fractal and fractional

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Qualitative Aspects of a Fractional-Order Integro-Differential Equation with a Quadratic Functional Integro-Differential Constraint

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Abstract: This manuscript investigates a constrained problem of an arbitrary (fractional) order quadratic functional integro-differential equation with a quadratic functional integro-differential constraint. We demonstrate that there is at least one solution $x \in C[0,T]$ to the problem. Moreover, we outline the necessary demands for the solution's uniqueness. In addition, the continuous dependence of the solution and the Hyers–Ulam stability of the problem are analyzed. In order to illustrate our results, we provide some particular cases and instances.

Keywords: constrained problem; functional integro-differential equation; fractional order; Schauder fixed-point theorem; continuous dependence

Citation: El-Sayed, A.M.A.; Alhamali, A.A.A.; Hamdallah, E.M.A.; Ebead, H.R. Qualitative Aspects of a Fractional-Order Integro-Differential Equation with a Quadratic Functional Integro-Differential Constraint. *Fractal Fract.* **2023**, *7*, 835. https://doi.org/10.3390/fractalfract7120835

Academic Editors: Carlo Cattani, Rajarama Mohan Jena and Snehashish Chakraverty

Received: 25 October 2023
Revised: 19 November 2023
Accepted: 22 November 2023
Published: 24 November 2023

1. Introduction

Fractional-order differential and integral equations have a wide range of applications across various fields with examples in physics, engineering, and biomedical engineering. The nonlocal conditions are often encountered in mathematical and physical problems, where the behavior of a system depends on different factors or parameters; see [1–10].

In recent years, several scholars have concentrated their efforts on constrained integral equations. Their findings about functional integral equations have been expanded to include a particular set of constrained integral equations on a bounded interval (see [11–13]) and unbounded intervals (see [14]). Constrained problems are essential in the mathematical depiction of real-world situations, where such problems are transformed into mathematical models [15–17]. The relevance of handling constraints or control variables arises from the unanticipated elements that persistently disrupt biological systems in the real world; biological traits like survival rates might change as a result. The question of whether an ecosystem can survive those erratic, disruptive occurrences that happen for a short while is of practical significance to ecology. The disturbance functions are what we refer to as control variables in the context of control variables. Numerous papers address this type of problem; for instance, in [18], the authors discussed a nonlinear constrained problem involving a nonlinear functional integral equation. They also examined the appropriate conditions for the solution's uniqueness and its continuous dependence on certain parameters. The authors applied Schauder's fixed-point theorem to prove the existence of solutions. In [14], the authors studied the solvability of a constrained problem involving a nonlinear-delay functional equation subject to a quadratic functional integral constraint. By applying the De Blasi measure of noncompactness, they studied nondecreasing solutions in the bounded interval $L_1[0,T]$ and nonincreasing solutions in the unbounded interval $L_1(R+)$.

Problems with a feedback control or control variable have great importance in numerous fields due to unforeseen factors that disrupt ecosystems in the real world. It could lead to changes in biological characteristics like survival rates; see [19–22]. Furthermore, ecology faces a practical challenge in determining whether an ecosystem can withstand unpredictable, disruptive events; see [15,23–25]. In addition, feedback control problems are crucial to establishing the solutions to delay population models; see [26–29]. In [23], the authors investigated the effect of feedback control on chemostat models; they studied a sufficient condition for the existence of a positive periodic solution tp the model. In [30], the author discussed a positive periodic solution to a nonlinear neutral delay population equation with feedback control. In [12], the authors studied fractional-order models of thermostats; they proved the existence of a solution and the continuous dependence of the unique solution on the control variable. In [13], the author investigated the solvability and the asymptotic stability of a class of nonlinear functional-integral equations with feedback control. For further relevant works, see [12,31–35].

Fixed-point theorems are a great tool for discussing the solvability of differential equation problems that have been studied in a number of monographs and publications; see [6,31,36–40].

Inspired by the above, we consider the constrained problem

$$\frac{dx}{dt} = f\left(t, g_1(t, D^{\zeta}x(t)) \cdot \int_0^{\vartheta(t)} g_2(s, D^{\gamma}x(s))ds\right), \qquad \zeta, \gamma \in (0,1), \;\; t \in (0,T] \tag{1}$$

with the quadratic functional integro-differential constrained

$$x(\tau) = x_0 + \int_0^{T-\tau} h\big(s, x(s) \cdot D^{\eta}x(s)\big)ds, \qquad \eta \in (0,1), \; \tau \in [0,T]. \tag{2}$$

Our aim in this paper is to examine the existence of a solution $x \in C(0,T]$ to the constrained problems (1) and (2). A sufficient hypothesis for the solution's uniqueness will be given. Furthermore, we prove the Hyers–Ulam stability of the problem. The continuous dependence of the solution on the fractional-order derivative $D^{\zeta}x(t)$, the parameter x_0, and the function h will be studied. To highlight our results, we present several examples and special cases. This study establishes conditions for the existence and uniqueness of the solution according to Schauder's fixed-point theorem.

2. Main Result

2.1. Formulation of the Problem

Consider the constrained problem (1) and (2) under the next hypothesis. Let $I = [0,T]$.

(i) $\vartheta : I \to I$ is continuous function such that $\vartheta(t) \leq t$.

(ii) f, h and g_i, $i = 1,2 : I \times R \to R$ are Caratheodory functions [41]. There exist bounded measurable functions [42] a and $a_i : I \to R$ and a positive constants b and b_i such that

$$\begin{aligned} |f(t,x)| &\leq |a(t)| + b|x| \leq a^* + b|x|, \quad a^* = \sup_{t\in I}|a(t)|. \\ |g_i(t,x)| &\leq |a_i(t)| + b_i|x| \leq a_i^* + b_i|x|, \quad a_i^* = \sup_{t\in I}|a_i(t)|, i = 1,2. \\ |h(t,x)| &\leq |a_3(t)| + b_3|x|, \quad \sup_{s\in I}\int_0^{T-\tau}|a_3(s)|ds \leq N. \end{aligned}$$

(iii) The following algebraic equation has a real positive root r_1.

$$bb_1b_2T^{2-\gamma}r_1^2 + \left(a^*b_2T^{2-\gamma} + ba_1^*b_2T^{2-\gamma} + bb_1a_2^*T^{2-\zeta} - 1\right)r_1 + a^*a_2^*T^{2-\zeta} + ba_1^*a_2^*T^{2-\zeta} = 0.$$

(iv) $r_1 b_3 T^{\zeta-\eta+1} < 1$.

The next lemma demonstrates the equivalence between the constrained problem (1) and (2) and its corresponding integral equations.

