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The Fractional Soliton Wave Propagation of Non-Linear Volatility and Option Pricing Systems with a Sensitive Demonstration

Muhammad Bilal Riaz [1,2], Ali Raza Ansari [3], Adil Jhangeer [4,*], Muddassar Imran [5] and Choon Kit Chan [1]

1 Faculty of Engineering and Quantity Surveying, INTI International University, Putra Nilai, Nilai 71800, Malaysia
2 Department of Computer Science and Mathematics, Lebanese American University, Byblos 1401, Lebanon
3 Centre for Applied Mathematics and Bioinformatics (CAMB), Gulf University for Science and Technology, Hawally 32093, Kuwait; ansari.a@gust.edu.kw
4 Department of Mathematics, Namal University, 30KM Talagang Road, Mianwali 42250, Pakistan
5 College of Humanities and Sciences, Ajman University, Ajman P.O. Box 346, United Arab Emirates; mudassar.imran@ajman.ac.ae
* Correspondence: adil.jhangeer@namal.edu.pk

Abstract: In this study, we explore a fractional non-linear coupled option pricing and volatility system. The model under consideration can be viewed as a fractional non-linear coupled wave alternative to the Black–Scholes option pricing governing system, introducing a leveraging effect where stock volatility corresponds to stock returns. Employing the inverse scattering transformation, we find that the Cauchy problem for this model is insolvable. Consequently, we utilize the Φ^6-expansion algorithm to generate generalized novel solitonic analytical wave structures within the system. We present graphical representations in contour, 3D, and 2D formats to illustrate how the system's behavior responds to the propagation of pulses, enabling us to predict suitable parameter values that align with the data. Finally, a conclusion is given.

Keywords: Φ^6-model expansion scheme; M-truncated fractional operator; analytical solution; coupled nonlinear volatility; option pricing model

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1. Introduction

Fractional calculus, initially developed to construct non-integer derivatives and integrals, constitutes a powerful mathematical framework for elucidating diverse phenomena across various scientific domains [1–4]. In the realm of scientific modeling, it has become increasingly valuable, particularly for simulating numerous physical processes characterized by power law behaviors, multi-scale media, or non-Gaussian statistics, as evidenced by various studies [5,6]. This increasing value stems from the demand for accurate simulations of both historical and contemporary physical phenomena, as highlighted in references [7,8]. It has been demonstrated that fractional operators are useful for modeling natural phenomena and that fractional-order models are more effective and productive than non-integer (classical) systems. Numerous fields, including applied mathematics, seismology, biology, control systems, engineering, mechanics, fluid mechanics, control systems, and control systems, have looked into the benefits of fractional derivatives [9–11]. Fractional nonlinear differential equations (fNLDEs) have garnered considerable attention due to their utility in addressing a wide spectrum of technical and engineering challenges. Extensive research efforts have been directed toward identifying optimal approaches for solving these equations, as highlighted in previous studies [12]. Moreover, the precise representation of intricate phenomena across diverse scientific domains, including signal processing, polymers, fluid dynamics, viscoelastic materials, control systems, and more, has further fueled the recent surge of interest in fNLDEs [13,14]. This growing interest underscores the importance of advancing our understanding and methods for tackling fNLDEs in various applications. In order to theoretically characterize the phenomenon and overcome the underlying limitations, it is possible to build the analytical wave solutions of these models. The analytical solutions of the fNLDEs are more beneficial for examining more complex problems when compared to the solutions of the integer-order equations.

Mathematical modeling is often the preferred method for representing physical phenomena, as it accommodates the inherently nonlinear nature of all physical occurrences [15–19]. The utilization of partial differential equations has proven to be a powerful tool in comprehending and unveiling the underlying characteristics of various physical processes. One significant challenge in this context pertains to obtaining analytical solutions for propagating waves within these models. Addressing this issue is of paramount importance in the realm of physics and mathematical modeling. Using an analytical approach for pricing and analyzing non-linear coupled options in a fractional volatility system offers several compelling motivations. Firstly, it provides a deeper understanding of complex financial instruments, enabling investors to make more informed decisions in volatile markets. Additionally, analytical methods can yield precise and efficient solutions, saving both time and computational resources compared to numerical alternatives. This approach empowers traders and risk managers to explore a broader range of scenarios, ultimately enhancing risk management and strategy development. By embracing analytical techniques in the context of fractional non-linear coupled option pricing and volatility, readers can unlock valuable insights and gain a competitive edge in today's intricate financial landscape. The recent advent of several advanced comprehensive techniques has enabled easier location of the analytical results of fNLDEs. Such methods include the auxiliary method [20], generalized Kudryashov's technique [21], the sine–cosine methodology [22], the first integral scheme [23], the sine–Gordon method [24], the Φ^6-expansion scheme [25], the $\left(\frac{G'}{G}\right)$ analytical technique [26], the modified tanh function methodology [27], the Hirota bilinear method [28], and many more. Utilizing the Φ^6-expansion scheme offers a broad spectrum of solutions, which can be highly advantageous in addressing various scenarios. As it covers different types of solutions, this may be helpful for researchers to discuss the model with various initial and boundary conditions more effectively.

The well-known Black–Scholes option pricing model, often called Black–Scholes–Merton, is described by the following equation: [29],

$$A_t + \frac{\sigma^2}{2}s^2 A_{ss} + rA_s - rA = 0. \tag{1}$$

The geometric Brownian motion (i.e., the stochastic differential equation) $dS = \mu sdt + \sigma dW(t)$ satisfied by the stock (asset) price S and the Ito lemma were presented [30], where μ is the instantaneous mean return, r is the risk-free interest rate, $A(s,t)$ is the value of the European call option on the asset price s at time t, σ is the stock volatility, and W is a Wiener process. This model has sparked significant interest, opening up an entirely new avenue of research within the domains of financial mathematics and financial engineering. While the Black–Scholes model serves as a widely utilized tool for pricing European-style options, it encounters limitations when applied to evaluate unconventional option types, such as American or Asian options. These limitations arise from the model's inability to accommodate exercise features and path dependencies [31–34]. As a result, there has been a growing need to develop alternative approaches that can encompass these complex financial instruments and their associated features.

Ivancevic recently presented a novel nonlinear option pricing model (called the Ivancevic option pricing model) [35] based on Lo's modern adaptive market hypothesis [36,37], the Elliott wave market theory [38,39], and the quantum neural computation approach [40].

$$iB_t + \frac{\sigma}{2}B_{ss} + \beta|B|^2 B = 0, \tag{2}$$

to satisfy behavioral and efficient markets, and the fundamental nonlinear complexities of those markets, where the dispersion frequency coefficient σ is the volatility (which can either be a constant or stochastic process itself), A(s,t) denotes the option–price wave function, and the Landau coefficient $\beta = \beta(r, w)$ represents the adaptive market potential. Ivancevic introduced a coupled nonlinear volatility and option pricing model, as documented in [41]. This model integrates controlled stochastic volatility into the adaptive-wave model (2):

$$\begin{aligned} i\mathfrak{Q}_t + \frac{1}{2}\mathfrak{Q}_{ss} + \beta\left(|\mathfrak{Q}|^2 + |\mathbb{V}|^2\right)\mathfrak{Q} = 0, \\ i\mathbb{V}_t + \frac{1}{2}\mathbb{V}_{ss} + \beta\left(|\mathfrak{Q}|^2 + |\mathbb{V}|^2\right)\mathbb{V} = 0, \end{aligned} \tag{3}$$

and transforms into a fractional model, such as

$$iD_t^{M,\alpha}\mathfrak{Q} + \frac{1}{2}\mathfrak{Q}_{ss} + \beta\left(|\mathfrak{Q}|^2 + |\mathbb{V}|^2\right)\mathfrak{Q} = 0. \tag{4}$$

$$iD_t^{M,\alpha}\mathbb{V} + \frac{1}{2}\mathbb{V}_{ss} + \beta\left(|\mathfrak{Q}|^2 + |\mathbb{V}|^2\right)\mathbb{V} = 0. \tag{5}$$

This transformation is executed to produce a leverage effect, i.e., stock volatility has been shown to be (negatively) correlated with stock returns [42,43]. In this context, Equations (4) and (5) are commonly referred to as the volatility model and the option pricing model, respectively. Additionally, $\mathfrak{Q}(s,t)$ represents the option pricing wave function, which serves as a nonlinear coefficient in the volatility model, while $\mathbb{V}(s,t)$ represents the volatility wave function, acting as a nonlinear coefficient in the option pricing model. In this context, both processes evolve within a unified self-organizing market heat potential, offering an accurate portrayal of adaptively controlled Brownian behavior within a hypothetical financial market.

As a future direction of this research, one may aim to extend the planar dynamical framework of the equation described above by exploring the potential of alternative transformation techniques beyond the Galilean transformation. Focus may be placed on investigating the emergence of chaotic and quasiperiodic behaviors, particularly for specific parameter values within the examined system. Conducting sensitivity and multistability analyses, while considering a range of initial conditions, could be the next point of consideration to gain deeper insights into the dynamics of periodic and quasiperiodic behaviors.

To the best of our knowledge, there is currently no existing literature addressing analytical solutions for the considered model. To bridge this gap, we employ the Φ^6-expansion method to derive analytical solutions based on Jacobi elliptic functions and provide a comprehensive demonstration of their sensitivity.

2. The Formation of Solitary Wave Solutions

2.1. The Description of the Φ^6-Expansion Approach

Consider a basic differential equation:

$$\mathfrak{R}(\mathfrak{Q}, \mathfrak{Q}_t, \mathfrak{Q}_s, \mathfrak{Q}_{tt}, \mathfrak{Q}_{ss}, \ldots) = 0. \tag{6}$$

It can be changed into an ordinary differential equation:

$$\mathbb{M}(\mathbb{P}, \mathbb{P}', \mathbb{P}'', \ldots) = 0, \tag{7}$$

by using the traveling-wave transformation,

$$\mathfrak{Q}(s,t) = \mathbb{P}(\varphi), \tag{8}$$

where $\varphi = k_1 x + k_2 t$. φ is particularly useful because it allows us to analyze the behavior of a wave-like phenomenon at a specific moment in time, without worrying about its entire time evolution. It also helps us to study and predict steady-state behavior, independent of transient effects.

Let us consider the solution of Equation (7),

$$\mathbb{P}(\varphi) = \sum_{q=0}^{2c} \left[a_q \mathfrak{R}^q(\varphi) \right]. \tag{9}$$

In the context of this study, we introduce a homogeneous balancing constant, denoted as 'c', and we examine the differential equation that $\mathfrak{R}(\varphi)$ satisfies, which is as follows:

$$\begin{gathered} \mathfrak{R}'^2(\varphi) = h_0 + h_2\mathfrak{R}^2(\varphi) + h_4\mathfrak{R}^4(\varphi) + h_6\mathfrak{R}^6(\varphi), \\ \mathfrak{R}''(\varphi) = h_2\mathfrak{R}(\varphi) + 2h_4\mathfrak{R}^3(\varphi) + 3h_6\mathfrak{R}^5(\varphi). \end{gathered} \tag{10}$$

Equation (10) satisfies

$$\mathfrak{R}(\varphi) = \frac{\Phi(\varphi)}{\sqrt{f\Phi^2(\varphi) + g}}, \tag{11}$$

where $f\Phi^2(\varphi) + g > 0$ and $\Phi(\varphi)$ represents the solution to the Jacobi elliptic equation:

$$\Phi'^2(\varphi) = l_0 + l_2\Phi^2(\varphi) + l_4\Phi^4(\varphi), \tag{12}$$

In the following discussion, we encounter unknown constants, denoted as l_0, l, and l_4, alongside given functions, f and g, which are defined as follows (Table 1):

$$f = \frac{h_4(l_2 - h_2)}{3l_0l_4 + (h_2^2 - l_2^2)}, \quad g = \frac{3h_4l_0}{3l_0l_4 + (h_2^2 - l_2^2)}, \tag{13}$$

along with,

$$3h_6[-l_2^2 + h_2^2 + 3l_0l_4]^2 + h_4^2(l_2 - h_2)[9l_0l_4 - (2l_2 + h_2)(l_2 - h_2)] = 0.$$

Table 1. Limiting cases for functions.

