figures, the significance of fractional order derivatives is clearly revealed. In order to show the significance of the fractional order derivative, the output responses of the considered systems with respect to the Riemann-Liouville, Caputo, and Hilfer derivative are graphically represented in Figures 3–5.

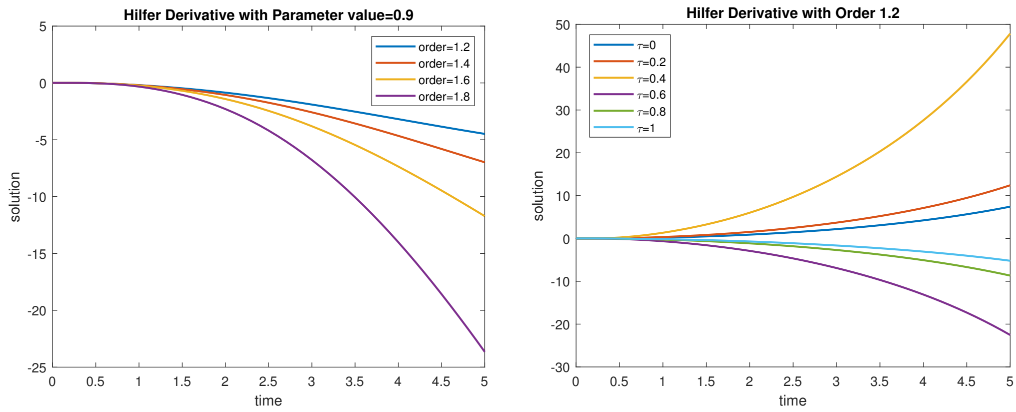
Notably, in Figure 3a, for the distinct values of order ω, $(\omega = 1.2, 1.4, 1.6, 1.8)$ with the parameter $\tau = 0$ is plotted. Similarly, for $\tau = 1$, it is plotted in Figure 3b. Figure 4a pictures $\tau = 0.9$. In Figure 4b, it should be noted that the Hilfer fractional order derivative is defined for different values of τ, which lies between 0 and 1.

In addition to this, a 3D plot with respect to the order ω, parameter τ and $y(t)$ is given in Figure 5. This figure clearly pictures the the impact of order ω and parameter τ for obtaining the solution of the considered systems. Overall, from the simulation result, the robustness of the developed methodology is validated.



(**a**) (**b**)

Figure 3. Different Fractional order of R-L and Caputo Derivative. (**a**) Riemann-Liouville Fractional Derivative, (**b**) Caputo Fractional Derivative.



(**a**) (**b**)

Figure 4. Hilfer Fractional Derivative. (**a**) Different Fractional Order, (**b**) Different Parameter Values.

The solution representation is modified when we change the order and parameter. One of the main benefits of our problem of non-local integro differential boundary value problems is that, while this changes the small size of the order and parameter values, it can have a major effect when applied to a real-world problem.

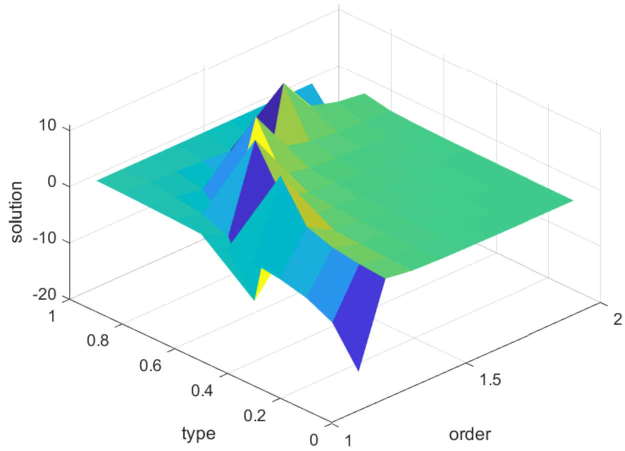


Figure 5. 3D-View of Hilfer Derivative of Different Orders and Parameter at $t = 2.5$.

Example 4. *Consider the following non-local boundary value problem with the integro-differential equation of the Hilfer fractional differential equation of the form.*

$$\begin{cases} {}^{H}D^{\omega,\tau}y(t) = \frac{4}{4t+7}\left(\frac{y^2(t)}{1+|y(t)|}+\frac{2}{3}\right)+\frac{1}{2}e^{-1/2}\int\limits_0^t y(s)ds, & t\in[0,3],\\ y(0)=0, \quad y(3)=0.8I^{0.75}y(0.5)+0.5I^{0.85}y(0.75)+0.48I^{0.54}y(0.25). \end{cases} \tag{23}$$

In Figure 6a, the solution is plotted for different values of $\omega = 1.2,\ 1.4,\ 1.6,\ 1.8$ with parameter $\tau = 0$, which is the Riemann-Liouville derivative. In Figure 6b, the solution is plotted for distinct values of the order $\omega = 1.2,\ 1.4,\ 1.6,\ 1.8$ with parameter $\tau = 1$, which is the Caputo derivative.