Lemma 1. *If the solution to (1) and (2) exists, then it can be expressed by*

$$x(t) = x_0 + \int_0^{T-\tau} \Big(h(s, x(s) \cdot I^{\zeta-\eta} y(s)) \Big) ds - I^{\zeta} y(\tau) + I^{\zeta} y(t) \tag{3}$$

and

$$y(t) = I^{1-\zeta} f\left(t, g_1(t, y(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y(s)) ds \right). \tag{4}$$

Proof. Let x be the solution to (1) and (2). Operating by $I^{1-\zeta}$ in on both sides of (1), we obtain

$$D^{\zeta} x(t) = I^{1-\zeta} \frac{dx}{dt} = I^{1-\zeta} \left(f(t, g_1(t, D^{\zeta} x(t)) \cdot \int_0^{\vartheta(t)} g_2(s, D^{\gamma} x(s)) ds \right).$$

Taking $D^{\zeta} x(t) = y(t)$, then,

$$x(t) = x(0) + I^{\zeta} y(t). \tag{5}$$

And we can deduce that

$$\begin{aligned} I^{\zeta-\gamma} y(t) &= I^{\zeta-\gamma} D^{\zeta} x(t) = I^{\zeta-\gamma} I^{1-\zeta} \frac{dx}{dt} \\ &= I^{1-\gamma} \frac{dx}{dt} = D^{\gamma} x(t), \end{aligned} \tag{6}$$

and similarly,

$$\begin{aligned} I^{\zeta-\eta} y(t) &= I^{\zeta-\eta} D^{\zeta} x(t) = I^{\zeta-\eta} I^{1-\zeta} \frac{dx}{dt} \\ &= I^{1-\eta} \frac{dx}{dt} = D^{\eta} x(t). \end{aligned} \tag{7}$$

Substituting from (5)–(7) in (1) and (2), we obtain (4) and (3). Conversely, let x be a solution to (3). Differentiating (3), we obtain

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} [x_0 + \int_0^{T-\tau} \Big(h(s, x(s) \cdot I^{\zeta-\eta} y(s)) \Big) ds - I^{\zeta} y(\tau) + I^{\zeta} y(t)]. \\ &= \frac{d}{dt} I^{\zeta} y = \frac{d}{dt} I^{\zeta} I^{1-\zeta} f(t, g_1(t, D^{\zeta} x(t)) \cdot \int_0^{\vartheta(t)} g_2(s, D^{\gamma} x(s)) ds \\ &= f(t, g_1(t, D^{\zeta} x(t)) \cdot \int_0^{\vartheta(t)} g_2(s, D^{\gamma} x(s)) ds \end{aligned}$$

This proves the equivalence between the two systems (1) and (2) and (3) to (4). □

2.2. Existence of the Solution

Here, we prove the existence of the continuous solution $x \in C(I)$ of (1) and (2). For this purpose, we present the next theorem.

Theorem 1. *Assume that the hypotheses (i)–(iv) are satisfied; then, the solution $x \in C(I)$ of (1) and (2) exists.*

Proof. Define the closed sphere Q_{r_1} and the operator F_1 with

$$Q_{r_1} = \{ y \in C(I) : \|y\| \le r_1 \}.$$

and

$$F_1y(t) = I^{1-\zeta} f\left(t, g_1(t, y(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y_2(s)) ds\right).$$

Let $y \in Q_{r_1}$; then, for $t \in [0, T]$, and assumptions (i)–(ii), we obtain

$$\begin{aligned}
|F_1y(t)| &= \left| I^{1-\zeta} f\left(t, g_1(t, y(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y_2(s)) ds\right) \right| \\
&\leq I^{1-\zeta}\left(a^* + b(a_1^* + b_1|y(s)|) \cdot \int_0^t (a_2^* + b_2 I^{\zeta-\gamma}|y(s)|) ds\right) \\
&\leq (a^* + b(a_1^* + b_1 r_1))(I^{2-\zeta} a_2^* + I^{2-\gamma} b_2 r_1) \\
&\leq (a^* + b(a_1^* + b_1 r_1))\left(\frac{a_2^* t^{2-\zeta}}{\Gamma(3-\zeta)} + \frac{b_2 r_1 t^{2-\gamma}}{\Gamma(3-\gamma)}\right) \\
&\leq (a^* + b(a_1^* + b_1 r_1))(a_2^* T^{2-\zeta} + b_2 r_1 T^{2-\gamma}) = r_1.
\end{aligned}$$

From assumption (iii), we obtain

$$\|F_1y\| \leq (a^* + b(a_1^* + b_1 r_1))(a_2^* T^{2-\zeta} + b_2 r_1 T^{2-\gamma}) = r_1.$$

This proves that $\{F_1y\}$ is uniformly bounded on Q_{r_1}. Let $y \in Q_{r_1}$, $t_1, t_2 \in I$ such that $t_2 > t_1$ and $| t_1 - t_2 | \leq \delta$. By using assumption (ii), then,

$$\begin{aligned}
&|F_1y(t_2) - F_1y(t_1)| = \\
&\left| \int_0^{t_2} \frac{(t_2-s)^{-\zeta}}{\Gamma(1-\zeta)} \left(f(s, g_1(s, y(s)) \cdot \int_0^{\vartheta(s)} g_2(\theta, I^{\zeta-\gamma} y(\theta)) d\theta\right) ds \right. \\
- &\left. \int_0^{t_1} \frac{(t_1-s)^{-\zeta}}{\Gamma(1-\zeta)} \left(f(s, g_1(s, y(s)) \cdot \int_0^{\vartheta(s)} g_2(\theta, I^{\zeta-\gamma} y(\theta)) d\theta\right) ds \right| \\
\leq &\left| \int_0^{t_1} \frac{(t_2-s)^{-\zeta}}{\Gamma(1-\zeta)} \left(f(s, g_1(s, y(s)) \cdot \int_0^{\vartheta(s)} g_2(\theta, I^{\zeta-\gamma} y(\theta)) d\theta\right) ds \right. \\
+ &\int_{t_1}^{t_2} \frac{(t_2-s)^{-\zeta}}{\Gamma(1-\zeta)} \left(f(s, g_1(s, y(s)) \cdot \int_0^{\vartheta(s)} g_2(\theta, I^{\zeta-\gamma} y(\theta)) d\theta\right) ds \\
- &\left. \int_0^{t_1} \frac{(t_1-s)^{-\zeta}}{\Gamma(1-\zeta)} \left(f(s, g_1(s, y(s)) \cdot \int_0^{\vartheta(s)} g_2(\theta, I^{\zeta-\gamma} y(\theta)) d\theta\right) ds \right| \\
\leq &\left| \int_0^{t_1} \frac{(t_2-s)^{-\zeta}}{\Gamma(1-\zeta)} - \frac{(t_1-s)^{-\zeta}}{\Gamma(1-\zeta)} (a^* + b(a_1^* + b_1 r_1))(a_2^* T^{2-\zeta} + b_2 r_1 T^{2-\gamma})) ds \right. \\
+ &\left. \int_{t_1}^{t_2} \frac{1}{\Gamma(1-\zeta)(t_2-s)^{\zeta}} (a^* + b(a_1^* + b_1 r_1))(a_2^* T^{2-\zeta} + b_2 r_1 T^{2-\gamma})) ds \right| \\
\leq &\int_0^{t_1} \left| \frac{(t_2-s)^{\zeta} - (t_1-s)^{\zeta}}{\Gamma(1-\zeta)(t_1-s)^{\zeta}(t_2-s)^{\zeta}} \right| (a^* + b(a_1^* + b_1 r_1))(a_2^* T^{2-\zeta} + b_2 r_1 T^{2-\gamma})) ds \\
+ &\int_{t_1}^{t_2} \frac{1}{\Gamma(1-\zeta)(t_2-s)^{\zeta}} (a^* + b(a_1^* + b_1 r_1))(a_2^* T^{2-\zeta} + b_2 r_1 T^{2-\gamma})) ds.
\end{aligned}$$

This proves that $F_1 : Q_{r_1} \to Q_{r_1}$ and that $\{F_1 y\}$ is equi-continuous on Q_{r_1}. From [41], $\{F_1 y\}$ is relatively compact. Hence, the operator F_1 is compact.
Let $\{y_n\} \subset Q_{r_1}$ be such that $y_n \to y$; then,

$$F_1 y_n(t) \quad = \quad I^{1-\zeta} f\left(t, g_1(t, y_n(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y_n(s)) ds\right),$$

Thus, by taking the limits for both sides and in view of Lebesgues dominated convergence Theorem [41] and assumption (ii), we obtain

$$\begin{aligned}
\lim_{n\to\infty} F_1 y_n(t) &= \lim_{n\to\infty} I^{1-\zeta} f\left(t, g_1(t, y_n(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y_n(s)) ds\right) \\
&= I^{1-\zeta} f\left(t, g_1(t, \lim_{n\to\infty} y_n(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} \lim_{n\to\infty} y_n(s)) ds\right) \\
&= I^{1-\zeta} f\left(t, g_1(t, y(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y(s)) ds\right) \\
&= F_1 y(t),
\end{aligned}$$

Hence, F_1 is continuous and the solution to (4) exists.