The Jacobi Elliptic Functions			
No.	**Functions**	$\boldsymbol{n \to 1}$	$\boldsymbol{n \to 0}$
1	$sn(\varphi, n)$	$\tanh(\varphi)$	$\sin(\varphi)$
2	$dn(\varphi, n)$	$\text{sech}(\varphi)$	1
3	$cn(\varphi, n)$	$\text{sech}(\varphi)$	$\cos(\varphi)$
4	$ns(\varphi, n)$	$\coth(\varphi)$	$\csc(\varphi)$
5	$ds(\varphi, n)$	$\text{csch}(\varphi)$	$\csc(\varphi)$
6	$cs(\varphi, n)$	$\text{csch}(\varphi)$	$\cot(\varphi)$
7	$sc(\varphi, n)$	$\sinh(\varphi)$	$\tan(\varphi)$
8	$sd(\varphi, n)$	$\sinh(\varphi)$	$\sin(\varphi)$
9	$cd(\varphi, n)$	1	$\cos(\varphi)$
10	$nc(\varphi, n)$	$\cosh(\varphi)$	$\sec(\varphi)$

2.2. *Analytical Traveling-Wave Solutions*

Using the following transformation:

$$\mathfrak{Q}(s,t) = \mathbb{P}(\varphi)e^{i(s+t+\theta)}, \mathbb{V}(s,t) = \mathfrak{L}(\varphi)e^{i(s+t+\theta)}, \text{ where } \varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^{\alpha}) \tag{14}$$

the system described by Equations (4) and (5) is converted (for more details, see [16,20] and the references therein):

$$
\begin{aligned}
-ic\mathbb{P}' + \frac{1}{2}\mathbb{P}'' + i\mathbb{P}' - \frac{3}{2}\mathbb{P} + \beta\left(|\mathbb{P}|^2 + |\mathfrak{L}|^2\right)\mathbb{P} = 0,\\
-ic\mathfrak{L}' + \frac{1}{2}\mathfrak{L}'' + i\mathfrak{L}' - \frac{3}{2}\mathfrak{L} + \beta\left(|\mathbb{P}|^2 + |\mathfrak{L}|^2\right)\mathfrak{L} = 0.
\end{aligned} \tag{15}
$$

The imaginary part of Equation (15),

$$
\begin{aligned}
-ic\mathbb{P}' + i\mathbb{P}' = 0, \implies c = 1,\\
-ic\mathfrak{L}' + i\mathfrak{L}' = 0 \implies c = 1.
\end{aligned} \tag{16}
$$

The real part of Equation (15),

$$
\begin{aligned}
\frac{1}{2}\mathbb{P}'' - \frac{3}{2}\mathbb{P} + \beta\left(|\mathbb{P}|^2 + |\mathfrak{L}|^2\right)\mathbb{P} = 0,\\
\frac{1}{2}\mathfrak{L}'' - \frac{3}{2}\mathfrak{L} + \beta\left(|\mathbb{P}|^2 + |\mathfrak{L}|^2\right)\mathfrak{L} = 0.
\end{aligned} \tag{17}
$$

Let us now contemplate a transformation, denoted as $\mathbb{P} = \mathfrak{L} + \kappa$, which results in the transformation of Equation (17) to the following form:

$$
\mathfrak{L}'' + 4\beta\mathfrak{L}^3 + 4\beta\kappa\mathfrak{L}^2 + (2\beta\kappa^2 - 2)\mathfrak{L} = 0. \tag{18}
$$

Equation (18) yields $j = 1$; thus,

$$
\mathcal{L}(s,t) = b_0 + b_1\mathfrak{R}(\varphi) + b_2\mathfrak{R}^2(\varphi), \tag{19}
$$

where

$$
\begin{aligned}
\mathfrak{R}'^2(\varphi) = h_0 + h_2\mathfrak{R}^2(\varphi) + h_4\mathfrak{R}^4(\varphi) + h_6\mathfrak{R}^6(\varphi),\\
\mathfrak{R}''(\varphi) = h_2\mathfrak{R}(\varphi) + 2h_4\mathfrak{R}^3(\varphi) + 3h_6\mathfrak{R}^5(\varphi).
\end{aligned} \tag{20}
$$

Equation (19) is substituted into Equation (18),

$$
\begin{aligned}
&\mathbb{L}^0 : 2\,b_0\beta\,\kappa^2 + 4\,\beta\,\kappa\,{b_0}^2 + 4\,\beta\,{b_0}^3 + 2\,b_2h_0 - 3\,b_0 = 0,\\
&\mathbb{L}^1 : 2\,\beta\,\kappa^2 b_1 + 8\,\beta\,\kappa\,b_0b_1 + 12\,\beta\,{b_0}^2 b_1 + b_1h_2 - 3\,b_1 = 0,\\
&\mathbb{L}^2 : 2\,\beta\,\kappa^2 b_2 + 8\,\beta\,\kappa\,b_0b_2 + 4\,\beta\,\kappa\,{b_1}^2 + 12\,\beta\,{b_0}^2 b_2 + 12\,\beta\,b_0{b_1}^2 + 4\,b_2h_2 - 3\,b_2 = 0,\\
&\mathbb{L}^3 : 8\,\beta\,\kappa\,b_1b_2 + 24\,\beta\,b_0b_1b_2 + 4\,\beta\,{b_1}^3 + 2\,b_1h_4 = 0,\\
&\mathbb{L}^4 : 4\,\beta\,\kappa\,{b_2}^2 + 12\,\beta\,b_0{b_2}^2 + 12\,\beta\,{b_1}^2 b_2 + 6\,b_2h_4 = 0,\\
&\mathbb{L}^5 : 12\,\beta\,b_1{b_2}^2 + 3\,b_1h_6 = 0,\\
&\mathbb{L}^6 : 4\,\beta\,{b_2}^3 + 8\,b_2h_6 = 0.
\end{aligned}
$$

In this study, we employ a computational tool known as `Maple software` to seek solutions for the previously mentioned system (21).
Solution set:

$$
\begin{aligned}
&\Bigg[b_0 = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right),\ b_1 = 0,\ b_2 = \pm 3\sqrt{-\frac{1}{\varpi}},\\
&h_0 = \frac{\varpi^2\left(2\sqrt{-\frac{1}{\varpi}}\beta\,\kappa + 1\right)\left(-\frac{10\beta^2\kappa^2}{\varpi} + \frac{27\beta}{\varpi} - 2\sqrt{-\frac{1}{\varpi}}\beta\,\kappa + 1\right)}{324\,h_4\beta^2},\ h_6 = \frac{9{h_4}^2\beta}{2\varpi}\Bigg].
\end{aligned} \tag{21}
$$

where $\varpi = 2\beta^2\kappa^2 + 12\beta h_2 - 9\beta$. One can obtain the general solution by plugging (21) into (19),

$$\mathcal{L}(\varphi) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\Re^2(\varphi), \tag{22}$$

Category 1. *if* $l_0 = 1$, $l_2 = -1 - s^2$, $l_4 = s^2$, $n \in (0,1)$, *then* $\Phi(\varphi) = sn(\varphi, n)$ *or* $\Phi(\varphi) = cd(\varphi, n)$*; thus, we have*

$$\mathbb{V}_1(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sn^2(\varphi,n)}{fsn^2(\varphi,n)+g}\right) \times e^{i(s+t+\theta)} \tag{23}$$

$$\mathfrak{Q}_1(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sn^2(\varphi,n)}{fsn^2(\varphi,n)+g}\right) + \kappa, \tag{24}$$

or

$$\mathbb{V}_2(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{cd^2(\varphi,n)}{fcd^2(\varphi,n)+g}\right), \tag{25}$$

$$\mathfrak{Q}_2(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{cd^2(\varphi,n)}{fcd^2(\varphi,n)+g}\right) + \kappa, \tag{26}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^\alpha)$, *and* f *and* g *are given below:*

$$f = \frac{h_4(n^2+h_2+1)}{n^4-n^2-{h_2}^2+1},\; g = -3\,\frac{h_4}{n^4-n^2-{h_2}^2+1},$$

along with the condition:

$$0 = {h_4}^2\left(-n^2-h_2-1\right)\left(9\,n^2-\left(-n^2-h_2-1\right)\left(-2\,n^2+h_2-2\right)\right) + \frac{27\,{h_4}^2\beta\left(3\,n^2-\left(-n^2-1\right)^2+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

The possibility of deriving a single wave analytical solution arises as we approach the limit $n \to 1$.

$$\mathbb{V}_{1,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\tanh^2(\varphi,n)}{f\tanh^2(\varphi,n)+g}\right), \tag{27}$$

$$\mathfrak{Q}_{1,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\tanh^2(\varphi,n)}{f\tanh^2(\varphi,n)+g}\right) + \kappa, \tag{28}$$

along with the condition:

$$0 = {h_4}^2(-2-h_2)(9-(-2-h_2)(-4+h_2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

The potential to formulate a single wave analytical solution emerges as we approach the limit $n \to 0$.

$$\mathbb{V}_{1,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sin^2(\varphi,n)}{f\sin^2(\varphi,n)+g}\right), \tag{29}$$

$$\mathfrak{Q}_{1,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sin^2(\varphi,n)}{f\sin^2(\varphi,n)+g}\right) + \kappa, \tag{30}$$

or

$$\mathbb{V}_{2,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cos^2(\varphi,n)}{f\cos^2(\varphi,n)+g}\right), \tag{31}$$

$$\mathfrak{Q}_{2,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cos^2(\varphi,n)}{f\cos^2(\varphi,n)+g}\right) + \kappa, \tag{32}$$

along with the condition:

$$0 = {h_4}^2(-h_2-1)(-(-h_2-1)(h_2-2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

In Figures 1–3, 3D, contour and 2D propagation of $\mathfrak{Q}_{1,2}(s,t)$ at different fractional-order α are given. While in Figures 4–6, 3D, contour and 2D propagation of $\mathbb{V}_{1,2}(s,t)$ at different fractional-order α are presented.