In Figure 7a, the solution is plotted for different fractional orders and the Hilfer derivative with parameter $\tau = 0.6$. In Figure 7b, the solution is plotted for fractional order $\omega = 1.2$ with different parameter values. In Figure 6, the solution is plotted for the parameter values $\tau = 1$ and $\tau = 0$, then it is referred to as the Caputo derivative and the R-L derivative, respectively. In Figure 7, the solution is plotted for the Hilfer derivative of various parameter values.

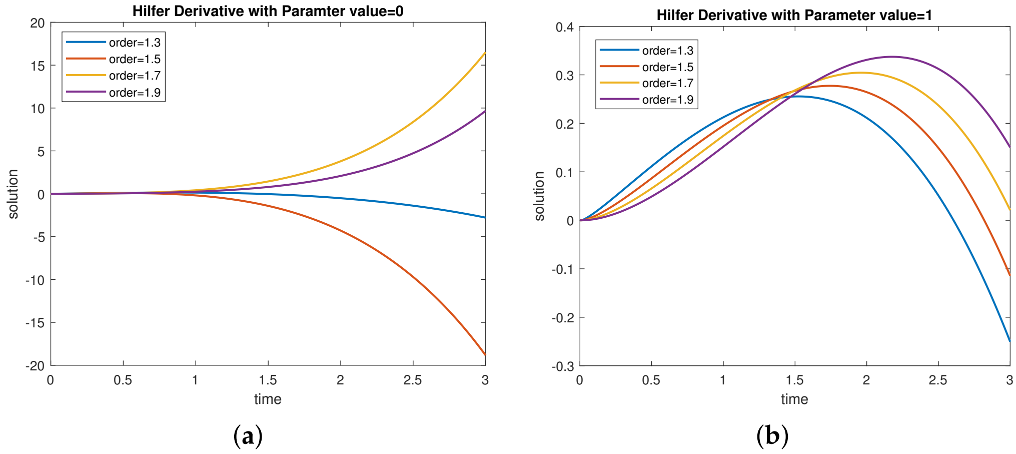


Figure 6. Various Fractional order of R-L and Caputo Derivative. (**a**) Riemann–Liouville Fractional Derivative, (**b**) Caputo Fractional Derivative.

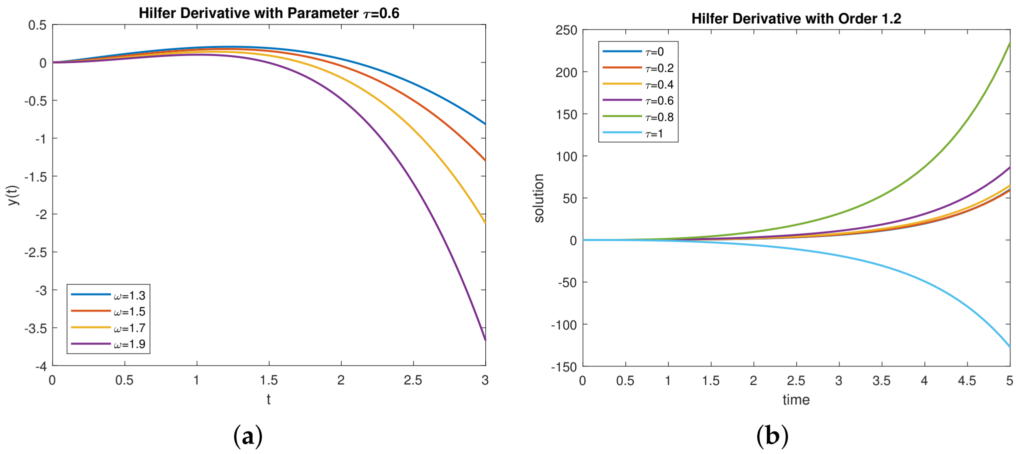


Figure 7. Hilfer Fractional Derivative. (**a**) Different Fractional Order, (**b**) Differential Parameter Values.

6. Conclusions

In this paper, the existence and uniqueness results of the RLC circuit equation are investigated utilizing Schaefer's fixed point theorem and Banach's contractions principle. The Ulam-type stability results for the Hilfer fractional integro-differential equations with non-local boundary conditions are studied for the RLC model. Finally, some numerical examples are provided for illustrating the theoretical results. A similar generalized system involving (k, ψ)-Hilfer fractional derivatives with particular multi-point boundary conditions, and various constant and distributed delays will be further studied in future works.

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Notations

The following abbreviations are used in this manuscript:

$\mathbb{I}(t)$	Current
$\mathbb{V}(t)$	charge at t
$\mathbb{E}(t)$	Supplied source (volt)
$\mathbb{C}$	Capacitance (farad)
$\mathbb{R}$	Resistance (ohms)
t	time

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