Now, for the validity of solutions $x \in C(I)$ of (3), let the assumptions (i)–(iv) be satisfied. Define Q_{r_2} as the closed sphere

$$Q_{r_2} = \{x \in C(I) : \|x\| \le r_2\}, \;\; r_2 \;=\; \frac{|x_0| + N + 2r_1 T^{\zeta}}{1 - b_3 r_1 T^{\zeta-\eta+1}}$$

and define the operator F_2 as

$$F_2 x(t) = x_0 + \int_0^{T-\tau} \left(h(s, x(s) \cdot I^{\zeta-\eta} y(s))\right) ds - I^{\zeta} y(\tau) + I^{\zeta} y(t).$$

Let $x \in Q_{r_2}$; then, by using assumption (ii), we obtain

$$\begin{aligned}
|F_2 x(t)| &= \left| x_0 + \int_0^{T-\tau} h(s, x(s) \cdot I^{\zeta-\eta} y(s)) ds - I^{\zeta} y(\tau) + I^{\zeta} y(t) \right| \\
&\le |x_0| + \int_0^{T-\tau} |h(s, x(s) \cdot I^{\zeta-\eta} y(s)| ds + I^{\zeta} |y(\tau)| + I^{\zeta} |y(t)| \\
&\le |x_0| + \int_0^{T-\tau} \left(|a_3(s)| + b_3(|x(s) I^{\zeta-\eta} y(s)| \right) ds + 2r_1 I^{\zeta} \\
&\le |x_0| + \int_0^{T-\tau} \left(a_3 + b_3 r_1 r_2 \frac{T^{\zeta-\eta}}{\Gamma(\zeta-\eta+1)} \right) ds + \frac{2r_1 T^{\zeta}}{\Gamma(\zeta+1)} \\
&\le |x_0| + N + \frac{r_1 r_2 b_3 T^{\zeta-\eta}}{\Gamma(\zeta-\eta+1)} + \frac{2r_1 T^{\zeta}}{\Gamma(\zeta+1)}
\end{aligned}$$

and from assumption (iv), we obtain

$$\|F_2 x\| \le |x_0| + N + r_1 r_2 b_3 T^{\zeta-\eta} + 2r_1 T^{\zeta} = r_2.$$

This shows that $\{F_2x\}$ is uniformly bounded on Q_{r_2}. Now, for $x \in Q_{r_2}$ and $t_1,\ t_2 \in I$, where $t_2 > t_1$ and $\mid t_1 - t_2 \mid \leq \delta$, we obtain

$$\begin{aligned}
|F_2x(t_2) - F_2x(t_1)| &= \Big| x_0 + \int_0^{T-\tau} h(s, x(s) \cdot I^{\zeta-\eta}y(s))ds - I^{\zeta}y(\tau) + I^{\zeta}y(t_2) \\
&- \ x_0 + \int_0^{T-\tau} h(s, x(s) \cdot I^{\zeta-\eta}y(s))ds - I^{\zeta}y(\tau) + I^{\zeta}y(t_1) \Big| \\
&\leq \int_0^{t_2} |f\left(s, g_1(s, y(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}y(s))ds\right)|ds \\
&- \int_0^{t_1} |f\left(s, g_1(s, y(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}y(s))ds\right)|ds \\
&\leq \int_{t_1}^{t_2} |f\left(s, g_1(s, y(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}y(s))ds\right)|ds.
\end{aligned}$$

This means that $F_2 : Q_{r_2} \to Q_{r_2}$ and that $\{F_2x\}$ is equi-continuous on Q_{r_2}. From [41], $\{F_2x\}$ is relatively compact. Hence, F_2 is compact.
Assuming that $\{x_n\} \subset Q_{r_2}$, where $x_n \to x$, then,

$$F_2x_n(t) = x_0 + \int_0^{T-\tau} h(s, x_n(s) \cdot I^{\zeta-\eta}y(s))ds - I^{\zeta}y(\tau) + I^{\zeta}y(t)$$

and by passing the limit, we have

$$\lim_{n\to\infty} F_2x_n(t) = \lim_{n\to\infty} \left(x_0 + \int_0^{T-\tau} h(s, x_n(s) \cdot I^{\zeta-\eta}y(s))ds - I^{\zeta}y(\tau) + I^{\zeta}y(t) \right)$$

Applying the Lebesgue dominated convergence Theorem [41], then,

$$\begin{aligned}
\lim_{n\to\infty} F_2x_n(t) &= x_0 + \int_0^{T-\tau} h(s, \lim_{n\to\infty} x(s) \cdot I^{\zeta-\eta}y(s))ds - I^{\zeta}y(\tau) + I^{\zeta}y(t) \\
&= x_0 + \int_0^{T-\tau} h(s, x(s) \cdot I^{\zeta-\eta}y(s))ds - I^{\zeta}y(\tau) + I^{\zeta}y(t) = F_2x(t).
\end{aligned}$$

This means that $F_2x_n(t) \to F_2x(t)$. Therefore, F_2 is continuous. From [41], the solution $x \in C(I)$ of (3) exists. As a result, the solution $x \in C[0, T]$ to Problem (1) and (2) exists. □

3. Uniqueness of the Solution

Consider the next additional hypothesis:

$(i)^*$ f, h and $g_i : I \times R \to R$ are measurable in $t \in I,\ \ \forall x \in R$ and satisfy the Lipschitz condition [43]

$$\begin{aligned}
|f(t,x) - f(t,y)| &\leq b|x-y|, \\
|g_i(t,x) - g_i(t,y)| &\leq b_i|x-y| \\
|h(t,x) - h(t,y)| &\leq b_3|x-y|
\end{aligned}$$

with Lipschitz constants $b,\ b_i,\ b_3 > 0$ and $t \in I,\ x,\ y \in R,\ i = 1,2$.

Remark 1. *From assumption $(i)^*$, we deduce assumption (ii) as follows:*

$$|f(t,x)| \leq |f(t,0)| + b|x|,$$

$$|f(t,x)| \leq a + b|x|, \quad \textit{where} \ \ a = \sup_{t\in I} |f(t,0)|.$$

Also,

$$|g_i(t,x)| \leq |g_i(t,0)| + b_i|x|,$$

$$|g_i(t,x)| \leq a_i + b_i|x|, \quad where \ \ a_i = \sup_{t\in I}|g_i(t,0)|, \quad i = 1,2.$$

and

$$|h(t,x)| \leq |h(t,0)| + b_3|x|,$$

$$|h(t,x)| \leq a_3 + b_3|x|, \quad where \ \ a_3 = \sup_{t\in I}|h(t,0)|.$$

Theorem 2. *Let the hypotheses (i)–(iv) and (i*) be valid. If*

$$(a^*b_2 + bb_2(a_1^* + b_1r_1))T^{2-\gamma} + bb_1(a_2^*T^{2-\zeta} + r_1b_2T^{2-\gamma}) < 1,$$

Hence, the solution to (1) and (2) is unique.