Category 2. *In the scenario where specific values are assigned to the constants, namely,* $l_0 = 1-s^2$, $l_2 = 2s^2-1$, $l_4 = -s^2$, *and* $n \in (0,1)$ *our expression* $\Phi(\varphi) = cn(\varphi,n)$. *Consequently, we obtain the following relationship:*

$$\mathbb{V}_3(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{cn^2(\varphi,n)}{fcn^2(\varphi,n)+g}\right), \tag{33}$$

$$\mathfrak{Q}_3(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{cn^2(\varphi,n)}{fcn^2(\varphi,n)+g}\right) + \kappa, \tag{34}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^\alpha)$, f, *and* g, *given below:*

$$f = -\frac{h_4\left(2\,n^2-h_2-1\right)}{n^4-n^2-{h_2}^2+1},\ g = 3\,\frac{\left(n^2-1\right)h_4}{n^4-n^2-{h_2}^2+1},$$

along with the condition:

$$\begin{aligned}0 =& {h_4}^2\left(2\,n^2-h_2-1\right)\left(-9\left(-n^2+1\right)n^2-\left(2\,n^2-h_2-1\right)\left(4\,n^2+h_2-2\right)\right)+\\ &\frac{27\,{h_4}^2\beta\left(-3\left(-n^2+1\right)n^2-\left(2\,n^2-1\right)^2+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.\end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{3,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{sech}^2(\varphi,n)}{f\operatorname{sech}^2(\varphi,n)+g}\right), \tag{35}$$

$$\mathfrak{Q}_{3,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{sech}^2(\varphi,n)}{f\operatorname{sech}^2(\varphi,n)+g}\right) + \kappa, \tag{36}$$

along with the condition:

$$0 = {h_4}^2(1-h_2)(-(1-h_2)(2+h_2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

Category 3. *if* $l_0 = s^2-1,\ l_2 = 2-s^2,\ l_4 = -1,\ n \in (0,1)$, *then* $\Phi(\varphi) = dn(\varphi,n)$; *thus, we have*

$$\mathbb{V}_4(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{dn^2(\varphi,n)}{f dn^2(\varphi,n)+g}\right), \tag{37}$$

$$\mathfrak{Q}_4(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{dn^2(\varphi,n)}{f dn^2(\varphi,n)+g}\right) + \kappa, \tag{38}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^\alpha)$, *and* f *and* g *are given below:*

$$f = \frac{h_4\left(n^2+h_2-2\right)}{n^4-n^2-{h_2}^2+1},\ g = -3\,\frac{\left(n^2-1\right)h_4}{n^4-n^2-{h_2}^2+1},$$

along with the condition:

$$\begin{aligned}0 =&{h_4}^2\left(-n^2-h_2+2\right)\left(-9\,n^2+9-\left(-n^2-h_2+2\right)\left(-2\,n^2+h_2+4\right)\right)+\\ &\frac{27\,{h_4}^2\beta\left(-3\,n^2+3-\left(-n^2+2\right)^2+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.\end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{4,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{sech}^2(\varphi,n)}{f\operatorname{sech}^2(\varphi,n)+g}\right), \tag{39}$$

$$\mathfrak{Q}_{4,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{sech}^2(\varphi,n)}{f\operatorname{sech}^2(\varphi,n)+g}\right) + \kappa, \tag{40}$$

along with the condition:

$$0 = {h_4}^2(-h_2+1)(-(-h_2+1)(2+h_2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

Category 4. *In the scenario where specific values are assigned to the constants, namely,* $l_0 = s^2$, $l_2 = -1-s^2,\ l_4 = 1,\ n \in (0,1)$, *then* $\Phi(\varphi) = ns(\varphi,n)$ *or* $\Phi(\varphi) = dc(\varphi,n)$; *thus, we have*

$$\mathbb{V}_5(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{ns^2(\varphi,n)}{f ns^2(\varphi,n)+g}\right), \tag{41}$$

$$\mathfrak{Q}_5(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{ns^2(\varphi,n)}{f ns^2(\varphi,n)+g}\right) + \kappa, \tag{42}$$

or

$$\mathbb{V}_6(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{dc^2(\varphi,n)}{f dc^2(\varphi,n)+g}\right), \tag{43}$$

$$\mathfrak{Q}_6(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{dc^2(\varphi,n)}{f dc^2(\varphi,n)+g}\right) + \kappa, \tag{44}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^{\alpha})$, *and* f *and* g *are given below:*

$$f = \frac{h_4\left(n^2+h_2+1\right)}{n^4-n^2-{h_2}^2+1},\; g = -3\,\frac{n^2 h_4}{n^4-n^2-{h_2}^2+1},$$

along with the condition:

$$\begin{aligned}0 =& {h_4}^2\left(-n^2-h_2-1\right)\left(9\,n^2-\left(-n^2-h_2-1\right)\left(-2\,n^2+h_2-2\right)\right)+\\ &\frac{27\,{h_4}^2\beta\left(3\,n^2-\left(-n^2-1\right)^2+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.\end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{5,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\coth^2(\varphi,n)}{f\coth^2(\varphi,n)+g}\right), \tag{45}$$

$$\mathfrak{Q}_{5,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\coth^2(\varphi,n)}{f\coth^2(\varphi,n)+g}\right) + \kappa, \tag{46}$$

along with the condition:

$$0 = {h_4}^2(-h_2-2)(9-(-2-h_2)(-4+h_2)) + \frac{27\,{h_4}^2\beta\left(3-4+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

We may be able to develop a single wave analytical solution if $n \to 0$,

$$\mathbb{V}_{5,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\csc^2(\varphi,n)}{f\csc^2(\varphi,n)+g}\right), \tag{47}$$

$$\mathfrak{Q}_{5,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\csc^2(\varphi,n)}{f\csc^2(\varphi,n)+g}\right) + \kappa, \tag{48}$$

or

$$\mathbb{V}_{6,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sec^2(\varphi,n)}{f\sec^2(\varphi,n)+g}\right), \tag{49}$$

$$\mathfrak{Q}_{6,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sec^2(\varphi,n)}{f\sec^2(\varphi,n)+g}\right) + \kappa, \tag{50}$$

along with the condition:

$$0 = {h_4}^2(-h_2-1)(-(-h_2-1)(h_2-2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

Category 5. *if* $l_0 = -s^2,\ l_2 = -1 + 2s^2,\ l_4 = 1 - s^2,\ n \in (0,1)$, *then* $\Phi(\varphi) = nc(\varphi, n)$; *thus, we have*

$$\mathbb{V}_7(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{nc^2(\varphi,n)}{f nc^2(\varphi,n) + g}\right), \tag{51}$$

$$\mathfrak{Q}_7(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{nc^2(\varphi,n)}{f nc^2(\varphi,n) + g}\right) + \kappa, \tag{52}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^{\alpha})$, *and* f *and* g *are given below:*

$$f = -\frac{h_4\left(2\,n^2 - h_2 - 1\right)}{n^4 - n^2 - {h_2}^2 + 1},\ g = 3\,\frac{n^2 h_4}{n^4 - n^2 - {h_2}^2 + 1},$$

along with the condition:

$$\begin{aligned}0 =& {h_4}^2\left(2\,n^2 - h_2 - 1\right)\left(-9\left(-n^2+1\right)n^2 - \left(2\,n^2 - h_2 - 1\right)\left(4\,n^2 + h_2 - 2\right)\right) + \\ & \frac{27\,{h_4}^2\beta\left(-3\left(-n^2+1\right)n^2 - \left(2\,n^2-1\right)^2 + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.\end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{7,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cosh^2(\varphi,n)}{f\cosh^2(\varphi,n) + g}\right), \tag{53}$$

$$\mathfrak{Q}_{7,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cosh^2(\varphi,n)}{f\cosh^2(\varphi,n) + g}\right) + \kappa, \tag{54}$$

along with the condition:

$$0 = {h_4}^2(1-h_2)(-(1-h_2)(2+h_2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

We may be able to develop a single wave analytical solution if $n \to 0$,

$$\mathbb{V}_{7,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sec^2(\varphi,n)}{f\sec^2(\varphi,n) + g}\right), \tag{55}$$

$$\mathfrak{Q}_{7,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sec^2(\varphi,n)}{f\sec^2(\varphi,n) + g}\right) + \kappa, \tag{56}$$

along with the condition:

$$0 = {h_4}^2(-h_2-1)(-(-h_2-1)(h_2-2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

Category 6. *In the scenario where specific values are assigned to the constants, namely,* $l_0 = -1$, $l_2 = 2 - s^2,\ l_4 = -1 + s^2,\ n \in (0,1)$, *then* $\Phi(\varphi) = nd(\varphi, n)$; *thus, we have*

$$\mathbb{V}_8(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{nd^2(\varphi,n)}{f nd^2(\varphi,n) + g}\right), \tag{57}$$

$$\mathfrak{Q}_8(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{nd^2(\varphi,n)}{f nd^2(\varphi,n)+g}\right) + \kappa, \tag{58}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^{\alpha})$*, and f and g are given below:*

$$f = \frac{h_4\left(n^2+h_2-2\right)}{n^4-n^2-{h_2}^2+1},\ g = 3\,\frac{h_4}{n^4-n^2-{h_2}^2+1},$$

along with the condition:

$$\begin{aligned}0 =& {h_4}^2\left(-n^2-h_2+2\right)\left(-9\,n^2+9-\left(-n^2-h_2+2\right)\left(-2\,n^2+h_2+4\right)\right)+\\ &\frac{27\,{h_4}^2\beta\left(-3\,n^2+3-\left(-n^2+2\right)^2+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.\end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{8,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cosh^2(\varphi,n)}{f\cosh^2(\varphi,n)+g}\right), \tag{59}$$

$$\mathfrak{Q}_{8,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cosh^2(\varphi,n)}{f\cosh^2(\varphi,n)+g}\right) + \kappa, \tag{60}$$

along with the condition:

$$0 = {h_4}^2(1-h_2)(-(1-h_2)(2+h_2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

Category 7. *In the scenario where specific values are assigned to the constants, namely,* $l_0 = 1$, $l_2 = 2-s^2$, $l_4 = 1-s^2$, $n \in (0,1)$, *then* $\Phi(\varphi) = sc(\varphi,n)$*; thus, we have*

$$\mathbb{V}_9(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sc^2(\varphi,n)}{f sc^2(\varphi,n)+g}\right), \tag{61}$$

$$\mathfrak{Q}_9(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sc^2(\varphi,n)}{f sc^2(\varphi,n)+g}\right) + \kappa, \tag{62}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^{\alpha})$*, and f and g are given below:*

$$f = \frac{h_4\left(n^2+h_2-2\right)}{n^4-n^2-{h_2}^2+1}, g = -3\,\frac{h_4}{n^4-n^2-{h_2}^2+1},$$

along with the condition:

$$\begin{aligned}0 =& {h_4}^2\left(-n^2-h_2+2\right)\left(-9\,n^2+9-\left(-n^2-h_2+2\right)\left(-2\,n^2+h_2+4\right)\right)+\\ &\frac{27\,{h_4}^2\beta\left(-3\,n^2+3-\left(-n^2+2\right)^2+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.\end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{9,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sinh^2(\varphi,n)}{f\sinh^2(\varphi,n)+g}\right), \tag{63}$$

$$\mathfrak{Q}_{9,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sinh^2(\varphi,n)}{f\sinh^2(\varphi,n)+g}\right) + \kappa, \tag{64}$$

along with the condition:

$$0 = h_4{}^2(1-h_2)(-(1-h_2)(2+h_2)) + \frac{27\,h_4{}^2\beta\left(-1+h_2{}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

One can develop a periodic solution if $n \to 0$,

$$\mathbb{V}_{9,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\tan^2(\varphi,n)}{f\tan^2(\varphi,n)+g}\right), \tag{65}$$

$$\mathfrak{Q}_{9,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\tan^2(\varphi,n)}{f\tan^2(\varphi,n)+g}\right) + \kappa, \tag{66}$$

along with the condition:

$$h_4{}^2(-h_2+2)(9-(-h_2+2)(h_2+4)) + \frac{27\,h_4{}^2\beta\left(3-4+h_2{}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

Category 8. *if $l_0 = 1,\ l_2 = 2s^2-1,\ l_4 = -s^2(1-s^2),\ n \in (0,1)$, then $\Phi(\varphi) = sd(\varphi,n)$; thus, we have*

$$\mathbb{V}_{10}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sd^2(\varphi,n)}{fsd^2(\varphi,n)+g}\right), \tag{67}$$

$$\mathfrak{Q}_{10}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sd^2(\varphi,n)}{fsd^2(\varphi,n)+g}\right) + \kappa, \tag{68}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^\alpha)$, and f and g are given below:

$$f = -\frac{h_4\left(2\,n^2-h_2-1\right)}{n^4-n^2-h_2{}^2+1},\ g = -3\,\frac{h_4}{n^4-n^2-h_2{}^2+1},$$

along with the condition:

$$\begin{aligned}0 =& h_4{}^2\left(2\,n^2-h_2-1\right)\left(-9\left(-n^2+1\right)n^2-\left(2\,n^2-h_2-1\right)\left(4\,n^2+h_2-2\right)\right)+\\ &\frac{27\,h_4{}^2\beta\left(-3\left(-n^2+1\right)n^2-\left(2\,n^2-1\right)^2+h_2{}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.\end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 0$,

$$\mathbb{V}_{10,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sin^2(\varphi,n)}{f\sin^2(\varphi,n)+g}\right), \tag{69}$$

$$\mathfrak{Q}_{10,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sin^2(\varphi,n)}{f\sin^2(\varphi,n)+g}\right) + \kappa, \tag{70}$$

along with the condition:

$$0 = {h_4}^2(-h_2-1)(-(-h_2-1)(+h_2-2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

Category 9. *In the scenario where specific values are assigned to the constants, namely* $l_0 = 1-s^2$, $l_2 = 2-s^2$, $l_4 = 1$, $n \in (0,1)$, *then* $\Phi(\varphi) = cs(\varphi, n)$; *thus, we have*

$$\mathbb{V}_{11}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{cs^2(\varphi,n)}{fcs^2(\varphi,n)+g}\right), \tag{71}$$

$$\mathfrak{Q}_{11}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{cs^2(\varphi,n)}{fcs^2(\varphi,n)+g}\right) + \kappa, \tag{72}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^\alpha)$, *and* f *and* g *are given below:*

$$f = \frac{h_4(n^2+h_2-2)}{n^4-n^2-{h_2}^2+1},\; g = 3\,\frac{(n^2-1)h_4}{n^4-n^2-{h_2}^2+1},$$

along with the condition:

$$\begin{aligned}0 =& {h_4}^2\left(-n^2-h_2+2\right)\left(-9\,n^2+9-\left(-n^2-h_2+2\right)\left(-2\,n^2+h_2+4\right)\right)+\\ &\frac{27\,{h_4}^2\beta\left(-3\,n^2+3-\left(-n^2+2\right)^2+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.\end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{11,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{csch}^2(\varphi,n)}{f\operatorname{csch}^2(\varphi,n)+g}\right), \tag{73}$$

$$\mathfrak{Q}_{11,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{csch}^2(\varphi,n)}{f\operatorname{csch}^2(\varphi,n)+g}\right) + \kappa, \tag{74}$$

along with the condition:

$$0 = {h_4}^2(1-h_2)(-(-h_2+1)(2+h_2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

One can develop a periodic solution if $n \to 0$,

$$\mathbb{V}_{11,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cot^2(\varphi,n)}{f\cot^2(\varphi,n)+g}\right), \tag{75}$$

$$\mathfrak{Q}_{11,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cot^2(\varphi,n)}{f\cot^2(\varphi,n)+g}\right) + \kappa, \tag{76}$$

along with the condition:

$$0 = {h_4}^2(-h_2+2)(9-(-h_2+2)(h_2+4)) + \frac{27\,{h_4}^2\beta\left(3-4+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

Category 10. *if* $l_0 = -s^2(1-s^2),\ l_2 = 2s^2-1,\ l_4 = 1,\ n \in (0,1)$, *then* $\Phi(\varphi) = ds(\varphi, n)$; *thus, we have*

$$\mathbb{V}_{12}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{ds^2(\varphi,n)}{f ds^2(\varphi,n)+g}\right), \tag{77}$$

$$\mathfrak{Q}_{12}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{ds^2(\varphi,n)}{f ds^2(\varphi,n)+g}\right) + \kappa, \tag{78}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^\alpha)$, *and* f *and* g *are given below:*

$$f = -\frac{h_4\left(2\,n^2-h_2-1\right)}{n^4-n^2-{h_2}^2+1},\ g = -3\,\frac{n^2\left(n^2-1\right)h_4}{n^4-n^2-{h_2}^2+1},$$

along with the condition:

$$\begin{aligned}0 =& {h_4}^2\left(2\,n^2-h_2-1\right)\left(-9\left(-n^2+1\right)n^2-\left(2\,n^2-h_2-1\right)\left(4\,n^2+h_2-2\right)\right)+\\ & \frac{27\,{h_4}^2\beta\left(-3\left(-n^2+1\right)n^2-\left(2\,n^2-1\right)^2+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.\end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{12,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{csch}^2(\varphi,n)}{f\operatorname{csch}^2(\varphi,n)+g}\right), \tag{79}$$

$$\mathfrak{Q}_{12,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{csch}^2(\varphi,n)}{f\operatorname{csch}^2(\varphi,n)+g}\right) + \kappa, \tag{80}$$

along with the condition:

$$0 = {h_4}^2(1-h_2)(-(1-h_2)(2+h_2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}$$

We may be able to develop a single wave analytical solution if $n \to 0$,

$$\mathbb{V}_{12,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\csc^2(\varphi,n)}{f\csc^2(\varphi,n)+g}\right), \tag{81}$$

$$\mathfrak{Q}_{12,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\csc^2(\varphi,n)}{f\csc^2(\varphi,n)+g}\right) + \kappa, \tag{82}$$

along with the condition:

$$0 = {h_4}^2(-h_2-1)(-(-h_2-1)(h_2-2)) + \frac{27\,{h_4}^2\beta\left(-1+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

Category 11. *if* $l_0 = \frac{1-s^2}{4}$, $l_2 = \frac{1+s^2}{2}$, $l_4 = \frac{1-s^2}{4}$, $n \in (0,1)$, *then* $\Phi(\varphi) = nc(\varphi, n) \pm sc(\varphi, n)$ *or* $\Phi(\varphi) = \frac{cn(\varphi,n)}{1 \pm sn(\varphi,n)}$; *thus, we have*

$$\mathbb{V}_{13}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(nc(\varphi,n) \pm sc(\varphi,n))^2}{f(nc(\varphi,n) \pm sc(\varphi,n))^2 + g}\right), \tag{83}$$

$$\mathfrak{Q}_{13}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(nc(\varphi,n) \pm sc(\varphi,n))^2}{f(nc(\varphi,n) \pm sc(\varphi,n))^2 + g}\right) + \kappa, \tag{84}$$

or

$$\mathbb{V}_{14}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{cn^2(\varphi,n)}{fcn^2(\varphi,n) + (1 \pm sn^2(\varphi,n))g}\right), \tag{85}$$

$$\mathfrak{Q}_{14}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{cn^2(\varphi,n)}{fcn^2(\varphi,n) + (1 \pm sn^2(\varphi,n))g}\right) + \kappa, \tag{86}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^\alpha)$, *and* f *and* g *are given below:*

$$f = \frac{h_4\left(\frac{1}{2}n^2 + \frac{1}{2} - h_2\right)}{3\left(-\frac{1}{4}n^2 + \frac{1}{4}\right)^2 - \left(\frac{1}{2}n^2 + \frac{1}{2}\right)^2 + {h_2}^2}, \; g = 3\,\frac{\left(-\frac{1}{4}n^2 + \frac{1}{4}\right)h_4}{3\left(-\frac{1}{4}n^2 + \frac{1}{4}\right)^2 - \left(\frac{1}{2}n^2 + \frac{1}{2}\right)^2 + {h_2}^2},$$

along with the condition:

$$\begin{aligned} 0 =& {h_4}^2\left(\frac{1}{2}n^2 + \frac{1}{2} - h_2\right)\left(9\left(-\frac{1}{4}n^2 + \frac{1}{2}\right)^2 - \left(\frac{1}{2}n^2 + \frac{1}{2} - h_2\right)\left(n^2 + h_2 + 1\right)\right) + \\ & \frac{27\,{h_4}^2\beta\left(3\left(-\frac{1}{4}n^2 + \frac{1}{4}\right)^2 - \left(\frac{1}{2}n^2 + \frac{1}{2}\right)^2 + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}. \end{aligned}$$

We might be able to develop a unified analytical wave solution when $n \to 1$

$$\mathbb{V}_{13,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(\cosh(\varphi,n) \pm \sinh(\varphi,n))^2}{f(\cosh(\varphi,n) \pm \sinh(\varphi,n))^2 + g}\right), \tag{87}$$

$$\mathfrak{Q}_{13,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(\cosh(\varphi,n) \pm \sinh(\varphi,n))^2}{f(\cosh(\varphi,n) \pm \sinh(\varphi,n))^2 + g}\right) + \kappa, \tag{88}$$

or

$$\mathbb{V}_{14,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{sech}^2(\varphi,n)}{f\operatorname{sech}^2(\varphi,n) + (1 \pm \tanh^2(\varphi,n))g}\right), \tag{89}$$

$$\mathfrak{Q}_{14,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\operatorname{sech}^2(\varphi,n)}{f\operatorname{sech}^2(\varphi,n) + (1 \pm \tanh^2(\varphi,n))g}\right) + \kappa, \tag{90}$$

along with the condition:

$$0 = {h_4}^2(1 - h_2)\left(\frac{9}{8} - (1 - h_2)(2 + h_2)\right) + \frac{27\,{h_4}^2\beta\left(-1 + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

One can develop a periodic solution if $n \to 0$,

$$\mathbb{V}_{13,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(\sec(\varphi,n) \pm \tan(\varphi,n))^2}{f(\sec(\varphi,n) \pm \tan(\varphi,n))^2 + g}\right), \tag{91}$$

$$\mathfrak{Q}_{13,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(\sec(\varphi,n) \pm \tan(\varphi,n))^2}{f(\sec(\varphi,n) \pm \tan(\varphi,n))^2 + g}\right) + \kappa, \tag{92}$$

or

$$\mathbb{V}_{14,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cos^2(\varphi,n)}{f\cos^2(\varphi,n) + (1 \pm \sin^2(\varphi,n))g}\right), \tag{93}$$

$$\mathfrak{Q}_{14,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\cos^2(\varphi,n)}{f\cos^2(\varphi,n) + (1 \pm \sin^2(\varphi,n))g}\right) + \kappa, \tag{94}$$

along with the condition:

$$0 = {h_4}^2\left(\frac{1}{2} - h_2\right)\left(\frac{9}{4} - \left(\frac{1}{2} - h_2\right)(h_2 + 1)\right) + \frac{27\,{h_4}^2\beta\left(\frac{1}{8} + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

In Figures 7–9, 3D, contour and 2D propagation of $\mathfrak{Q}_{13,2}(s,t)$ at different fractional-order α are given. While in Figures 10–12, 3D, contour and 2D propagation of $\mathbb{V}_{13,2}(s,t)$ at different fractional-order α are presented. In Figures 13–15, 3D, contour and 2D propagation of $\mathfrak{Q}_{14,2}(s,t)$ at different fractional-order α are given. While in Figures 16–18, 3D, contour and 2D propagation of $\mathbb{V}_{14,2}(s,t)$ at different fractional-order α are presented.