Proof. It is clear that all hypotheses of Theorem 1 are valid, and thus, the solution to (4) exists. Now, assume that y_1, y_2 are two solutions of (4); then,

$$\begin{aligned}
& |y_2(t) - y_1(t)| \\
= & \left| I^{1-\zeta} f\left(t, g_1(t, y_2(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}y_2(s))ds\right)\right. \\
- & \left. I^{1-\zeta} f\left(t, g_1(t, y_1(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}y_1(s))ds\right)\right| \\
= & \left| I^{1-\zeta} f\left(t, g_1(t, y_2(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}y_2(s))ds\right)\right. \\
- & I^{1-\zeta} f\left(t, g_1(t, y_2(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}y_1(s))ds\right) \\
+ & I^{1-\zeta} f\left(t, g_1(t, y_2(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}y_1(s))ds\right) \\
- & \left. I^{1-\zeta} f\left(t, g_1(t, y_1(s)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}y_1(s))ds\right)\right| \\
\leq & \ I^{1-\zeta}|f(t, g_1(t, y_2(t))| \cdot \int_0^{\vartheta(t)} |g_2(s, I^{\zeta-\gamma}y_2(s)) - g_2(s, I^{\zeta-\gamma}y_1(s))|ds \\
+ & \ I^{1-\zeta}\Big(|f(t, g_1(t, y_2(t)) - f(t, g_1(t, y_1(t))|\Big) \cdot \int_0^{\vartheta(t)} |g_2(s, I^{\zeta-\gamma}y_1(s))|ds \\
\leq & \ I^{1-\zeta}(a^* + b(a_1^* + b_1r_1)) \cdot b_2 \int_0^t I^{\zeta-\gamma}|y_2(s) - y_1(s)|ds \\
+ & \ I^{1-\zeta}\left(bb_1|y_2(s) - y_1(s)| \int_0^t (a_2^* + b_2I^{\zeta-\gamma}|y(s)|)ds\right) \\
\leq & \ (a^* + b(a_1^* + b_1r_1)) \cdot b_2I^{2-\gamma}\|y_2 - y_1\| \ + \ bb_1(a_2^*T^{2-\zeta} + r_1b_2T^{2-\gamma})\|y_2 - y_1\|.
\end{aligned}$$

Hence,

$$\|y_2 - y_1\|\left(1 - \left[(a^*b_2 + bb_2(a_1^* + b_1r_1))T^{2-\gamma} + bb_1(a_2^*T^{2-\zeta} + r_1b_2T^{2-\gamma})\right]\right) \leq 0.$$

Since

$$(a^*b_2 + bb_2(a_1^* + b_1r_1))T^{2-\gamma} + bb_1(a_2^*T^{2-\zeta} + r_1b_2T^{2-\gamma}) < 1.$$

Then, the solution of (4) is unique.

Now, for every solution $y \in C(I)$ to (4), there exists a unique solution $x \in C(I)$ of (3). Let $y \in C(I)$ be a solution to (4), and let x_1, x_2 be two solutions to Equation (3); then,

$$\begin{aligned}
|x_2(t) - x_1(t)| &= \Big| x_0 + \int_0^{T-\tau} h\big(s, x_2(s) \cdot I^{\zeta-\eta} y(s)\big) ds - I^{\zeta} y(\tau) + I^{\zeta} y(t) \\
&- \quad x_0 - \int_0^{T-\tau} h\big(s, x_1(s) \cdot I^{\zeta-\eta} y(s)\big) ds + I^{\zeta} y(\tau) - I^{\zeta} y(t) \Big|. \\
&\leq \int_0^{T-\tau} \Big| h\big(s, x_2(s) \cdot I^{\zeta-\eta} y(s)\big) - h\big(s, x_1(s). I^{\zeta-\eta} y(s)\big) \Big| ds \\
&\leq r_1 b_3 \int_0^{T-\tau} |x_2(s) - x_1(s)| \, I^{\zeta-\eta} ds, \\
&\leq r_1 b_3 \|x_2 - x_1\| \frac{T^{\zeta-\eta+1}}{\Gamma(1+\zeta-\eta)},
\end{aligned}$$

from assumption (iv), we obtain

$$\|x_2 - x_1\|(1 - (\frac{r_1 b_3 T^{\zeta-\eta+1}}{\Gamma(1+\zeta-\eta)})) \leq 0.$$

Thus, there is only one solution to (3). As a result, there is only one solution to (1) and (2). □

4. Hyers–Ulam Stability

Definition 1. [44] *Let the solution to (1) and (2) exist. The constrained problem (1) and (2) is Hyers–Ulam-stable if* $\forall \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ *such that, for any solution* $x_s \in C[0, T]$ *of (1) and (2) satisfying*

$$\left| \frac{dx_s}{dt} - f(t, g_1(t, D^{\zeta} x_s(t)) \cdot \int_0^{\vartheta(t)} g_2(s, D^{\gamma} x_s(s))) \right| \leq \delta. \tag{8}$$

Then

$$\|x - x_s\|_c \leq \epsilon.$$

Theorem 3. *Assume that the hypothesis of Theorem 2 is satisfied; then, problem (1) and (2) is Hyers–Ulam-stable.*

Proof. Let the condition of Equation (8) be satisfied; then, we have

$$\begin{aligned}
-\delta &\leq \frac{dx_s(t)}{dt} - f(t, g_1(t, D^{\zeta} x_s(t)) \cdot \int_0^{\vartheta(t)} g_2(s, D^{\gamma} x_s(s)) ds) \leq \delta, \\
-\tfrac{T^{1-\zeta}\delta}{\Gamma(2-\zeta)} &\leq I^{1-\zeta} \frac{dx_s(t)}{dt} - I^{1-\zeta} f(t, g_1(t, D^{\zeta} x_s(t)) \cdot \int_0^{\vartheta(t)} g_2(s, D^{\gamma} x_s(s)) ds) \leq \frac{T^{1-\zeta}\delta}{\Gamma(2-\zeta)}, \\
-\delta_1 &\leq y_s(t) - I^{1-\zeta} f(t, g_1(t, y_s(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y_s(s)) ds) \leq \delta_1.
\end{aligned}$$