Category 12. *If* $l_0 = -\frac{(1-s^2)^2}{4}$, $l_2 = \frac{1+s^2}{2}$, $l_4 = -\frac{1}{4}$, $n \in (0,1)$, *then* $\Phi(\varphi) = s\,cn(\varphi,n) \pm dn(\varphi,n)$; *thus, we have*

$$\mathbb{V}_{15}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(n\,cn(\varphi,n) \pm dn(\varphi,n))^2}{f(n\,cn(\varphi,n) \pm dn(\varphi,n))^2 + g}\right), \tag{95}$$

$$\mathfrak{Q}_{15}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(n\,cn(\varphi,n) \pm dn(\varphi,n))^2}{f(n\,cn(\varphi,n) \pm dn(\varphi,n))^2 + g}\right) + \kappa, \tag{96}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^{\alpha})$, *and* f *and* g *are given below:*

$$f = -8\,\frac{h_4\left(n^2 - 2\,h_2 + 1\right)}{n^4 + 14\,n^2 - 16\,{h_2}^2 + 1},\; g = 12\,\frac{\left(n^2 - 1\right)^2 h_4}{n^4 + 14\,n^2 - 16\,{h_2}^2 + 1},$$

along with the condition:

$$0 = {h_4}^2\left(\frac{1}{2}\,n^2 + \frac{1}{2} - h_2\right)\left(\frac{9\left(-n^2 + 1\right)^2}{16} - \left(\frac{1}{2}\,n^2 + \frac{1}{2} - h_2\right)\left(n^2 + h_2 + 1\right)\right) + \frac{27\,{h_4}^2\beta\left(\frac{3}{16}\left(-n^2 + 1\right)^2 - \left(\frac{1}{2}\,n^2 + \frac{1}{2}\right)^2 + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{15,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(n\ \mathrm{sech}(\varphi,n) \pm \mathrm{sech}(\varphi,n))^2}{f(n\ \mathrm{sech}(\varphi,n) \pm \mathrm{sech}(\varphi,n))^2 + g}\right), \quad (97)$$

$$\mathfrak{Q}_{15,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(n\ \mathrm{sech}(\varphi,n) \pm \mathrm{sech}(\varphi,n))^2}{f(n\ \mathrm{sech}(\varphi,n) \pm \mathrm{sech}(\varphi,n))^2 + g}\right) + \kappa, \quad (98)$$

along with the condition:

$$0 = {h_4}^2(1-h_2)(-(1-h_2)(h_2+1)) + \frac{27\,{h_4}^2\beta\left(-\frac{1}{16} + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

We may be able to develop a single wave analytical solution if $n \to 0$,

$$\mathbb{V}_{15,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(n\ \cos(\varphi,n) \pm 1)^2}{f(n\ \cos(\varphi,n) \pm 1)^2 + g}\right), \quad (99)$$

$$\mathfrak{Q}_{15,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{(n\ \cos(\varphi,n) \pm 1)^2}{f(n\ \cos(\varphi,n) \pm 1)^2 + g}\right) + \kappa, \quad (100)$$

along with the condition:

$$0 = {h_4}^2\left(\frac{1}{2} - h_2\right)\left(\frac{9}{16} - \left(\frac{1}{2} - h_2\right)(h_2+1)\right) + \frac{27\,{h_4}^2\beta\left(-\frac{1}{16} + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

Category 13. *if $l_0 = \frac{1}{4}$, $l_2 = \frac{1-2s^2}{2}$, $l_4 = \frac{1}{4}$, $n \in (0,1)$, then $\Phi(\varphi) = \frac{sn(\varphi,n)}{1 \pm cn(\varphi,n)}$; thus, we have*

$$\mathbb{V}_{16}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sn^2(\varphi,n)}{fsn^2(\varphi,n) + (1 \pm cn^2(\varphi,n))g}\right), \quad (101)$$

$$\mathfrak{Q}_{16}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sn^2(\varphi,n)}{fsn^2(\varphi,n) + (1 \pm cn^2(\varphi,n))g}\right) + \kappa, \quad (102)$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^{\alpha})$, and f and g are given below:

$$f = 8\,\frac{h_4(2\,n^2 + 2\,h_2 - 1)}{16\,n^4 - 16\,n^2 - 16\,{h_2}^2 + 1},\ g = -12\,\frac{h_4}{16\,n^4 - 16\,n^2 - 16\,{h_2}^2 + 1},$$

along with the condition:

$$\begin{aligned} 0 =& {h_4}^2\left(-n^2 + 1/2 - h_2\right)\left(\frac{9}{16} - \left(-n^2 + 1/2 - h_2\right)\left(-2\,n^2 + h_2 + 1\right)\right) + \\ & \frac{27\,{h_4}^2\beta\left(3/16 - \left(-n^2 + 1/2\right)^2 + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}. \end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{16,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\tanh^2(\varphi,n)}{f\tanh^2(\varphi,n) + (1 \pm \mathrm{sech}^2(\varphi,n))g}\right), \quad (103)$$

$$\mathfrak{Q}_{16,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\tanh^2(\varphi,n)}{f\tanh^2(\varphi,n) + (1 \pm \operatorname{sech}^2(\varphi,n))g}\right) + \kappa, \tag{104}$$

along with the condition:

$$0 = {h_4}^2\left(-\frac{1}{2} - h_2\right)\left(\frac{9}{16} - \left(-\frac{1}{2} - h_2\right)(-1 + h_2)\right) + \frac{27\,{h_4}^2\beta\left(-\frac{1}{16} + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

One can develop a periodic solution if $n \to 0$,

$$\mathbb{V}_{16,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sin^2(\varphi,n)}{f\sin^2(\varphi,n) + (1 \pm \cos^2(\varphi,n))g}\right), \tag{105}$$

$$\mathfrak{Q}_{16,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sin^2(\varphi,n)}{f\sin^2(\varphi,n) + (1 \pm \cos^2(\varphi,n))g}\right) + \kappa, \tag{106}$$

along with the condition:

$$0 = {h_4}^2\left(\frac{1}{2} - h_2\right)\left(\frac{9}{16} - \left(\frac{1}{2} - h_2\right)(h_2 + 1)\right) + \frac{27\,{h_4}^2\beta\left(-\frac{1}{16} + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}.$$

Category 14. *if $l_0 = \frac{1}{4}$, $l_2 = \frac{1+s^2}{2}$, $l_4 = \frac{(1-s^2)^2}{4}$, $n \in (0,1)$, then $\Phi(\varphi) = \frac{sn(\varphi,n)}{cn(\varphi,n) \pm dn(\varphi,n)}$; thus, we have*

$$\mathbb{V}_{17}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sn^2(\varphi,n)}{fsn^2(\varphi,n) + g(cn(\varphi,n) \pm dn(\varphi,n))^2}\right), \tag{107}$$

$$\mathfrak{Q}_{17}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{sn^2(\varphi,n)}{fsn^2(\varphi,n) + g(cn(\varphi,n) \pm dn(\varphi,n))^2}\right) + \kappa, \tag{108}$$

where $\varphi = s - \frac{\Gamma(\gamma+1)}{\alpha}(ct^{\alpha})$, and f and g are given below:

$$f = -8\,\frac{h_4(n^2 - 2\,h_2 + 1)}{n^4 + 14\,n^2 - 16\,{h_2}^2 + 1},\ g = -12\,\frac{h_4}{n^4 + 14\,n^2 - 16\,{h_2}^2 + 1},$$

along with the condition:

$$\begin{aligned} 0 =& {h_4}^2\left(\frac{1}{2}\,n^2 + \frac{1}{2} - h_2\right)\left(\frac{9\left(-n^2+1\right)^2}{16} - \left(\frac{1}{2}\,n^2 + \frac{1}{2} - h_2\right)\left(n^2 + h_2 + 1\right)\right) + \\ & \frac{27\,{h_4}^2\beta\left(3/16\left(-n^2+1\right)^2 - \left(\frac{1}{2}\,n^2 + \frac{1}{2}\right)^2 + {h_2}^2\right)^2}{4\,\beta^2\kappa^2 + 24\,\beta\,h_2 - 18\,\beta}. \end{aligned}$$

We may be able to develop a single wave analytical solution if $n \to 1$,

$$\mathbb{V}_{17,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\tanh^2(\varphi,n)}{f\tanh^2(\varphi,n) + g(\operatorname{sech}(\varphi,n) \pm \operatorname{sech}(\varphi,n))^2}\right), \tag{109}$$

$$\mathfrak{Q}_{17,1}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\tanh^2(\varphi,n)}{f\tanh^2(\varphi,n) + g(\operatorname{sech}(\varphi,n) \pm \operatorname{sech}(\varphi,n))^2}\right) + \kappa, \tag{110}$$

along with the condition:

$$0 = {h_4}^2(1-h_2)(-(1-h_2)(2+h_2)) + \frac{27\,{h_4}^2\beta\left(-1^2+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$

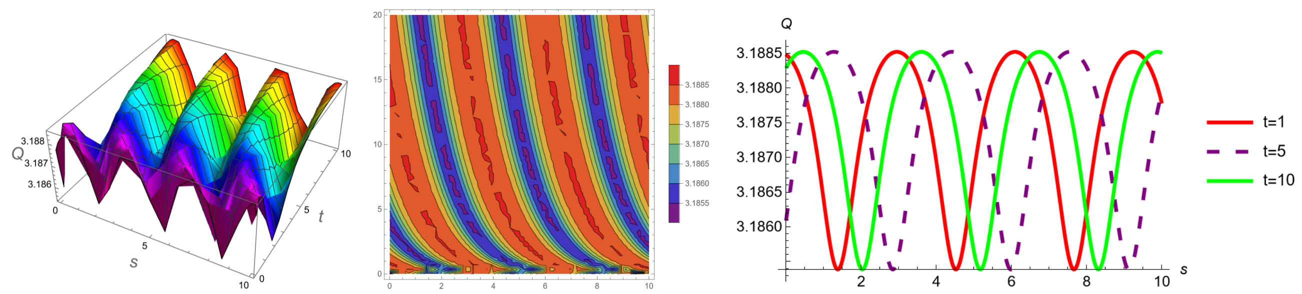
One can develop a periodic solution if $n \to 0$,

$$\mathbb{V}_{17,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sin^2(\varphi,n)}{f\sin^2(\varphi,n)+g(\cos(\varphi,n)\pm 1)^2}\right), \tag{111}$$

$$\mathfrak{Q}_{17,2}(s,t) = \pm\left(\frac{1}{3}\kappa + \frac{\sqrt{-\varpi}}{6\beta}\right) \pm 3\sqrt{-\frac{1}{\varpi}}\left(\frac{\sin^2(\varphi,n)}{f\sin^2(\varphi,n)+g(\cos(\varphi,n)\pm 1)^2}\right)+\kappa, \tag{112}$$

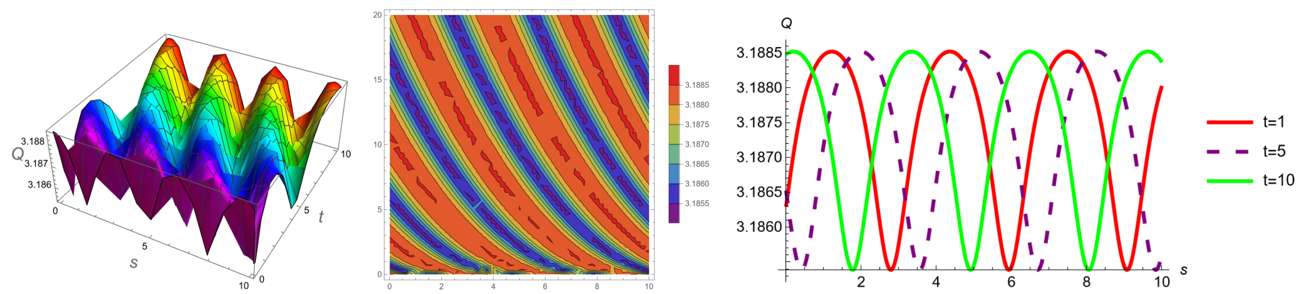
along with the condition:

$$0 = {h_4}^2\left(\frac{1}{2}-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-h_2\right)(h_2+1)\right) + \frac{27\,{h_4}^2\beta\left(-\frac{1}{16}+{h_2}^2\right)^2}{4\,\beta^2\kappa^2+24\,\beta\,h_2-18\,\beta}.$$



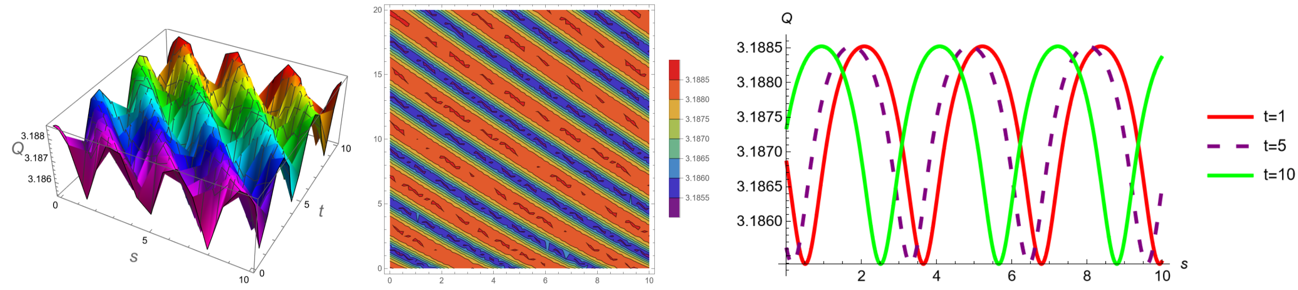
(**a**) (**b**) (**c**)

Figure 1. (**a**) A 3D propagation of $\mathfrak{Q}_{1,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**b**) Contour propagation of $\mathfrak{Q}_{1,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**c**) A 2D propagation of $\mathfrak{Q}_{1,2}(s,t)$ at fractional-order $\alpha = 0.1$.



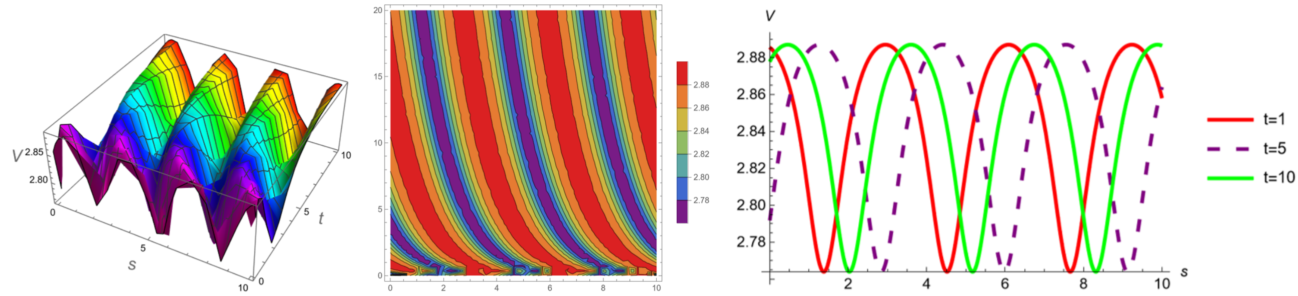
(**a**) (**b**) (**c**)

Figure 2. (**a**) A 3D propagation of $\mathfrak{Q}_{1,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**b**) Contour propagation of $\mathfrak{Q}_{1,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**c**) A 2D propagation of $\mathfrak{Q}_{1,2}(s,t)$ at fractional-order $\alpha = 0.5$.



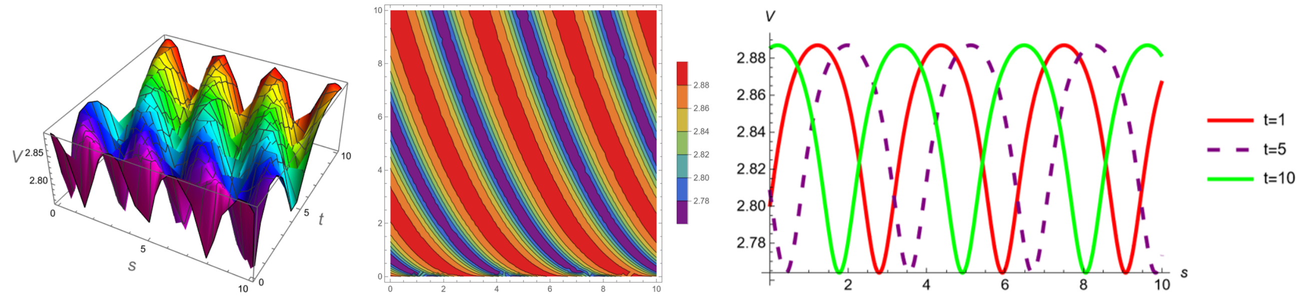
(**a**) (**b**) (**c**)

Figure 3. (**a**) A 3D propagation of $\mathfrak{Q}_{1,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**b**) Contour propagation of $\mathfrak{Q}_{1,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**c**) A 2D propagation of $\mathfrak{Q}_{1,2}(s,t)$ at fractional-order $\alpha = 0.9$.



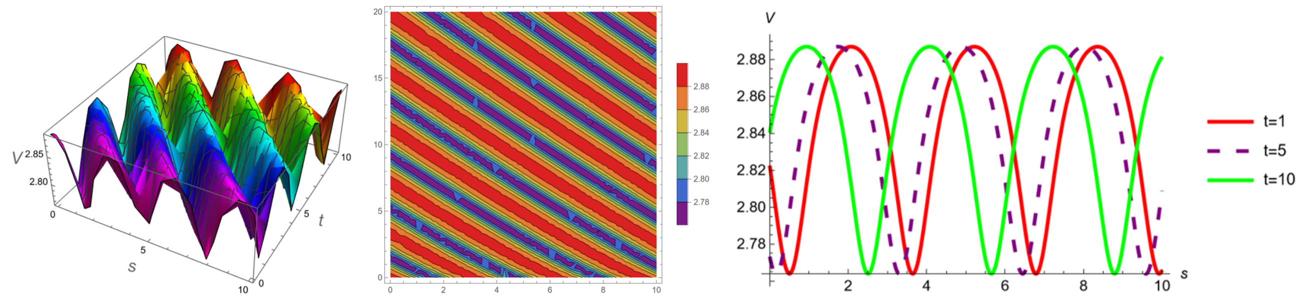
(**a**) (**b**) (**c**)

Figure 4. (**a**) A 3D propagation of $\mathbb{V}_{1,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**b**) Contour propagation of $\mathbb{V}_{1,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**c**) A 2D propagation of $\mathbb{V}_{1,2}(s,t)$ at fractional-order $\alpha = 0.1$.



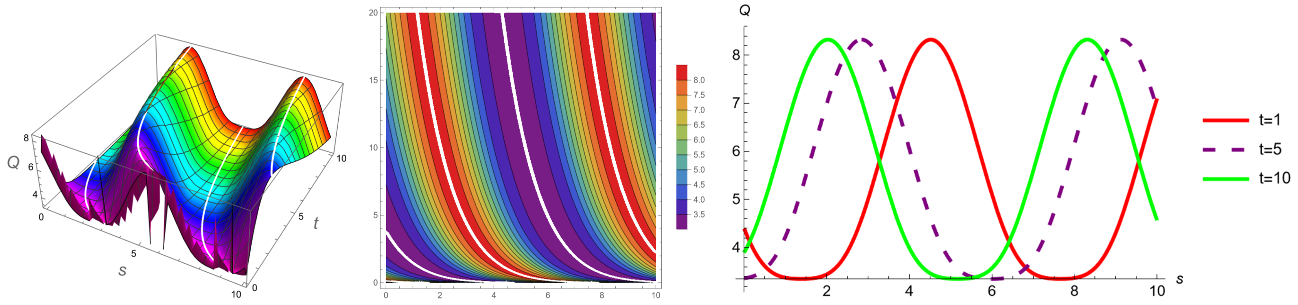
(**a**) (**b**) (**c**)

Figure 5. (**a**) A 3D propagation of $\mathbb{V}_{1,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**b**) Contour propagation of $\mathbb{V}_{1,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**c**) A 2D propagation of $\mathbb{V}_{1,2}(s,t)$ at fractional-order $\alpha = 0.5$.



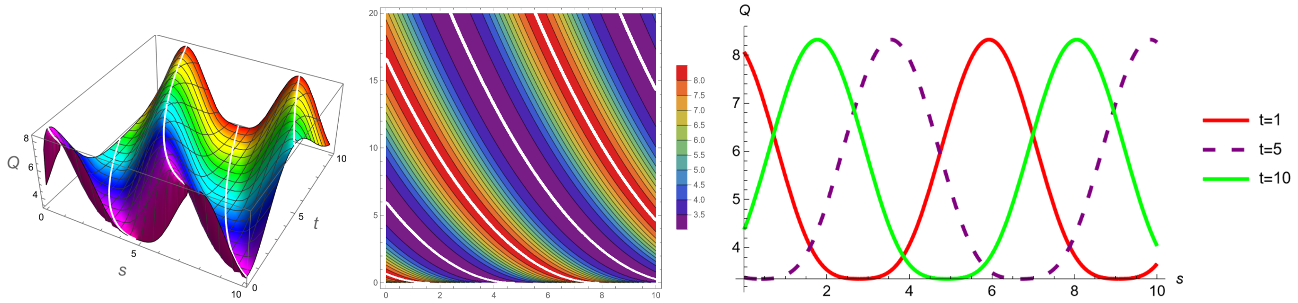
(**a**) (**b**) (**c**)

Figure 6. (**a**) A 3D propagation of $\mathbb{V}_{1,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**b**) Contour propagation of $\mathbb{V}_{1,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**c**) A 2D propagation of $\mathbb{V}_{1,2}(s,t)$ at fractional-order $\alpha = 0.9$.



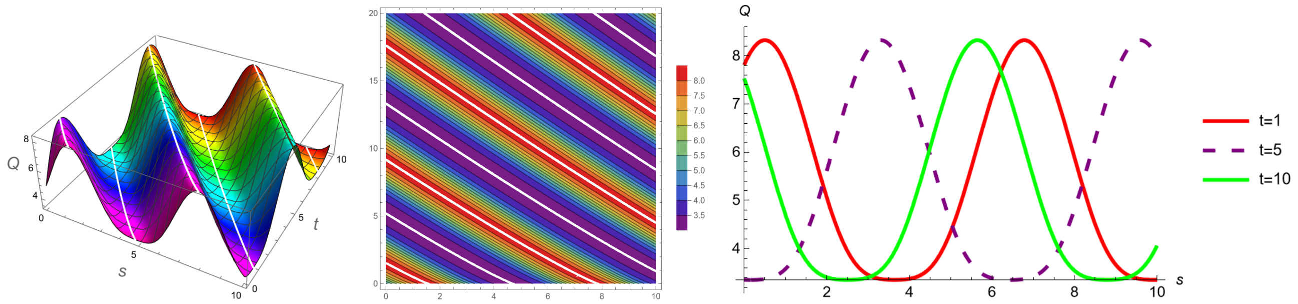
(**a**) (**b**) (**c**)

Figure 7. (**a**) A 3D propagation of $\mathfrak{Q}_{13,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**b**) Contour propagation of $\mathfrak{Q}_{13,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**c**) A 2D propagation of $\mathfrak{Q}_{13,2}(s,t)$ at fractional-order $\alpha = 0.1$.



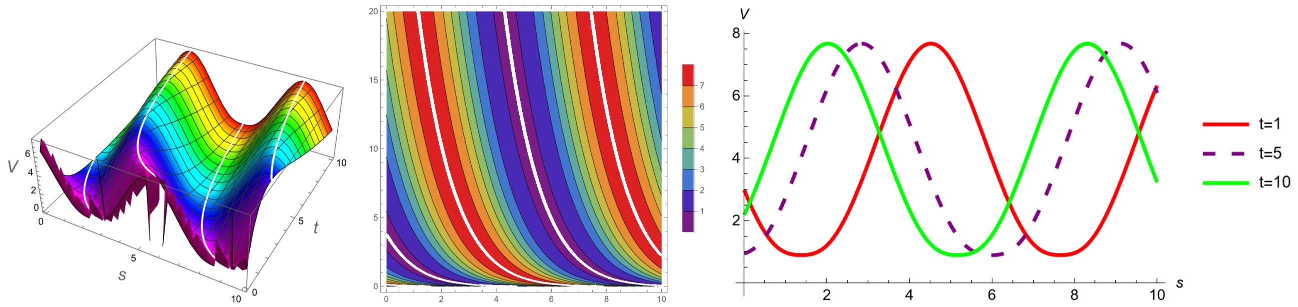
(**a**) (**b**) (**c**)

Figure 8. (**a**) A 3D propagation of $\mathfrak{Q}_{13,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**b**) Contour propagation of $\mathfrak{Q}_{13,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**c**) A 2D propagation of $\mathfrak{Q}_{13,2}(s,t)$ at fractional-order $\alpha = 0.5$.