Now,

$$
\begin{aligned}
&|y(t) - y_s(t)| = \\
&\left| I^{1-\zeta} f(t, g_1(t, y(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y(s)) ds) - y_s(t) \right. \\
- \quad & I^{1-\zeta} f(t, g_1(t, y_s(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y_s(s)) ds) \\
+ \quad & \left. I^{1-\zeta} f(t, g_1(t, y_s(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y_s(s)) ds) \right| \\
\leq \quad & \left| I^{1-\zeta} f(t, g_1(t, y(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y(s)) ds) \right. \\
- \quad & \left. I^{1-\zeta} f(t, g_1(t, y_s(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y_s(s)) ds) \right| \\
+ \quad & \left| I^{1-\zeta} f(t, g_1(t, y_s(t)) \cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma} y_s(s)) ds) - y_s(t) \right| \\
\leq \quad & I^{1-\zeta} \left| f(t, g_1(t, y(t)) \right| \cdot \int_0^{\vartheta(t)} |g_2(s, I^{\zeta-\gamma} y(s)) - g_2(s, I^{\zeta-\gamma} y_s(s)))| ds \\
+ \quad & I^{1-\zeta} \left| f(t, g_1(t, y(t)) - f(t, g_1(t, y_s(t)) \right| \cdot \int_0^{\vartheta(t)} |g_2(s, I^{\zeta-\gamma} y_s(s))| ds + \delta_1 \\
\leq \quad & I^{1-\zeta} (a^* + b(a_1^* + b_1 r_1)) \cdot b_2 \int_0^t I^{\zeta-\gamma} |y(s) - y_s(s)| ds. \\
+ \quad & I^{1-\zeta} b b_1 |y(s) - y_s(s)| \cdot \int_0^t (a_2^* + b_2 I^{\zeta-\gamma} |y_s|) ds + \delta_1 \\
\leq \quad & (a^* + b(a_1^* + b_1 r_1)) \cdot b_2 T^{2-\gamma} \|y - y_s\| + b_1 b(a_2^* T^{2-\zeta} + r_1 . b_2 T^{2-\gamma}) \|y - y_s\| + \delta_1.
\end{aligned}
$$

Hence,

$$
\|y - y_s\| \left(1 - \left[(a^* + b(a_1^* + b_1 r_1)) \cdot b_2 T^{2-\gamma} + b_1 b(a_2^* T^{2-\zeta} + r_1 \cdot b_2 T^{2-\gamma}) \right] \right) \leq \delta_1
$$

and

$$
\|y - y_s\| \leq \frac{\delta_1}{1 - \left[(a^* + b(a_1^* + b_1 r_1)) \cdot b_2 T^{2-\gamma} + b_1 b(a_2^* T^{2-\zeta} + r_1 \cdot b_2 T^{2-\gamma}) \right]}.
$$

Since

$$
(a^* + b(a_1^* + b_1 r_1)) \cdot b_2 T^{2-\gamma} + b_1 b(a_2^* T^{2-\zeta} + r_1 \cdot b_2 T^{2-\gamma}) < 1,
$$

then

$$
\|y - y_s\| < \epsilon.
$$

Also, using assumption (iv), we obtain

$$
\begin{aligned}
|x(t)-x_s(t)| &= \Big| x_0 + \int_0^{T-\tau} h(s, x(s)\cdot I^{\zeta-\eta}y(s))ds - I^{\zeta}y(\tau) + I^{\zeta}y(t) \\
&- \quad x_0 - \int_0^{T-\tau} h(s, x_s(s)\cdot I^{\zeta-\eta}y_s(s))ds + I^{\zeta}y_s(\tau) - I^{\zeta}y_s(t)\Big| \\
&\leq \int_0^{T-\tau} \big|h(s, x(s)\cdot I^{\zeta-\eta}y(s)) - h(s, x_s(s)\cdot I^{\zeta-\eta}y_s(s))\big|ds + 2I^{\zeta}\|y-y_s\| \\
&\leq b_3\int_0^{T-\tau} (|x(s)-x_s(s)|\, I^{\zeta-\eta}|y(s)| + I^{\zeta-\eta}|y(s)-y_s(s)|\,|x_s|)ds + \frac{2T^{\zeta}}{\Gamma(\zeta+1)}\epsilon \\
&\leq \frac{r_1 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)}\|x-x_s\| + \frac{r_2 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)}\epsilon + \frac{2T^{\zeta}}{\Gamma(\zeta+1)}\epsilon,
\end{aligned}
$$

$$
\|x-x^*\| \leq \frac{\left(\frac{r_2 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)} + \frac{2T^{\zeta}}{\Gamma(\zeta+1)}\right)\epsilon}{1-\left(\frac{r_1 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)}\right)}.
$$

Since

$$
\frac{r_1 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)} < 1,
$$

thus,

$$
\|x-x^*\| \leq \epsilon.
$$

Then, the problem (1) and (2) is Hyers–Ulam-stable. □

5. Continuous Dependence

Definition 2. *The solution to (1) and (2) depends continuously on $y = D^{\zeta}x$, h and x_0, and if $\forall\, \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ such that*

$$
\max\{\|y-\check{y}\|, \|h-\check{h}\|, |x_0-\check{x}_0| \leq \delta\} \;\Rightarrow \|x-\check{x}\| \leq \epsilon,
$$

where $\check{x}$ and $\check{y}$ are the solutions to

$$
\begin{aligned}
\check{x}(t) &= \check{x}_0 + \int_0^{T-\tau} \check{h}(s, \check{x}(s)\cdot I^{\zeta-\eta}\check{y}(s))ds - I^{\zeta}\check{y}(\tau) + I^{\zeta}\check{y}(t), &(9)\\
\check{y}(t) &= I^{1-\zeta} f\Big(t, g_1(t,\check{y}(t))\cdot \int_0^{\vartheta(t)} g_2(s, I^{\zeta-\gamma}\check{y}(s))ds\Big). &(10)
\end{aligned}
$$

Theorem 4. *Suppose that the hypotheses of Theorem 2 are satisfied; then, the solution to (1) and (2) depends continuously on y, h, and x_0.*

Proof. If $x(t)$ and $\check{x}(t)$ are the solutions to (3) and (9), respectively, using assumption $(i)^*$, we obtain

$$\begin{aligned}
& |x(t)-\check{x}(t)| \\
= & \left| x_0 + \int_0^{T-\tau} h(s, x(s)\cdot I^{\zeta-\eta}y(s))ds - I^{\zeta}y(\tau) + I^{\zeta}y(t) \right. \\
- & \left. \check{x}_0 - \int_0^{T-\tau} \check{h}(s, \check{x}(s) I^{\zeta-\eta}\check{y}(s))ds + I^{\zeta}\check{y}(\tau) - I^{\zeta}\check{y}(t) \right| \\
\leq & |x-\check{x}_0| + \left| \int_0^{T-\tau} \left(h(s, x(s)\cdot I^{\zeta-\eta}y(s)) - \check{h}(s, \check{x}(s)\cdot I^{\zeta-\eta}\check{y}(s))ds \right) ds \right. \\
+ & \left. I^{\zeta}(y(\tau)-\check{y}(\tau)) + I^{\zeta}(y(t)-\check{y}(t)) \right| \\
\leq & |x-\check{x}_0| + \int_0^{T-\tau} \left| h(s, x(s)\cdot I^{\zeta-\eta}y(s)) - h(s, \check{x}(s)\cdot I^{\zeta-\eta}\check{y}(s)) \right. \\
+ & \left. h(s, \check{x}(s)\cdot I^{\zeta-\eta}\check{y}(s)) - \check{h}(s, \check{x}(s)\cdot I^{\zeta-\eta}\check{y}(s)) \right| ds + 2I^{\zeta}\|y-\check{y}\| \\
\leq & |x-\check{x}_0| + b_3 \int_0^{T-\tau} |x(s)I^{\zeta-\eta}y(s) - \check{x}(s)I^{\zeta-\eta}\check{y}(s)|ds \\
+ & b_3 \int_0^{T-\tau} \|h-\check{h}\|ds + 2I^{\zeta}\|y-\check{y}\| \\
\leq & |x-\check{x}_0| + b_3 \int_0^{T-\tau} |x(s)I^{\zeta-\eta}y(s) - x(s)I^{\zeta-\eta}\check{y}(s) \\
+ & x(s)I^{\zeta-\eta}\check{y}(s) - \check{x}(s)I^{\zeta-\eta}\check{y}(s)|ds + b_3\|h-\check{h}\|T + \|y-\check{y}\|\frac{2T^{\zeta}}{\Gamma(\zeta+1)} \\
\leq & \delta + b_3 \int_0^{T-\tau} |x(s)|I^{\zeta-\eta}|y(s)-\check{y}(s)|ds \\
+ & b_3 \int_0^{T-\tau} |x(s)-\check{x}(s)|I^{\zeta-\eta}|\check{y}(s)|ds + b_3 T\delta + \frac{2T^{\zeta}\delta}{\Gamma(\zeta+1)} \\
\leq & \delta + b_3\, r_2\|y-\check{y}\|\frac{T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)} + b_3\, r_1\|x-\check{x}\|\frac{T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)} + b_3\, T\delta + \frac{2\delta T^{\zeta}}{\Gamma(\zeta+1)} \\
\leq & \delta + b_3\, r_2\delta\frac{T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)} + b_3\, r_1\|x-\check{x}\|\frac{T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)} + b_3\, T\delta + \frac{2\delta T^{\zeta}}{\Gamma(\zeta+1)}.
\end{aligned}$$