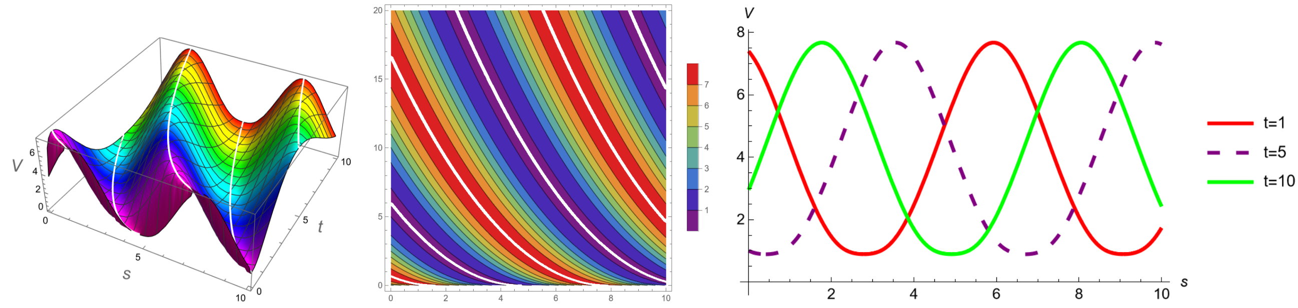
(**a**) (**b**) (**c**)

Figure 9. (**a**) A 3D propagation of $\mathfrak{Q}_{13,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**b**) Contour propagation of $\mathfrak{Q}_{13,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**c**) A 2D propagation of $\mathfrak{Q}_{13,2}(s,t)$ at fractional-order $\alpha = 0.9$.



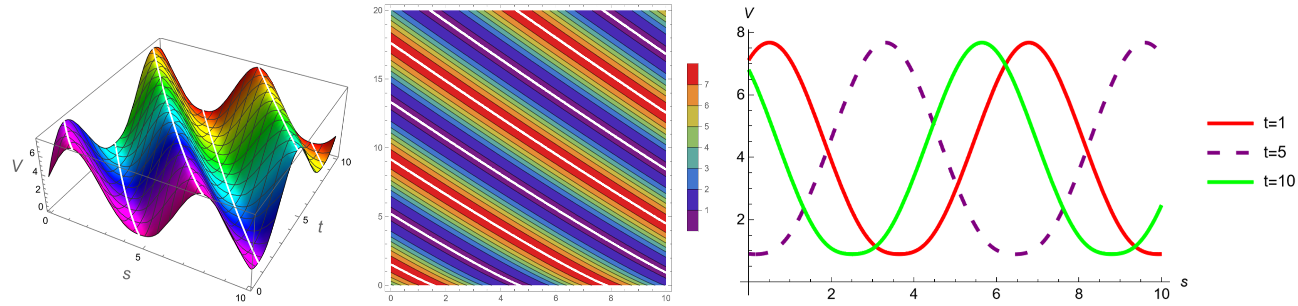
(**a**) (**b**) (**c**)

Figure 10. (**a**) A 3D propagation of $\mathbb{V}_{13,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**b**) Contour propagation of $\mathbb{V}_{13,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**c**) A 2D propagation of $\mathbb{V}_{13,2}(s,t)$ at fractional-order $\alpha = 0.1$.



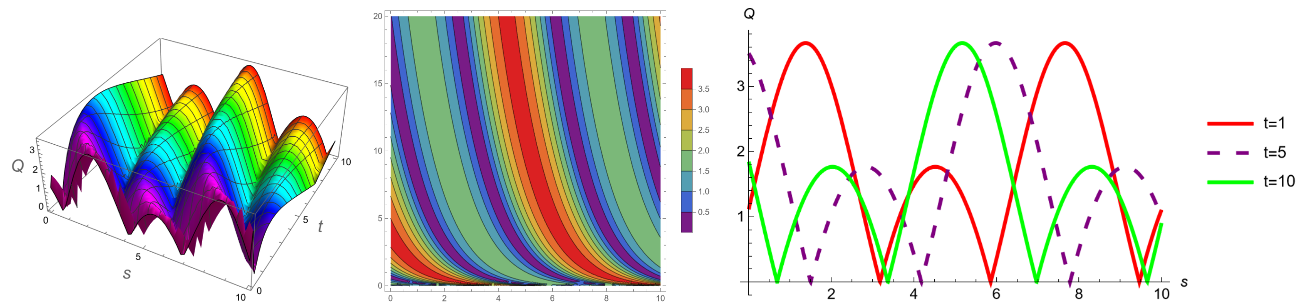
(**a**) (**b**) (**c**)

Figure 11. (**a**) A 3D propagation of $\mathbb{V}_{13,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**b**) Contour propagation of $\mathbb{V}_{13,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**c**) A 2D propagation of $\mathbb{V}_{13,2}(s,t)$ at fractional-order $\alpha = 0.5$.



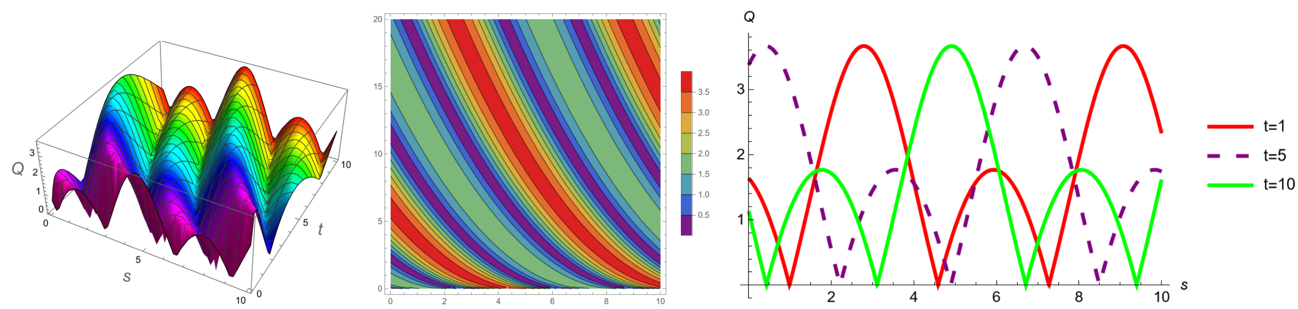
(**a**) (**b**) (**c**)

Figure 12. (**a**) A 3D propagation of $\mathbb{V}_{13,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**b**) Contour propagation of $\mathbb{V}_{13,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**c**) A 2D propagation of $\mathbb{V}_{13,2}(s,t)$ at fractional-order $\alpha = 0.9$.



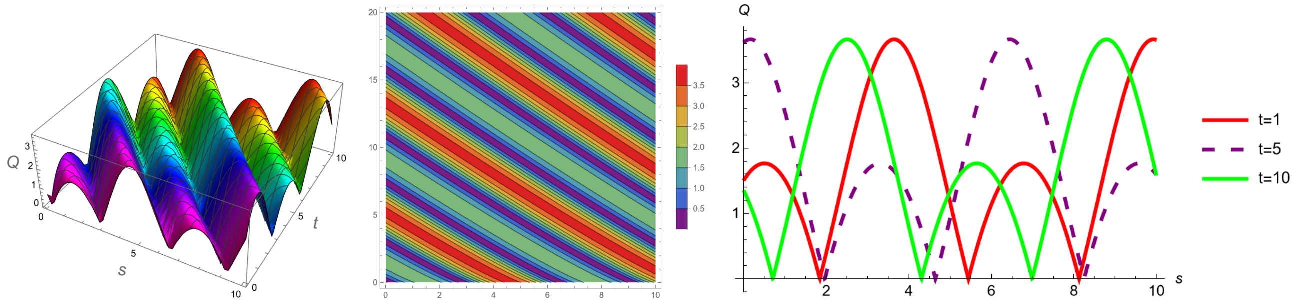
(**a**) (**b**) (**c**)

Figure 13. (**a**) A 3D propagation of $\mathfrak{Q}_{14,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**b**) Contour propagation of $\mathfrak{Q}_{14,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**c**) A 2D propagation of $\mathfrak{Q}_{14,2}(s,t)$ at fractional-order $\alpha = 0.1$.



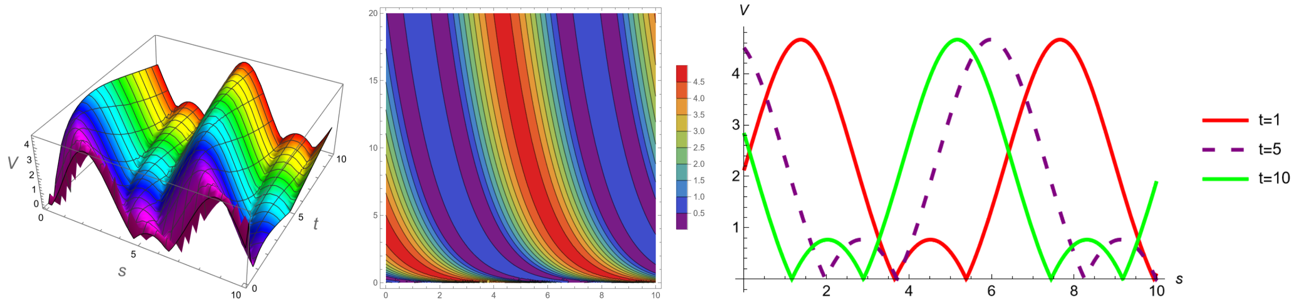
(**a**) (**b**) (**c**)

Figure 14. (**a**) A 3D propagation of $\mathfrak{Q}_{14,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**b**) Contour propagation of $\mathfrak{Q}_{14,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**c**) A 2D propagation of $\mathfrak{Q}_{14,2}(s,t)$ at fractional-order $\alpha = 0.5$.



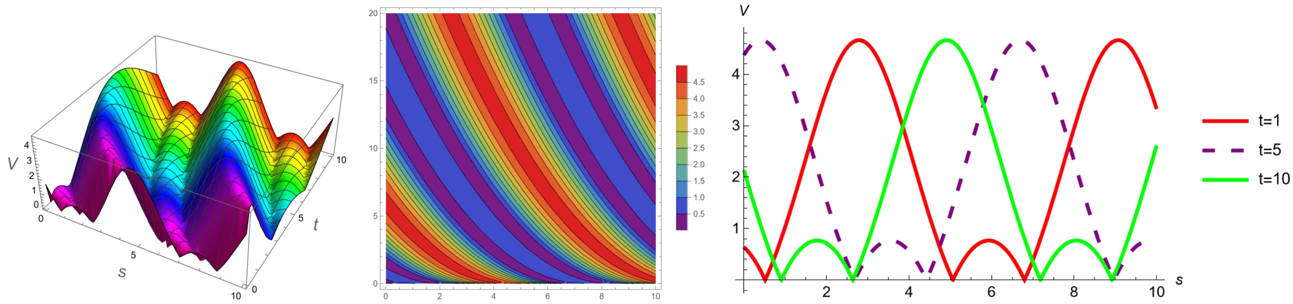
(**a**) (**b**) (**c**)

Figure 15. (**a**) A 3D propagation of $\mathfrak{Q}_{14,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**b**) Contour propagation of $\mathfrak{Q}_{14,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**c**) A 2D propagation of $\mathfrak{Q}_{14,2}(s,t)$ at fractional-order $\alpha = 0.9$.



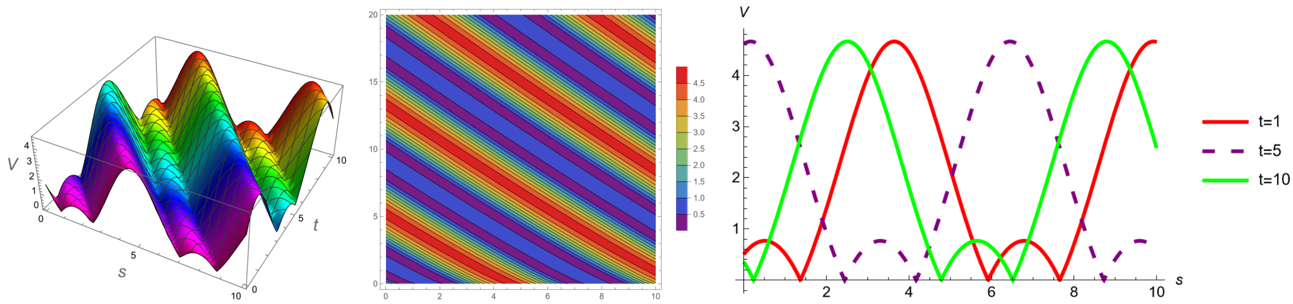
(**a**) (**b**) (**c**)

Figure 16. (**a**) A 3D propagation of $\mathbb{V}_{14,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**b**) Contour propagation of $\mathbb{V}_{14,2}(s,t)$ at fractional-order $\alpha = 0.1$. (**c**) A 2D propagation of $\mathbb{V}_{14,2}(s,t)$ at fractional-order $\alpha = 0.1$.