Hence,

$$\|x-\check{x}\|(1-\frac{r_1 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)}) \leq \left(1+\frac{r_2 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)} + \frac{2T^{\zeta}}{\Gamma(\zeta+1)} + b_3 T\right)\delta$$

and

$$\|x-\check{x}\| = \frac{\left(1+\frac{r_2 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)} + \frac{2T^{\zeta}}{\Gamma(\zeta+1)} + b_3 T\right)\delta}{1-\frac{r_1 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)}} = \epsilon.$$

Since $\frac{r_1 b_3 T^{\zeta-\eta+1}}{\Gamma(\zeta-\eta+1)} < 1$, therefore, the solution to (3) depends continuously on y, h, x_0. Consequently, the solution $x \in C[0, T]$ of (1) and (2) depends continuously on y, h, x_0. □

6. Special Cases and Examples

Corollary 1. *Let the hypothesis of Theorem 1 be valid; if we put $\tau = T$ in (2), then the backward problem*

$$\frac{dx}{dt} = f(t, g_1(t, D^{\zeta}x(t)) \int_0^{\vartheta(t)} g_2(s, D^{\gamma}x(s)ds)), \;\; \zeta, \gamma \in (0,1), \; t \in (0,T],$$

$$x(T) = x_0,$$

has a solution $x \in C[0,T]$. Consequently, if the hypotheses of Theorem 2 are valid, it has a unique solution $x \in C[0,T]$.

Corollary 2. *Let the hypothesis of Corollary 1 be valid. If $\tau = T$, $\gamma = 1 - \zeta$, then the backward problem*

$$\frac{dx}{dt} = f(t, g_1(t, D^{\zeta}x(t)) \int_0^{\vartheta(t)} g_2(s, D^{1-\zeta}x(s)ds)), \;\; \zeta \in (\frac{1}{2}, 1), \; t \in (0,T],$$

$$x(T) = x_0.$$

has a solution $x \in C[0,T]$. Consequently, if the hypotheses of Theorem 2 are valid, it has a unique solution $x \in C[0,T]$.

Example 1. *Consider the next problem,*

$$\frac{dx}{dt} \;=\; \frac{1}{2}(\frac{e^{-t}}{1+e^{-t}}) + \frac{1}{8}(\frac{t^2}{2} + \frac{1}{6}D^{\frac{1}{2}}x(t)) \cdot \int_0^{\rho t} (\frac{s^3}{4} + \frac{1}{3}D^{\frac{1}{2}}x(s))ds, \;\; t \in (0,1], \quad (11)$$

$$x(\tau) \;=\; \frac{1}{4} + \int_0^{1-\tau} (\frac{sins}{6} + \frac{1}{2}x(s) \cdot D^{\frac{1}{2}}x(s))ds, \quad (12)$$

where

$$\zeta = \eta = \gamma = \frac{1}{2}, \;\; \rho \in (0,1), \;\; x(0) = \frac{1}{4}.$$

Then

$$f\left(t, g_1(t, D^{\zeta}x(t)) \cdot \int_0^{\vartheta(t)} g_2(s, D^{\gamma}x(s)ds)\right)$$
$$= \frac{1}{2}(\frac{e^{-t}}{1+e^{-t}}) + \frac{1}{8}\left(\frac{t^2}{2} + \frac{1}{6}D^{\frac{1}{2}}x(t)\right) \cdot \int_0^{\rho t} (\frac{s^3}{4} + \frac{1}{3}D^{\frac{1}{2}}x(s))ds.$$

Set

$$g_1(t, D^{\zeta}x(t)) \;=\; \frac{t^2}{2} + \frac{1}{6}D^{\frac{1}{2}}x(t)$$
$$g_2(t, D^{\gamma}x(t)) \;=\; \frac{t^3}{4} + \frac{1}{3}D^{\frac{1}{2}}x(t)$$
$$h(t, x(t) \cdot D^{\eta}x(t)) \;=\; \frac{sint}{6} + \frac{1}{2}x(t) \cdot D^{\frac{1}{2}}x(t).$$

Here, we have

$$a^* = a_1^* = \frac{1}{2}, \; a_2^* = \frac{1}{4}, \; b = \frac{1}{8}, \; b_1 = \frac{1}{6}, \; b_2 = \frac{1}{3}, \; b_3 = \frac{1}{2}$$
$$N = \frac{1}{6}, \; r_1 \approx 0.2, \; and \; r_2 \approx 0.9.$$

It is obvious that all the hypotheses of Theorem 1 are valid. Hence there exists at least one solution $x \in [0,1]$ of (15)–(12). Moreover, we have

$$a^* b_2 + bb_2(a_1^* + b_1 r_1)T^{2-\gamma} + bb_1(a_2^* T^{2-\zeta} + r_1 b_2 T^{2-\gamma}) = 0.1950 < 1.$$

Thus, all the hypotheses of Theorem 2 are valid, so the solution of Problem (15)–(12) is unique.

Example 2. *Consider the problem*

$$\frac{dx}{dt} = \frac{1}{4}e^{-t^2}cos(2t) + \frac{1}{3}\left(\frac{1}{3}(\frac{1}{5-t} + D^{\frac{1}{3}}x(t)) \cdot \int_0^{\frac{1}{2}t} \frac{1}{5}(\frac{e^{-s}}{s+2} + D^{\frac{1}{4}}x(s))ds\right) \quad t \in [0,1], \tag{13}$$

$$x(\tau) = \frac{1}{5} + \int_0^{1-\tau} \left(\frac{1}{18}s^2(sin(2s+1)) + \frac{1}{6}x(s)D^{\frac{1}{5}}x(s)\right)ds, \tag{14}$$

where

$$\zeta = \frac{1}{3},\ \eta = \frac{1}{5},\ \gamma = \frac{1}{4},\ t \in (0,1],\ x(0) = \frac{1}{5}.$$