(**a**) (**b**) (**c**)

Figure 17. (**a**) A 3D propagation of $\mathbb{V}_{14,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**b**) Contour propagation of $\mathbb{V}_{14,2}(s,t)$ at fractional-order $\alpha = 0.5$. (**c**) A 2D propagation of $\mathbb{V}_{14,2}(s,t)$ at fractional-order $\alpha = 0.5$.



(**a**) (**b**) (**c**)

Figure 18. (**a**) A 3D propagation of $\mathbb{V}_{14,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**b**) Contour propagation of $\mathbb{V}_{14,2}(s,t)$ at fractional-order $\alpha = 0.9$. (**c**) A 2D propagation of $\mathbb{V}_{14,2}(s,t)$ at fractional-order $\alpha = 0.9$.

3. Dynamical Analysis

In this section, we delve into the dynamics of the newly-emerged coupled nonlinear volatility and option pricing model. To achieve this objective, we employ a sensitivity analysis and construct a Hamiltonian function. Furthermore, to obtain a dynamical system, we apply the Galilean transformation to Equation (18).

$$\begin{cases} \frac{d\mathfrak{L}}{d\varphi} = \mathcal{P}, \\ \frac{d\mathcal{P}}{d\varphi} = -(4\beta\mathfrak{L}^3 + 4\beta\kappa\mathfrak{L}^2 + (2\beta\kappa^2 - 2)\mathfrak{L}). \end{cases} \tag{113}$$

The planer dynamical system (113) is a planer Hamiltonian system, which needs to be mentioned. One can ensure that a dynamical system's Hamiltonian function exists by integrating (113).

$$\mathfrak{H}(\mathfrak{L}, \mathcal{P}) = \frac{\mathcal{P}^2}{2} + \beta\mathfrak{L}^4 + \frac{4}{3}\beta\kappa\mathfrak{L}^3 + (\beta\kappa^2 - 1)\mathfrak{L}^2 = h. \tag{114}$$

One can verify from (114),

$$\frac{d\mathfrak{L}}{d\varphi} = \frac{\partial\mathfrak{H}}{\partial\mathcal{P}} \text{ and } \frac{d\mathcal{P}}{d\varphi} = -\frac{\partial\mathfrak{H}}{\partial\mathfrak{L}}. \tag{115}$$

The set of equations in Equation (113), as indicated by Equation (115), constitutes a Hamiltonian system. This observation leads us to conclude that the system, as described by (113), possesses conservative properties. Consequently, the phase trajectories generated by the vector field of this system will encompass all of the traveling-wave solutions to Equation (18).

Sensitive Analysis

The ordinary differential equation is transformed into a first-order dynamical system through the Galilean transformation. This transformation aims to assess the model's sensitivity.

As depicted in the aforementioned graphs, the system's behavior exhibits notable variations, even in cases where initial conditions remain relatively stable. This observation highlights the model under investigation's high sensitivity to initial conditions, especially when parameters $\beta = 0.9$ and $\kappa = 1$ are employed. In Figure 19, curve 1 corresponds to $(0.8, 0.03)$ and curve 2 to $(0.9, 0.02)$ while in Figure 20, curve 1 corresponds to $(1.08, 0.03)$ and curve 2 to $(0.1, 0.02)$, in Figure 21 curve 1 corresponds to $(1.08, 0.03)$ and curve 2 to $(0.1, 0.02)$ and finally Figure 22 is the representation of various initial conditions for sensitivity analysis.

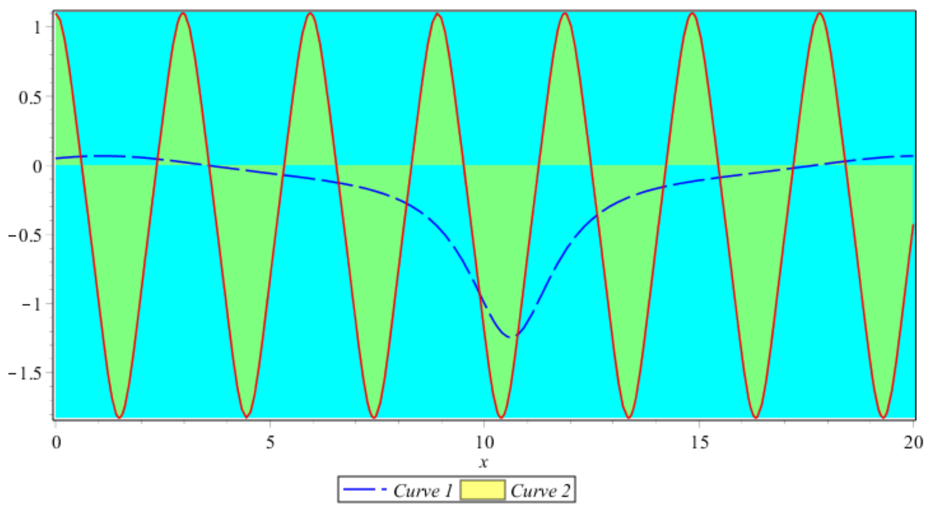


Figure 19. Graph of sensitivity: Curve 1 corresponds to $(0.8, 0.03)$ and curve 2 to $(0.9, 0.02)$.

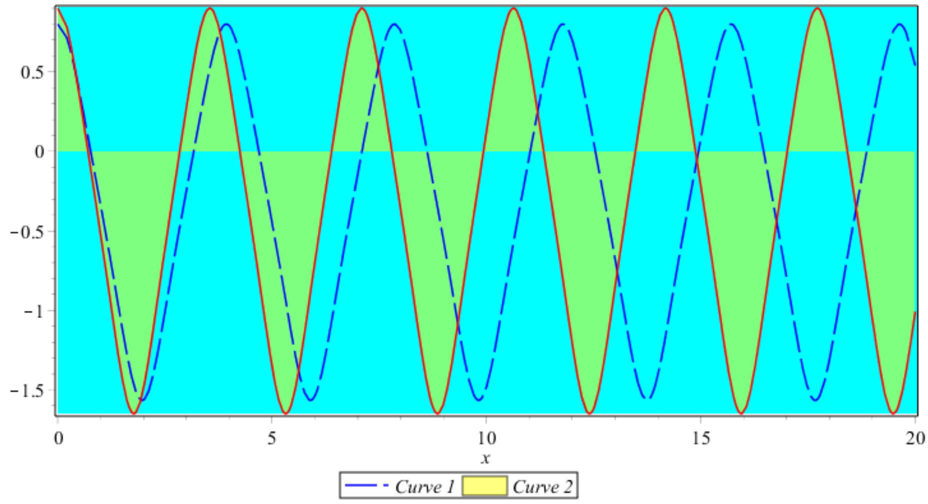


Figure 20. Graph of sensitivity: Curve 1 corresponds to (0.08, 0.03) and curve 2 to (0.1, 0.02).

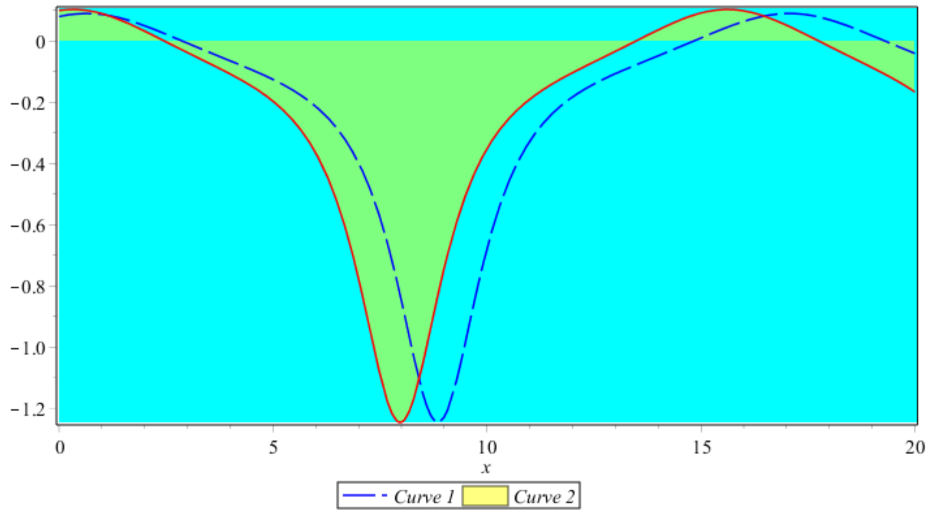


Figure 21. Graph of sensitivity: Curve 1 corresponds to (1.08, 0.03) and curve 2 to (0.1, 0.02).

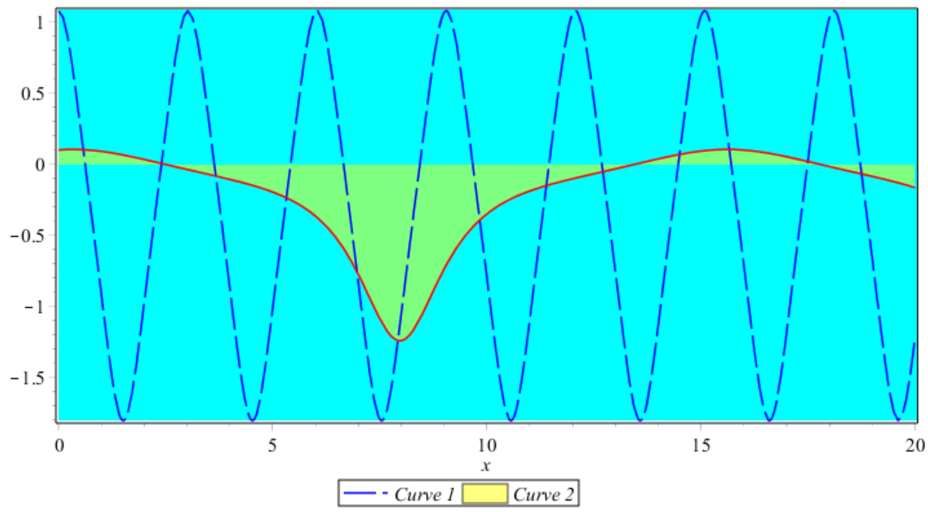


Figure 22. Various initial conditions for sensitivity analysis.

4. Conclusions

In conclusion, this study successfully developed numerous novel solitons within the context of the nonlinear fractional coupled option pricing and volatility governing system, employing a generalized expansion strategy, specifically the Φ^6-expansion algorithm. This algorithm has yielded a rich array of fourteen distinct families of soliton structures, resulting in a total of twenty-eight solutions. These solutions exhibit solitary wave patterns based on Jacobi elliptic functions, transitioning into hyperbolic solutions as the limit approaches $n \to 1$, and trigonometric solutions as the limit approaches $n \to 0$. The inclusion of constraints with each result ensures the validity of these solutions. To visualize the dynamic propagation properties of these acquired solutions, in Fig we presented 2D, 3D, and contour graphics alongside the relevant parametric values, which were carefully chosen to meet the established criteria. These parametric values, assigned to the linked free parameters, facilitate the description of the graphical behavior of optical pulses. This work provides a deeper understanding of the physical perspective of the nonlinear model through the admissible solutions presented. The Φ^6-model process emerges as a potent and effective mathematical technique that can be readily applied to furnish analytical solutions for a wide range of challenging mathematical problems.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of Open Access Journals
TLA	three-letter acronym
LD	linear dichroism

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