Then,

$$\begin{aligned} & f(t, g_1(t, D^{\zeta}x(t)) \int_0^{\vartheta(t)} g_2(s, D^{\gamma}x(s)ds)) \\ = \ & \frac{1}{4}e^{-t^2}cos(2t) + \frac{1}{3}(\frac{1}{3}\frac{1}{5-t} + D^{\frac{1}{3}}x(t)) \cdot \int_0^{\frac{1}{2}t} \frac{1}{5}(\frac{e^{-s}}{s+2} + D^{\frac{1}{4}}x(s))ds. \end{aligned}$$

Set

$$\begin{aligned} g_1(t, D^{\zeta}x(t)) &= \frac{1}{3}(\frac{1}{5-t} + D^{\frac{1}{3}}x(t)) \\ g_2(t, D^{\gamma}x(t)) &= \frac{1}{5}(\frac{e^{-t}}{t+2} + D^{\frac{1}{4}}x(t)) \\ h(t, x(t) \cdot D^{\eta}x(t)) &= \frac{1}{6}(\frac{1}{3}t^2(sin(2t+1)) + x(t)D^{\frac{1}{5}}x(t)). \end{aligned}$$

Here, we have

$$\begin{aligned} & a^* = \frac{1}{4},\ a_1^* = \frac{1}{12},\ a_2^* = \frac{1}{10},\ b = b_1 = \frac{1}{3},\ b_2 = \frac{1}{5},\ b_3 = \frac{1}{6}, \\ & N = \frac{1}{18},\ r_1 = 0.3,\ r_2 = 0.05. \end{aligned}$$

It is obvious that all the hypotheses of Theorem 1 are valid. Hence the solution $x \in [0,T]$ of (13) and (14) exists. Moreover, we have

$$a^* b_2 + bb_2(a^* + b_1 r_1)t^{2-\gamma} + bb_1(a_2^* t^{2-\gamma} + r_1 b_2 t^{2-\gamma}) = 0.2314 < 1,$$

Thus, all the hypotheses of Theorem 2 are valid, and then the solution to (13)–(14) is unique.

Example 3. *Consider the next problem*

$$\frac{dx}{dt} = \frac{1}{4}(\frac{t}{t^3+1}) + \frac{1}{3}(\frac{7+3t}{16} + \frac{\ln(1+|D^{\frac{1}{5}}x(t)|)}{5t+7}) \cdot \int_0^{t^4} (\frac{1}{9-s} + \frac{(D^{\frac{1}{7}}x(s))^2}{6(1+|D^{\frac{1}{7}}x(s)|)})ds, \quad t \in (0, \frac{1}{3}], \tag{15}$$

$$x(\tau) = \frac{1}{4} + \int_0^{\frac{1}{3}-\tau} (\frac{s^2}{s^2+1} + \frac{\ln(1+|x(s) \cdot D^{\frac{1}{4}} x(s)|)}{8+s^2}) ds. \quad (16)$$

Here, we have

$$x_0 = \frac{1}{4}, \zeta = \frac{1}{5}, \eta = \frac{1}{7}, \gamma = \frac{1}{4}, \vartheta(t) = t^4,$$

and

$$f\left(t, g_1(t, D^\zeta x(t)) \cdot \int_0^{\vartheta(t)} g_2(s, D^\eta x(s) ds)\right)$$
$$= \frac{1}{4}(\frac{t}{t^3+1}) + \frac{1}{3}(\frac{7+3t}{16} + \frac{\ln(1+|D^{\frac{1}{5}} x(t)|)}{5t+7}) \cdot \int_0^{t^4} (\frac{1}{9-s} + \frac{(D^{\frac{1}{7}} x(s))^2}{6(1+|D^{\frac{1}{7}} x(s)|)}) ds.$$

Set

$$\begin{aligned}
\vartheta(t) &= t^4, \\
g_1(t, D^\zeta x(t)) &= \frac{7+3t}{16} + \frac{\ln(1+|D^{\frac{1}{5}} x(t)|)}{5t+7}, \\
g_2(t, D^\gamma x(t)) &= \frac{1}{9-s} + \frac{(D^{\frac{1}{7}} x(s))^2}{6(1+|D^{\frac{1}{7}} x(s)|)}, \\
h(t, x(t) \cdot D^\eta x(t)) &= \frac{t^2}{t^2+1} + \frac{\ln(1+|x(t) \cdot D^{\frac{1}{4}} x(t)|)}{8+t^2}.
\end{aligned}$$

Thus, we obtain

$$a^* = \frac{1}{12}, a_1^* = \frac{1}{2}, a_2^* = \frac{3}{26}, N = \frac{1}{9}, b = \frac{1}{3}, b_1 = \frac{1}{7}, b_2 = \frac{1}{6}, b_3 = \frac{1}{8},$$
$$r_1 \approx 0.03, \text{ and } r_2 \approx 0.42.$$

It is clear that all assumptions of Theorem 1 are satisfied. Hence, there exists at least one solution $x \in [0, T]$ of (12)–(15). Moreover, we have

$$a^* b_2 + b b_2 (a_1^* + b_1 r_1) t^{2-\gamma} + b b_1 (a_2^* t^{2-\gamma} + r_1 b_2 t^{2-\gamma}) = 0.043 < 1.$$

Thus, all assumptions of Theorem 2 are satisfied, and then the solution of the problem (12)–(15) is unique.

7. Conclusions

In this manuscript, we considered the constrained problem of the fractional-order integro-differential equation (1) under the quadratic functional integro-differential constraint (2). We proved the existence of solutions to the problem (1) and (2). The sufficient conditions for the uniqueness of the solution have been presented. The Hyers–Ulam stability of the problem (1) and (2) has been analyzed. The continuous dependence of the unique solution on its fractional-order derivative $D^\zeta x(t)$, the parameter x_0, and the function h has been studied. We introduced several examples and special cases.

Author Contributions: A.M.A.E.-S., Formal analysis, Supervision; A.A.A.A., Methodology; E.M.A.H., Formal analysis; H.R.E., Methodology. All authors have read and agreed to the published version of this manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Baleanu, D.; Etemad, S.; Rezapour, S. On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators. *Alex. Eng. J.* **2020**, *59*, 3019–3027. [CrossRef]
2. Banaś, J.; Zajac, T. A new approach to the theory of functional integral equations of fractional order. *J. Math. Anal. Appl.* **2011**, *375*, 375–387. [CrossRef]
3. Boucherif, A.; Precup, R. On the nonlocal initial value problem for first order differential equations. *Fixed Point Theory* **2003**, *4*, 205–212.
4. Caponetto, R. Fractional order systems. In *Modeling and Control Applications*; World Scientific: Singapore, 2010, Volume 72.
5. Caputo, M. Linear model of dissipation whose Q is almost frequency independent-II. *Geophys. J. R. Astr. Soc.* **1967**, *13*, 529–539. [CrossRef]
6. Kazemi, M.; Yaghoobnia, A.R. Application of fixed point theorem to solvability of functional stochastic integral equations. *Appl. Math. Comput.* **2022**, *417*, 126759. [CrossRef]
7. Monje, C.A.; Chen, Y.; Vinagre, B.M.; Xue, D.; Feliu-Batlle, V. *Fractional-Order Systems and Controls: Fundamentals and Applications*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2010.
8. Podlubny, I. Fractional-order systems and fractional-order controllers. *Inst. Exp. Phys. Slovak Acad. Sci. Kosice* **1994**, *12.3*, 1–18.
9. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Integrals and Derivatives of the Fractional Order and Some of Their Applications*; Nauka i Tehnika: Minsh, Belarus, 1987. (In Russian English Translation)
10. Schneider, W.R.; Wyss, W. Fractional diffusion and wave equations. *J. Math. Phys.* **1989**, *30*, 134. [CrossRef]
11. El-Sayed, A.M.A.; Alrashdi, M.A.H. On a functional integral equation with constraint existence of solution and continuous dependence. *Int. J. Differ. Eq. Appl.* **2019**, *18*, 37–48.
12. Al-Issa, S. M.; El-Sayed, A.M.A; Hashem, H.H.G. An Outlook on Hybrid Fractional Modeling of a Heat Controller with Multi-Valued Feedback Control. *Fractal Fract.* **2023**, *7*, 759. [CrossRef]
13. Nasertayoob, P. Solvability and asymptotic stability of a class of nonlinear functional-integral equation with feedback control. *Commun. Nonlinear Anal.* **2018**, *5*, 19–27.
14. El-Sayed, A.M.; Ba-Ali, M.M.; Hamdallah, E.M. An Investigation of a Nonlinear Delay Functional Equation with a Quadratic Functional Integral Constraint. *Mathematics* **2023**, *11*, 4475. [CrossRef]
15. De la Sen, M.; Alonso-Quesada, S. Control issues for the Beverton–Holt equation in ecology by locally monitoring the environment carrying capacity: Non-adaptive and adaptive cases. *Appl. Math. Comput.* **2009**, *215*, 2616–2633. [CrossRef]
16. Rezapour, S.; Etemad, S.; Agarwal, R.P.; Nonlaopon, K. On a Lyapunov-Type Inequality for Control of a y-Model Thermostat and the Existence of Its Solutions. *Mathematics* **2022**, *10*, 4023. [CrossRef]
17. Zhao, K. Stability of a nonlinear Langevin system of ML-Type fractional derivative affected by time-varying delays and differential feedback control. *Fractal Fract.* **2022**, *6*, 725. [CrossRef]
18. El-Sayed, A.M.A.; Hamdallah, E.A.; Ahmed, R.G. On a nonlinear constrained problem of a nonlinear functional integral equation. *Appl. Anal. Optim.* **2022**, *6*, 95–107.
19. Cosentino, C.; Bates, D. *Feedback Control in Systems Biology*; CRC Press: Boca Raton, FL, USA, 2011.
20. Cowan, N.J.; Ankarali, M.M.; Dyhr, J.P.; Madhav, M.S.; Roth, E.; Sefati, S.; Sponberg, S.; Stamper, S.A.; Fortune, E.S.; Daniel, T.L. Feedback control as a framework for understanding tradeoffs in biology. *Am. Zool.* **2014**, *54*, 223–237. [CrossRef] [PubMed]
21. Del Vecchio, D.; Dy, A. J.; Qian, Y. Control theory meets synthetic biology. *J. R. Soc. Interface* **2016** , *13*, 20160380. [CrossRef] [PubMed]
22. Peiffer, S.; Dan, T.B.B.; Frevert, T.; Cavari, B.Z. Survival of *E. coli* and enterococci in sediment-water systems of Lake Kinneret under (Feedback) controlled concentrations of hydrogen sulfide. *Water Res.* **1988**, *22*, 233–240. [CrossRef]
23. De Leenheer, P.; Smith, H. Feedback control for chemostat models. *J. Math. Biol.* **2003**, *46*, 48–70. [CrossRef]
24. Marshall, S.H.; Smith, D.R. Feedback control and the distribution of prime numbers. *Math. Mag.* **2013**, *86*, 189–203. [CrossRef]
25. Stewart, G.E.; Dimitry, M.G.; Dumont, G.A. Feedback controller design for a spatially distributed system: The paper machine problem. *IEEE Trans. Control. Syst. Technol.* **2003**, *11.5*, 612–628. [CrossRef]
26. Chen, F.; Lin, F.; Chen, X. Sufficient conditions for the existence positive periodic solutions of a class of neutral delay models with feedback control. *Appl. Math. Comput.* **2004**, *158*, 45–68. [CrossRef]
27. Matveeva, I. Exponential stability of solutions to nonlinear time-varying delay systems of neutral type equations with periodic coefficients. *Electron. J. Differ. Eq.* **2020**, *20*, 1–12. [CrossRef]
28. Saker, S.H. Periodic solutions, oscillation and attractivity of discrete nonlinear delay population model. *Math. Comput. Model.* **2008**, *47*, 278–297. [CrossRef]
29. Yang, Z. Positive periodic solutions of a class of single-species neutral models with state-dependent delay and feedback control. *Eur. J. Appl. Math.* **2006**, *17*, 735–757. [CrossRef]
30. Nasertayoob, P.; Vaezpour, S.M. Positive periodic solution for a nonlinear neutral delay population equation with feedback control. *J. Nonlinear Sci. Appl.* **2013**, *6*, 152–161. [CrossRef]
31. Ahmed, E.; El-Sayed, A.M.A.; El-Mesiry, A.E.M.; El-Saka, H.A.A. Numerical solution for the fractional replicator equation. *Int. J. Mod. Phys. C* **2005**, *16*, 1017–1026. [CrossRef]
32. Banaś, J.; Caballero, J.; Rocha, J.; Sadarangani, K. Monotonic solutions of a class of quadratic integral equations of Volterra type. *Comput. Math. Appl.* **2005**, *49*, 943–952. [CrossRef]

33. El-Sayed, A.M.A.; Gaafar, F.M. Fractional calculus and some intermediate physical processes. *J. A. Math.Comp.* **2033**, *144*, 117–126. [CrossRef]
34. Failla, G.; Zingales, M. Advanced materials modelling via fractional calculus: Challenges and perspectives. *Phil. Trans. R. Soc.* **2020**, *378*, 20200050. [CrossRef] [PubMed]
35. Sumelka, W. Fractional viscoplasticity. *Mech. Res. Commun.* **2014**, *56*, 31–36. [CrossRef]
36. Bader, R.; Papageorgiou, N.S. Nonlinear multivalued boundary value problems. *Discuss. Math. Differ. Inclusions Control. Optim.* **2001**, 21, 127–148. [CrossRef]
37. Debnath, P.; Srivastava, H.M.; Kumam, P.; Hazarika, B. *Fixed Point Theory and Fractional Calculus*; Springer Nature: Singapore, 2022.
38. Gasiński, L.; Papageorgiou, N.S. Nonlinear second-order multivalued boundary value problems. *Proc. Indian Acad. Sci. (Math. Sci.)* **2003**, *113*, 293–319. [CrossRef]
39. Kamenskii, M.; Obukhovskii, V.; Petrosyan, G.; Yao, J. Existence and approximation of solutions to nonlocal boundary value problems for fractional differential inclusions. *Fixed Point Theory Appl.* **2019**, *2019*, 2. [CrossRef]
40. Tidke, H.L.; Kharat, V.V.; More, G.N. Some results on nonlinear mixed fractional integro differential equations. *J. Adv. Math. Stud.* **2022**, *15*, 274–287.
41. Curtain, R.F.; Pritchard, A.J. *Functional Analysis in Modern Applied Mathematics*; Academic Press: Cambridge, MA, USA, 1977.
42. Dunford, N.; Schwartz, J.T. *Linear Operators, (Part 1), General Theory*; NewYork Interscience: New York, NY, USA, 1957.
43. Cobzaş, Ş.; Miculescu, R.; Nicolae, A. *Lipschitz Functions*; Springer International Publishing: Berlin/Heidelberg, Germany, 2019.
44. Jung, S.M. *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2011; Volume 48.